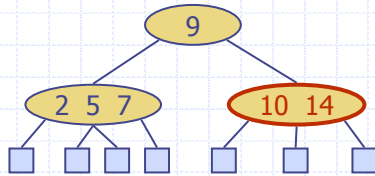# (2,4) Trees

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Multi-Way Search Tree

- ◈ A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1 v_2 \ldots v_d$ storing keys $k_1 k_2 \ldots k_{d-1}$
    - ◆ keys in the subtree of v_1 are less than k_1
    - ◆ keys in the subtree of v_i are between k_{i-1} and k_i $(i = 2, \ldots, d-1)$
    - ◆ keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders
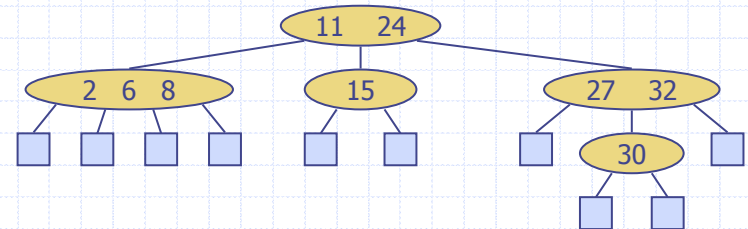

Multi-Way Inorder Traversal

- ◈ We can extend the notion of inorder traversal from binary trees to multi-way search trees
- ◈ Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- ◈ An inorder traversal of a multi-way search tree visits the keys in increasing order
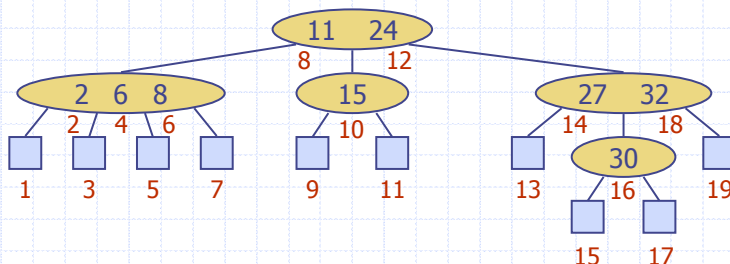

Multi-Way Searching

- ◈ Similar to search in a binary search tree
- ◈ A each internal node with children $v_1 v_2 \ldots v_d$ and keys $k_1 k_2 \ldots k_{d-1}$
  - $k = k_i$ $(i = 1, \ldots, d-1)$: the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ $(i = 2, \ldots, d-1)$: we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- ◈ Reaching an external node terminates the search unsuccessfully
- ◈ Example: search for 30
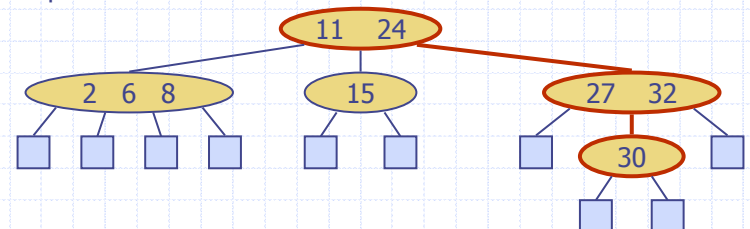
(2,4) Trees

- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - **Node-Size Property**: every internal node has at most four children
  - **Depth Property**: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node

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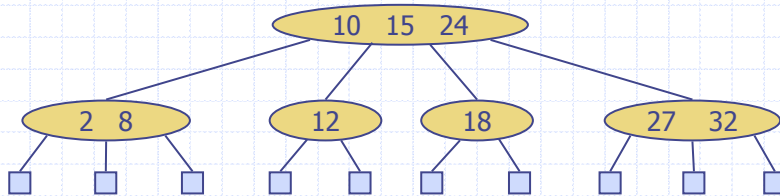
Height of a (2,4) Tree

- **Theorem:** A (2,4) tree storing n items has height $O(\log n)$
  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \ldots, h-1$ and no items at depth h, we have
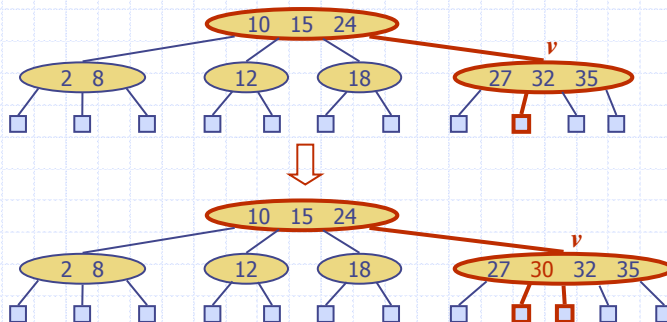 $$n \geq 1 + 2 + 4 + \ldots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log(n+1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time

| depth	items
0	1
1	2
$h-1$	2^{h-1}
h	0
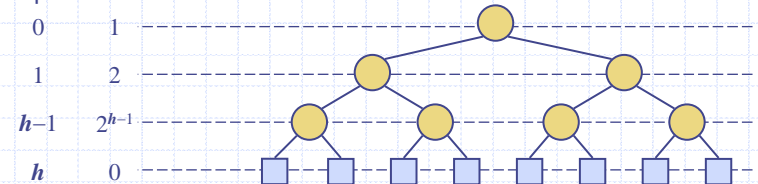
Insertion

- We insert a new item (k, o) at the parent v of the leaf reached by searching for k
  - We preserve the depth property but
  - We may cause an overflow (i.e., node v may become a 5-node)
- Example: inserting key 30 causes an overflow

Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced nodes v' and v''
    - v' is a 3-node with keys $k_1 k_2$ and children $v_1 v_2 v_3$
    - v'' is a 2-node with key k_4 and children $v_4 v_5$
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u
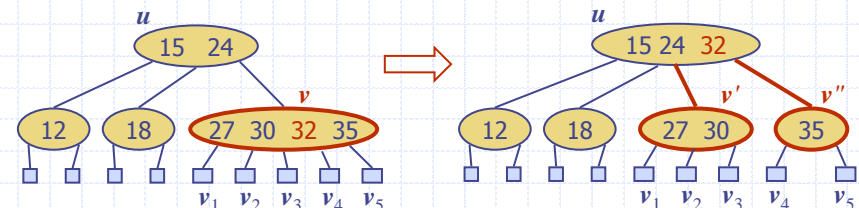
Analysis of Insertion

Algorithm *put*(k, o)

1. We search for key k to locate the insertion node v

2. We add the new entry (k, o) at node v

3. **while** *overflow*(v)

   if *isRoot*(v)

       create a new empty root above v

   $v \leftarrow split(v)$

♦ Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Step 2 takes $O(1)$ time
  - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits

♦ Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

Deletion

♦ We reduce deletion of an entry to the case where the item is at the node with leaf children
♦ Otherwise, we replace the entry with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter entry
♦ Example: to delete key 24, we replace it with 27 (inorder successor)
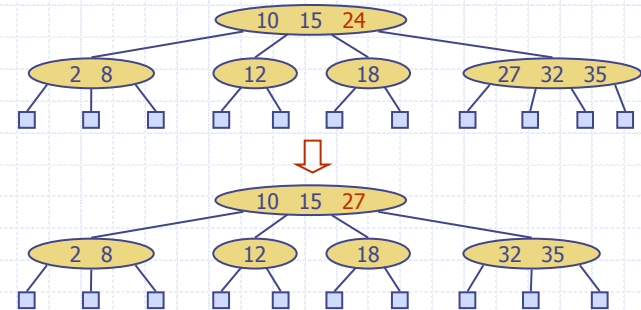
Underflow and Fusion

♦ Deleting an entry from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
♦ To handle an underflow at node v with parent u, we consider two cases
♦ **Case 1:** the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an entry from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u
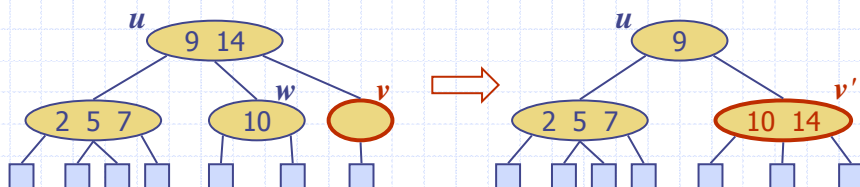
Underflow and Transfer

♦ To handle an underflow at node v with parent u, we consider two cases
♦ **Case 2:** an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - After a transfer, no underflow occurs
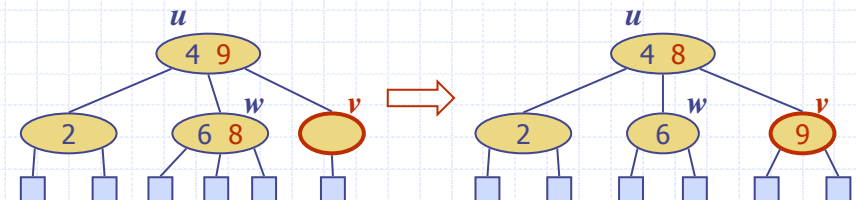
Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the entry
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time

Comparison of Map Implementations

	Get	Put	Delete	Notes
Hash Table	1 expected	1 expected	1 expected	o no ordered map methods o simple to implement
Skip List	$\log n$ high prob.	$\log n$ high prob.	$\log n$ high prob.	o randomized insertion o simple to implement
AVL and (2,4) Tree	$\log n$ worst-case	$\log n$ worst-case	$\log n$ worst-case	o complex to implement