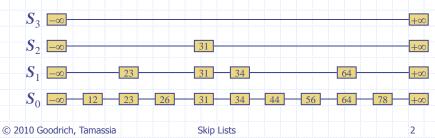


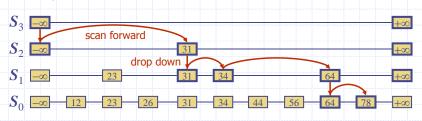
What is a Skip List

- riangle A skip list for a set S of distinct (key, element) items is a series of lists S_0, S_1, \ldots, S_h such that
 - Each list S, contains the special keys $+\infty$ and $-\infty$
 - List S_0 contains the keys of S in nondecreasing order
 - Each list is a subsequence of the previous one, i.e., $S_0 \supset S_1 \supset ... \supset S_h$
 - List S_h contains only the two special keys
- We show how to use a skip list to implement the dictionary ADT



Search

- \Box We search for a key x in a a skip list as follows:
 - We start at the first position of the top list
 - At the current position p, we compare x with $y \leftarrow key(next(p))$
 - x = y: we return element(next(p))
 - x > y: we "scan forward"
 - x < y: we "drop down"
 - If we try to drop down past the bottom list, we return *null*
- Example: search for 78



Randomized Algorithms

- A randomized algorithm performs coin tosses (i.e., uses random bits) to control its execution
- It contains statements of the type

 $b \leftarrow random()$ **if** b = 0do A ... **else** { b = 1 }

do B ...

 Its running time depends on the outcomes of the coin tosses

- We analyze the expected running time of a randomized algorithm under the following assumptions
 - the coins are unbiased, and
 - the coin tosses are independent
- The worst-case running time of a randomized algorithm is often large but has very low probability (e.g., it occurs when all the coin tosses give "heads")
- We use a randomized algorithm to insert items into a skip list

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Skip Lists

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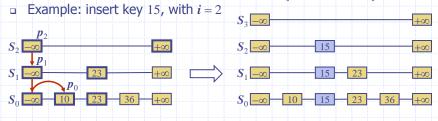
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Skip Lists

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Insertion

- - We repeatedly toss a coin until we get tails, and we denote with i
 the number of times the coin came up heads
 - If $i \ge h$, we add to the skip list new lists S_{h+1}, \ldots, S_{i+1} , each containing only the two special keys
 - We search for x in the skip list and find the positions $p_0, p_1, ..., p_i$ of the items with largest key less than x in each list $S_0, S_1, ..., S_i$
 - For $j \leftarrow 0, ..., i$, we insert item (x, o) into list S_i after position p_i



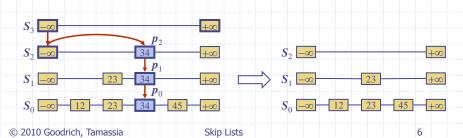
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Skip Lists

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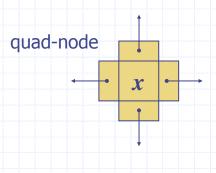
Deletion

- □ To remove an entry with key *x* from a skip list, we proceed as follows:
 - We search for x in the skip list and find the positions $p_0, p_1, ..., p_i$ of the items with key x, where position p_i is in list S_i
 - We remove positions $p_0, p_1, ..., p_i$ from the lists $S_0, S_1, ..., S_i$
 - We remove all but one list containing only the two special keys
- Example: remove key 34



Implementation

- We can implement a skip list with quad-nodes
- A quad-node stores:
 - entry
 - link to the node prev
 - link to the node next
 - link to the node below
 - link to the node above
- Also, we define special keys PLUS_INF and MINUS_INF, and we modify the key comparator to handle them



Space Usage

- The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm
- We use the following two basic probabilistic facts:
 - Fact 1: The probability of getting *i* consecutive heads when flipping a coin is $1/2^i$
 - Fact 2: If each of *n* entries is present in a set with probability *p*, the expected size of the set is *np*

- Consider a skip list with n entries
 - By Fact 1, we insert an entry in list S, with probability 1/2ⁱ
 - By Fact 2, the expected size of list S_i is n/2ⁱ
- The expected number of nodes used by the skip list is

$$\sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i} < 2n$$

Thus, the expected space usage of a skip list with n items is O(n)

Height

- The running time of the search an insertion algorithms is affected by the height h of the skip list
- We show that with high probability, a skip list with n items has height $O(\log n)$
- We use the following additional probabilistic fact:

Fact 3: If each of *n* events has probability p, the probability that at least one event occurs is at most np

- Consider a skip list with n entires
 - By Fact 1, we insert an entry in list S, with probability $1/2^i$
 - By Fact 3, the probability that list S, has at least one item is at most $n/2^i$
- \square By picking $i = 3\log n$, we have that the probability that $S_{3\log n}$ has at least one entry is at most

$$n/2^{3\log n} = n/n^3 = 1/n^2$$

 Thus a skip list with n entries has height at most $3\log n$ with probability at least $1 - 1/n^2$

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Skip Lists

Search and Update Times

- The search time in a skip list is proportional to
 - the number of drop-down steps, plus
 - the number of scan-forward steps
- The drop-down steps are bounded by the height of the skip list and thus are $O(\log n)$ with high probability
- To analyze the scan-forward steps, we use yet another probabilistic fact:

Fact 4: The expected number of coin tosses required in order to get tails is 2

- When we scan forward in a list, the destination key does not belong to a higher list
 - A scan-forward step is associated with a former coin toss that gave tails
- By Fact 4, in each list the expected number of scanforward steps is 2
- Thus, the expected number of scan-forward steps is $O(\log n)$
- We conclude that a search in a skip list takes $O(\log n)$ expected time
- The analysis of insertion and deletion gives similar results

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Skip Lists

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Summary

- A skip list is a data structure for dictionaries that uses a randomized insertion algorithm
- \Box In a skip list with nentries
 - The expected space used is O(n)
 - The expected search. insertion and deletion time is $O(\log n)$
- Using a more complex probabilistic analysis, one can show that these performance bounds also hold with high probability
- Skip lists are fast and simple to implement in practice