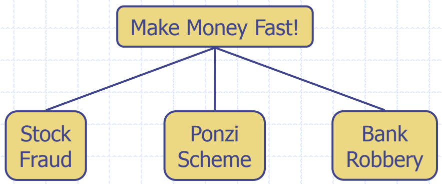
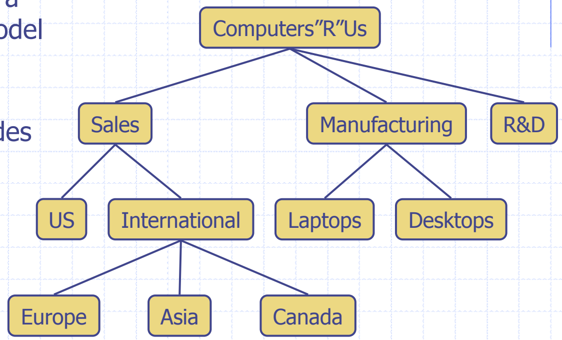
Trees



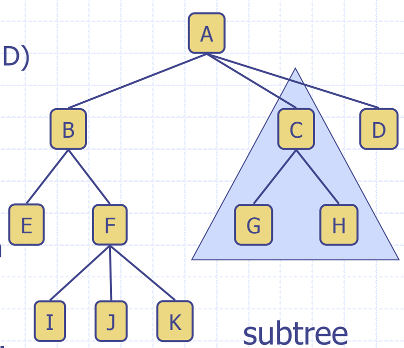
What is a Tree

- In computer science, a tree is an abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation
- Applications:
  - Organization charts
  - File systems
  - Programming environments



Tree Terminology

- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- External node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, grand-grandparent, etc.
- Depth of a node: number of ancestors
- Height of a tree: maximum depth of any node (3)
- Descendant of a node: child, grandchild, grand-grandchild, etc.
- Subtree: tree consisting of a node and its descendants



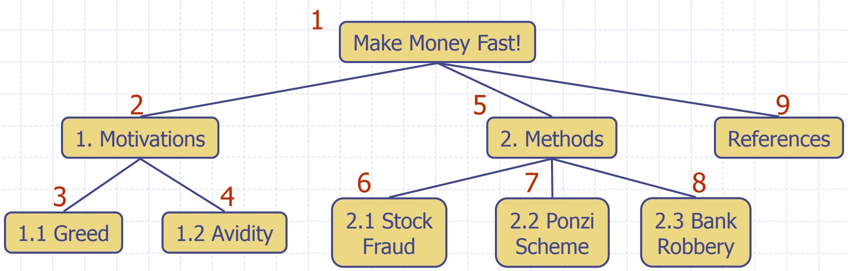
Tree ADT

- We use positions to abstract nodes
- Generic methods:
  - integer size()
  - boolean isEmpty()
  - Iterator iterator()
  - Iterable positions()
- Accessor methods:
  - position root()
  - position parent(p)
  - Iterable children(p)
- Query methods:
  - boolean isInternal(p)
  - boolean isExternal(p)
  - boolean isRoot(p)
- Update method:
  - element replace (p, o)
- Additional update methods may be defined by data structures implementing the Tree ADT

Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

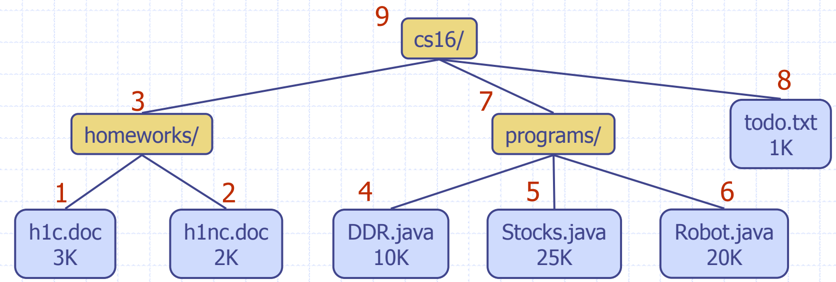
Algorithm *preOrder*(v)
    visit(v)
    for each child w of v
        preorder (w)



Postorder Traversal

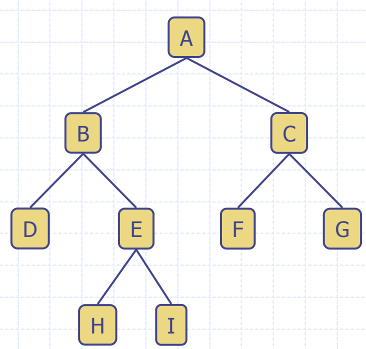
- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

Algorithm *postOrder*(v)
    for each child w of v
        postOrder (w)
    visit(v)



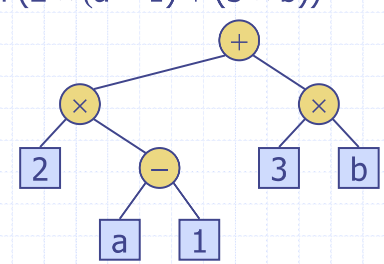
Binary Trees

- A binary tree is a tree with the following properties:
  - Each internal node has at most two children (exactly two for proper binary trees)
  - The children of a node are an ordered pair
- We call the children of an internal node left child and right child
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree
- Applications:
  - arithmetic expressions
  - decision processes
  - searching



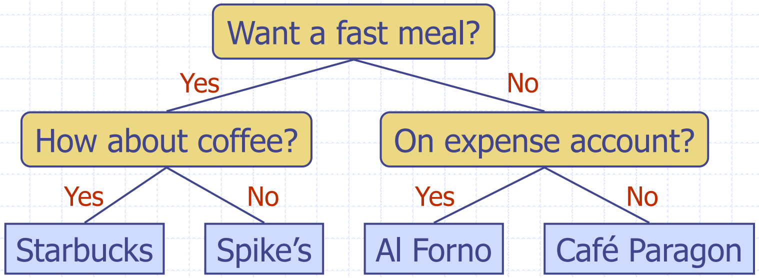
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



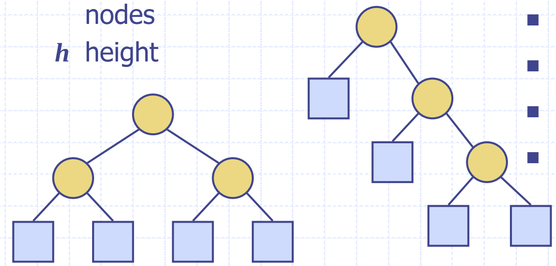
Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision



Properties of Proper Binary Trees

- Notation
  - n number of nodes
  - e number of external nodes
  - i number of internal nodes
  - h height



- Properties:
  - $e = i + 1$
  - $n = 2e - 1$
  - $h \leq i$
  - $h \leq (n - 1)/2$
  - $e \leq 2^h$
  - $h \geq \log_2 e$
  - $h \geq \log_2 (n + 1) - 1$

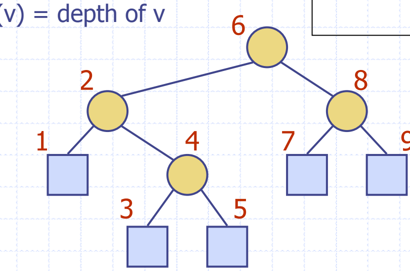
BinaryTree ADT

- The BinaryTree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional methods:
  - position left(p)
  - position right(p)
  - boolean hasLeft(p)
  - boolean hasRight(p)
- Update methods may be defined by data structures implementing the BinaryTree ADT

Inorder Traversal

- In an inorder traversal a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - x(v) = inorder rank of v
  - y(v) = depth of v

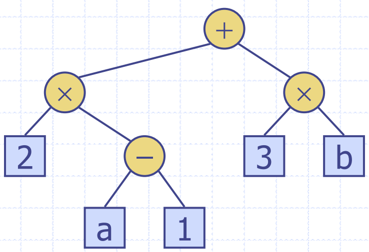
Algorithm *inOrder*(v)
  if *hasLeft* (v)
    inOrder (*left* (v))
  visit(v)
  if *hasRight* (v)
    inOrder (*right* (v))



Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree

Algorithm *printExpression*(*v*)
  if *hasLeft* (*v*)
    print("(")
    inOrder (*left*(*v*))
  print(*v.element* ())
  if *hasRight* (*v*)
    inOrder (*right*(*v*))
    print (")")

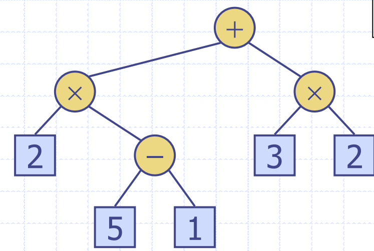


((2 × (a – 1)) + (3 × b))

Evaluate Arithmetic Expressions

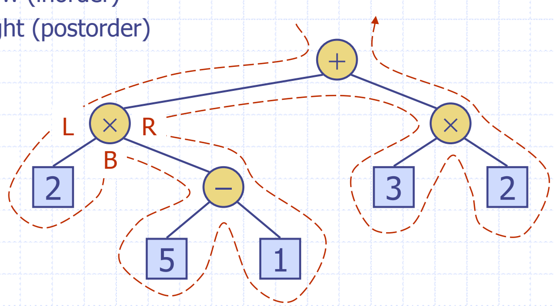
- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees

Algorithm *evalExpr*(*v*)
  if *isExternal* (*v*)
    return *v.element* ()
  else
    x ← *evalExpr*(*leftChild* (*v*))
    y ← *evalExpr*(*rightChild* (*v*))
    ◊ ← operator stored at *v*
    return *x* ◊ *y*



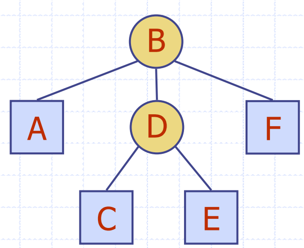
Euler Tour Traversal

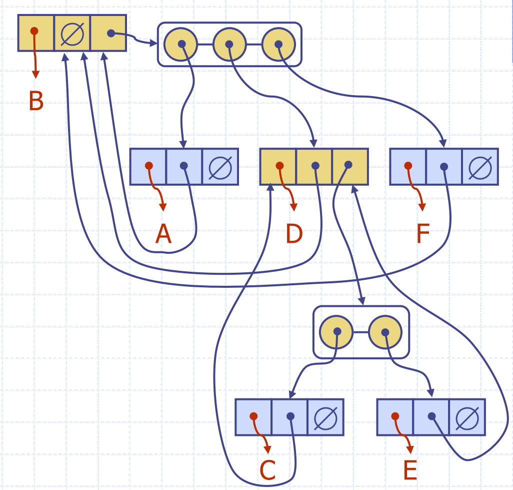
- Generic traversal of a binary tree
- Includes a special cases the preorder, postorder and inorder traversals
- Walk around the tree and visit each node three times:
  - on the left (preorder)
  - from below (inorder)
  - on the right (postorder)



Linked Structure for Trees

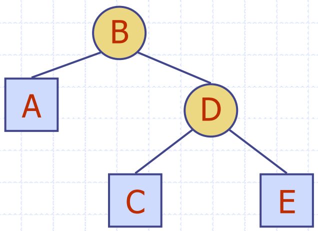
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT

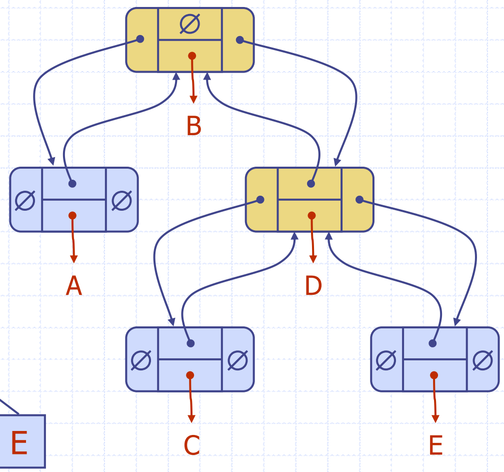




Linked Structure for Binary Trees

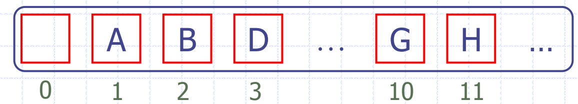
- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node
- Node objects implement the Position ADT

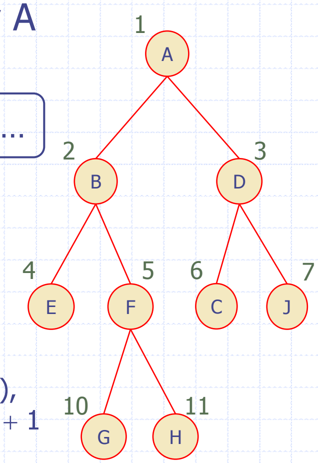




Array-Based Representation of Binary Trees

- Nodes are stored in an array A





- Node v is stored at A[rank(v)]
  - rank(root) = 1
  - if node is the left child of parent(node), rank(node) = 2 · rank(parent(node))
  - if node is the right child of parent(node), rank(node) = 2 · rank(parent(node)) + 1

Template Method Pattern

- Generic algorithm
- Implemented by abstract Java class
- Visit methods redefined by subclasses
- Template method eulerTour
  - Recursively called on left and right children
  - A TourResult object with fields left, right and out keeps track of the output of the recursive calls to eulerTour

```
public abstract class EulerTour <E, R> {
    protected BinaryTree<E> tree;
    public abstact R execute(BinaryTree<E> T);
    protected void init(BinaryTree<E> T) { tree = T; }
    protected R eulerTour(Position<E> v) {
        TourResult<R> r = new TourResult<R>();
        visitLeft(v, r);
        if (tree.hasLeft(p))
            { r.left=eulerTour(tree.left(v)); }
        visitBelow(v, r);
        if (tree.hasRight(p))
            { r.right=eulerTour(tree.right(v)); }
        return r.out;
    }
    protected void visitLeft(Position<E> v, TourResult<R> r) {}
    protected void visitBelow(Position<E> v, TourResult<R> r) {}
    protected void visitRight(Position<E> v, TourResult<R> r) {}
}
```


Specializations of EulerTour

- Specialization of class EulerTour to evaluate arithmetic expressions
- Assumptions
  - Nodes store ExpressionTerm objects with method getValue
  - ExpressionVariable objects at external nodes
  - ExpressionOperator objects at internal nodes with method setOperands(Integer, Integer)

```
public class EvaluateExpressionTour
        extends EulerTour<ExpressionTerm, Integer> {
    public Integer execute
            (BinaryTree<ExpressionTerm> T) {
        init(T);
        return eulerTour(tree.root());
    }
    protected void visitRight
            (Position<ExpressionTerm> v,
             TourResult<Integer> r) {
        ExpressionTerm term = v.element();
        if (tree.isInternal(v)) {
            ExpressionOperator op = (ExpressionOperator) term;
            op.setOperands(r.left, r.right); }
        r.out = term.getValue();
    }
}
```