EME5943
현대제어시스템

김종현 교수

성균관대학교
SUNG KYUN KWAN UNIVERSITY(SKKU)

Lyapunov stability analysis

▪ Finding Lyapunov functions

- not always easy to find
- not unique
- If a system is stable → a Lyapunov function exists

- Sum-of-squares technique : a systematic approach
  - ✓ If need, see ref. in textbook

- Systematic method for linear system?

$$\frac{dx}{dt} = Ax$$

$$V(x) = x^T P x$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix ($P = P^T$)

Lyapunov stability analysis

candidate Lyapunov function $V(x) = x^T P x$

$$\begin{aligned}\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P A x \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x =: -x^T Q x\end{aligned}$$

$$\frac{dx}{dt} = Ax$$

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

Lyapunov equation

$$A^T P + PA = -Q \qquad Q > 0$$

- Always has a solution if all of the eigenvalues of A are in the left half-plane
- $P > 0$ if $Q > 0$

Lyapunov stability analysis

▪ Finding Lyapunov function

- Stability of nonlinear system
  with finding Lyapunov function of linear system

$$\frac{dx}{dt} = F(x) =: Ax + \tilde{F}(x)$$

$$F(0) = 0$$

$\tilde{F}(x)$ contains terms that are second order and higher

Theorem 5.3. *Consider the dynamical system* (5.18) *with* $F(0) = 0$ *and* $\tilde{F}$ *such that* $\lim \|\tilde{F}(x)\|/\|x\| \to 0$ *as* $\|x\| \to 0$. *If the real parts of all eigenvalues of* A *are strictly less than zero, then* $x_e = 0$ *is a locally asymptotically stable equilibrium point of equation* (5.18).

현대제어시스템

Linear systems (1)

Basic definition

$$\frac{dx}{dt} = f(x,u), \qquad y = h(x,u)$$

convenient $x = 0,\ u = 0$ $\Rightarrow$ $\dot{x} = 0$ $h(0,0) = 0$

real $(x_e, u_e) \neq (0,0)$ $y_e = h(x_e, u_e)$

Translation

Basic definition

- **Linearity**

$$\frac{dx}{dt} = f(x,u), \qquad y = h(x,u)$$

- Linear input/output system

$$\begin{aligned}
&\text{(i)} \quad y(t;\alpha x_1 + \beta x_2, 0) = \alpha y(t;x_1,0) + \beta y(t;x_2,0) \\
&\text{(ii)} \quad y(t;\alpha x_0, \delta u) = \alpha y(t;x_0,0) + \delta y(t;0,u), \\
&\text{(iii)} \quad y(t;0,\delta u_1 + \gamma u_2) = \delta y(t;0,u_1) + \gamma y(t;0,u_2).
\end{aligned}$$

Three solutions

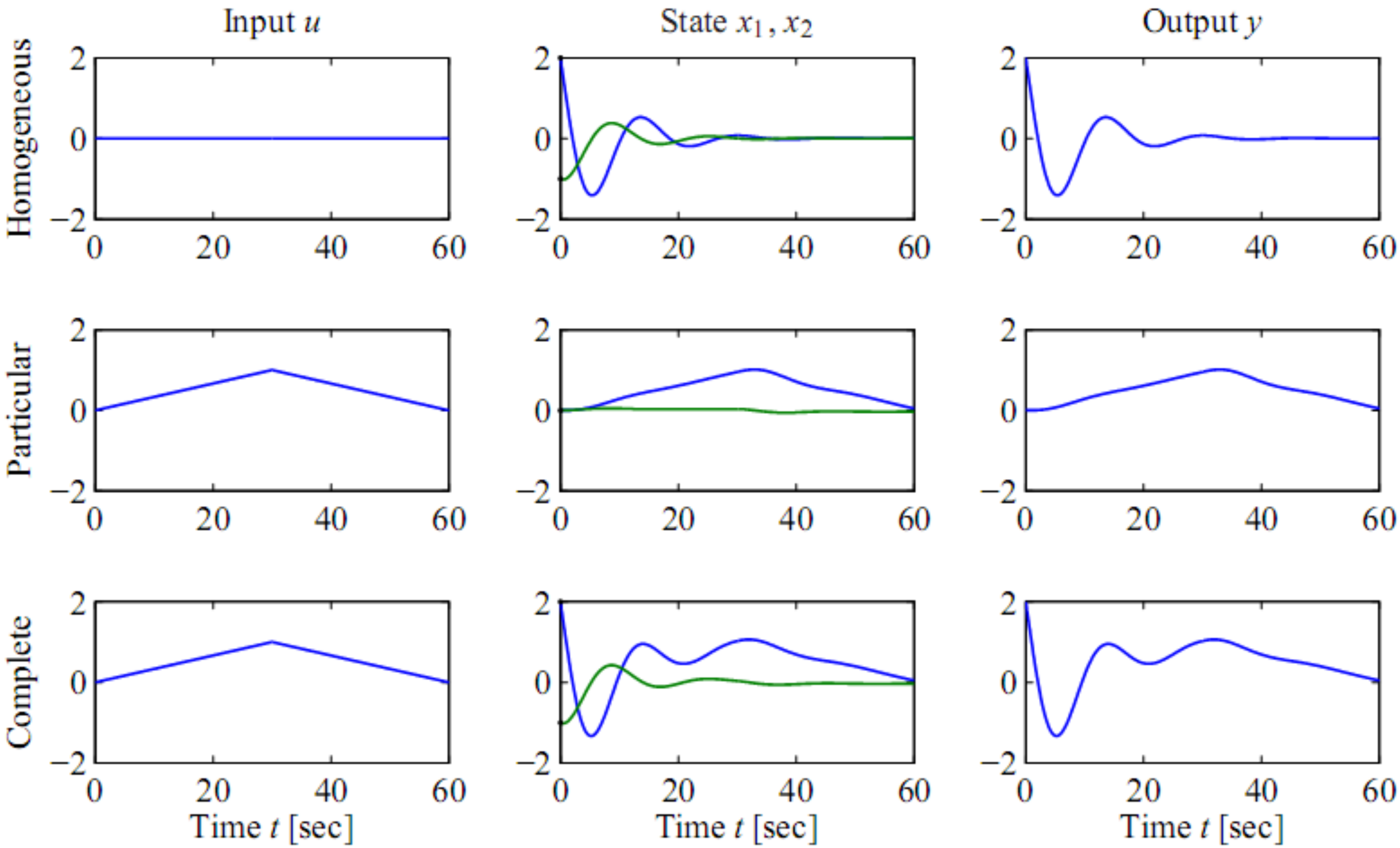
$x_h(t)$ to be the solution with zero input

$x_p(t)$ to be the solution with zero initial condition

Homogeneous solution + Particular solution =

Basic definition

Homogeneous solution + Particular solution = Complete solution



$$\text{(ii)} \quad y(t; \alpha x_0, \delta u) = \alpha y(t; x_0, 0) + \delta y(t; 0, u)$$

Basic definition

(i) $y(t;\alpha x_1+\beta x_2,0)=\alpha y(t;x_1,0)+\beta y(t;x_2,0)$

(ii) $y(t;\alpha x_0,\delta u)=\alpha y(t;x_0,0)+\delta y(t;0,u)$,

(iii) $y(t;0,\delta u_1+\gamma u_2)=\delta y(t;0,u_1)+\gamma y(t;0,u_2)$.

Ex) A scalar system

$$\frac{dx}{dt}=ax+u, \qquad y=x \qquad \text{with } x(0)=x_0$$

$$u_1=A\sin\omega_1 t \qquad u_2=B\cos\omega_2 t$$

Homogeneous solution

Particular solutions

$$x_{p1}(t)=-A\frac{-\omega_1 e^{at}+\omega_1\cos\omega_1 t+a\sin\omega_1 t}{a^2+\omega_1^2}$$

$$x_{p2}(t)=B\frac{ae^{at}-a\cos\omega_2 t+\omega_2\sin\omega_2 t}{a^2+\omega_2^2}$$

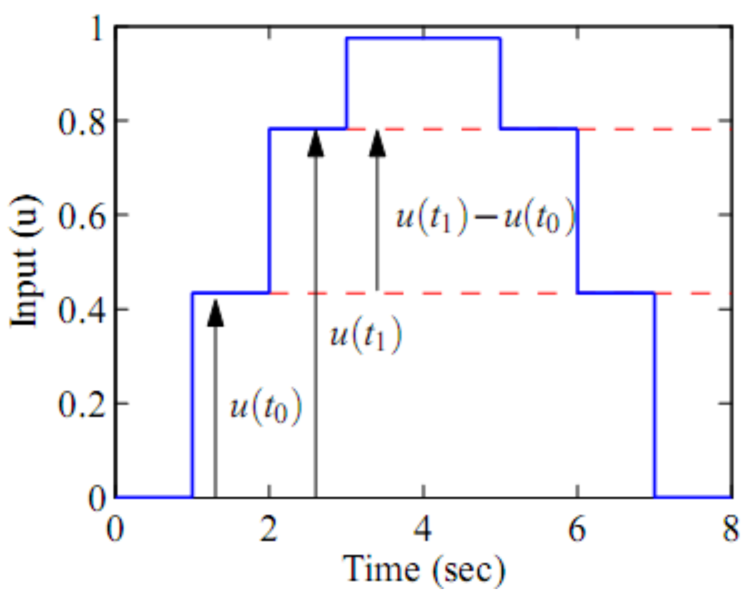
$$=e^{at}\left(\alpha x_0+\frac{A\omega_1}{a^2+\omega_1^2}+\frac{Ba}{a^2+\omega_2^2}\right)-A\frac{\omega_1\cos\omega_1 t+a\sin\omega_1 t}{a^2+\omega_1^2}+B\frac{-a\cos\omega_2 t+\omega_2\sin\omega_2 t}{a^2+\omega_2^2}$$

Basic definition

- **Time invariance**
  - System do not change with time

$$u(t) \Rightarrow y(t)$$

$$u(t+a) \Rightarrow y(t+a)$$



$H(t)$: response to a unit step applied at time 0

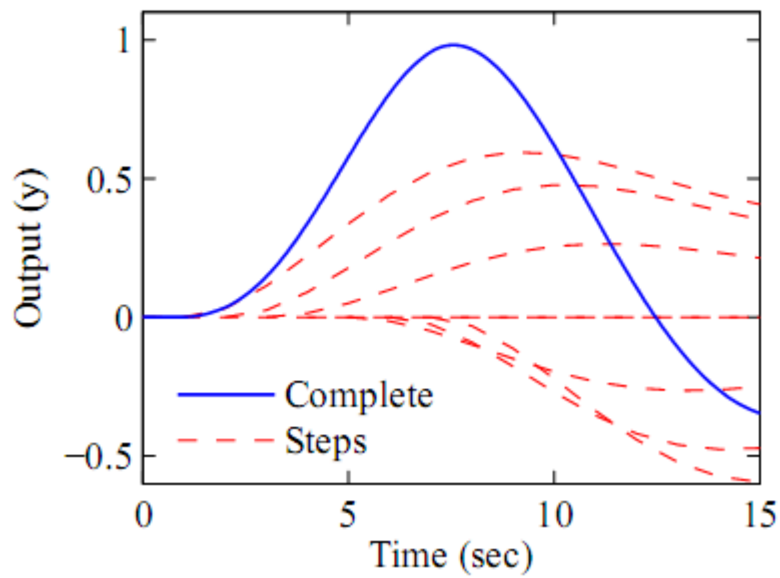
$$H(t-t_0)u(t_0)$$

$$H(t-t_1)\big(u(t_1)-u(t_0)\big)$$

Basic definition

$$H(t-t_0)u(t_0) \qquad H(t-t_1)\big(u(t_1)-u(t_0)\big)$$

$$\begin{aligned}
y(t) &= H(t-t_0)u(t_0) + H(t-t_1)\big(u(t_1)-u(t_0)\big) + \cdots \\
&= \big(H(t-t_0) - H(t-t_1)\big)u(t_0) + \big(H(t-t_1) - H(t-t_2)\big)u(t_1) + \cdots \\
&= \sum_{n=0}^{t_n<t} \big(H(t-t_n) - H(t-t_{n+1})\big)u(t_n) \\
&= \sum_{n=0}^{t_n<t} \frac{H(t-t_n) - H(t-t_{n+1})}{t_{n+1}-t_n} u(t_n)\,(t_{n+1}-t_n).
\end{aligned}$$



Matrix exponential

- **Initial condition response**

$$\frac{dx}{dt} = ax$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\frac{dx}{dt} = Ax$$

matrix exponential

$$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}X^k$$

$$X^0 = I \qquad X^2 = XX$$

$$X^n = X^{n-1}X$$

Matrix exponential

$$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}X^k$$

$\Downarrow$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^kt^k$$

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{1}{2}A^3t^2 + \cdots =$$

$\Downarrow$

Proposition 6.1. *The solution to the homogeneous system of differential equations (6.6) is given by*

$$x(t) = e^{At}x(0).$$

$$\frac{dx}{dt} = Ax$$

Matrix exponential

Ex) Double integrator

$$\ddot{q} = u, \qquad y = q. \qquad e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k$$

$$x = (q, \dot{q})$$

$\Rightarrow$

Matrix exponential

Ex) Undamped oscillator

$$\ddot{q} + \omega_0^2 q = u$$

$$A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

Matrix exponential

Ex) Undamped oscillator

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k$$

$$\ddot{q} + \omega_0^2 q = u$$

$$A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix}$$

$$Ae^{At}$$

$$\frac{d}{dt}e^{At} = \begin{pmatrix} -\omega_0 \sin\omega_0 t & \omega_0 \cos\omega_0 t \\ -\omega_0 \cos\omega_0 t & -\omega_0 \sin\omega_0 t \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix}$$

$$x(t) = e^{At}x(0) = \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

Matrix exponential

- **Diagonalization**

$$\frac{dx}{dt} = Ax$$

invertible matrix T such that TAT^{-1} is diagonal

$$z = Tx \quad \Rightarrow \quad \frac{dz}{dt} = T\frac{dx}{dt} = TAx = TAT^{-1}z$$

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \qquad (At)^k = \begin{pmatrix} \lambda_1^k t^k & & & 0 \\ & \lambda_2^k t^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k t^k \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Matrix exponential

- **Jordan form**
  - Diagonalization
    - ✓ cannot for matrices with equal eigenvalues

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}.$$

$$\begin{aligned} \chi(\lambda) &= \det(\lambda I - A) \\ &= (\lambda-1)(\lambda-2)(\lambda-4)^2. \end{aligned}$$

$$A\vec{\alpha}_i = \lambda_i \vec{\alpha}_i \qquad (i = 1, 2, \cdots, n).$$

$$P = \begin{pmatrix} \vec{\alpha}_1 & \vec{\alpha}_2 & \cdots & \vec{\alpha}_n \end{pmatrix},$$

???

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

Matrix exponential

- **Jordan form**

$$\begin{bmatrix} \lambda_1 & 1 & & & & & & & \\ & \lambda_1 & 1 & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & 1 & & & & \\ & & & & \lambda_2 & & & & \\ & & & & & \lambda_3 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_n & 1 \\ & & & & & & & & \lambda_n \end{bmatrix}$$

Ex)

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}$$

Matrix exponential

- **Jordan form**

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & J_{k-1} & 0 \\ 0 & 0 & \dots & 0 & J_k \end{pmatrix}$$

$$\text{where} \quad J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{pmatrix}$$

$$\begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix} \lambda_n & 1 \\ & \lambda_n \end{matrix}} \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}.$$

$$P = \left[v \middle| w \middle| x \middle| y \right] = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \quad P^{-1}AP = J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Matrix exponential

Theorem 6.2 (Jordan decomposition). *Any matrix $A \in \mathbb{R}^{n\times n}$ can be transformed into Jordan form with the eigenvalues of A determining λ_i in the Jordan form.*

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & J_{k-1} & 0 \\ 0 & 0 & \dots & 0 & J_k \end{pmatrix}$$

$$e^{Jt} = \begin{pmatrix} e^{J_1 t} & & & 0 \\ & e^{J_2 t} & & \\ 0 & & \ddots & \\ & & & e^{J_k t} \end{pmatrix}$$

$$e^{J_i t} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & & 1 & \ddots & \vdots \\ & & & \ddots & t \\ 0 & \dots & & 0 & 1 \end{pmatrix} e^{\lambda_i t}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Matrix exponential

Theorem 5.1 (Stability of a linear system). *The system*

$$\frac{dx}{dt} = Ax$$

is asymptotically stable if and only if all eigenvalues of A have a strictly negative real part and is unstable if any eigenvalue of A has a strictly positive real part.

Proof of Theorem 5.1. Let $T \in \mathbb{C}^{n\times n}$ be an invertible matrix that transforms A into Jordan form, $J = TAT^{-1}$. Using coordinates $z = Tx$, we can write the solution $z(t)$ as

$$z(t) = e^{Jt} z(0),$$

where $z(0) = Tx(0)$, so that $x(t) = T^{-1} e^{Jt} z(0)$.

The solution $z(t)$ can be written in terms of the elements of the matrix exponential. From equation (6.11) these elements all decay to zero for arbitrary $z(0)$ if and only if $\operatorname{Re}\lambda_i < 0$ for all i. Furthermore, if any λ_i has positive real part, then there exists an initial condition $z(0)$ such that the corresponding solution increases without bound. Since we can scale this initial condition to be arbitrarily small, it follows that the equilibrium point is unstable if any eigenvalue has positive real part. $\square$

Matrix exponential

$$e^{J_i t} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & & 1 & \ddots & \vdots \\ & & & \ddots & t \\ 0 & \dots & & 0 & 1 \end{pmatrix} e^{\lambda_i t}$$

$\Downarrow$

Proposition 6.3. *Suppose that the system*

$$\frac{dx}{dt} = Ax$$

has no eigenvalues with strictly positive real part and one or more eigenvalues with zero real part. Then the system is stable (in the sense of Lyapunov) if and only if the Jordan blocks corresponding to each eigenvalue with zero real part are scalar (1×1) blocks.

Matrix exponential

Ex) Linear model of a vectored thrust aircraft

$$\frac{dz}{dt} = \begin{pmatrix} z_4 \\ z_5 \\ z_6 \\ -g\sin z_3 - \frac{c}{m} z_4 \\ g(\cos z_3 - 1) - \frac{c}{m} z_5 \\ 0 \end{pmatrix}$$

where $z = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$

$$-g\sin z_{3,e} = 0$$
$$g(\cos z_{3,e} - 1) = 0$$

$$z_{3,e} = \theta_e = 0$$

equilibrium points

$$A = \left.\frac{\partial F}{\partial z}\right|_{z_e} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & -c/m & 0 & 0 \\ 0 & 0 & 0 & 0 & -c/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda(A) = \{0, 0, 0, 0, -c/m, -c/m\}$$

Matrix exponential

$$A = \frac{\partial F}{\partial z}\bigg|_{z_e} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & -c/m & 0 & 0 \\ 0 & 0 & 0 & 0 & -c/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J = \left(\begin{array}{c|ccc|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -c/m & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -c/m \end{array}\right)$$

Unstable

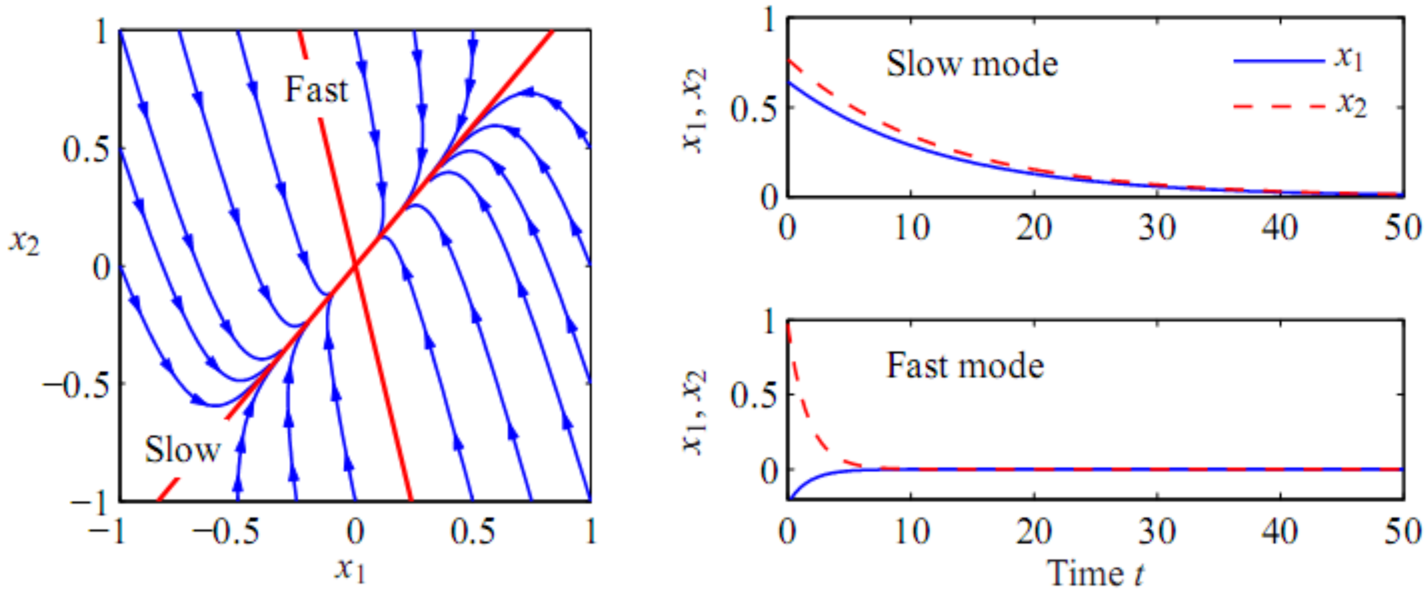
Matrix exponential

- **Eigenvalues and modes**

$$Av = \lambda v$$

solution of the differential equation for $x(0) = v$:

Matrix exponential



Input/output response

- **Convolution equation**

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du.$$

Theorem 6.4. *The solution to the linear differential equation* (6.13) *is given by*

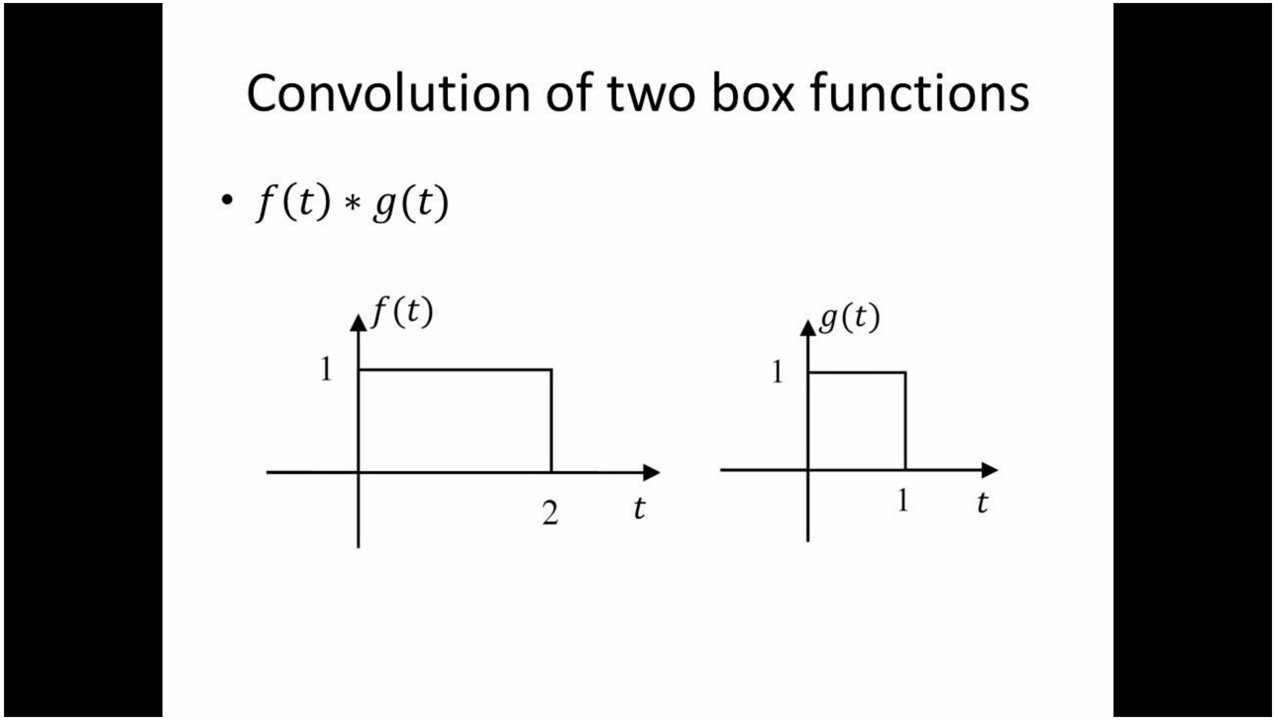
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{6.14}$$

Definition of convolution

$$(f * g)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau.$$

Input/output response

$$(f * g)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau.$$



http://youtube.com/watch?v=C1N55M1VD2o

Differentiation of convolution

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = f * \frac{dg}{dx}$$

Input/output response

$$\frac{dx}{dt} = Ax + Bu$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\frac{dx}{dt} = Ae^{At}x(0) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)$$

$$= Ax + Bu$$

Theorem 6.4. *The solution to the linear differential equation* (6.13) *is given by*

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{6.14}$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Input/output response

- **Coordinate invariance**

$$z = Tx$$

$$\frac{dx}{dt} = Ax + Bu \quad \Rightarrow \quad \frac{dz}{dt} = T(Ax + Bu) = TAT^{-1}z + TBu$$
$$=: \tilde{A}z + \tilde{B}u$$

$$y = Cx + Du = CT^{-1}z + Du =: \tilde{C}z + Du$$

Input/output response

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB, \qquad \tilde{C} = CT^{-1}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$e^{TST^{-1}} = Te^{S}T^{-1}$$

$$\Downarrow$$

$$x(t) = T^{-1}z(t) = T^{-1}e^{\tilde{A}t}Tx(0) + T^{-1}\int_0^t e^{\tilde{A}(t-\tau)}\tilde{B}u(\tau)d\tau$$