Jump Diffusion Model

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Why Jump Diffusion?

- Real asset prices exhibit sudden jumps (news, crises, earnings reports).
- The Black-Scholes model assumes continuous price paths $\Rightarrow$ unrealistic in practice.
- Jump diffusion models introduce **discrete jumps** on top of Brownian motion.
- Widely used in option pricing, credit risk, insurance, and cryptocurrencies.

Mathematical Formulation

Jump Diffusion SDE:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + Jump$$

- μ: expected return (drift)
- σ: volatility (diffusion)
- W_t: Brownian motion
- N_t: Poisson process with intensity λ

Jump Diffusion Model

Step 1: Start with Continuous Model (GBM)

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- The relative jump size is given by $(J - 1)$, hence:

$$S_t - S_{t^-} = S_{t^-}(J - 1)$$

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Step 3: Combine Continuous and Jump Components

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_{t^-}(J - 1)dN_t$$
$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (J - 1)dN_t$$

Jump Diffusion Model

Jump can be divided into two parts

$$\text{Jump} = \begin{cases} \text{Arrival} & \sim \text{Poisson}(\lambda) \\ \text{Size} & \sim \text{Log-Normal}(\mu_J, \sigma_J^2) \end{cases}$$

Consider expected value of $(J-1)dN_t$

$$\begin{aligned} \mathbb{E}[(J-1)dN_t] &= \mathbb{E}[(J-1)]E[dN_t] \\ &= (e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1)(\lambda dt) \end{aligned}$$

Adding this part, it changes the drift. so, we subtract it with the expected value of jump for making the jump to be pure-jump

Jump Diffusion Model

Drift adjusted equation:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sigma dW_t + (J-1)dN_t - (e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1)(\lambda dt) \\ &= (\mu - A)dt + \sigma dW_t + (J-1)dN_t\end{aligned}$$

Note that: $\mathrm{A} = (e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1)(\lambda)$

When time interval is not too small, more than 1 jump can occur.

price jump from $S_t \Rightarrow S_t(J_1 J_2 J_3 \cdots J_{dN_t})$

$$\begin{aligned}dS_t &= S_t(\mu - A)dt + \sigma S_t dW_t + (J_1 J_2 J_3 \cdots J_{dN_t} - 1)S_t dN_t \\ &= (\mu - A)S_t dt + \sigma dW_t + (\prod_{i=1}^{dN_t} J_i - 1)dN_t\end{aligned}$$

Jump model in Option Pricing

From Ito's lemma

$$df = \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2} dS_t^2 + \underbrace{\left(f\left(S_t \prod_{i=1}^{dN_t} J_i\right) - f(S)\right)}_{\text{Jump contribution to total change in } f(S_t)}$$

Substitute $f(S) = ln(S)$

$$d(ln(S_t)) = \frac{dS_t}{S_t} - \frac{dS_t^2}{2S_t^2} + ln(\prod_{i=1}^{dN_t} J_i)$$

Jump model in Option Pricing

Compute the derivative in the equation, From Ito's product rule we obtain

$$\begin{aligned} dS_t^2 = (dS_t)^2 &= (\mu S_t dt + \sigma S_t dW_t)^2 \\ &= \sigma^2 S_t^2 dt \end{aligned}$$

note that

- $dt^2 = 0$
- $dt\, dW = 0$
- $(dW)^2 = dt$

$$d(ln(S_t)) = (\mu - A)dt + \sigma dW_t - \frac{\sigma^2 dt}{2} + ln(\prod_{i=1}^{dN_t} J_i)$$

Jump model in Option Pricing

$$d(ln(S_t)) = (\mu - A - \frac{\sigma^2}{2})dt + \sigma dW_t + ln(\prod_{i=1}^{dN_t} J_i)$$

$$ln(S_t) - ln(S_0) = (\mu - A - \frac{\sigma^2}{2})t + \sigma(W_t - W_0) + ln(\prod_{i=1}^{N_t - N_0} J_i)$$

$$= (\mu - A - \frac{\sigma^2}{2})t + \sigma(W_t) + ln(\prod_{i=1}^{N_t} J_i)$$

Jump model in Option Pricing

$$\sigma W_t \sim \mathcal{N}\left(0, \sigma^2 t\right)$$

$$\sum_{i=1}^{n} \ln J_i \sim \mathcal{N}\left(n\mu_j, n\sigma_j^2\right)$$

$$\sigma W_t + \sum_{i=1}^{n} \ln J_i \sim \mathcal{N}\left(0 + n\mu_j, \sigma^2 t + n\sigma_j^2\right)$$

Jump model in Option Pricing

$$ln(S_t) - ln(S_0) = (\mu - A - \frac{\sigma^2}{2})t + n\mu_j + \sqrt{\sigma^2 + \frac{n\sigma_j^2}{t}}\sqrt{t}Z$$

$$= (\mu - A - \frac{\sigma^2}{2})t + n\mu_j + \sqrt{\sigma^2 + \frac{n\sigma_j^2}{t}}W_t$$

$ln(S_t) - ln(S_0) = (\mu - \frac{\sigma^2}{2})t + \sigma W_t$ Geometric Brownian motion Sol

$$\text{Let } \sqrt{\sigma^2 + \frac{n\sigma_j^2}{t}} = \sigma_n \Rightarrow \frac{\sigma^2}{2} = \frac{\sigma_n^2}{2} - \frac{n\sigma_j^2}{2t}$$

$$ln(S_t) - ln(S_0) = (\mu - A - (\frac{\sigma^2}{2} + \frac{n\sigma_j^2}{2t}) + \frac{n\sigma_j^2}{2t})t + n\mu_j + \sqrt{\sigma^2 + \frac{n\sigma_j^2}{t}}W_t$$

Jump model in Option Pricing

$$ln(S_t) - ln(S_0) = (-At + \frac{n\sigma_j^2}{2t} + n\mu_j) + (\mu - \frac{\sigma_n^2}{2}) + \sigma_n W_t$$

$$S_t = S_0(e^{(-At + \frac{n\sigma_j^2}{2t} + n\mu_j) + (\mu - \frac{\sigma_n^2}{2}) + \sigma_n W_t})$$

Let $S_0^{(n)}$ be $S_0(e^{(-At + \frac{n\sigma_j^2}{2t} + n\mu_j)}) \Rightarrow S_t = S_0^{(n)} e^{(\mu - \frac{\sigma_n^2}{2}) + \sigma_n W_t}$

$$S_t = S_0^{(n)} e^{(r - \frac{\sigma_n^2}{2}) + \sigma_n W_t^Q}$$; Girsanov's Theorem

$$C\left(S_0^{(n)}, T\right) = \sum_{n=0}^{\infty} C\left(S_0^{(n)}, T \mid N_T = n\right) \mathbb{P}[N_T = n]$$

$$= \sum_{n=0}^{\infty} C\left(S_0^{(n)}, T \mid N_T = n\right) \left(\frac{(\lambda T)^n}{n!} e^{-\lambda T}\right)$$

Comparison: Black-Scholes vs. Jump Diffusion

Feature	**Black-Scholes**	**Jump Diffusion**
Price Path	Continuous	Discontinuous (Jumps)
Randomness	Brownian Motion	Brownian + Poisson
Extreme Events	Not captured	Captured via jumps

Example Simulation

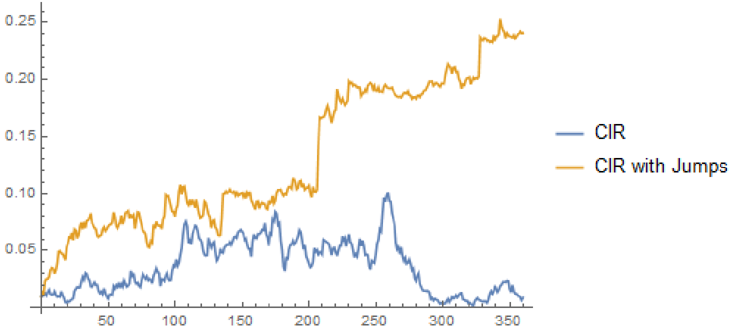

Figure 1.

Key Takeaways

- Jump Diffusion combines continuous motion (Brownian) and discrete jumps (Poisson).
- Useful for modeling sudden price changes in financial markets.
- Flexible: captures volatility smiles and tail risks.
- Foundation for advanced models like Bates, Kou, and stochastic volatility with jumps.

Thank You

QnA

Appendix: Girsanov Theorem

Girsanov's Theorem allows us to change the probability measure (from real-world $\mathbb{P}$ to risk-neutral $\mathbb{Q}$) so that we can use no-arbitrage pricing.

$$\text{Under } \mathbb{P} : dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}}$$
$$\text{Then under } \mathbb{Q} : dS_t = r S_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}}$$
$$\text{Where:} dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \theta \, dt \text{with} \theta = \frac{\mu - r}{\sigma} \text{being the market price of risk.}$$