Question 1

Part (a)

$$A = \begin{bmatrix} 0.03 & 21.2 \\ 2.18 & -0.30 \end{bmatrix} \quad b = \begin{bmatrix} 21.5 \\ 21.5 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - \frac{2.18}{0.03} R_1 \quad A = \begin{bmatrix} 0.03 & 21.2 \\ 0 & -1540.833333 \end{bmatrix} \quad b = \begin{bmatrix} 21.5 \\ -1540.833333 \end{bmatrix}$$

Now use backward substitution

$$-1540.833333\, x_2 = -1540.833333$$
$$x_2 = 1$$

$$0.03x_1 + 21.2 \times 1 = 21.5$$
$$0.03x_1 = 0.3$$
$$x_1 = 10 \quad \text{Therefore } x = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

Part (b)

$$R_2 \leftarrow R_2 - \frac{2.18\times10^0}{3.00\times10^{-2}} R_1 \quad A = \begin{bmatrix} 3.00\times10^{-2} & 2.12\times10^1 \\ 0 & -1.54\times10^3 \end{bmatrix} \quad b = \begin{bmatrix} 2.15\times10^1 \\ -1.54\times10^3 \end{bmatrix}$$

$$-1.54\times10^3 x_2 = -1.54\times10^3$$
$$x_2 = 1.00\times10^0$$

$$3.00\times10^{-2} x_1 + 2.12\times10^1 x_2 = 2.15\times10^1$$
$$3.00\times10^{-2} x_1 = 2.15\times10^1 - 2.12\times10^1$$
$$3.00\times10^{-2} x_1 = 3.00\times10^{-1}$$
$$x_1 = 1.00\times10^1$$

Part (c)

$$A = \begin{bmatrix} 2.18 & -0.30 \\ 0.03 & 21.2 \end{bmatrix} \qquad b = \begin{bmatrix} 21.5 \\ 21.5 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - \frac{3.00\times10^{-2}}{2.18\times10^{0}} R_1$$

$$A = \begin{bmatrix} 2.18\times10^{0} & -3.00\times10^{-1} \\ 0 & 2.12\times10^{1} - 1.37\times10^{-2} \times -3.00\times10^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 2.18\times10^{0} & -3.00\times10^{-1} \\ 0 & 2.12\times10^{1} \end{bmatrix}$$

$$2.12\times10^{1} + 4.11\times10^{-3} = 2.12\times10^{1}$$

$$b = \begin{bmatrix} 2.15\times10^{1} \\ 2.15\times10^{1} - 1.37\times10^{-2}\times 2.15\times10^{1} \end{bmatrix}$$

$$= \begin{bmatrix} 2.15\times10^{1} \\ 2.12\times10^{1} \end{bmatrix}$$

$$2.12\times10^{1}\, x_2 = 2.12\times10^{1}$$
$$x_2 = 1.00\times10^{0}$$

$$2.18\times10^{0} x_1 + (-3.00\times10^{-1} x_2) = 2.15\times10^{1}$$
$$2.18\times10^{0} x_1 = 2.15\times10^{1} + 3.00\times10^{-1}$$
$$2.18\times10^{0} x_1 = 2.18\times10^{1}$$
$$x_1 = 1.00\times10^{1}$$

Both close to part (a)

Question 2

Part (a)

proof: $\|A\|_\infty = \max\limits_{\underline{x} \neq 0} \dfrac{\|Ax\|_\infty}{\|x\|_\infty}$ (1)

Suppose $Ax = b$, then $b_i = \sum\limits_{k=1}^{n} a_{ik} x_k$

$$\|b\|_\infty = \max_i |b_i| = \max_i \left| \sum_{k=1}^{n} a_{ik} x_k \right|$$

$$\|Ax\|_\infty = \|b\|_\infty = \max_i \left| \sum_{k=1}^{n} a_{ik} x_k \right| \leq \max_i \sum_{k=1}^{n} |a_{ik}| |x_k|$$

$$\leq \|x\|_\infty \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$(1) \rightarrow \max_{\underline{x} \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_{\underline{x} \neq 0} \frac{\|x\|_\infty \max_i \sum_{j=1}^{n} |a_{ij}|}{\|x\|_\infty}$$

$$= \max_i \sum_{j=1}^{n} |a_{ij}|$$

Part (b)

Claim: AB^{-1} is also well-conditioned.

$$\begin{aligned} cond(AB^{-1}) &= \|AB^{-1}\| \cdot \|BA^{-1}\| \\ &\leq \|A\| \cdot \|B^{-1}\| \cdot \|B\| \cdot \|A^{-1}\| \\ &= \|A\| \cdot \|A^{-1}\| \cdot \|B\| \cdot \|B^{-1}\| \\ &= cond(A) \cdot cond(B) \end{aligned}$$

The condition number of AB^{-1} is bounded by the product of cond (A) and cond (B), which are both small. Therefore the cond (AB^{-1}) is also small $\Rightarrow$ well-conditioned.

Part (c)

Question 3

Part (a)

proof: ① $\forall \underline{x} \in \mathbb{R}^n$, $\|x\|_A = (x^TAx)^{\frac{1}{2}} \geq 0$

since A is positive definite, $x^TAx > 0$, so $(x^TAx)^{\frac{1}{2}} > 0$

Also $\|x\|_A = 0$ iff $\underline{x} = 0$ since $\underline{0}^TA\underline{0} = 0$

② $\forall \underline{x} \in \mathbb{R}^n$, $c \in \mathbb{R}$

$$\|c\underline{x}\|_A = (cx^TAcx)^{\frac{1}{2}} = (c^2x^TAx)^{\frac{1}{2}} = |c| \cdot (x^TAx)^{\frac{1}{2}} = |c| \cdot \|x\|_A$$

③ $\forall x, y \in \mathbb{R}^n$, $\|x+y\|_A = [(x+y)^TA(x+y)]^{\frac{1}{2}}$

Question 4

(b) The reason behind these inaccurate results lies on the limitation of python's floating point system. The 0.999... in u and v has more than 16 digits, this will cause the value got rounded to even in binary, which introduce rounding error that leads to inaccurate result.

(c) proof: ① Suppose $A-B$ is rank 1

then $A-B = xy^T$

Notice $A-xy^T = A-(A-B) = B$

$$(A-xy^T)^{-1} = A^{-1}+A^{-1}x(1-y^TA^{-1}x)^{-1}y^TA^{-1}$$

$$B^{-1} = A^{-1}+A^{-1}x(1-y^TA^{-1}x)^{-1}y^TA^{-1}$$

$$A^{-1}-B^{-1} = \underbrace{-(1-y^TA^{-1}x)^{-1}}_{\text{constant}}A^{-1}xy^TA^{-1}$$

Now let $u = -(1-y^TA^{-1}x)^{-1}A^{-1}x$

$v^T = [(A^{-1})^Ty]^T = y^TA^{-1}$

$u: n\times 1$

$A^{-1}: n\times n$ $x: n\times 1$

$A^{-1}: n\times n$ $y^T: 1\times n$

$v^T: 1\times n$ $v: n\times 1$

Then $A^{-1}-B^{-1} = uv^T$

Both u, v are column vector.

$A^{-1}-B^{-1}$ is rank 1

② Suppose $A^{-1}-B^{-1}$ is rank 1, then $A^{-1}-B^{-1} = xy^T$

$$B^{-1}-A^{-1} = -xy^T$$

$$B^{-1} = A^{-1}-xy^T$$

$$B = (A^{-1}-xy^T)^{-1}$$

$$B = A+Ax(1-y^TAx)^{-1}y^TA$$

Similar to ①, with some substitution we can get $A-B=uv^T$ for some column vector u, v. $A-B$ is rank 1

Question 5

proof: Want to show $HH^T = I$ by definition of orthogonal matrix.

$$HH^T = \left(I - 2\frac{v \cdot v^T}{v^T \cdot v}\right)\left(I - 2\frac{v \cdot v^T}{v^T v}\right)^T$$

$$= \left(I - 2\frac{vv^T}{v^T v}\right)\left(I^T - 2\left(\frac{vv^T}{v^T v}\right)^T\right)$$

$$= \left(I - 2\frac{vv^T}{v^T v}\right)\left(I - 2\frac{vv^T}{v^T v}\right)$$

$$= I - 2\frac{vv^T}{v^T v} - 2\frac{vv^T}{v^T v} + 4\frac{vv^T vv^T}{(v^T v)^2}$$ $v^T v$ is constant

$$= I - 4\frac{vv^T}{v^T v} + 4\frac{vv^T}{v^T v}$$

$$= I$$