

Multi-Objective Archiving

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Abstract—Most multi-objective optimisation algorithms maintain an archive explicitly or implicitly during their search. Such an archive can be solely used to store high-quality solutions presented to the decision maker, but in many cases may participate in the search process (e.g., as the population in evolutionary computation). Over the last two decades, archiving, the process of comparing new solutions with previous ones and deciding how to update the archive/population, stands as an important issue in evolutionary multi-objective optimisation (EMO). This is evidenced by constant efforts from the community on developing various effective archiving methods, ranging from conventional Pareto-based methods to more recent indicator-based and decomposition-based ones. However, the focus of these efforts is on empirical performance comparison in terms of specific quality indicators; there is lack of systematic study of archiving methods from a general theoretical perspective. In this paper, we attempt to conduct a systematic overview of multi-objective archiving, in the hope of paving the way to understand archiving algorithms from a holistic perspective of theory and practice, and more importantly providing a guidance on how to design theoretically desirable and practically useful archiving algorithms. In doing so, we also present that archiving algorithms based on weakly Pareto compliant indicators (e.g., ϵ -indicator), as long as designed properly, can achieve the same theoretical desirables as archivers based on Pareto compliant indicators (e.g., hypervolume indicator). Such desirables include the property *limit-optimal*, the limit form of the possible optimal property that a bounded archiving algorithm can have with respect to the most general form of superiority between solution sets.

Index Terms—Multi-objective optimisation, evolutionary computation, archive, archiving methods, population maintenance, environmental selection

I. INTRODUCTION

MULTI-objective optimisation refers to an optimisation scenario where several conflicting objectives are optimized simultaneously. A prominent feature of a multi-objective optimisation problem (MOP) is that, in contrast to its single-objective counterpart, it does not have a single optimal solution, but rather a set of trade-off solutions, called Pareto-optimal solutions or the Pareto front in the objective space, whose size is usually prohibitively large or even infinite.

A common way to solve an MOP is to find a good, but smaller size, approximation to its Pareto front and present it to the decision maker, who chooses one solution from the approximation to deploy. Multi-objective optimisers designed for this purpose maintain an *archive*, i.e., a set of high-quality solutions discovered during the search. In this context,

archiving is the process of updating the archive by comparing new solutions with those already in the archive and deciding which ones to keep and which ones to discard [1, 2].

Archiving becomes even more relevant in evolutionary multi-objective optimisation (EMO), in which population-based evolutionary search is performed. The population maintenance (i.e., environmental selection) in EMO can be seen as an archiving process [3, 4], where the population is regarded as an archive updated at each generation; that is, new offspring solutions are compared with the ones already in the population, either chunk by chunk, e.g., in the generational evolution of NSGA-II [5] and SPEA2 [6] or one-by-one, e.g., in the steady-state evolution of SMS-EMOA [7] and MOEA/D [8].

Ideally, the optimiser would store all Pareto-optimal solutions ever evaluated in an unbounded archive [9]. In practice, however, the computational cost of maintaining an unbounded archive often makes this approach infeasible. Thus, all archivers share the common purpose of preserving a solution set of bounded size. Thus, the core issue in archiving is to decide which solutions are kept and which are removed when new solutions arrive. Many archiving algorithms (*archivers*) have been proposed in the literature using different rules (or selection criteria in the EMO), ranging from traditional Pareto-based criteria, which use Pareto dominance to distinguish between solutions, to more recent criteria such as indicator-based and decomposition-based ones, which rely on a quality indicator and a set of weight vectors, respectively.

Although there are a number of works dedicated to the topic of archiving, e.g., [1, 3, 4, 10], most archiving methods have been proposed and studied as part of complete MOEAs and not as independent algorithmic components; which is at odds with the fact that the key distinctive characteristic of many MOEAs is their population maintenance (i.e., archiving) rule. For example, the main innovation of NSGA-III [11] is the method for selecting the solutions that will form the population in the next generation, and all of its other components (e.g., mating selection, crossover and mutation) follow common practice.

Moreover, most studies of archiving from a theoretical perspective have focused on specific quality indicators [1, 12–14]. Except for few exceptions [4], it remains largely unclear, what are the theoretical properties held by the archiving algorithms used within the state-of-the-art MOEAs; for example, whether their population may suffer from deterioration in terms of Pareto-optimality, or whether they are able to return a maximal subset of the Pareto-optimal solutions discovered so far. Such properties matter, as the decision maker certainly would be unhappy if she is forced to select an inferior solution from the final archive/population instead of a better solution, in terms of Pareto-optimality, that was generated but later removed from the archive. Many unwelcome phenomena caused by the lack

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of such properties have been reported in the literature, on synthetic point sequences [2, 4, 10], benchmark test problems [9, 15, 16] and real-life scenarios [17–19].

In this paper, we aim to conduct a comprehensive review of archiving in multi-objective optimisation that includes theoretical studies as well as archivers proposed as part of complete MOEAs. We begin by providing background knowledge of multi-objective optimisation (Section II). We then define the archiving problem and summarise its history (Section III). In Section IV, we focus on theory of archiving and give desirable properties for archiving algorithms to hold. Our proposed formulation of these properties covers, in an uniform manner, both bounded-size archivers, which only store nondominated solutions, and fixed-size archivers, which may store dominated solutions. Moreover, we prove that archiving algorithms based on weakly Pareto compliant indicators (e.g., ϵ -indicator [20]) can achieve the same theoretical desirability as those based on Pareto-compliant indicators (e.g., hypervolume [21]). Next, based upon those theoretical properties, we classify well-known archiving algorithms into four classes (Section V). Afterwards, we discuss important issues in multi-objective archiving (Section VI) and suggest several future research lines which deserve more attention (Section VII). Lastly, we conclude the paper in Section VIII.

II. PRELIMINARIES

In multi-objective archiving, we are interested in point vectors (i.e., objective vectors) in finite and multidimensional objective spaces. For simplicity, we always use the term “solutions” to refer to points in the objective space, even though this term is often reserved for points in the decision space. For any finite and multidimensional objective space, an order relation can be defined as follows (w.l.o.g., we consider a minimisation scenario throughout the paper).

Definition 1 (Pareto dominance relation). *Let $Y \subset \mathbb{R}^d$ be a finite, d -dimensional objective space ($d > 1$). For two solutions $y, y' \in Y$, y is said to weakly dominate y' ($y \preceq y'$), iff $\forall i \in 1, \dots, d$, $y_i \leq y'_i$. More strictly, y is said to dominate y' ($y \prec y'$), iff $y \preceq y'$ and $y \neq y'$. In addition, we say that two solutions are (mutually) nondominated iff $y \not\prec y'$ and $y' \not\prec y$.*

The Pareto dominance relation immediately leads to the notion of optimality in multi-objective optimisation. Thus, we can define the set of minimal elements of a given subset $P \subseteq Y$ as [4]:

$$\min(P, \prec) = \{y \in P \mid \nexists y' \in P, y' \prec y\}. \quad (1)$$

From Eq. (1), we can define the Pareto front of Y as the set of minimal elements of Y .

Definition 2 (Pareto-optimal solution, Pareto-optimal set and Pareto front). *A solution $y^* \in Y$ is called Pareto optimal iff $\nexists y \in Y$, $y \prec y^*$. The set of all Pareto optimal solutions of Y is called its Pareto optimal set (or Pareto front), i.e., $Y^* = \min(Y, \prec) = \{y \in Y \mid \nexists y' \in Y, y' \prec y^*\}$.*

We use the term *nondominated set* for any set $P \subseteq Y$ with the property $P = \min(P, \prec)$.

The order relations between solutions can be readily extended to sets of solutions.

Definition 3 (Pareto dominance relation between sets). *For two solution sets $A, B \subset Y$, A is said to weakly dominate B ($A \preceq B$), iff $\forall b \in B, \exists a \in A$, $a \preceq b$. More strictly, A is said to dominate B ($A \prec B$), iff $\forall b \in B, \exists a \in A$, $a \prec b$.*

It can be seen that the set-based weak-dominance relation cannot rule out the equivalence between two sets, while the set-based dominance relation does not allow the equality between any two solutions. Therefore, Zitzler et al. [20] proposed another order relation between sets, called *better* relation, which represents the most general and weakest form of superiority between two sets.

Definition 4 (Better relation between sets [20]). *For two solution sets $A, B \subset Y$, A is said to be better than B ($A \triangleleft B$), iff $A \preceq B$ and $\exists a \in A, \forall b \in B$, $b \not\preceq a$.*

In other words, $A \triangleleft B$ means that A is at least as good as B , but B is not as good as A , i.e., $A \preceq B$ but $B \not\preceq A$.

Unfortunately, the above set order relations are in general not sufficient to distinguish between solution sets. Two sets are incomparable as soon as there exist two mutually nondominated solutions from different sets. This is particularly the case in many-objective optimisation [22] as the chance of two solutions being nondominated increases exponentially with the number of objectives [23].

A total order between sets can be defined by means of a quality indicator I that maps a set to a real number, formally, $I: \mathcal{P}(Y) \setminus \emptyset \rightarrow \mathbb{R}$, where $\mathcal{P}(Y)$ denotes the power set of Y . Yet, mapping a set of solution vectors to a real number inevitably results in information loss. We certainly hope that the mapping of a quality indicator is always compliant with the \triangleleft -relation, the most general form of superiority between two solution sets.

Definition 5 (weakly Pareto-compliant indicator [20]). *A quality indicator I is called weakly Pareto compliant iff $\forall A, B \subset Y$, if $A \triangleleft B$ then $I(A) \leq I(B)$ (assuming w.l.o.g. that smaller values of I are preferable).*

An indicator being weakly Pareto-compliant implies that $I(A) < I(B) \implies B \not\triangleleft A$, that is, if the quality indicator says that A is better than B , then B cannot be better than A in terms of Pareto optimality, which is the weakest requirement of an indicator. This property prevents a solution set being evaluated better than another by the quality indicator, yet the former will never be preferred by the decision maker according to Pareto-optimality. This unwelcome situation can happen in application scenarios if the indicator used does not hold this property [24]. Fortunately, there are several weakly Pareto-compliant indicators in the literature, such as the ϵ -indicator [20], $R2$ [25] and IGD^+ [26] (see [27] for a review of quality indicators). However, indicators holding this property may still fail to distinguish between two solution sets that satisfy the \triangleleft -relation, that is, it may happen that $I(A) = I(B)$ given $A \triangleleft B$. Thus, a more strict property is useful.

Definition 6 (Pareto-compliant indicator [20]). *A quality in-*

indicator I is called *Pareto compliant* iff $\forall A, B \subset Y$, if $A \triangleleft B$ then $I(A) < I(B)$, which implies $I(A) \leq I(B) \implies B \not\triangleleft A$.

An indicator being Pareto compliant means that if a solution set is better in terms of Pareto optimality, then its quality value must be strictly lower. This implies that only the Pareto-optimal set achieves the minimum value of the indicator. This property is very strict and very few quality indicators hold it (the hypervolume indicator [28] is one).¹

III. THE ARCHIVING PROBLEM

A. Brief History

Study on the archiving problem starts from the convergence analysis of MOEAs in the late 90s [3, 30]. Unlike the single-objective case where constructing a convergent EA is generally straightforward (via the elitist-preserving rule), constructing a convergent MOEA (i.e., the sequence of the populations produced by the MOEA converges into a subset of the Pareto front [31]) is non-trivial. The main difficulty is that, in contrast to the single-objective case where there is a total order relation between solutions, Pareto dominance is a partial order, which leads to solutions (and solution sets) being incomparable. An MOEA using a fixed-size population needs to decide which nondominated solutions are removed from the current population to allow the entry of newly generated ones. This decision is taken by the environmental selection mechanism, which can be seen as an archiving method that receives new solutions as input, compares them with the ones in the population (archive), and decides which solutions are kept and which ones are thrown away.

Environmental selection is usually designed on the basis of two principles: (1) dominated solutions should be removed earlier than nondominated ones and (2) solutions in crowded regions should be removed earlier than ones in sparse regions when all of them are mutually nondominated. Different ways of implementing these two principles resulted in many successful MOEAs during the period of 1999–2002, such as SPEA [21], PAES [32], PESA-II [33], NSGA-II [5] and SPEA2 [6]. However, such a “Pareto dominance + density” criterion does not guarantee a convergent MOEA. Solutions in the population can deteriorate with time since the population may accept solutions that are dominated by a solution removed previously, provided that these solutions are not dominated within the current population and are located in a less crowded region (an illustration will be given in the next subsection).

In the meanwhile, researchers attempted to develop MOEAs with guaranteed convergence [3, 30, 34, 35] by dropping the density criterion and removing solutions only if they are dominated by newcomers. This criterion ensures the monotonicity of the populations with respect to Pareto dominance, but the final population returned may end up crowding a small region of the Pareto front.

To address the above issues, in 2002, Laumanns et al. [15, 36] proposed the concept of ϵ -approximation in archiving, aiming to bridge the gap of MOEAs between theoretical

desirability and practical performance. The idea is to ensure that every point in the Pareto front can be represented (i.e., ϵ -dominated) by at least a solution in an archive of (polynomially) bounded size. However, the choice of the parameter ϵ becomes critical and it may not be practical to set an appropriate ϵ value for a problem whose Pareto front is unknown, while adapting ϵ on the fly may easily end up with too few solutions in the archive [2].

By 2003, Knowles and Corne formalised the archiving problem and separated it from EMO as an independent research topic [1, 2]. They highlighted the importance of archiving in multi-objective optimisation and showed that, from the perspective of the no-free-lunch theorem [37], the archiving method is a critical component that distinguishes between MOEAs [38]. They also listed several desirable properties of archivers [2] and then investigated several representative archivers and their convergence properties [1, 39]. Moreover, they proved that in general no archiving algorithm is able to maintain an “optimal approximation” of the Pareto front (see Def. 8 on page 5) of the sequence points at every timestep [2].

Since then, more archiving algorithms with desirable theoretical properties, such as solution monotonicity (Prop. 2 on page 5) have emerged. These archivers either were based on existing concepts such as the ϵ -dominance [40, 41] and hypervolume [42], or developed new archiving criteria such as the open rectangle [43] and multi-level grid [12]. In 2011, López-Ibáñez et al. [4] systematically analysed representative archiving algorithms and presented several properties desirable for an archiver to hold, including \triangleleft -monotonicity which is based on the *better* relation defined above (Def. 4), the weakest form of superiority between two solution sets. They also showed empirically that archiving methods used in well-known MOEAs, such as NSGA-II and SPEA2, do not hold this property and, thus, they may produce a population that is worse, in terms of Pareto-optimality, than a previous one.

More recent studies have focused on the convergence of archiving algorithms with respect to specific quality indicators such as the hypervolume [14, 44], ϵ -box [45] and Hausdorff metrics [46–48] (see [49] for a review study). In addition, several studies have empirically investigated archivers in isolation using artificial sequences of solutions [4, 10, 50], using solution neighborhoods in combinatorial problems [51] and the practical effectiveness of satisfying theoretical properties [16].

In contrast to the relatively few theoretical results, developing practically effective population maintenance (i.e., archiving) methods, regardless of their theoretical properties, has become the most active direction in EMO research, resulting in numerous MOEAs. These methods can mainly be categorised into three mainstream selection paradigms [52, 53]: the Pareto-based (Pareto dominance + density) [54], the indicator-based [55] and the decomposition-based [8] paradigms. Some researchers have also introduced an external archive to guide the evolution of the population [56–62]. To complement a non-Pareto selection criterion (e.g., the decomposition-based criterion) in the evolutionary population, the external archive is mainly based on the pattern of “Pareto dominance + density”, with the exception of some work using an indicator (e.g., hypervolume) as the criterion in archiving (see [63]). In those

¹In the literature, Pareto compliance is sometimes called strong Pareto compliance and weak Pareto compliance is called Pareto compliance [27, 29].

Pareto-based archivers, the density estimators more frequently used are crowding distance [56, 59, 64], niching [60, 65] and grid techniques [66–68]. Experimental work has shown that adding an external archive is often beneficial [69], specially if the parameters of the MOEAs are configured after adding the archive [63]. Recently, there is a trend in MOEA design that considers two archives, that is, two simultaneous populations that participate in solution generation, each of them updated by a different selection criterion [70–73]; for instance, one archive for promoting convergence and the other for promoting diversity [70, 71]. Such a two-archive approach is particularly suitable for multi-objective problems with additional features, e.g., with many objectives), as one archive can be designed specifically for dealing with those features. This is why the two-archive approach has now been used in various challenging multi-objective scenarios, such as many-objective optimisation [74–77], constrained optimisation [78–80], dynamic optimisation [81], multi-model optimisation [82, 83], expensive optimisation [84], and real-world problems [85–87].

Finally, some archivers [47, 88] aim to preserve not only Pareto-optimal solutions but also *nearly-optimal* ones, also called non-epsilon dominated or ϵ -efficient, that is, solutions that are not too far from being Pareto-optimal. There are also archivers that maximise diversity in the decision space, typically for multimodal problems [71] and a number of recent archivers (see [89] for a survey) consider both near-optimality and diversity in decision space. We will not study these types of archivers in this paper since their aims are rather different from most other archivers that focus on the objective space.

B. Formal Definition

The archiving process can be described as updating a set of solutions A , an *archive*, by an *input sequence* $S = \langle S^{(1)}, S^{(2)}, \dots, S^{(t)}, \dots \rangle$, which may be generated by a solution generator (e.g., an evolutionary algorithm) iteratively. At one iteration t , the generator may generate one or multiple solutions, i.e., $\forall t, |S^{(t)}| \geq 1$, possibly using the contents of the old archive $A^{(t-1)}$, where $A^{(t-1)}$ denotes the archive *after* updating it with $S^{(t-1)}$, and $A^{(0)}$ is the empty set. Solutions may be fed to the archive one-at-a-time as in ϵ -MOEA [90] and SMS-EMOA [7] or many-at-a-time as in NSGA-II [5] and SPEA2 [6]. There is no requirement that the elements in the sequence are unique.

We are interested here in archives of bounded capacity, i.e., $\forall t, |A^{(t)}| \leq N$ for some constant $N \in \mathbb{N}^+$, smaller than the number of Pareto-optimal solutions in the input sequence, $N < |Y^*|$. Thus, the archive $A^{(t)}$ stores a subset of the solutions in the input sequence up to time t . Now we can define an archiving algorithm as follows.

Definition 7 (Archiving algorithm or Archiver). *An archiving algorithm takes as input the previous archive $A^{(t-1)}$ and the current set in the sequence $S^{(t)}$ and returns the updated archive $A^{(t)}$, i.e., $A^{(t)} \leftarrow \text{Archiver}(A^{(t-1)}, S^{(t)})$, where $A^{(t)} = \{A \subseteq A^{(t-1)} \cup S^{(t)} \mid 1 \leq |A| \leq N\}$ and $S^{(t)} \subset Y$, where $Y \subset \mathbb{R}^d$ is a finite, d -dimensional objective space from which the solutions are generated.*

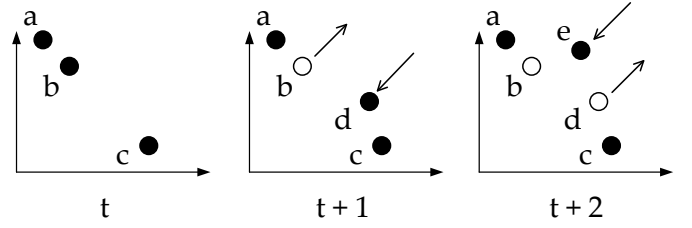


Figure 1. Illustration of an archiver based on the archiving rule “Pareto dominance + density” deteriorating. The capacity of the archive is 3. Black circles denote solutions in the archive and hollow circles denote solutions removed. At the timestep $t + 1$, solution d enters the archive and b is removed since d has less crowding degree than b . At the timestep $t + 2$, solution e enters the archive and edges out d since e has less crowding degree than d . Now the archive consists of $\{a, e, c\}$, which is worse than the archive of $\{a, b, c\}$ at timestep t .

The condition $A^{(t)} \subseteq A^{(t-1)} \cup S^{(t)}$ implies that the archiving algorithm is not allowed to revisit nor store previous solutions in the input sequence beyond those present in $A^{(t-1)}$ or duplicated in $S^{(t)}$.

In EMO, there are generally three ways to use the archive. Firstly, the archive may be used solely to store high-quality solutions found by a search algorithm but it does not influence the generation of solutions (i.e., the sequence), such as in PAES [66], MOEA/D [8] and others [9, 63, 91]. This is often called an *external* archive.

In the second way, the archive not only stores high-quality solutions but also participates in some way in the generation of new solutions; for example, in the crossover operation, one parent solution is from the population and the other solution is from the archive, such as in SPEA [21], ϵ -MOEA [90] and others [61].

The third way is what the vast majority of MOEAs follow, in which the archive is essentially the evolutionary population of the MOEA and new solutions are generated solely from it, that is: $S^{(t)} \leftarrow \text{Generator}(A^{(t-1)})$; $A^{(t)} \leftarrow \text{Archiver}(A^{(t-1)}, S^{(t)})$. Different MOEAs use different terms for this type of archive, e.g., NSGA-II [5] calls it “population” whereas SPEA2 [6] calls it “archive”. However, not all populations are archives, e.g., the (offspring) populations in NSGA-II and SPEA2, which temporarily store newly generated solutions, are elements of the input sequence.

An undesirable property of most MOEAs is that their archive/population may deteriorate in quality, i.e., the final archive returned by the MOEA may be worse than an archive in a previous step. This issue has been reported very early in the study of archiving [2, 9, 31, 92]. Figure 1 illustrates how an archiver based on the “Pareto dominance + density (crowding distance)” criterion may result in an archive at timestep t that is *better* (see Def. 4) than the archive two timesteps later. Not only density-based archivers suffer from deterioration, but also some indicator-based (e.g., in SMS-EMOA [7] as shown later in Fig. 4) and decomposition-based archivers (e.g., in NSGA-III [11] as shown by [17]).

IV. THEORY

When designing an archiver, one may wish it to hold some theoretical properties. For example, one may wish its archive

to consist of only Pareto-optimal points with respect to the input sequence; not to deteriorate (i.e., the current archive cannot be worse than at a previous timestep); to be able to converge with sufficient timesteps; and to contain as many nondominated solutions as possible within its capacity.

Unfortunately, archivers in most well-known MOEAs do not hold such theoretical desirables. For example, a large portion of their final population are not Pareto optimal with respect to the solutions generated; e.g., as reported in [16], the majority of solutions in the final population of NSGA-II, SPEA2 and MOEA/D are dominated by other solutions generated during the search on some problems, such as FON [93]. This not only fails to return the decision maker the best solutions found, which could be remedied by using an external unbounded archive, but may also affect the search progress since the evolutionary population cannot represent the best solutions discovered.

In most MOEAs, the population/archive has two roles: (1) storing the best solutions found for the decision-maker and (2) maintaining a solution set as the source for generating new solutions during the search. Here, we focus on the properties of archivers that are desirable for the first role, but not necessarily desirable for the second one.

A. Properties

In general, there are two types of properties that an archiving algorithm may have: anytime properties and limit properties. Anytime properties must hold at any timestep t , whereas limit properties must hold after a finite number of timesteps under that assumption that any solution may appear an infinite number of times in the input sequence [30]. In the following, we introduce six properties from the literature [2, 4, 39] but formulated in a manner that is applicable to both bounded-size archivers, which only store nondominated solutions, and fixed-size archivers, which may store dominated solutions. The first three properties are anytime properties, and the last three ones are limit properties. Any anytime property that holds for one pass over a finite sequence should also hold in the limit, that is, unlimited passes over a finite sequence or an infinite sequence drawn from a finite set Y . We denote the union of all points seen up to time t by $Y^{(t)} = \bigcup_{i=1}^t S^{(i)}$.

Property 1 (Pareto-subset [39]). *An archiver has the Pareto-subset property if no nondominated solution in its archive at any timestep is dominated by a solution in the input sequence seen so far: $\forall t \in \mathbb{N}^+, \forall a \in \min(A^{(t)}, \prec), \nexists s \in Y^{(t)}, s \prec a$.*

Property 2 (point-monotone [4]). *An archiver has the point-monotone property if $\forall t, \forall i \in \mathbb{N}^+$, there does not exist a pair of solutions $a \in \min(A^{(t)}, \prec)$ and $a' \in \min(A^{(t+i)}, \prec)$, such that $a \prec a'$. An archiver that does not have this property is said to point-deteriorate.*

Property 3 (set-monotone [4]). *An archiver has the set-monotone property if $\forall t, \forall i \in \mathbb{N}^+$, there does not exist a pair of archives $A^{(t)}, A^{(t+i)}$, such that $A^{(t)} \triangleleft A^{(t+i)}$, i.e., $A^{(t)}$ is better in terms of Pareto optimality than $A^{(t+i)}$. An archiver that does not have this property is said to set-deteriorate.*

Amongst the three properties, Property 1 is the strictest one to hold for an archiver and implies Property 2 which implies Property 3. Property 2 requires the archiver to have a rule prohibiting solutions from entering the archive if they are dominated by solutions eliminated previously. Such a rule can be implemented by setting a box (defined by ϵ) for every solution in the current archive and rejecting any oncoming solution in those boxes [15, 36].

Property 3 is more attainable; for example, if the sequence of archives never decreases the value of a Pareto compliant indicator, such as the multi-level grid archiver [12], the hypervolume-based archiver proposed by Knowles [39] or the hypervolume-based environmental selection of SMS-EMOA [7] (on the condition that the reference point used in the calculation of hypervolume does not change).

Now, we introduce three limit properties:

Property 4 (limit-stable [4]). *An archiver has the limit-stable property if for any sequence there exists a timestep $t \in \mathbb{N}^+$ such that $\forall i \in \mathbb{N}^+, A^{(t)} = A^{(t+i)}$. That is, the archive converges to a stable solution set in finite time.*

Being *limit-stable* has some practical benefits, e.g., used as a stop condition during the search. Moreover, *set-monotone* implies *limit-stable*, thus an archiver that is not *limit-stable* cannot be *set-monotone*. But it may be of more interest if all solutions of the converged archive are Pareto-optimal solutions of the sequence:

Property 5 (limit-Pareto-subset). *An archiver has the limit-Pareto-subset property if for any sequence there exists a timestep $t \in \mathbb{N}^+$ such that $\forall i \in \mathbb{N}^+, A^{(t)} = A^{(t+i)}$ and $\min(A^{(t)}, \prec) \subseteq Y^*$, where $Y^* = \min(Y, \prec)$. That is, the archive converges to a subset of the Pareto optimal set of Y .*

Converging to a stable Pareto subset is desirable, but one may also care about the number of Pareto optimal solutions in the converged archive. The decision-maker may not be very happy if an archive of capacity $N = 100$ ends up with only one Pareto-optimal solution after being feed with hundreds of them. This intuition leads to the following definition and property.

Definition 8 (optimal approximation of bounded size [4]). *Let a solution set $A \subseteq Y$, $1 \leq |A| \leq N$, be a nondominated set, i.e., $A = \min(A, \prec)$. If $\nexists B \subseteq Y$, $|B| \leq N$, such that $B \triangleleft A$, then A is called an optimal approximation with bounded size N of Y^* , where $Y^* = \min(Y, \prec)$.*

An optimal approximation of bounded size N of the nondominated solutions in the sequence seen so far is the best possible archive that a bounded archiver can produce with respect to Pareto optimality. Unfortunately, as proved by Knowles and Corne [2], no archiver can guarantee to store at least N nondominated solutions (or the number of nondominated solutions in the sequence seen so far, if the latter is smaller than N). As a consequence, no archiver can guarantee to store an optimal approximation of bounded size N at every timestep and for any finite sequence [4]. Having said that, its limit form may be achievable.

Algorithm 1: Archiving algorithm based on a weakly Pareto-compliant indicator

Input: $A^{(t-1)}, s^{(t)}$

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1 if  $\exists a \in A^{(t-1)}, a \preceq s^{(t)}$  then           // Rule 1: if the new solution is weakly dominated by a solution in the archive.
2   |  $A^{(t)} \leftarrow A^{(t-1)}$ 
3 else if  $|\min(A^{(t-1)} \cup \{s^{(t)}\}, \prec)| \leq N$  then           // Rule 2: if the number of all nondominated solutions is less than or
4   |  $A^{(t)} \leftarrow \min(A^{(t-1)} \cup \{s^{(t)}\}, \prec)$            // equal to the archive capacity after adding the new solution.
5 else if  $I(A^{(t-1)}) \leq \min_{a \in A^{(t-1)}} \{I(A^{(t-1)} \cup \{s^{(t)}\} \setminus \{a\})\}$  then           // Rule 3: if the new solution cannot lead to a better
6   |  $A^{(t)} \leftarrow A^{(t-1)}$                                    // indicator value.
7 else                                           // Rule 4: the new solution can lead to a better indicator value.
8   |  $a' \leftarrow \arg \min_{a \in A^{(t-1)}} \{I(A^{(t-1)} \cup \{s^{(t)}\} \setminus \{a\})\}$ 
9   |  $A^{(t)} \leftarrow A^{(t-1)} \cup \{s^{(t)}\} \setminus \{a'\}$ 
10 end
Output:  $A^{(t)}$ 

```

Property 6 (limit-optimal [4]). *An archiver has the limit-optimal property if for any sequence there exists a timestep $t \in \mathbb{N}^+$ such that $\forall i \in \mathbb{N}^+, A^{(t)} = A^{(t+i)}$ and $\min(A^{(t)}, \prec)$ is an optimal approximation of size N of Y^* , where $Y^* = \min(Y, \prec)$ and N is the capacity of the archive.*

An archiver that optimizes a Pareto-compliant indicator when updating the archive will be *limit-optimal* [4]. Holding this property can bring practical benefits, as shown in [16], where a hypervolume-based archiver significantly outperforms other non-convergence-guaranteed archivers. In the next subsection, we prove that an archiver optimising a weakly Pareto-compliant indicator as the archiving criterion may also hold this property. That means that some popular indicators in the EMO area, such as the ϵ -indicator [20] and IGD^+ [26], can also lead to convergence-guaranteed archivers.

B. Convergence-Guaranteed Archivers with Weakly-Pareto-Compliant Indicators

Weak Pareto compliance is a weaker version of Pareto compliance for quality indicators. If an indicator is weakly Pareto compliant, then it may not be able to distinguish between two sets, even if one is better than the other with respect to Pareto optimality (Def. 4). For example, given the sets $A = \{(0, 1), (0.5, 0.5), (1, 0)\}$ and $B = \{(0, 1), (1, 0)\}$, then $A \triangleleft B$ (A provides the decision maker with one more option than B). A weakly Pareto compliant indicator may evaluate them to be the same, but a Pareto compliant indicator will evaluate A to be better than B .

We show here that, with respect to the theoretical properties for archiving presented above, weakly-Pareto-compliant indicators are as desirable as Pareto-compliant ones, if the archiving rules are designed properly. Until now, only archivers based on a Pareto-compliant indicator were known to hold properties such as *limit-optimal* (see Table I in the next section).

In this section, in accordance with previous studies [2, 4, 36, 38], we assume the input sequence is a sequence of individual solutions presented to the archiving algorithm one at a time, that is, $\forall t, |S^{(t)}| = 1$ (Section III-B). We can always convert a many-at-a-time sequence into a one-at-a-time sequence, thus properties that hold for archivers that handle the latter also

hold for archivers that handle the former. The opposite is not true, however, as we will discuss later in Section VI-C. To make explicit when we refer to a one-at-a-time sequence, we denote its elements by $s^{(t)}$ and the sequence up to the time t by $\mathcal{S}^{(t)} = \langle s^{(1)}, s^{(2)}, \dots, s^{(t)} \rangle$.

We consider a generic archiving algorithm (Algorithm 1) based on a weakly Pareto-compliant indicator I . The archiving rules we propose are very general and similar to those used in most indicator-based archivers: (1) uses weak Pareto dominance relation to compare solutions in the archive with the new solution; (2) checks if the nondominated set after adding the solution exceeds the archive capacity; (3) checks if adding the new solution does not lead to a better indicator value; otherwise (4) removes the archived solution that contributes the least to the indicator value after adding the new solution.

We now prove that, assuming the indicator I is weakly Pareto compliant, the archiver in Algorithm 1 holds the three limit properties: *limit-stable*, *limit-Pareto-subset* and *limit-optimal*. To do so, we first need to introduce several auxiliary properties of the archiver.

Lemma 1. *The I value of the archive under Algorithm 1 never degrades: $\forall t \in \mathbb{N}^+, I(A^{(t+1)}) \leq I(A^{(t)})$.*

Proof: Rules 1 and 3 will not change the I value as the archive remains the same. Rule 4 will lead to a better I value according to the definition. For Rule 2, $A^{(t)} = \min(A^{(t-1)} \cup \{s^{(t)}\}, \prec)$ and the nondominated set of the union of the archive and the new solution is *better* than the archive, i.e., $A^{(t)} \triangleleft A^{(t-1)}$, which implies $I(A^{(t)}) \leq I(A^{(t-1)})$ because of the definition of a weakly Pareto-compliant indicator (Definition 5). Thus, the I value of the archive will never degrade. ■

Lemma 2. *Under Algorithm 1, if the archive is different at two different timesteps, t and $t+i$, then for any timestep after $t+i$, the archive is always different from the archive at timestep t : $\forall t, i \in \mathbb{N}^+, A^{(t)} \neq A^{(t+i)} \implies \forall j \in \mathbb{N}^+, A^{(t)} \neq A^{(t+i+j)}$.*

Proof: By contradiction. Assume $A^{(t)} \neq A^{(t+i)}$ and $\exists j \in \mathbb{N}^+$ such that $A^{(t)} = A^{(t+i+j)}$. Due to $I(A^{(t)}) = I(A^{(t+i+j)})$ and since the I value of the archive never degrades (Lemma 1), we have $I(A^{(t)}) = I(A^{(t+i)}) = I(A^{(t+i+j)})$. This implies that, from timestep t to $t+i+j$, the archiving process never

goes through Rule 4, since Rule 4 necessarily leads to a better I value.

Consider the archiving process from the timestep t to $t+i$. Since $A^{(t)} \neq A^{(t+i)}$ and Rules 1 and 3 do not change the archive, the archiving process must go through Rule 2 at least once, where it accepts a new solution a that is not dominated by any solution in $A^{(t)}$.

To satisfy our assumption that $A^{(t)} = A^{(t+i+j)}$, the archiving process must eliminate solution a between timestep $t+i$ and $t+i+j$, but without accepting any new solution. This must happen at Rule 2 since Rules 1 and 3 will not change the archive. But Rule 2 will only eliminate solution a in the archive if it is dominated by the new solution added to the archive. Likewise, this new solution added to the archive cannot be eliminated through Rule 2 without accepting a newer and better solution. Therefore, the archiving process cannot remove a without accepting a new solution that dominates it and $A^{(t+i+j)}$ can never go back to $A^{(t)}$, thus the assumption cannot hold. ■

Lemma 2 means that Algorithm 1 cannot revisit again a previous archive after the archive has changed. With that in mind, we are in a position to prove that the three limit desirable properties hold for Algorithm 1.

Theorem 1. *Algorithm 1 is limit-stable:*

$$\exists t, \forall i \in \mathbb{N}^+, A^{(t)} = A^{(t+i)}.$$

Proof: By contradiction. Assume the archive never converges, i.e. $\forall t, \exists i \in \mathbb{N}^+, A^{(t)} \neq A^{(t+i)}$. This implies that there are an infinite number of different archives since none can be revisited (Lemma 2). However, since input solutions are drawn from the finite set Y , there must be a finite number of different archives, thus a contradiction. ■

Theorem 2. *Algorithm 1 is limit-Pareto-subset:*

$\exists t, \forall i \in \mathbb{N}^+, A^{(t)} = A^{(t+i)}$ and $\min(A^{(t)}, \prec) \subseteq Y^*$, where $Y^* = \min(Y, \prec)$.

Proof: According to Theorem 1, we have $\exists t, \forall i \in \mathbb{N}^+, A^{(t)} = A^{(t+i)}$. Moreover, the archive is always a nondominated set, i.e., $A^{(t)} = \min(A^{(t)}, \prec)$, because Rule 1 prevents adding a new solution that is weakly dominated by any solution in the archive and Rule 2 removes archived solutions that are dominated by the new solution. Thus, we only need to prove $A^{(t)} \subseteq Y^*$.

By contradiction: Let us assume $A^{(t)} \not\subseteq Y^*$. As Y^* is the set of all the Pareto-optimal solutions of Y , there exists at least one solution in $A^{(t)}$ that is dominated by at least one solution $y^* \in Y^*$. Since all solutions have a non-zero probability of being generated in a future timestep, then $\exists i \in \mathbb{N}^+$, such that the archiver receives $s^{(t+i)} = y^*$. Then, the algorithm must go to Rule 2 since there is no solution in $A^{(t+i-1)}$ weakly dominating y^* , which is Pareto optimal. Since we assumed that there is at least one solution in $A^{(t)} = A^{(t+i-1)}$ dominated by y^* , then $|\min(A^{(t+i-1)} \cup \{y^*\}, \prec)| \leq N$ and Rule 2 will accept y^* , which implies that $A^{(t)} \neq A^{(t+i)}$, thus contradicting Theorem 1. ■

Theorem 3. *Algorithm 1 is limit-optimal:*

$\exists t, \forall i \in \mathbb{N}^+, A^{(t)} = A^{(t+i)}$ and $\min(A^{(t)}, \prec)$ is an optimal

approximation of size N of Y^* , where $Y^* = \min(Y, \prec)$ and N is the capacity of the archive.

Proof: According to Theorem 1, we have $\exists t, \forall i \in \mathbb{N}^+, A^{(t)} = A^{(t+i)}$. Its proof also shows that $A^{(t)} = \min(A^{(t)}, \prec)$. Thus, we need to prove that $A^{(t)}$ is an optimal approximation of bounded size (Definition 8), i.e., $\nexists B \subseteq Y, |B| \leq N$ such that $B \prec A^{(t)}$.

It is easy to see that $|A^{(t)}| \leq \min\{N, |Y^*|\}$ because N is the capacity of the archive and $A^{(t)} \subseteq Y^*$ (Theorem 2). Now we prove the theorem by considering three cases: (i) $|A^{(t)}| = |Y^*|$, (ii) $|A^{(t)}| = N$, and (iii) $|A^{(t)}| < \min\{N, |Y^*|\}$.

Let us first consider the case $|A^{(t)}| = |Y^*|$. Rule 1 in Algorithm 1 forbids duplicated solutions in $A^{(t)}$. Thus, $|A^{(t)}| = |Y^*|$ implies $A^{(t)} = Y^*$ (Theorem 2) and the archive is optimal (it contains the complete Pareto front). Thus, $\nexists B \subseteq Y$ such that $B \prec A^{(t)}$.

Let us now consider the case $|A^{(t)}| = N$. According to Rule 1 and $A^{(t)} \subseteq Y^*$ (Theorem 2), we know that all solutions in $A^{(t)}$ are unique elements of Y^* . Assume that $\exists B \subseteq Y, |B| \leq N$, such that $B \prec A^{(t)}$, which implies $B \preceq A^{(t)} \wedge A^{(t)} \not\subseteq B$. Since $B \preceq A^{(t)}$, B should contain all solutions in $A^{(t)}$ as they are unique elements of Y^* , i.e., $A^{(t)} \subseteq B$. In addition, $A^{(t)} \not\subseteq B$ implies that $A^{(t)} \neq B$, thus $A^{(t)} \subset B$ and $|B| > N = |A^{(t)}|$, a contradiction with $|B| \leq N$, thus there is not such $B \prec A^{(t)}$.

Lastly, let us consider the case $|A^{(t)}| < \min\{N, |Y^*|\}$. $|A^{(t)}| < \min\{N, |Y^*|\}$ implies that $A^{(t)}$ is missing at least one solution from Y^* . If the missed solution(s) are already duplicated in $A^{(t)}$, the archive is optimal. Let us assume one of the missed solutions is not duplicated; then Rule 2 will accept the solution because $|A^{(t)}| < N$, contradicting the initial assumption that $A^{(t)} = A^{(t+i)}$. Therefore, when $|A^{(t)}| < \min\{N, |Y^*|\}$, $A^{(t)}$ must consist of all unique elements of Y^* , thus $\nexists B \subseteq Y$ such that $B \prec A^{(t)}$. ■

In summary, an archiver based on a weakly Pareto-compliant indicator can respect the three limit properties. In addition, Algorithm 1 also respects the *set-monotone* property (Property 3); the proof is straightforward given the definition of a weakly Pareto-compliant indicator. The overall conclusion is that archivers based on a weakly Pareto-compliant indicator can hold the same theoretical desirables as archivers based on a Pareto-compliant indicator. This conclusion may explain recent empirical observations [94] showing no significant difference between MOEAs guided by either weakly Pareto-compliant indicators or Pareto-compliant indicators. These theoretical and empirical results should encourage the study of archivers based on weakly Pareto-compliant indicators, since many indicators meet the condition of being weakly Pareto compliant, including ϵ -indicator [20], IGD⁺ [26], R2 [95], PCI [96], IPF [97] and others [98–101].

V. CLASSIFICATION OF EXISTING ARCHIVERS

In this section, we review existing archivers in the literature on the basis of the theoretical properties in the previous section. Apart from those theoretical desirables, there may also exist practical desirables for archivers to respect.

Table I
CLASSIFICATION OF REPRESENTATIVE ARCHIVING ALGORITHMS AND THEIR THEORETICAL AND PRACTICAL DESIRABLES.

Class	Archiver	Theoretical desirables						Practical desirables			
		<i>Pareto subset</i>	<i>Point-monotone</i>	<i>Set-monotone</i>	<i>Limit-stable</i>	<i>Limit-Pareto subset</i>	<i>Limit-optimal</i>	<i>Diversifies</i>	<i>Controllable size</i>	<i>Polynomial time</i>	<i>No problem-specific parameter</i>
I	NSGA-II [5]							+	+	+	+
	SPEA2 [6]							+	+	+	+
	NSGA-III [11]							+	+	+	+
II	A_{dom} [30]		+	+	+	+	+		+	+	+
	ϵ -approx [15]		+	+	+			+		+	
	ϵ -Pareto [15]	+	+	+	+	+		+		+	
III	MOEA/D-PBI [8]				+			+	+	+	+
	MOEA/D-TCH [8]		–	–	+	+		+	+	+	+
	$A_{R_2}^*$ [102]		–	–	+	+		+	+	+	+
IV	A_{HV} [1]			+	+	+	+	+	+		
	SMS-EMOA [7]			–	–	–	–	+	+		+
	MGA [12]			+	+	+	+	+	+	+	+

*Here A_{R_2} is slightly different from [102], in which the new solution will be rejected if it has the same lowest fitness as the old ones (see Algorithm 6). “+” indicates that the archiver can fully respect the specified desirable and “–” indicates that the archiver can respect the desirable under certain condition.

For example, one may wish that (1) an archiver *diversifies*, i.e., avoids convergence to a small region of the Pareto front; (2) the size of its archive is controllable, not only respecting any user-defined maximum capacity, but being as full of nondominated solutions as possible; (3) the archiving operation does not take too much time, e.g., not exponentially increasing with the number of objectives; and (4) the archiving process does not need any problem-dependent parameter set by the user. Non-diversifying archivers include efficiency preserving archivers [3] that, when full, only accept solutions that dominate an archived solution, thus they often converge to one or few small regions of the Pareto front. Non-efficiency preserving archivers may also fail to diversify; for example, an archiver that removes the solution farthest away from an ideal point or an archiver that select solutions according to their distance to reference vectors, if the vectors used are not well-distributed along the Pareto front.

According to these properties, archivers can be categorised into four classes. The first class (I) refers to archiving algorithms (or selection criteria) that do not hold any theoretical desirables. Archivers in many well-established MOEAs belong to this class. The second class (II) refers to those that never deteriorate (i.e., hold the *point-monotone* property) but are not very useful in practice due to failing to diversify or not using their full capacity to store nondominated solutions. The third class (III) refers to those that have some good theoretical and practical properties, but are not *limit-optimal*. The fourth class (IV) refers to those that, in addition to having have good theoretical and practical properties, are also *limit-optimal* (under certain conditions).

Table I shows several representative archivers in the four classes and their theoretical and practical properties. For some classes, there are numerous archivers in the area (such as Class I and III) and we only consider representative algorithms. For classes where there are very few archivers (such as Class IV), we aim to list them completely provided that they are significantly different in terms of the archiving criteria used.

Algorithm 2: Archiver based on NSGA-II’s selection rules.

Input: $A^{(t-1)}, s^{(t)}$
 // Partition all the solutions into different nondominated fronts
 // and identify the last front F_l .
 1 $(F_1, F_2, \dots, F_l) \leftarrow \text{nondom_sorting}(A^{(t-1)} \cup s^{(t)})$
 // Find solution in F_l with the minimum crowding distance.
 2 $a \leftarrow \arg \min_{a \in F_l} \text{crowding_distance}(F_l)$
 3 $A^{(t)} \leftarrow A^{(t-1)} \cup \{s^{(t)}\} \setminus \{a\}$
Output: $A^{(t)}$

A. Class I: Archivers Holding No Theoretical Properties

The first class contains archivers from many well-established MOEAs that do not hold any theoretical desirables. It includes all Pareto-based (Pareto dominance + density) algorithms and some archivers in other types of algorithms (e.g., decomposition-based ones).

In general, the archiving procedure (i.e., environmental selection procedure) in Pareto-based MOEAs, such as NSGA-II [5] and SPEA2 [6], consists of two steps: considering Pareto dominance first and then solutions’ density. As an example, in NSGA-II, first a nondominated sorting procedure divides the archive into different nondominated fronts, and then a density metric (crowding distance) is used to select among solutions in the last front; Algorithm 2 gives the procedure of an archiver based on NSGA-II’s selection rules. In such archivers, the density-based rules are the cause of set-deterioration² because they may eliminate a nondominated, or even Pareto-optimal, solution that may end up dominating another solution later accepted. As shown in Figure 1 previously, an inferior solution can enter the archive provided that it is located in a sparser region and there is no solution in the current archive dominating it.

Set-deterioration not only arises with density-based rules but also with the decomposition-based rules in NSGA-III [11] and the rules that combine solutions’ density with proximity to the

²Set-deterioration (see Prop. 3) implies point-deterioration (Prop. 2).

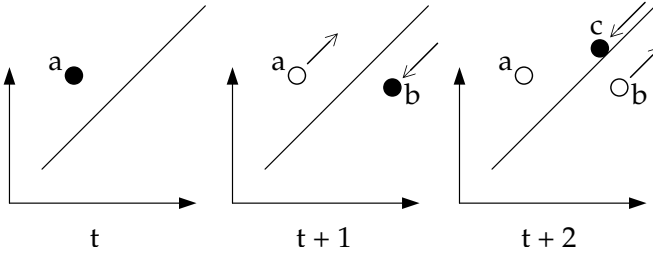


Figure 2. Illustration of deterioration when using NSGA-III's archiving rules (adapted from [17]). Black circles denote solutions in the archive, hollow circles denote solutions removed, and the solid line is the weight vector considered. At the timestep t , the archive contains solution a . At the timestep $t+1$, solution b replaces a since b is closer than a to the line. At the timestep $t+2$, solution c replaces b since c is closer to the line. However, c is dominated by a and the archive at t was better ($<$) than the one at $t+2$, thus the archive both *set-deteriorates* and *point-deteriorates*.

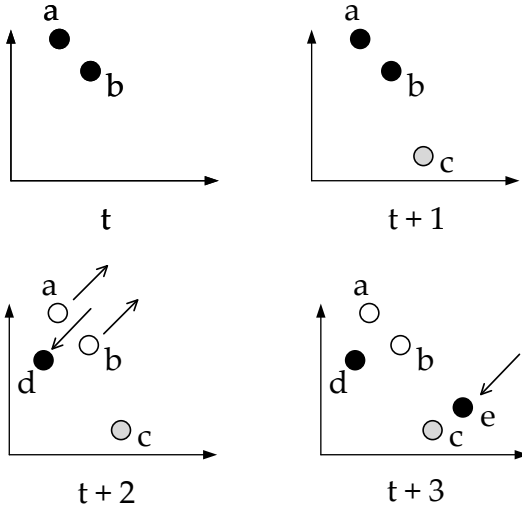


Figure 3. Illustration that the archiver A_{dom} does not hold the property *Pareto-subset* (Property 1). The capacity of the archive is 2. Here, black circles denote solutions in the archive, hollow circles denote solutions removed from the archive and grey circles denote solutions that are not allowed to enter the archive. At the timestep $t+1$, solution c cannot enter the archive because it does not dominate any solution in the current archive and the archive is at full capacity. At the timestep $t+2$, solution d enters the archive and solutions a and b are removed since they are dominated by d . At the timestep $t+3$, solution e enters the archive since it is not dominated by d and the archive is not at full capacity. However, solution e is dominated by a solution of the sequence (i.e., c), thus the archiver does not hold the *Pareto-subset* property.

Pareto front in SDE [103]. Figure 2 illustrates how NSGA-III's archiving rules produce set-deterioration. The rules in NSGA-III first consider the Pareto dominance relation between solutions and, if they are nondominated, then compare their closeness to the weight vectors. As can be seen in Figure 2, the archive eliminates solution a , but after two timesteps it accepts solution c , which is dominated by a , thus the archive at $t+2$ is worse than at t .

B. Class II: Archivers Holding Some Theoretical Properties but not Useful in Practice

This class comprises archivers holding some theoretical desirables but not being very useful in practice. It includes (1) archivers that do not diversify, e.g., the *dominating* archiver

Algorithm 3: A_{dom} Archiver

Input: $A^{(t-1)}, s^{(t)}$

```

1 if  $\exists a \in A^{(t-1)}, a \preceq s^{(t)}$  then
2    $A^{(t)} \leftarrow A^{(t-1)}$ 
3 else if  $|\min(A^{(t-1)} \cup \{s^{(t)}\}, \prec)| \leq N$  then
4    $A^{(t)} \leftarrow \min(A^{(t-1)} \cup \{s^{(t)}\}, \prec)$ 
5 else
6    $A^{(t)} \leftarrow A^{(t-1)}$ 
7 end
Output:  $A^{(t)}$ 

```

(A_{dom}) found in the AR1 algorithm [30], which when full only accepts a new solution if it dominates an archived one, and (2) archivers that do not “use” their full capacity to store nondominated solutions, e.g., ϵ -approx and ϵ -Pareto [15] whose archive size cannot be controlled if the parameter ϵ is pre-defined, and adapting ϵ to bound the maximum size tends to archive too few points [2], thus they are not limit-optimal.

As the earliest archiver that holds theoretical desirables, A_{dom} is set to only accept a new solution if it dominates some solution in the archive (at the time when the archive is at full capacity). Algorithm 3 gives the procedure of A_{dom} . Since a solution in the archive cannot be removed unless a dominating one arrives, the archiver respects the property *point-monotone*, which implies *set-monotone*, and the property *limit-optimal*, which implies all the other limit properties. Yet, A_{dom} does not respect the property *Pareto-subset*. An example is given in Figure 3, where the archive rejects solution c but later accepts a solution (i.e., solution e) dominated by c . This is because when c arrives, the archive is full, but after several timesteps when e arrives, there is a slot available in the archive.

Another type of archivers in this class are those that are not able to control its archive size: they either cannot respect a fixed maximum capacity N or archive too few nondominated solutions. In such archivers, the objective space is divided into regions and only one solution can be accepted within each region. Such a region can be a hyper-box [15, 40, 68, 90, 104] (defined by a parameter ϵ or its variant [43]). Since the superiority relation between nondominated solutions in such archivers is not changed (old one always being regarded better than new one if they are in the same region), they can hold many theoretical desirables including *point-monotone*³, *set-monotone* and *limit-stable*. They can also be extended to hold the *Pareto-subset* and *limit-Pareto-subset* properties, provided that Pareto dominance is considered in the archiving update. However, they do not respect the property *limit-optimal* as a nondominated solution may not be allowed to enter the archive even if the archive is not full, because there is already one solution in the same region.

One of the most representative archivers in this class is ϵ -Pareto [15]. It is the only known archiver that guarantees storing a subset of the Pareto-optimal solutions seen so far (*Pareto-subset*), while also capable of diversifying. Algorithm 4 gives the procedure of ϵ -Pareto. A new solution is accepted if it meets one of the three conditions: (1) it *box*-dominates some

³In [4], the archiver ϵ -approx [15] was said to not hold the *point-monotone* property, which is not the case.

Algorithm 4: ϵ -Pareto Archiver

Input: $A^{(t-1)}$, $s^{(t)}$, ϵ
 /* $\text{box}(a)$ is the box index vector of a that discretises the space into boxes based on ϵ , where $\text{box}_i(a) = \left\lfloor \frac{\log a_i}{\log(1+\epsilon)} \right\rfloor$ for $i = 1, \dots, d$ and d is the number of objectives. */

```

1  $D \leftarrow \{a \in A^{(t-1)} \mid \text{box}(s^{(t)}) \prec \text{box}(a)\}$ 
2 if  $D \neq \emptyset$  then
3    $A^{(t)} \leftarrow A^{(t-1)} \cup \{s^{(t)}\} \setminus D$ 
4 else if  $\exists a \in A^{(t-1)}, \text{box}(a) = \text{box}(s^{(t)}) \wedge s^{(t)} \prec a$  then
5    $A^{(t)} \leftarrow A^{(t-1)} \cup \{s^{(t)}\} \setminus \{a\}$ 
6 else if  $\nexists a \in A^{(t-1)}, \text{box}(a) \preceq \text{box}(s^{(t)})$  then
7    $A^{(t)} \leftarrow A^{(t-1)} \cup \{s^{(t)}\}$ 
8 else
9    $A^{(t)} \leftarrow A^{(t-1)}$ 
10 end
Output:  $A^{(t)}$ 

```

solution in the archive (lines 1–3), (2) it is located in the same box as another solution but dominates the latter (lines 4–5), or (3) there is no any other occupied box weakly dominating it (lines 6–7).

A practical weakness of ϵ -Pareto is that the size of the archive is not controllable (despite bounded [105]), but determined by the interplay between the value of ϵ , the optimisation problem and the search algorithm. Even if ϵ is adapted such that the archive size never surpasses a given capacity, the actual number of solutions at the end of the archiving process are often much fewer than this capacity [2]. In some cases, it is not even possible to find a value of ϵ such that the size of the archive approximates the given capacity [106]. Since an optimal set of size close to the user-specified maximum capacity is of primary interest, an archive of uncontrollable size is not very practical.

C. Class III: Archivers Holding Theoretical Properties and Being of Practical Use, but not Limit-Optimal

This class includes archivers that perform well in practice and also hold some theoretical properties (under certain conditions). They can further be divided into two types of methods, decomposition-based methods and indicator-based methods.

The decomposition-based archiving methods, represented by MOEA/D [8], decompose the original multi-objective problem into a number of single-objective subproblems through a set of weight vectors and a scalarising function. Algorithm 5 gives the procedure of the archiving algorithm based on MOEA/D, which follows a rather different template than other archivers. The archiver In MOEA/D manipulates a set of N weights (rather than N solutions), and each weight is associated with a solution, which has the best value on that weight. Since a solution may associate with multiple weights, the total number of unique nondominated solutions in the archive may be significantly less than N , particularly on problems with irregular Pareto fronts [107].

Depending on the scalarising function used (e.g., TCH or PBI [8]), MOEA/D archivers may hold different theoretical properties. For example, MOEA/D+PBI holds none of the three desirable anytime properties, since the PBI scalarising

Algorithm 5: Archiver based on MOEA/D's selection rules

Input: $A^{(t-1)}$, $s^{(t)}$, $W = \{w_1, w_2, \dots, w_N\}$ (set of weights), r (reference point)
 // $A^{(t-1)}$ is the set of solutions associated with each weight,
 // i.e., $A^{(t-1)} = \{a_{w_1}^{(t-1)}, \dots, a_{w_N}^{(t-1)}\}$.

```

1  $r \leftarrow \text{update\_refpoint}(r, s^{(t)})$  // Update the reference point.
2 foreach  $w_i \in W$  do
3   if  $\text{Scalarize}(s^{(t)}, w_i, r) < \text{Scalarize}(a_{w_i}^{(t-1)}, w_i, r)$  then
4     /* Replace the current solution associated with  $w_i$  with the new solution  $s^{(t)}$  if  $s^{(t)}$  has better scalar value on  $w_i$ . */
5      $a_{w_i}^{(t)} \leftarrow s^{(t)}$ 
6   else
7      $a_{w_i}^{(t)} \leftarrow a_{w_i}^{(t-1)}$ 
8   end
end
Output:  $A^{(t)}$ ,  $r$ 

```

function, which is an aggregation of the distance of a solution to the weight vector and the distance of its projection on the vector, may regard a dominated solution as better than the solution dominating it. In contrast, the Tchebycheff scalarising function, which is weakly in line with Pareto dominance, i.e., $a \prec b \implies \text{TCH}(a) \leq \text{TCH}(b)$, makes MOEA/D+TCH hold the *point-monotone* and *set-monotone* properties conditionally, i.e., as long as the ideal point used for the calculation of the Tchebycheff function does not actually change during the archiving process. As for the limit properties, since the ideal point can always be settled in the limit sense, the two archivers MOEA/D+PBI and MOEA/D+TCH are *limit-stable* and the latter is also *limit-Pareto-subset*. Yet, they are not *limit-optimal* as a nondominated solution may not be able to enter the archive even if the archive is not full, because it cannot lead to a better scalarising function value on any weight vector.

Another type of archivers in this class are indicator-based archivers. They use a quality indicator to measure the quality of the whole archive, such that the quality contribution of a solution is the difference of the indicator values between the archive with and without the solution. Most existing indicator-based archivers belong to this class, but not the one presented in the original IBEA [55]. This is because IBEA does not use an indicator to measure the quality of the whole set, but rather uses an indicator to define a measure (e.g., based on the ϵ -indicator) between two solutions, hence not holding these theoretical properties.

Indicator-based archivers, such as those found in SMS-EMOA [7], MO-CMA-ES [108] (hypervolume-based) and R2-EMOA [102], typically follow a two-step process: solutions are first ranked based on Pareto dominance and ties (nondominated solutions) are then broken based on a quality indicator (instead of the density metric in Pareto-based archivers). Algorithm 6 gives the procedure of indicator-based archivers from the literature. As can be seen from the algorithm, the archiver determines the set D of solutions least contributing to the indicator value, that is, solutions whose removal from the considered nondominated set would lead to the best indicator value compared to the removal of any other solution (line 2).

Algorithm 6: Archiver based on a common indicator-based MOEA's selection rules

Input: $A^{(t-1)}, s^{(t)}$
 // Partition all the solutions into different nondominated fronts and identify the last front F_l .
 1 $(F_1, F_2, \dots, F_l) \leftarrow \text{nondom_sorting}(A^{(t-1)} \cup s^{(t)})$
 // Find the solutions whose removal would minimize the indicator value of the nondominated set F_l .
 2 $D \leftarrow \arg \min_{a \in F_l} I(F_l \setminus \{a\})$
 3 **if** $s^{(t)} \in D$ **then**
 4 $A^{(t)} \leftarrow A^{(t-1)}$
 5 **else**
 6 $a' \leftarrow \text{sample}(D)$ // Draw a solution randomly from D .
 7 $A^{(t)} \leftarrow A^{(t-1)} \cup \{s^{(t)}\} \setminus \{a'\}$
 8 **end**
Output: $A^{(t)}$

Afterwards, if the new solution belongs to the set D of least-contributing solutions, then the archive is unchanged (line 4); otherwise, a solution from D is removed randomly (lines 6–7). In some indicator-based MOEAs, the new solution and old ones are not distinguished, e.g., in R2-EMOA [102]. That is, one of the least-contributing solutions will be randomly selected to remove, whether it is the new solution or the old one. Removing an old solution when it has the same indicator value as the new one may cause a cyclic behavior (i.e., solutions may enter and exit the archive many times during the archiving process) for some indicators like R2 [95]. The cyclic behavior prevents convergence and, thus, any limit properties. Moreover, randomly removing solutions with the same value of a weakly Pareto compliant indicator prevents set-monotonicity even when the archiver considers the dominance relation between solutions first (like in most indicator-based archivers). For example, let us consider three solutions $\{a, b, c\}$ with the same indicator value, and a dominates c but b is mutually nondominated with a and c . Imagine an archive of capacity one that at $t = 1$ only contains a . At $t = 2$, the archiver receives b and (randomly) removes a . At $t = 3$, the archiver receives c and (randomly) removes b . As result, the archive at $t = 3$ is dominated by the archive at $t = 1$.

Indicator-based archivers have different properties depending on whether the indicator used is Pareto compliant, weakly Pareto compliant or neither. In any case, all of them hold the *limit-stable* property since they always maximise/minimise the indicator value, as long as they remove the newest solution when they have the same indicator contribution. If the indicator used is not Pareto compliant, then the archiver will not have any Pareto dominance-related properties like *point-monotone*, *set-monotone* and *limit-Pareto-subset*. Representative example are IGD-based [109] archivers, e.g., [110, 111]. If the indicator is Pareto compliant like hypervolume, the archiver may hold most of the theoretical properties including *limit-optimal*, thus we will discuss them in the next section.

If the indicator is weakly Pareto compliant, the archiver (based on Algorithm 6) may hold many properties but not *limit-optimal* since the indicator may not be able to distinguish between solution sets subject to the \triangleleft -relation (Definition 5). Such an example is A_{R2} [102] in Table I. When the ideal

point is unchanged, A_{R2} holds the anytime properties *point-monotone* and *set-monotone*. Since the ideal point can always be settled in the limit sense, A_{R2} always holds the limit property *limit-Pareto-subset*.

Lastly, it is worth noting that despite using a weakly Pareto compliant indicator, the archiver proposed in Algorithm 1 is *limit-optimal*, thus it belongs to the class discussed next. The essential difference between Algorithm 1 and existing indicator-based archivers (presented in Algorithm 6) is that Algorithm 1 does not accept duplicate solutions in its archive, which makes the archive always “tight” and have room for accommodating different nondominated solutions. In contrast, in Algorithm 6 the archiver allows duplicate solutions. A duplicate solution may not be able to be replaced by a new nondominated solution since adding that nondominated solution into the archive may not necessarily lead to a better indicator value of the archive for a weakly Pareto compliant indicator.

D. Class IV: Archivers Holding the Limit-Optimal Property and also Being of Practical Use

Archivers in this class hold the critical property *limit-optimal* as well as being capable of diversifying their solutions. There do not exist many known archivers having these two desirable properties. Three representatives are A_{HV} [1], SMS-EMOA [7], and MGA [12], though one may expect more to emerge in the future since an archiver based on a weakly Pareto compliant indicator can also hold these desirables (if designed properly), as we proved previously.

The first two archivers A_{HV} and SMS-EMOA are both based on the hypervolume indicator, which is Pareto compliant. The difference between them is that in A_{HV} the reference point is fixed during the archiving process, whereas in SMS-EMOA the reference point is adaptive.

The archiver A_{HV} [1, 42] is arguably the earliest archiving algorithm using the hypervolume indicator. It presets a reference point for the hypervolume calculation and eliminates the solution with the least hypervolume contribution. The greedy nature of its update in the one-at-a-time case means that the resulting archive cannot maximise the hypervolume in the anytime scenario [112], so it does not meet *Pareto-subset* nor *point-monotone*. In the limit case, however, the archive will converge to a set of maximum hypervolume among all sets of size N , which implies that all its elements will be Pareto-optimal [113]. Nevertheless, its use in practice is not without challenges since setting an appropriate reference point a priori may require problem-specific knowledge.

In SMS-EMOA, like many well-established hypervolume-based archivers, the reference point is adapted, usually set to be a slightly worse vector than the nadir point of the nondominated set obtained. However, changing the reference point may lead to the archiver *set-deteriorating*. Figure 4 gives an example of the deterioration of the hypervolume-based archiving with an unfixed reference point. As the figure shows, the archive at the timestep $t + 3$ is dominated by its past version at timestep t , in which the reference point is determined adaptively by solutions in the archive and the

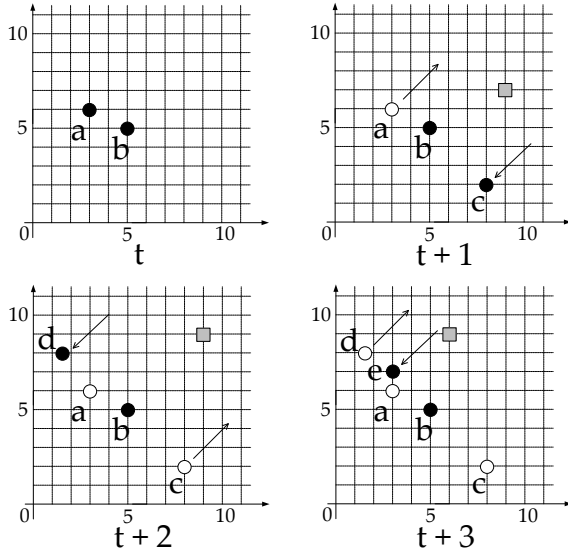


Figure 4. Illustration of the deterioration of hypervolume-based archiving with adaptive reference point. In this example, the reference point is the worst objective values in all solutions at the current timestep increased by one, as in SMS-EMOA [7]) and the capacity of the archive is $N = 2$. Black circles denote solutions in the archive, and hollow circles denote solutions removed from the archive. The grey square denotes the reference point used in the hypervolume calculation. At the timestep $t + 1$, solution c arrives and a is removed since its hypervolume contribution (HVC) is the lowest: $HVC(a) = HV\{a, b, c\} - HV\{b, c\} = 2$, $HVC(b) = HV\{a, b, c\} - HV\{a, c\} = 3$, $HVC(c) = HV\{a, b, c\} - HV\{a, b\} = 3$. At the timestep $t + 2$, solution d arrives and solution c is removed since it now has the lowest HV contribution: $HVC(b) = 9$, $HVC(c) = 3$, $HVC(d) = 3.5$. At the timestep $t + 3$, solution e arrives and solution d is removed since it now has the lowest HV contribution: $HVC(b) = 2$, $HVC(d) = 1.5$, $HVC(e) = 2$. However, e is dominated by a , thus the archive at the timestep $t + 3$ is dominated by the one at the timestep t , i.e., the archive shows *set-deterioration*.

new arrival. Such a HV-based archiver holds the three limit properties only if the nadir point settles down in the limit, which may or may not happen depending on the problem.

MGA or multi-level grid archiver [12] can be seen as an improved version of the ϵ -Pareto archiver [15]. It compares solutions using a hierarchy of boxes of different coarseness over the space. Algorithm 7 gives the procedure of MGA. As can be seen in the algorithm, the standard Pareto dominance relation is first used to compare solutions (lines 1–6), and, when the number of nondominated solutions exceeds the capacity of the archive, solutions are compared using *box-dominance*, i.e., applying the Pareto dominance relation to their box indices (lines 7–23). The size of the boxes is not set by a parameter, unlike the ϵ -approx and ϵ -Pareto archivers, but determined by the smallest coarseness level β that leads to at least one solution being weakly box-dominated (lines 7–14). If the new solution $s^{(t)}$ belongs to such weakly dominated boxes, then it is rejected (lines 15–16), otherwise an arbitrary solution from such boxes is eliminated (lines 17–20).

MGA does not respect the property *point-monotone*, as shown by [4], since any nondominated solution can enter the archive if the archive is not full, even if this solution was dominated by a solution previously removed. However, MGA cannot *set-deteriorate* because it implicitly optimises a Pareto compliant indicator such that accepting a new solution

Algorithm 7: Multi-level Grid Archiver (MGA)

Input: $A^{(t-1)}, s^{(t)}$

```

1 if  $\exists a \in A^{(t-1)}, a \prec s^{(t)}$  then
2    $A^{(t)} \leftarrow A^{(t-1)}$ 
3 else
4    $A' \leftarrow \min(A^{(t-1)} \cup \{s^{(t)}\}, \prec)$ 
5   if  $|A'| \leq N$  then
6      $A^{(t)} \leftarrow A'$ 
7   else
8     //  $\bar{b}$  is the largest box index possible in  $A'$ .
9      $\bar{b} \leftarrow \lfloor \log_2(\max_{a \in A'} \max_{i \in \{1, \dots, d\}} |a_i|) \rfloor + 1$ 
10     $\mathcal{Z} \leftarrow \{b \in \mathbb{Z}, b \leq \bar{b} \mid \exists a, a' \in A,$ 
11       $\text{box}^{(b)}(a) \preceq \text{box}^{(b)}(a') \wedge a \neq a'\}$ 
12    /*  $\text{box}^{(b)}(a)$  is the box index vector of  $a$  at the coarseness level  $b \in \mathbb{Z}$ , i.e.,  $\text{box}^{(b)}(a)_i = \lfloor a_i \cdot 2^{-b} \rfloor$ , for  $i = 1, \dots, d$  and  $d$  is the number of objectives. */
13    if  $\mathcal{Z} = \emptyset$  then
14       $A^{(t)} \leftarrow A^{(t-1)}$ 
15    else
16       $\beta \leftarrow \min \mathcal{Z}$ 
17       $D \leftarrow \{a \in A' \mid \exists a' \in A',$ 
18         $\text{box}^{(\beta)}(a') \preceq \text{box}^{(\beta)}(a) \wedge a' \neq a\}$ 
19      if  $s^{(t)} \in D$  then
20         $A^{(t)} \leftarrow A^{(t-1)}$ 
21      else // Draw a solution randomly from  $D$ .
22         $a \leftarrow \text{sample}(D)$ 
23         $A^{(t)} \leftarrow A' \setminus \{a\}$ 
24    end
25  end
26 end

```

Output: $A^{(t)}$

into the archive, possibly replacing an existing one, will always lead to a better value of the indicator [12]. It also eventually converges to an archive that minimises this indicator value, i.e., an optimal approximation of bounded size, hence, it is *limit-optimal*, which implies all other limit properties. Despite having the same properties as hypervolume-based archivers, MGA has less computational cost in general; its time complexity is $O(mN^2L)$, where L is the length of the binary encoded input [4].

Despite the above desirable properties, a recent study has shown that MGA is unlikely to preserve boundary solutions [10]. This is a common problem of all archivers using box- or ϵ -based dominance, such as ϵ -MOEA [90] and GrEA [114]. In addition, the archive maintained by MGA is not uniformly distributed along the Pareto front [10]. This occurrence can be attributed to the facts that (1) MGA picks one solution randomly to remove when there are multiple solutions at the β level, and (2) the new solution is not allowed to enter the archive if it is at the same level as some of the solutions in the archive (lines 15–16 in Algorithm 7).

VI. IMPORTANT ISSUES IN ARCHIVING

In this section, we discuss several important issues of archiving, including its performance, various attributes as well as connection with research topics in other fields.

A. Theoretical Desirables vs Practical Use

It is certainly helpful that archivers hold desirable theoretical properties, but it is more important that archivers are of practical use. In particular, avoiding convergence to a small region of the Pareto front is a critical, practical desirable (i.e., the first practical desirable in Table I). Archivers that fail to diversify are not very useful in practice, since they produce a poor approximation of the actual Pareto front, even if they may hold most theoretical desirables, e.g., A_{dom} [30]. Having an archive of controllable size, that is, with a user-specified maximum capacity and that stores as many nondominated solutions as can fit in that capacity, is also an important practical desirable, since one may not want to end up with a population during the search that is too small or too big. The lack of this property may explain why archivers in Class II, e.g., ϵ -Pareto [15], are not widely used in practice. In addition, archivers may need to compromise practical desirables in order to meet theoretical ones. For example, in contrast to SMS-EMOA [7] which adaptively sets the reference point according to the input sequence, the archiver A_{HV} [42] requires setting a fixed reference point a priori, which may lead to poor diversification if the reference point is either too far or too close to the nadir point of the Pareto front [115]. In short, we cannot say that an archiver without any theoretical property (i.e., those in Class I) performs worse than those with some of them (Classes II–IV) in practice.

Yet, equally, we would never say that an archiver without any theoretical property is the best in practice, even when used to manage the population in an MOEA. Indeed, a lack of theoretical properties may harm the search progress, as reported on various synthetic and practical scenarios [9, 15, 17, 19, 116]. The *set-monotone* property, in particular, prevents the oscillation of the archive’s performance [16] and leads to the eventual convergence of the archive, which can be used as a stopping condition of an MOEA. Finally, it is worth mentioning that one may not be able to say that archivers in the Class III are certainly better in practice (e.g., have better values on an indicator) than those in Class IV, though most of the archivers in the former are popular in the area whereas some in the latter are not (e.g., A_{HV} and MGA [12]). Archivers with the *limit-optimal* property under any condition are surely preferable provided that they do not sacrifice much practical performance (e.g., convergence and diversity), such as MGA.

B. What Is An Ideal Archiver?

Apart from holding theoretical desirables, one may ask what an ideal archiver is in practice. In general, an archiver can be called “ideal” if it can maintain a representative subset of all Pareto-optimal solutions of any sequence at any time. There are three major attributes with respect to sequences that can affect the performance of an archiver: the dimensionality of solution vectors in the sequence, the shape of the Pareto-optimal solutions of the sequence, and the order of the solutions in the sequence. As such, an ideal archiver needs to work well on various sequences and be (almost) unaffected by the dimensionality, shape and order of solutions.

In this sense, existing archivers unfortunately are far from being ideal. Pareto-based archivers, which use Pareto dominance and a density estimator as the selection rules, fail to scale up with the number of objectives [52, 117]. In contrast, some modifications, which aim to make Pareto-based archivers work in a high-dimensional objective space, may be detrimental to their performance in a low-dimensional space. For example, shift-based density estimation (SDE) [103], which enables Pareto-based archivers to work well in many-objective optimisation, may affect their ability to maintain the boundary solutions when dealing with bi- or tri-objective problems [118–120].

Indicator-based and decomposition-based archivers are more effective in dealing with increasing number of objectives as a result of making nondominated solutions comparable by an indicator or using the weight vectors, respectively. However, such a mechanism may make them sensitive to the shape of the sequence’s Pareto front. It is known that decomposition-based methods (e.g., MOEA/D) may not be able to maintain a well-distributed archive for irregular Pareto front shapes [121]. Indicator-based methods may also struggle on some shapes, depending on the characteristics of their indicators. For example, SMS-EMOA has been found to be less effective on problems with inverted simplex-like Pareto front shapes [122], with highly degenerate Pareto fronts [123], or with many dominance resistance solutions [123].⁴

The effect of the order of solutions fed to the archive has been rarely studied empirically. A recent study has shown that the order matters in the sense that different sequences of the same set of solutions can produce very different archiving results [10]. Archivers, not only from Class I (e.g., NSGA-II and NSGA-III) but also from Class III (e.g., MOEA/D+TCH) and Class IV (e.g., SMS-EMOA and MGA), may struggle to maintain a well-distributed archive when facing sequences of “interesting” orders of solutions even on low-dimensional sequences with regular Pareto front shapes (i.e., simplex shapes) [10].

C. Batch Size

Batch size in the archiving process refers to the number of solutions fed to the archive at one step. It is often set to either one (e.g., in most theoretical studies [1, 4] and some MOEAs [7, 8]) or to the archive/population size (e.g., in many MOEAs [5, 11, 55]). In the context of evolutionary computation, the former is called steady-state evolution mode (i.e., $\mu+1$) and the latter is called generational evolution mode (i.e., $\mu+\mu$), where μ denotes the archive/population size. It is worth noting that the evolution mode of an optimiser is orthogonal to the batch-size of the archiver: a $(\mu+1)$ archiver can always handle a $(\mu+\mu)$ evolution mode by processing the μ offspring one at time, but doing so the $(\mu+1)$ archiver will not gain any of the properties of a $(\mu+\mu)$ archiver. Similarly, a $(\mu+\mu)$ archiver can always be combined with a $(\mu+1)$ evolution mode, but at the cost of losing all the properties

⁴Dominance resistant solutions are those with an extremely poor value in one objective but with (near) optimal values in the others [124].

that result from updating the archive with many solutions at a time.

For a given sequence of solutions, the question of which size is better may be trivial since a bigger batch size always gives the archiver more knowledge about future input, so that the archiver can make more informed decision. Yet, when being used in the process of generating offspring solutions in MOEAs, a $(\mu + 1)$ archiver can be more suitable in some cases since an instant update of the source archive forming the mating pool may be helpful in generating better offspring, particularly when evaluating solutions is expensive.

The batch size has important effects of the properties of archivers. Zitzler et al. [125] proved that there is no $(\mu + 1)$ archiver that never decreases the hypervolume of the archive, which implies *set-monotone* (Property 3), and ends up with an archive of maximum hypervolume when given the best possible input sequence starting from any sub-optimal archive. Bringmann and Friedrich [14, Th. 2] confirmed this result and extended it to $(\mu + \lambda)$ with $\lambda < \mu$ [14, Th. 5]. There are $(\mu + \mu)$ archivers, however, that are able to reach the maximum hypervolume when given the best possible input sequence [125, Th. 3.4], [14, Th. 3].

One of the main practical reasons for preferring a $(\mu + 1)$ archiver instead of a $(\mu + \mu)$ one is the additional computational cost of choosing the optimal subset from all $\binom{2\mu}{\mu}$ subsets, however, there are efficient algorithms available for both hypervolume and ϵ -indicators on bi-objective problems [126] and further improvements in higher dimensions are possible [127].

D. Unbounded Archive

Archivers discussed so far maintain an archive of bounded capacity, that is, when the number of nondominated solutions exceeds the capacity of the archive, the archiver needs to remove a solution. Since the archiver does not know the future input, it is an online algorithm whose decisions cannot be guaranteed to be optimal [1], thus no archiver can guarantee an optimal approximation of bounded size for any finite sequence [2]. Furthermore, most archivers will deliver a final archive that consists of many solutions which are dominated by solutions removed previously (*point-deteriorate*), as shown in [16].

An unbounded archive that stores all nondominated solutions ever generated does not have the same limitations as an online archiver of bounded size. If at each timestep an archiver selects a small subset of the Pareto optimal solutions (e.g., to present to the DM) from an unbounded archive of the input sequence seen so far, then the selected subset would never point-deteriorate. For some problems, modern computers may be able to keep hundreds of thousands of their solutions in memory, thus an unbounded archive becomes increasingly viable for some applications [9, 128, 129]. Research involving an unbounded archive includes directly using it to store high-quality solutions generated by an MOEA [46, 63, 130–133], incorporating it into an MOEA as an important algorithm component [134], benchmarking various MOEAs [69, 128, 135], benchmarking bounded versus unbounded archivers [63], and

designing efficient data structures for it [9, 136–140]. In addition, selecting from an unbounded memory can be seen as an offline archiving algorithm, in particular, a subset selection problem, where the archiver knows the whole input sequence and its task is to select a specified number of solutions to represent the whole archive. Several subset selection methods [141, 142], along with benchmarking test data [143], have been proposed. They consider various indicators as selection criteria, ranging from common ones used in the area such as hypervolume [141, 144–146], ϵ -indicator [141] and IGD⁺ [147] to similarity-based metrics such as distance-based [142, 148] and clustering-based ones [149].

Despite the capacity of modern computers, a downside of using an unbounded external archive is still its computational cost. When the optimisation problem is computationally hard but solution evaluation is fast, an algorithm may produce millions of nondominated solutions thus leading to a very slow archiving process, particularly for continuous MOPs which typically have infinitely many Pareto optimal solutions. On the other hand, costly solution evaluations often imply a simulation process that generates large amounts of data on top of the decision and objective vectors, which increases the memory requirements for storing such solutions. In some real-world problems, solutions may actually map to a particular chemical or physical object, whose construction is economically costly, and thus the archive is bounded by how many of those objects it can store in the real-world [150].

An unbounded archive is mainly used for storing the best solutions found so far. It is rarely used as a population to generate new solutions, except in very few cases [151]. An unbounded population made up of all nondominated solutions generated may cause harmful genetic drift phenomenon due to over-representation of some areas in the search space, especially when the mapping of the search space to the objective space is not uniform.

E. Related Problems in Theoretical Computer science

Speaking of subset selection, there is a similar research problem in the field of theoretical computer science: given a specified accuracy ϵ , determine a minimum set of solutions such that any solution of a given set (or of a multi-objective problem) can be ϵ -dominated by at least one of its solutions [105, 152–155]. Its dual problem tends to be more relevant (essentially, an offline archiving problem): given a set of nondominated solutions, find a specified number of N solutions that provide the best approximation to the Pareto optimal set with respect to the ϵ -dominance. In contrast to using the hypervolume as the selection criterion, using the ϵ -dominance provides the decision-maker with a measure of the approximation error ϵ of the subset selected. However, like hypervolume-based subset selection [156], solving this problem is difficult [152, 155]. When the number of objectives is two, the problem is already NP-hard despite having a polynomial time approximation [126, 152]; when the number of objectives is larger than two, any multiplicative approximation is impossible, unless P=NP [155].

VII. FUTURE RESEARCH DIRECTIONS

After providing an overview of important issues in archiving in the previous section, this section suggests several research directions that deserve attention in coming years.

A. Developing Archivers with Theoretical Desirables and Practical Use

Most existing work in the EMO area focuses on the practical performance of archiving algorithms (e.g., with respect to the hypervolume and IGD indicators), ignoring their theoretical properties. However, an archiving algorithm with any of the limit properties avoids that the same solutions enter and exit the archive repeatedly, which causes fluctuation of the quality of the archive [9], while the point- and set-monotonicity properties avoid that the quality of the archive/population deteriorates over time [10].

Fortunately, archivers in Classes III and IV (cf. Table I) have the potential to hold both theoretical and practical desirables, in contrast to those in Classes I and II where either of them is missing. Despite their practical shortcomings, archivers in Class II are the only ones respecting the property *point-monotone*. Within Class II, ϵ -Pareto and ϵ -approx tend to result in an archive size far smaller than the capacity available, thus preventing them from being limit-optimal. It remains an open question whether there exist archivers with similar theoretical properties and whose archive size is always close to the given capacity or, if possible, limit-optimal. Some recent studies are interesting attempts to address this question [48, 104].

Archivers in Class III strike a good balance between theoretical and practical desirables, as evidenced by their wide use in the EMO area. However, they may not hold *set-monotonicity* fully, which has the risk of the archive/population deteriorating over time. Class IV is a class having high potential to be explored. It is the only class that guarantees limit-optimal, diversification and a controllable size. Given that archivers based on weakly Pareto compliant indicators can hold the same theoretical desirables as those based on Pareto compliant indicators, we expect that more archivers from Class IV will emerge in the near future.

B. Order of Solutions Arriving

In contrast to extensive archiving studies on the effect of the number of objectives and the Pareto front shape, there are very few works studying how the order of solutions in the sequence affects archivers. It has been shown [4, 10] that commonly-used archivers, such as NSGA-II, SPEA2, MOEA/D, SMS-EMOA and NSGA-III, may not be reliable on even the simplest Pareto fronts (i.e., 1D/2D simplex shapes) if solutions arrive one-at-a-time ($\mu + 1$) in pathological orders.

Although solutions generated by a search algorithm are expected to get better over time, input sequences may greatly differ in practice depending on the optimisation problem and search algorithm.

In particular, the landscape of optimisation problems may produce quite different patterns of solution sequences. For example, well-established test problems KUR [157] and

UF [158] typically lead MOEAs to start their search from a particular region and gradually move to others. Problems involving many local optima in the search space (e.g., DTLZ3 [159], ML-DMP [123] and MNK-landscapes [160]) easily lead to MOEAs generating dominance resistant solutions during the search. In many real-world problems, particularly problems with strict constraints or prioritised objectives, the search often starts from a tiny feasible region and then gradually expands to large regions, such as in the test suite generation for software product line [161] and in resource allocation for software testing [162].

On top of various sequences resulting from optimisation problems, there exist many multi-objective optimisers that tend to search for solutions in a certain order. For example, the algorithm in [163], developed for the bi-objective TSP problem, starts the search from an extreme solution and then gradually moves to the other extreme. The algorithm presented in [164] generates search directions that aim to fill the largest gap in the current approximation of the PF. The algorithms in [165, 166] search first for all extreme solutions of the Pareto front and then trade-off solutions between those. Pareto local search algorithms generate solutions that are neighbours in the decision space of a single solution taken from its archive, and thus the generated sequences often consist of very similar solutions [51, 167, 168]. Similar search strategies are also common in conventional mathematical optimisation [169, 170].

In short, the variety of optimisation problems' nature and search algorithms' behaviour (as well as the stochasticity of MOEAs) may lead to different types of solution sequences. Investigating their effect on archivers and, hence, developing reliable algorithms on various sequences are a potential direction waiting to be explored.

C. Archiving Based on Specific Indicators

Archivers based on a specific indicator represent the archive's quality through a scalar value and aim to find the archive that maximises/minimises that value. Frequently used indicators for this end include hypervolume [28], IGD [109], ϵ -indicator [20], Hausdorff indicator [171], R2 [95], and IGD⁺ [26], which can cover both the proximity to the Pareto front and the diversity along the front. As discussed in Section V-C, such archivers hold (or can be modified to hold) the limit properties and (weakly) Pareto compliant indicators enable the archiver to hold the three limit desirables. Indicators that are not compliant with Pareto dominance may enable the design of an archiver that is *limit-stable*, as long as the archive is monotone with respect to the indicator value.

Since an indicator-based archiver aims to maximise (or minimise) the indicator value, it is always of interest to know how good a value can be achieved by the archiver theoretically. There are several studies on this topic [14, 44, 46]. However, far more work is needed to understand the theoretical limitations of such archivers.

In addition, a well-established concept in the theory of online algorithms, called competitive analysis [172, 173], fits nicely in evaluating indicator-based archivers (as well as other archivers as long as a scalar quality indicator is used to

perform competitive analysis). Competitive analysis compares the relative performance of an online algorithm and an offline algorithm for the same sequence, i.e., how much worse the online algorithm performs due to not knowing the future input. Specifically, the competitive ratio of an algorithm is defined as the worst-case ratio of its quality divided by the optimal quality, over all possible sequences. Here, the optimal quality can be defined by using the unbounded archive (like in [4]) or using the best possible bounded size archive or perhaps something else that is actually achievable. López-Ibáñez et al. [4] suggested “*to use competitive analysis techniques from the field of online algorithms to obtain worst-case bounds, in terms of a measure of ‘regret’ for archivers*”, yet, to the best of our knowledge, the only analysis available is the work of Bringmann and Friedrich [14], who defined the competitive ratio based on the hypervolume metric, proved upper and lower bounds of this competitive ratio for different classes of hypervolume-based archivers and presented a computationally-efficient hypervolume-based archiver with a constant competitive ratio. Their analysis is based on the best-case and worst-case input sequences and they pointed out that an average-case analysis may lead to a different choice of the archiver. A similar analysis for other types of archivers and competitive ratios based on other quality metrics, such as the ϵ -indicator or IGD^+ , remains to be done.

Theoretical analysis on competitive ratios or regret according to various quality indicators could be complemented by empirical analysis that is not restricted to archivers explicitly using the quality indicators being measured. López-Ibáñez et al. [4] measured the ratio between the quality, in terms of hypervolume and ϵ -indicator, of various archivers and of the unbounded archiver.

Further theoretical development would be welcome. For example, the limit properties (Properties 4, 5 and 6) are not very useful in practice unless the time to converge to the limit is tractable. Thus, bounds on the number of steps or input points required to reach the limit would be of practical interest.

D. Internal Archive vs External Archive

Two major roles of archiving in EMO are to (1) store a set of representative Pareto optimal solutions for a *posteriori* decision-making and (2) maintain a set of high quality solutions as the source to generate offspring. The different purposes of the two roles may need different archiving algorithms, though existing work usually does not distinguish them, e.g., the hypervolume-based archiver is widely used for both roles. A recent study has shown that a combination of relatively small internal archive/population of Class I and a large external archive of Class IV may be a good choice [63]. The internal archive is focused on searching for promising solutions, while the external archive is focused on storing the best solutions found. In this setup, it does not matter if the internal archive *set-deteriorates* or it is not *limit-optimal* (an invariant population is not helpful for the search) as long as the external archive is. Nevertheless, much more studies are needed to investigate which are the best combinations of internal and external archivers.

An external archive can also be used to monitor the evolutionary status of the internal archive/population. For example, in [60] an external archive based on Pareto dominance and density criteria is used to check if the decomposition-based internal archive is trapped in partial regions of the optimal front. In this regard, it is beneficial that the different archivers consider complementary archiving criteria. This is one of the major reasons behind the development of various two-archive MOEAs [73, 74, 78, 81, 82, 174]. Moreover, automatically-designed MOEAs show that diverse choices of archiving criteria for environmental selection and external archiving often outperform well-known popular MOEAs, even after tuning the parameters of the latter [175, 176].

E. Interplay between archiving and solution generation

As aforementioned, in the absence of an external archive that does not participate in the search, the population of an MOEA is used not only to store the best solutions found so far but also to generate new solutions. This is also the case for some multi-objective local search algorithms using bounded archives [51, 177]. That means that the sequence of solutions fed to the population is generated by itself. In this case, we may need to consider other factors in the archiving operation on top of solutions’ quality, e.g., the life cycle of solutions in the archive. We want to exploit the very best solutions in the archive but may also want to explore new areas in which newly-generated, next-best solutions are located. This is essentially a problem of balancing exploration and exploitation. In multi-objective local search, solutions are marked as “explored” after their neighborhood is (partially or fully) explored [167, 168, 178]. Some MOEAs introduce the concept of “ageing” to prevent old high-quality solutions in the population from generating new solutions in order to help the search jump out of local optima [179].

Given the above, perhaps using a population serving both roles of storing and generating solutions is not ideal, despite the fact that it is the common practice in the EMO area. The final population returned to the decision-maker may contain many dominated solutions with respect to the sequence generated (i.e., all the solutions generated), while nondominated solutions may be discarded in the middle of the search [16]. Therefore, an external archive that stores best solutions found is desirable. It is worth mentioning that this observation also applies to the two-archive/population approach since both archives/populations participate in the search.

VIII. CONCLUSION

Bounded archiving, i.e., storing a bounded set of representative high-quality solutions is of good use in multi-objective optimisation. Not only is unbounded archiving computationally impractical in many scenarios, but also a bounded archive or population may help the search. In this paper, we conducted a systematic survey of multi-objective archiving, including

- We formalised the archiving problem and introduced six theoretical properties desirable for bounded archivers to hold.

- We showed analytically that archivers based on a weakly Pareto compliant indicator (e.g., ϵ -indicator) can achieve the same theoretical properties as archivers based on a Pareto compliant indicator (e.g., hypervolume).
- We exemplified representative archivers (including those in well-established MOEAs) and classified these archivers into four classes based on their theoretical and practical properties.
- We discussed important issues in designing and analysing multi-objective archivers.
- We suggested future research lines and pointed out several open questions that require further theoretical and empirical research.

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