Question 1

Part (a)

1. $\text{Abs C.N.} = \frac{1}{|f_1'(x)|} = \frac{1}{\left|\frac{\cos(0)}{100}\right|} = \frac{1}{\frac{1}{100}} = 100$

2. $\text{Abs C.N.} = \frac{1}{|f_2'(2)|} = \frac{1}{|18-20+8|} = \frac{1}{4}$

$f_2'(x) = 2x^3 - 10x + 8$

3. $\text{Abs. C.N} = \frac{1}{|f_3'(0)|} = \frac{1}{|2\cdot 0^2|}$ => undefined, mutiple roots.

Part (b)

1. $\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|} = \frac{|10^{-2n-2}|}{|10^{-2n}|^r} = 10^{-2} = c$ for $r=1$ => linear

2. $\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|} = \frac{|2^{-(n+1)^2}|}{|2^{-n^2}|^r}$

We have $x_1 = 2^{-1}$ $x_2 = 2^{-4}$ $x_3 = 2^{-9}$ $x_4 = 2^{-16}$

This is not quadratic but superlinear. If quadratic, $2^{-1} \to 2^{-2} \to 2^{-4} \ldots$

3. $x_1 = 2^{-\log 1} = 2^0$ $x_2 = 2^{-2\log 2}$ $x_3 = 2^{-3\log 3}$ $x_4 = 2^{-4\log 4}$

$\times 2^{-0.602}$ $\times 2^{-0.829}$ $\times 2^{-0.977}$

No constant covergence rate, this is something else.

Question 2

part (b)

Suppose we start with integers a, b. Since every interval is half the size of the previous, one of the endpoint will get one more decimal places per iteration. Therefore the minimum number of interval bisection is 10 iterations.

Question 3

Part (b) g_1, g_2, g_3 are all smooth (polynomials)

1. $g_1'(x) = \frac{2}{5}x$ since $|g_1'(1)| = \frac{2}{5} < 1$, the fixed point iteration will converge to 1.

2. $g_2'(x) = \frac{5}{2\sqrt{5x-4}}$ since $|g_2'(1)| = \frac{5}{2} > 1$, it will not converge to 1.

3. $g_3'(x) = \frac{4}{x^2}$ since $|g_3'(1)| = 4 > 1$, it will not converge to 1

Part (c)

1. The iteration still converges. The approximate convergence rate is linear.

2. The iteration converges. The approximate convergence rate is still linear.

3. The iteration converges. The approximate convergence rate is linear.

Question 4

Part (c)

1. $h_1(x) = (x-2)(x-5)(x-1)$ $h_1'(x) = 3x^2 - 16x + 17$

$$x_{k+1} = x_k - \frac{h(x_k)}{h_1'(x_k)} = x_k - \frac{(x_k-2)(x_k-5)(x_k-1)}{3x_k^2 - 16x_k + 17}$$

$x = 3.0$ $\left|\frac{h_1(x) \cdot h_1''(x)}{h_1'(x)^2}\right| = \left|\frac{-8}{16}\right| = \frac{1}{2} < 1$

It will converge.

2. $h_2(x) = x\cos(\pi x) - x$ $\quad h_2'(x) = \cos(\pi x) - \pi x \sin(\pi x) - 1$

$h_2''(x) = -\pi^2 x \cos(\pi x) - 2\pi \sin(\pi x)$

$$x_{k+1} = x_k - \frac{x_k \cos(\pi x_k) - x_k}{\cos(\pi x_k) - \pi x_k \sin(\pi x_k) - 1}$$

$$\left| \frac{h_2(x) \cdot h_2''(x)}{h_2'(x)^2} \right| \approx 0.511 < 1$$ so it will converge.

at $x = 2.1$

3. $h_3(x) = e^{-2x+4} + e^{x-2} - x$ $\quad h_3'(x) = -2e^{-2x+4} + e^{x-2} - 1$

$h_3''(x) = 4e^{-2x+4} + e^{x-2}$

$$x_{k+1} = x_k - \frac{h_3(x_k)}{h_3'(x_k)} = x_k - \frac{e^{-2x_k+4} + e^{x_k-2} - x_k}{-2e^{-2x_k+4} + e^{x_k-2} - 1}$$

$$\left| \frac{h_3(x) \cdot h_3''(x)}{h_3'(x)^2} \right| \approx 0.3285 < 1$$ so it will converge.

at $x = 2.1$

Question 5

Part (c)

Secant Method: $x_{k+1} = x_k - f(x_k) \dfrac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$

$$= \frac{x_k f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})} - \frac{f(x_k) x_k - f(x_k) x_{k-1}}{f(x_k) - f(x_{k-1})}$$

$$= \frac{x_{k-1} f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$ as needed.

Part (d)

Advantage: ① Avoid potential cancellation (when x_k close to x_{k-1})

Disadvantage: ① Need to evaluate f more. (More computational heavy)

Question 7

Part (a)

$$\nabla f(\underline{x}) = \begin{bmatrix} 8x_0^3 - 2x_0 x_1^2 - 2x_0 \\ 12x_1^3 - 2x_0^2 x_1 - 8x_1 \end{bmatrix}$$

Part (b)

$$H_f(\underline{x}) = \begin{bmatrix} 24x_0^2 - 2x_1^2 - 2 & -4x_0 x_1 \\ -4x_0 x_1 & 36x_1^2 - 2x_0^2 - 8 \end{bmatrix}$$

Outputs for Q3 partC

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g1 [3, 2.6, 2.152, 1.7262208, 1.3959676500705283, 1.1897451360086866, 1.0830986977312658, 1.0346205578054328, 1.014087939726725, 1.0056748698998388, 1.0022763887896116]
g2 [3, 3.3166247903554, 3.547269929364975, 3.7062581732557267, 3.8119930307227263, 3.880717092705114, 3.9247401778366897, 3.9526827458301597, 3.970316577950781, 3.981404637782237, 3.
g3 [3, 3.666666666666667, 3.909090909090909, 3.9767441860465116, 3.9941520467836256, 3.998535871156662, 3.9996338337605275, 3.9999084500595075, 3.999977111991028, 3.9999942779650155,
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