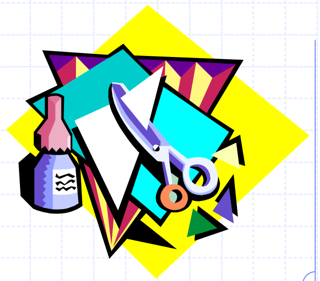
Union-Find Partition Structures

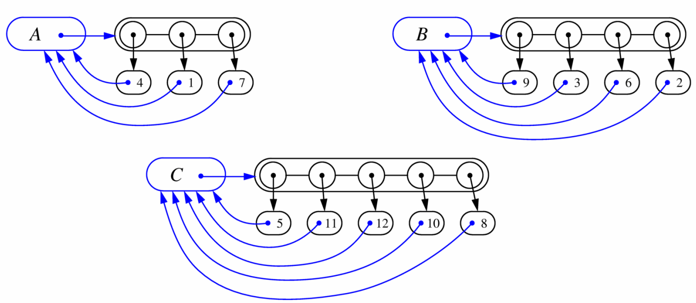


Partitions with Union-Find Operations

- makeSet(x): Create a singleton set containing the element x and return the position storing x in this set
- union(A,B): Return the set A U B, destroying the old A and B
- find(p): Return the set containing the element at position p

List-based Implementation

- Each set is stored in a sequence represented with a linked-list
- Each node should store an object containing the element and a reference to the set name

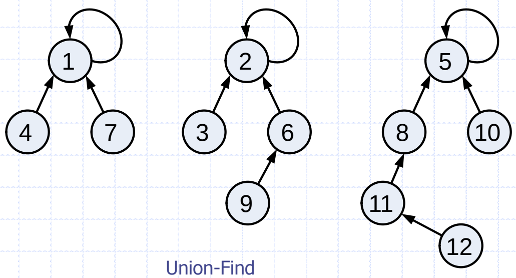


Analysis of List-based Representation

- When doing a union, always move elements from the smaller set to the larger set
  - Each time an element is moved it goes to a set of size at least double its old set
  - Thus, an element can be moved at most O(log n) times
- Total time needed to do n unions and finds is O(n log n).

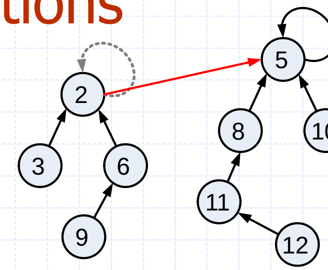
Tree-based Implementation

- Each element is stored in a node, which contains a pointer to a set name
- A node v whose set pointer points back to v is also a set name
- Each set is a tree, rooted at a node with a self-referencing set pointer
- For example: The sets “1”, “2”, and “5”:

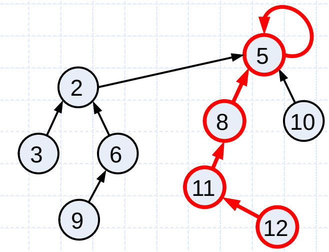


Union-Find Operations

- To do a union, simply make the root of one tree point to the root of the other

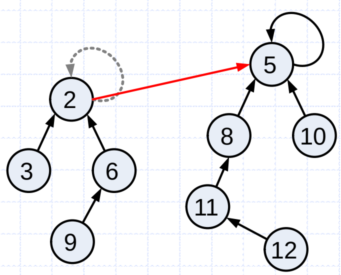


- To do a find, follow set-name pointers from the starting node until reaching a node whose set-name pointer refers back to itself



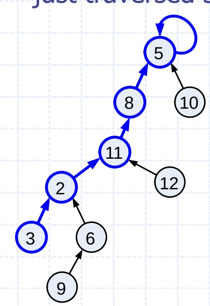
Union-Find Heuristic 1

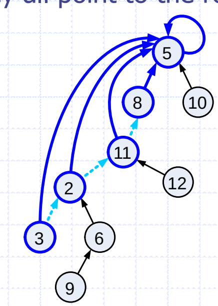
- Union by size:
  - When performing a union, make the root of smaller tree point to the root of the larger
- Implies O(n log n) time for performing n union-find operations:
  - Each time we follow a pointer, we are going to a subtree of size at least double the size of the previous subtree
  - Thus, we will follow at most O(log n) pointers for any find.



Union-Find Heuristic 2

- Path compression:
  - After performing a find, compress all the pointers on the path just traversed so that they all point to the root





- Implies O(n $\log^* n$) time for performing n union-find operations:
  - Proof is somewhat involved… (and not in the book)

Proof of log* n Amortized Time

- For each node v that is a root
  - define n(v) to be the size of the subtree rooted at v (including v)
  - identified a set with the root of its associated tree.
- We update the size field of v each time a set is unioned into v. Thus, if v is not a root, then n(v) is the largest the subtree rooted at v can be, which occurs just before we union v into some other node whose size is at least as large as v 's.
- For any node v, then, define the rank of v, which we denote as r (v), as r (v) = [log n(v)]:
- Thus, $n(v) \geq 2^{r(v)}$.
- Also, since there are at most n nodes in the tree of v, r (v) = [log n], for each node v.

Proof of log* n Amortized Time (2)

- For each node v with parent w:
  - r (v) > r (w)
- Claim: There are at most $n/ 2^s$ nodes of rank s.
- Proof:
  - Since r (v) < r (w), for any node v with parent w, ranks are monotonically increasing as we follow parent pointers up any tree.
  - Thus, if r (v) = r (w) for two nodes v and w, then the nodes counted in n(v) must be separate and distinct from the nodes counted in n(w).
  - If a node v is of rank s, then $n(v) \geq 2^s$.
  - Therefore, since there are at most n nodes total, there can be at most $n/ 2^s$ that are of rank s.

Proof of log* n Amortized Time (3)

- Definition: Tower of two's function:
  - $t(i) = 2^{t(i-1)}$
- Nodes v and u are in the same rank group g if
  - g = log*(r(v)) = log*(r(u)):
- Since the largest rank is log n, the largest rank group is
  - log*(log n) = (log* n) - 1

Proof of log* n Amortized Time (4)

- Charge 1 cyber-dollar per pointer hop during a find:
  - If w is the root or if w is in a different rank group than v, then charge the find operation one cyber-dollar.
  - Otherwise (w is not a root and v and w are in the same rank group), charge the node v one cyber-dollar.
- Since there are most (log* n)-1 rank groups, this rule guarantees that any find operation is charged at most log* n cyber-dollars.

Proof of log* n Amortized Time (5)

- After we charge a node v then v will get a new parent, which is a node higher up in v 's tree.
- The rank of v 's new parent will be greater than the rank of v 's old parent w.
- Thus, any node v can be charged at most the number of different ranks that are in v 's rank group.
- If v is in rank group g > 0, then v can be charged at most t(g)-t(g-1) times before v has a parent in a higher rank group (and from that point on, v will never be charged again). In other words, the total number, C, of cyber-dollars that can ever be charged to nodes can be bounded by

$$C \le \sum_{g=1}^{\log^* n - 1} n(g) \cdot (t(g) - t(g-1))$$

Proof of log* n Amortized Time (end)

- Bounding $n(g)$:

$$\begin{aligned} n(g) &\le \sum_{s=t(g-1)+1}^{t(g)} \frac{n}{2^s} \\ &= \frac{n}{2^{t(g-1)+1}} \sum_{s=0}^{t(g)-t(g-1)-1} \frac{1}{2^s} \\ &< \frac{n}{2^{t(g-1)+1}} \cdot 2 \\ &= \frac{n}{2^{t(g-1)}} \\ &= \frac{n}{t(g)} \end{aligned}$$

- Returning to C:

$$\begin{aligned} C &< \sum_{g=1}^{\log^* n - 1} \frac{n}{t(g)} \cdot (t(g) - t(g-1)) \\ &\le \sum_{g=1}^{\log^* n - 1} \frac{n}{t(g)} \cdot t(g) \\ &= \sum_{g=1}^{\log^* n - 1} n \\ &\le n \log^* n \end{aligned}$$