Automatic control exercises

Matteo Cocetti

May 13, 2018

1 Instructions

In the following section two exercises are proposed. Only ONE needs to be solved for the final project, so please pick the one that you prefer. Each exercise is divided into FOUR questions, each of which is worth a point. Thus, the maximum mark of a correct exercise is FOUR points. The maximum grade for the Automatic Control course is 30/30 and the exercise is worth four of these points. The resolution of one exercise is COMPULSORY to REGISTER the automatic control exam AND to be admitted to the ORAL examination. The exercise should be sent at matteo.cocetti@unitn.it the first day of each month. Please provide a small report (1-2 pages) explaining what you did and attach the MATLAB code. Academic credit awarded to an individual should represent the work of that individual. Therefore, students are expected to produce THEIR OWN original work. Collaboration or assistance is NOT permitted. The citation of all sources is required. Failure to do so is dishonest and is the basis for a charge of cheating, plagiarism, or unauthorized assistance. Such charges are subject to severe disciplinary action and to a final mark of zero.

2 Exercises

2.1 Exercise 1 - The vibration absorber

The vibration absorber [1][Ch. 9.11], also called dynamic vibration absorber, is a mechanical device which is used to reduce or eliminate unwanted vibration. It consists of a mass and a spring attached to the main (or original) mass that needs to be protected from vibration. Vibration absorbers are commonly used in a variety of applications which include sanders, saws, internal combustion engines and high-voltage transmission lines. A sketch of the system is pictured in Figure 1.

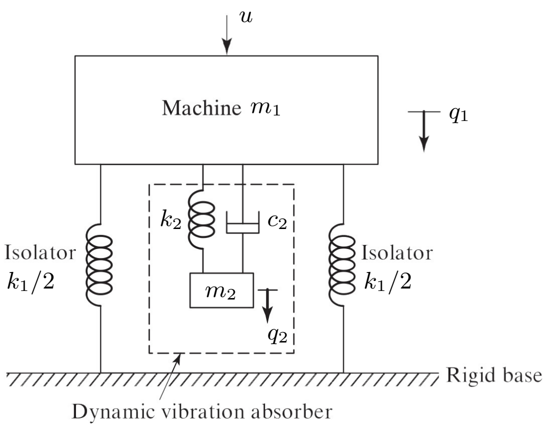

Figure 1: A graphical representation of the harmonic absorber.

The equations of motion for this two-degrees of freedom system can easily obtained using a Lagrange formulation and can be conveniently written in the following matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}. \tag{1}$$

Here m_1 and m_2 are the masses, c_1 and c_2 are the viscous damping coefficients, k_1 and k_2 are the stiffnesses associated to he springs. To describe the motion of the system we use q_1 and q_2 as generalized coordinates representing the relative displacements of the masses with respect to an equilibrium position. We also consider the presence of an input u representing a disturbance force acting on the system. This force can represent the effect of a rotating unbalanced mass, the force of the wind, or any other force disturbance that you might think of.

For this particular setup we are interested in studying the damping effect of the dynamic vibration absorber over the machine, thus we define as measured output of the system the position of mass m_1, i.e., $y = q_1$.

Because the equations of motions (1) are linear we can represent the system in the following state space form

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{2}$$

where $x \in \mathbb{R}^4$ is the state of the system comprising generalized positions and velocities and $A \in \mathbb{R}^{4\times4}$, $B \in \mathbb{R}^{4\times1}$ and $C \in \mathbb{R}^{1\times4}$ are properly defined matrices. To (2) we can associate a transfer function of the following form

$$G(s) := \frac{y(s)}{u(s)} = C(sI - A)^{-1}B, \tag{3}$$

where $s \in \mathbb{C}$ represents a complex variable in the Laplace domain. We can verify that the LTI system (10) with $u = 0$ is globally exponentially stable and satisfies the following Lyapunov inequality,

$$A^\top P + PA < 0, \tag{4}$$

for some $0 < P = P^\top \in \mathbb{R}^{4\times4}$. Suppose now that $u \neq 0$ then we immediately realize that in general the machine position q_1 does not converges anymore to zero but we could still be interested in understanding how a disturbance force u affects the machine position. One possible way to look at the problem is to estimate the $\mathcal{L}_2$ gain of (2). This $\mathcal{L}_2$ gain intuitively represents how much the $\mathcal{L}_2$ norm of an input disturbance signal u is amplified by the dynamics of the system. Formally $u \in \mathcal{L}_2$ if the following functional norm is well defined and has finite value

$$||u||_{\mathcal{L}_2} := \left(\int_0^\infty u(t)^\top u(t)\ \mathrm{d}t \right)^{\frac{1}{2}} < \infty. \tag{5}$$

This $\mathcal{L}_2$ norm can be interpreted as a sort of energy of the signal. A dynamical system of the form (2) takes the signal u as input and provides a signal y as output, and these signals in general may have different $\mathcal{L}_2$ norms. The minimum value $\gamma \in \mathbb{R}_{\geq 0}$ such that

$$||y||_{\mathcal{L}_2} \leq \gamma ||u||_{\mathcal{L}_2}, \tag{6}$$

for any $u \in \mathcal{L}_2$ is called the $\mathcal{L}_2$ gain of the system. The $\mathcal{L}_2$ for (10) can be estimated numerically solving the following convex optimization problem (LMI),

$$\min_{P,\gamma} \gamma \quad \text{subject to:}$$

$$P = P^\top > 0 \tag{7a}$$

$$\begin{bmatrix} A^\top P + PA & PB & C^\top \\ B^\top P & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} < 0. \tag{7b}$$

For this exercise please write a brief report (1-2 pages) answering and commenting the following questions:

1. Provide a state space representation (C, A, B) for the equations of motions in (1).
2. Setting $u = 0$, please check that (2) is globally exponentially stable through an eigenvalue test and solve numerically the Lyapunov inequality (4) providing a numerical value for $P = P^\top > 0$.
3. Use the LMI formulation in (7) to estimate numerically the $\mathcal{L}_2$ gain γ of the system (2).

4. Plot the Bode diagram of the transfer matrix (3) and try to find a relationship between the obtained $\mathcal{L}_2$ gain of the system and the peak value.

For the numerical part please use the following values $m_1 = 100$ kg, $m_2 = 15$ kg, $c_2 = 30$ Ns/m, $k_1 = 15$ kN/m and $k_2 = 2$ kN/m.

2.2 Exercise 2 - Sallen-Key low pass filter

The Sallen-Key filter, also known as a voltage control voltage source (VCVS), was first introduced in 1955 by R. P. Sallen and E. L. Key of MIT's Lincoln Labs [2] and still is one of the most widely used filter topology.

The main reason for his popularity is that the performance of this configuration depend very little on the characteristics of the operational amplifier. Therefore for a given operational amplifier, one can design a higher frequency filter than with other topologies. Another advantage of this configuration is that the ratio of the largest resistor value to the smallest resistor value, and the ratio of the largest capacitor value to the smallest capacitor value (component spread) are low, which is beneficial for manufacturability. The topology of the Sallen-Key filter in low pass configuration is pictured in Figure (2). Under the standard hypothesis of ideal operational amplifier we can easily

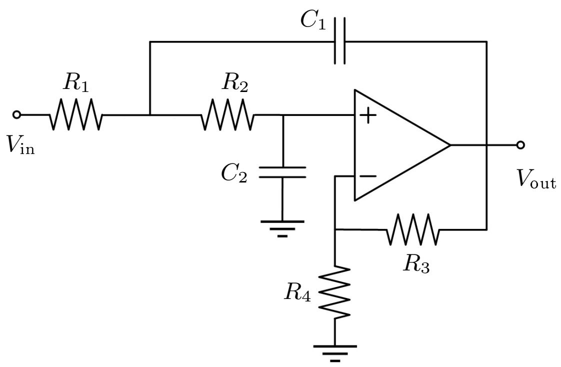

Figure 2: Low pass configuration of the Sallen-Key filter.

derive the following transfer function between the input voltage $V_{\text{in}} \in \mathbb{R}$ and the output voltage $V_{\text{out}} \in \mathbb{R}$,

$$G(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{k}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2} := \frac{y(s)}{u(s)}, \tag{8}$$

where $\omega_n \in \mathbb{R}_{\geq 0}$ is the cutoff pulsation of the filter, $Q \in \mathbb{R}_{\geq 0}$ is called quality factor and $s \in \mathbb{C}$ represents the complex variable in the Laplace domain.

The parameters of the filter are related to resistances and capacities as follows

$$\begin{cases} k = \dfrac{R_4 + R_3}{R_1 R_2 R_4 C_1 C_2} \\ Q = \dfrac{R_4\sqrt{R_1 R_2 \frac{C_1}{C_2}}}{R_1 R_4 + R_2 R_4 - R_1 R_3 \frac{C_1}{C_2}} \\ \omega_n = \dfrac{1}{\sqrt{R_1 R_2 C_1 C_2}} \end{cases} \tag{9}$$

The transfer function (8) can be represented in state space form as follows

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{10}$$

where $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$, $C \in \mathbb{R}^{1\times 2}$ are properly defined matrices. The linear system (10) is globally exponentially stable and for $u = 0$ satisfies the following Lyapunov inequality,

$$A^\top P + PA < 0, \tag{11}$$

for some $P = P^\top > 0$. Suppose now that $u \neq 0$, then we immediately realize that in general the output voltage $V_{\text{out}} = y$ cannot converge to zero. However we are interested in understanding how a input voltage $V_{\text{in}} = u$ is amplified by the filter. One possible way to look at the problem is to estimate the $\mathcal{L}_2$ gain of (10). This $\mathcal{L}_2$ gain intuitively represents how much the $\mathcal{L}_2$ norm of an input signal u is amplified by the dynamics of the system. Formally $u \in \mathcal{L}_2$ if the following functional norm is well defined and has finite value

$$||u||_{\mathcal{L}_2} := \left(\int_0^\infty u(t)^\top u(t) \, \mathrm{d}t \right)^{\frac{1}{2}} < \infty. \tag{12}$$

This $\mathcal{L}_2$ norm can be interpreted as a sort of energy of the signal. A dynamical system of the form (10) takes the signal u as input and provides a signal y as output, and these signals in general may have different $\mathcal{L}_2$ norms. The minimum value $\gamma \in \mathbb{R}_{\geq 0}$ such that

$$||y||_{\mathcal{L}_2} \leq \gamma ||u||_{\mathcal{L}_2}, \tag{13}$$

for any $u \in \mathcal{L}_2$ is called the $\mathcal{L}_2$ gain of the system. The $\mathcal{L}_2$ for (10) can be estimated numerically solving the following convex optimization problem (LMI),

$$\begin{aligned}
&\min_{P,\gamma} \gamma \quad \text{subject to:} \\
&P = P^\top > 0 && \text{(14a)} \\
&\begin{bmatrix} A^\top P + PA & PB & C^\top \\ B^\top P & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} < 0. && \text{(14b)}
\end{aligned}$$

For this exercise please write a brief report (1-2 pages) answering and commenting the following questions:

1. Provide a state space representation (C, A, B) for the transfer function (8).
2. Set $u = 0$ and check that (10) is globally exponentially stable through an eigenvalue test and solving the Lyapunov inequality (4). Provide a numerical value for matrix $P = P^\top > 0$ in (11). (Hint: The canonical observability and controllability realization are very fragile from a numerical point of view and are not recommended for LMIs optimizations. For this reason I suggest you to convert your representation into a balanced one using the Matlab command balreal).
3. Use the LMI formulation in (14) to estimate numerically the $\mathcal{L}_2$ gain γ of the system (10).
4. Study how γ changes as a function of R_3, and provide a plot of γ vs. R_3.

For the numerical part please use the following values $R_1 = R_2 = 1$ kΩ, $C_1 = C_2 = 0.1$ μF, $R_3 = 0$ Ω and $R_4 = 10$ kΩ. Regarding point 4., please consider values for $R_3 \in [0, 1]$ kΩ.

References

[1] Singiresu S Rao and Fook Fah Yap. *Mechanical vibrations*, volume 4. Prentice Hall Upper Saddle River, 2011.

[2] Roy Pines Sallen and Edwin L Key. A practical method of designing rc active filters. *IRE Transactions on Circuit Theory*, 2(1):74–85, 1955.