Lecture3

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Laplace equation

- Laplacian operator
  - A second-order differentiation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Laplace equation
  - For a function, $\phi(\mathbf{r})$, the Laplace equation reads

$$\nabla^2 \phi(\mathbf{r}) = 0$$

  - Of course, we need boundary conditions.
- Its application?
  - In this course, it is closely related to the Poisson equation.

Analytic solution

- Consider the 1D structure, $[0, a]$.
  - The boundary conditions for $\phi(x)$ are given by $\phi(0) = 0$ and $\phi(a) = 1$.
- It is easy to solve it.
  - In the 1D structure, $\frac{d^2}{dx^2}\phi = 0$.
  - The solution has the form of $\phi(x) = C_1 x + C_0$.
  - Using the boundary conditions, it is found that $\phi(x) = \frac{x}{a}$.

Solution vector

- Let us construct a matrix equation corresponding to the Laplace equation.
  - We assign N points. They are uniformly distributed. For the i-th point, it has the coordinate of

$$x_i = \frac{i-1}{N-1} a$$

  - The solution function, $\phi(x)$, can be approximated by

$$[\phi(x_1) \quad \phi(x_2) \quad \cdots \quad \phi(x_{N-1}) \quad \phi(x_N)]^T$$

  - Therefore, N equations are needed to determine the solution vector.
  - Remember that the Laplacian operator is involved.

Discretization

- How to assign N equations
  - For $2 \leq i \leq N-1$, the discretized Laplacian operator is used.

$$\left.\frac{d^2\phi}{dx^2}\right|_{x=x_i} \approx \frac{\phi(x_{i+1}) - 2\phi(x_i) + \phi(x_{i-1})}{\Delta x^2} = 0$$

  - The Laplace equation at $x = x_i$ can be written as

$$\frac{1}{(\Delta x)^2}\begin{bmatrix} 0 & \cdots & 1 & -2 & 1 & \cdots & 0 \end{bmatrix}\begin{bmatrix} \phi(x_1) \\ \vdots \\ \phi(x_{i-1}) \\ \phi(x_i) \\ \phi(x_{i+1}) \\ \vdots \\ \phi(x_N) \end{bmatrix} = [0]$$

Boundary condition

- How to assign N equations (continued)
  - For $i = 1$ or $i = N$, the boundary condition is applied.
  - For example, for $i = N$, the boundary condition can be written in a matrix form:

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi(x_1) \\ \vdots \\ \phi(x_{i-1}) \\ \phi(x_i) \\ \phi(x_{i+1}) \\ \vdots \\ \phi(x_N) \end{bmatrix} = [1]$$

$N = 5$ example

- Let us collect the previous results explicitly.
  - It is simply given by

$$Ax = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \\ \phi(x_3) \\ \phi(x_4) \\ \phi(x_5) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = b$$

  - How can we solve it numerically?

MATLAB example

- Step-by-step procedure
  - First, set the matrix, A.

```
A = zeros(5,5);
A(1,1) = 1.0;
A(2,1) = 1.0; A(2,2) = -2.0; A(2,3) = 1.0;
A(3,2) = 1.0; A(3,3) = -2.0; A(3,4) = 1.0;
A(4,3) = 1.0; A(4,4) = -2.0; A(4,5) = 1.0;
A(5,5) = 1.0;
```

  - Next , set the vector, b.

```
b = zeros(5,1); b(5,1) = 1.0;
```

  - Finally, get the solution vector, x.

```
x = A \ b
```

Generalized Poisson equation?

- Laplace equation

$$\nabla^2 \phi(\mathbf{r}) = 0$$

- Poisson equation

$$\nabla^2 \phi(\mathbf{r}) = \rho(\mathbf{r})$$

  – Here, $\rho(\mathbf{r})$ is the source function.
  – Therfore, the source-free Poisson equation is the Laplace equation.
- Then, why do we consider the “generalized” Poisson equations?
  – What is the exact meaning of the generalization?

Derivation

- One of the four Maxwell equations

$$\nabla \cdot \mathbf{D} = \rho(\mathbf{r})$$

$\mathbf{D}(\mathbf{r})$: Displacement vector, which is given by $\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$

$\mathbf{E}(\mathbf{r})$: Electric field

$\epsilon(\mathbf{r})$: Permittivity (assumed as a scalar)

$\rho(\mathbf{r})$: Net charge density

- Under the electrostatic approximation, $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$.
  - Using the above expression, it is found that

$$\nabla \cdot [\epsilon(\mathbf{r})\nabla\phi(\mathbf{r})] = -\rho(\mathbf{r})$$

  - The above equation is (incorrectly) called as the “Poisson equation.”