Algorithms and Data Structures

Graph Traversal

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HS 2012/13

Outline

Definitions

DFS Depth First Search

Applications of DFS

BFS Breath First Search

Applications of BFS

DFS vs. BFS

Decorator Pattern for Graphs

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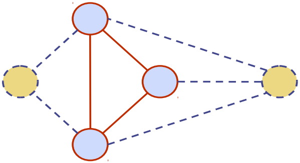
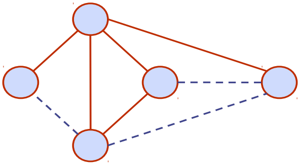
Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



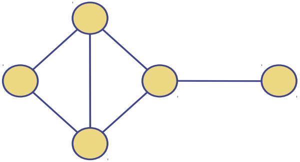
Subgraph



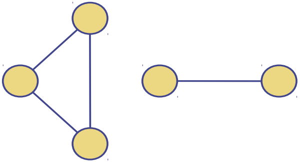
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



Connected graph

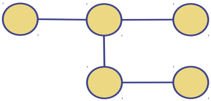


Non connected graph with two connected components

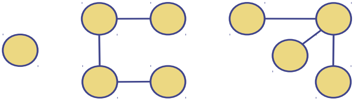
Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles

  This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



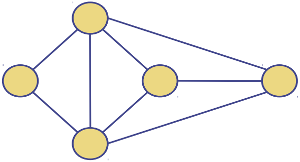
Tree



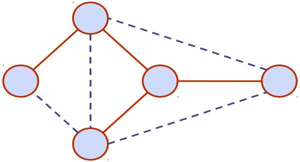
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

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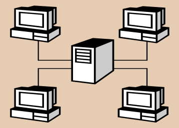
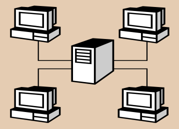
Depth-First Search



- Depth-first search (DFS) is a general technique for traversing a graph
- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- DFS on a graph with n vertices and m edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph
- Depth-first search is to graphs what Euler tour is to binary trees



DFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

Algorithm ***DFS(G)***
Input graph *G*
Output labeling of the edges of *G* as discovery edges and back edges
for all $u \in G.vertices()$
  $setLabel(u, UNEXPLORED)$
for all $e \in G.edges()$
  $setLabel(e, UNEXPLORED)$
for all $v \in G.vertices()$
  if $getLabel(v) = UNEXPLORED$
    $DFS(G, v)$

Algorithm ***DFS(G, v)***
Input graph *G* and a start vertex *v* of *G*
Output labeling of the edges of *G* in the connected component of *v* as discovery edges and back edges
$setLabel(v, VISITED)$
for all $e \in G.incidentEdges(v)$
  if $getLabel(e) = UNEXPLORED$
    $w \leftarrow G.opposite(v,e)$
    if $getLabel(w) = UNEXPLORED$
      $setLabel(e, DISCOVERY)$
      $DFS(G, w)$
    else
      $setLabel(e, BACK)$

Example

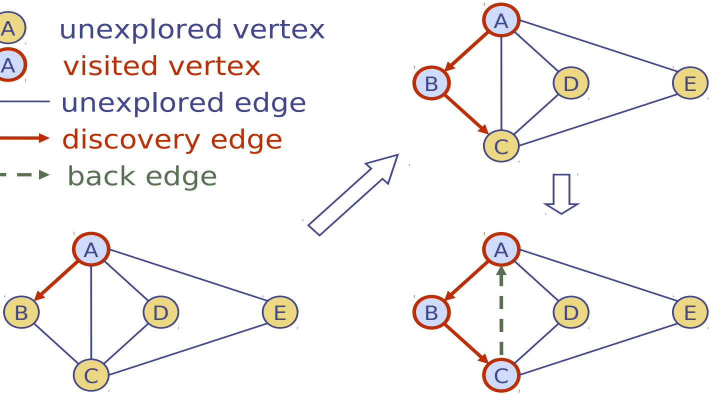
A unexplored vertex

A visited vertex

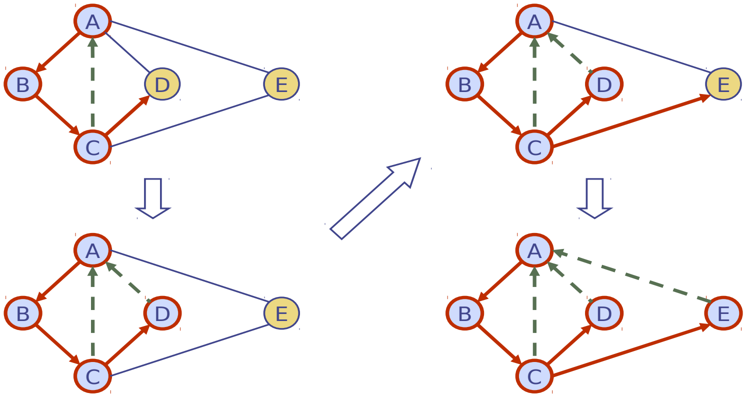
unexplored edge

discovery edge

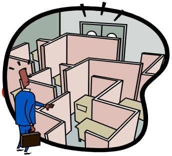
back edge



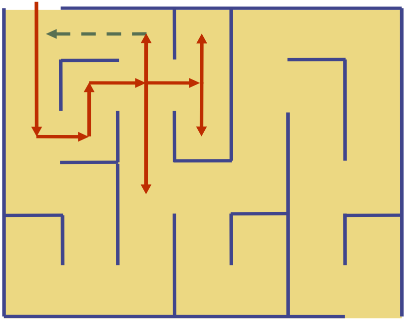
Example (cont.)



DFS and Maze Traversal



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



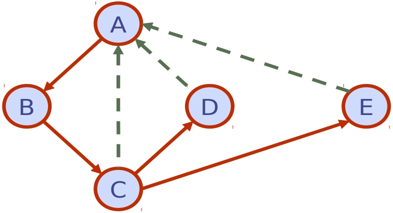
Properties of DFS

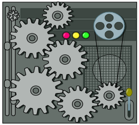
Property 1

$DFS(G, v)$ visits all the vertices and edges in the connected component of v

Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of v





Analysis of DFS

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n+m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

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Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $DFS(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v, e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop()          { e gets popped }
      else
        setLabel(e, BACK)
  S.pop()                { v gets popped }
```

Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
        w ← opposite(v,e)
        S.push(e)
        if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            pathDFS(G, w, z)
            S.pop()
        else
            C ← new empty stack
            repeat
                o ← S.pop()
                C.push(o)
            until o = w
            return C.elements()
  S.pop()
```

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Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

BFS Algorithm

- The algorithm uses a mechanism for setting and getting “labels” of vertices and edges

```
Algorithm BFS(G)
    Input graph G
    Output labeling of the edges
        and partition of the
        vertices of G
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
      BFS(G, v)
```

```
Algorithm BFS(G, s)
  L0 ← new empty sequence
  L0.insertLast(s)
  setLabel(s, VISITED)
  i ← 0
  while ¬Li.isEmpty()
    Li+1 ← new empty sequence
    for all v ∈ Li.elements()
      for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
          w ← opposite(v,e)
          if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            setLabel(w, VISITED)
            Li+1.insertLast(w)
          else
            setLabel(e, CROSS)
    i ← i +1
```

Example

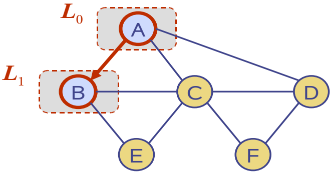
unexplored vertex

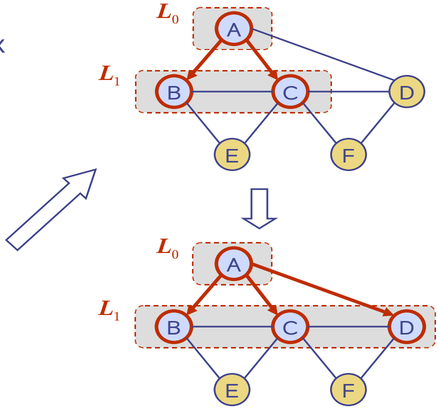
visited vertex

unexplored edge

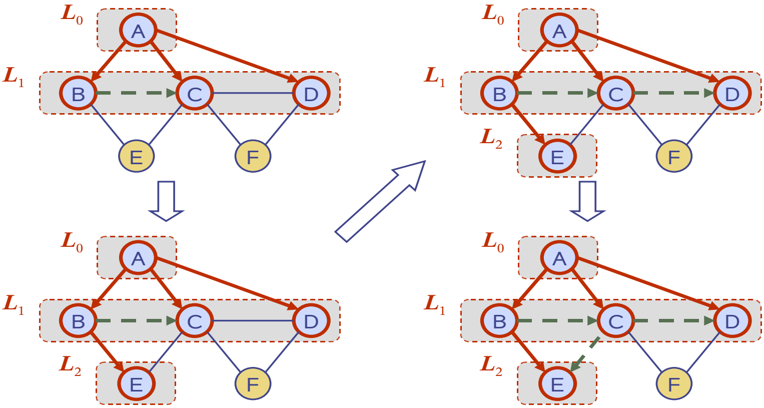
discovery edge

cross edge

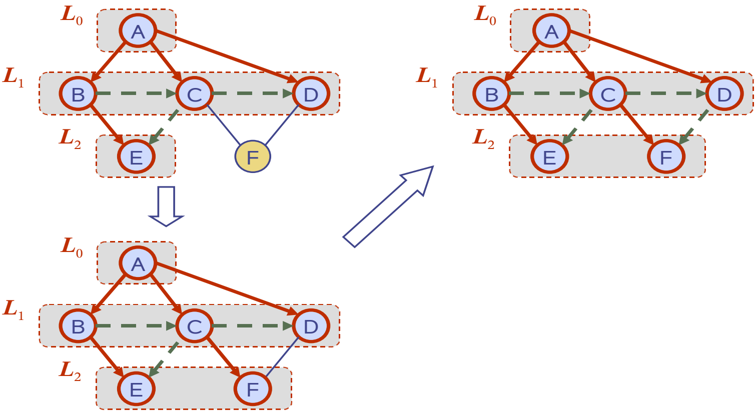




Example (cont.)



Example (cont.)



Properties

Notation

G_s: connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

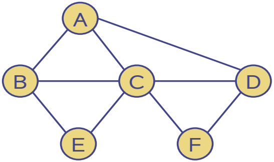
Property 2

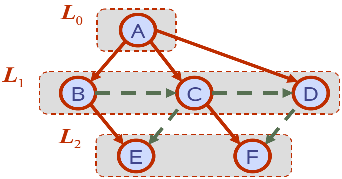
The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- The path of T_s from s to v has i edges
- Every path from s to v in G_s has at least i edges





Analysis

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

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Applications

- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n+m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

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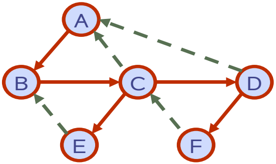
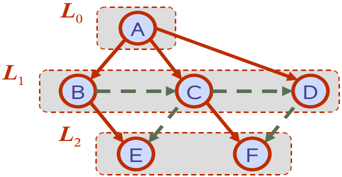
Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



BFS

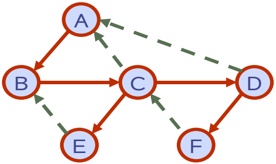
DFS vs. BFS (cont.)

Back edge (v,w)

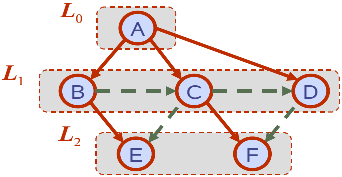
- w is an ancestor of v in the tree of discovery edges

Cross edge (v,w)

- w is in the same level as v or in the next level in the tree of discovery edges



DFS



BFS

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Decorations

What are decorations of vertices and edges??

- We want to "Mark" visited vertices.
- We want to save the number of unvisited in-incident edges.
- We want to mark in any way vertices or edges.

see also Chapter 12, page 602 (3rd edition)
Chapter 13, page 597 (4th edition)

Decorations

How to do this ?

- Add "attributes" or decorations to existing objects
- Each attribute is identified by a specific key (its name for instance).
- We allow this attribute to take different values for different objects
- One object may have more than one attribute

Note: the way explained below does not exactly match the notion of the "Decorator Pattern" you (will) see in Software Engineering!

DecorablePosition ADT

- `element()` Returns the element stored at this position
- `put(k,x)` Sets to `x` the value of attribute `k`.
  Returns the old value or null if this is a new attribute
- `get(k)` Returns the value of attribute `k` or null is this attribute has no value.
- `remove(k)` Remove the attribute `k`
  Returns the old value of `k` or null if there is no value
- `keySet()` Returns all keys for this position
- `values()` Returns all values for this position

A DecorablePosition interface

```
public interface DecorablePosition<E>
    extends Position<E>, Map<Object,Object> {
}
```

No new methods added!

Map allows for any Object to be used as keys and values, but not known in advance.

A GraphPosition abstract class

```
public abstract class GraphPosition<E>
   implements DecorablePosition<E>{

    private Map<Object,Object> map;
    private E element;

    public GraphPosition(E e){
        map = new HashMap<Object,Object>();
        element = e;
    }
    public E element(){
        return element;
    }
```

```
    public Object get(Object attribute){
        return map.get(attribute);
    }
    public Object put(Object attribute, Object value){
        return map.put(attribute,value);
    }
    public Object remove(Object attribute){
        return map.remove(attribute);
    }
    public Collection<Object> values(){
        return map.values();
    }
    public Set<Objects> keySet(){
        return map.keys();
    }
    ...
}
```