EME5943

현대제어시스템

김종현 교수

성균관대학교
SUNGKYUN KWAN UNIVERSITY(SKKU)

State-space model

▪ Difference equations

- Not continuously in time, but discrete instants of time

  → Discrete-time system

$$\frac{dx}{dt} = f(x,u) \quad \Rightarrow \quad x[k+1] = f(x[k],u[k])$$

$$y = h(x,u) \quad \Rightarrow \quad y[k] = h(x[k],u[k])$$

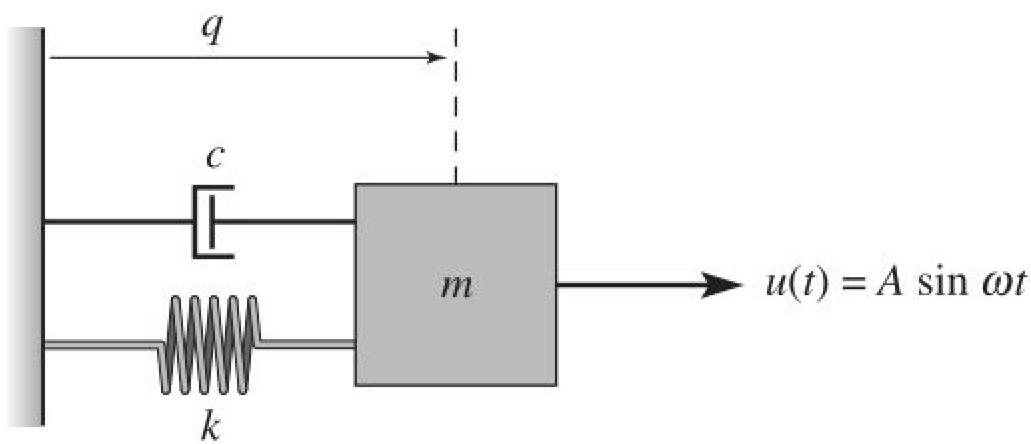
Linear cases

State-space models

▪ Simulation & Analysis

- Predicting evolution of system state from an initial condition
  - ✓ in closed form
  - ✓ through computer simulation
- Analyzing overall behavior of system without simulation

ex) mass-spring-damper system



$$m\ddot{q} + c\dot{q} + kq = u$$

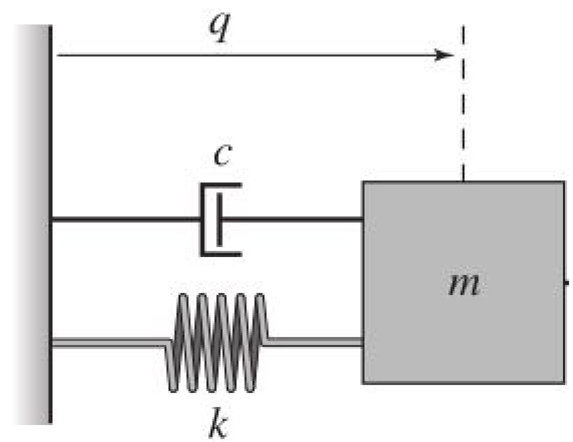
State-space models

▪ Simulation & Analysis

- Stability of an equilibrium point

ex) Equation of motion of
mass-spring-damper system with no input

$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ -\frac{c}{m}x_2 - \frac{k}{m}x_1 \end{pmatrix}$$



If the initial state of the system is away from the rest position, the system will return to the rest position eventually…

the rest position is *asymptotically stable*

Solving differential equations

- **ODEs**

$$\frac{dx}{dt} = f(x,u) \qquad\qquad y = h(x,u)$$

$$u \in \mathbb{R}^p \qquad y \in \mathbb{R}^q$$

$p = q = 1$ → single-input, single-output (SISO) systems

Solution?

$$\frac{dx(t)}{dt} = F(x(t)) \quad \text{for all } t_0 < t < t_f$$

Many solutions

Initial value problem

A unique solution

Solving differential equations

- **Lipschitz continuity**
  - For guaranteeing existence & uniqueness $\dfrac{dx}{dt} = F(x)$

$$\|F(x) - F(y)\| < c\|x - y\| \quad \text{for all } x, y$$

  - Sufficient condition

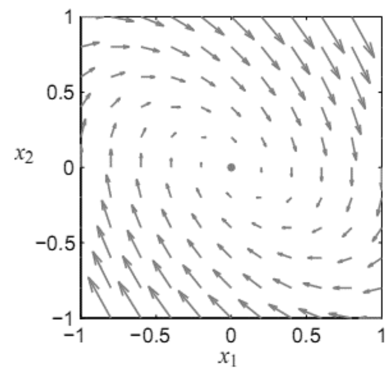
$$\partial F / \partial x \quad \text{uniformly bounded for all } x$$

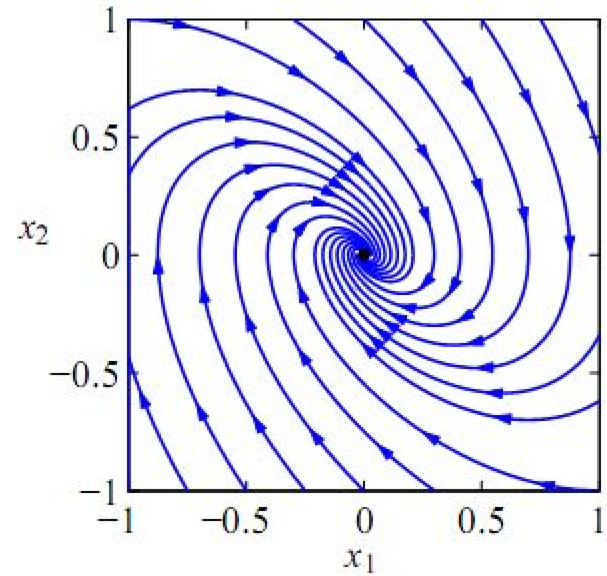
$$\frac{dx}{dt} = x^2 \quad \Rightarrow$$

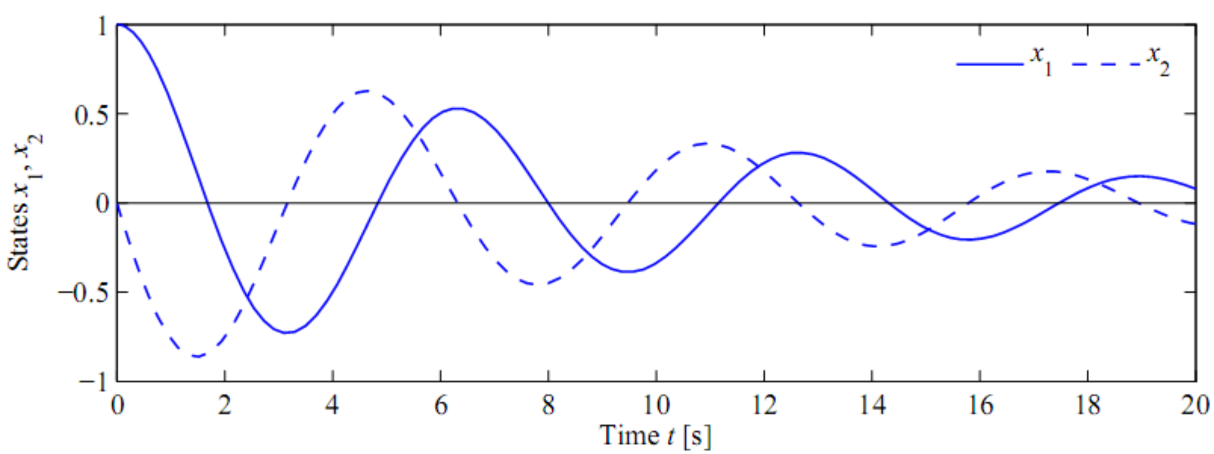
$$\frac{dx}{dt} = 2\sqrt{x} \quad \Rightarrow$$

Qualitative analysis

- **Phase portraits**
  - Plotting solution of differential equation
  - Solutions (streamlines) from different initial conditions







Qualitative analysis

- **Equilibrium points & Limit cycles**
  - Equilibrium points
    - ✓ Stationary conditions for the dynamics

$$\text{state } x_e \qquad \frac{dx}{dt} = F(x)$$

    - ✓ How many?

현대제어시스템

Dynamic behavior (2)

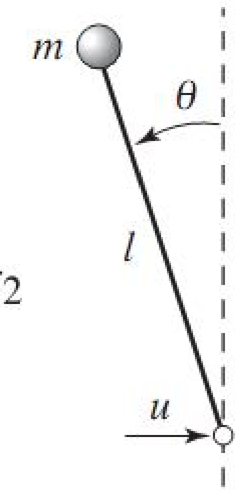
Qualitative analysis

- **Equilibrium points & Limit cycles**
  - Ex) Inverted pendulum

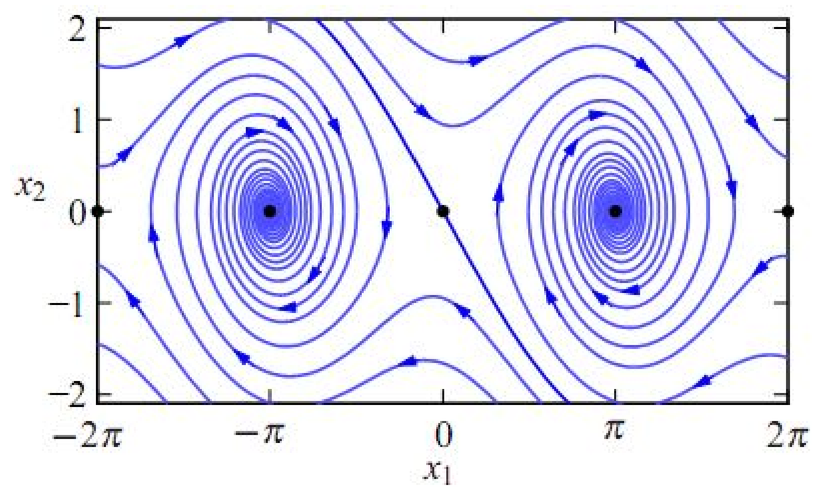
$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ \sin x_1 - c x_2 + u \cos x_1 \end{pmatrix}$$

$$\theta = x_1$$

$$d\theta/dt = x_2$$



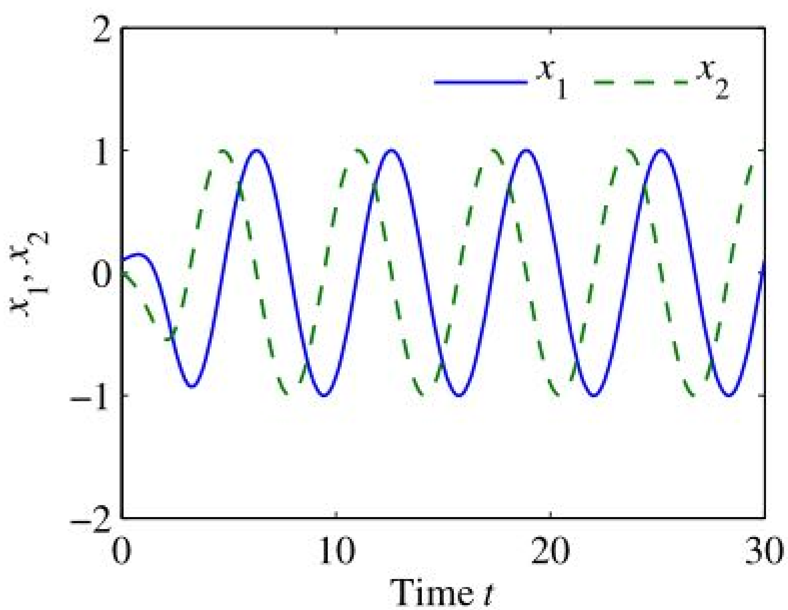
open loop dynamics

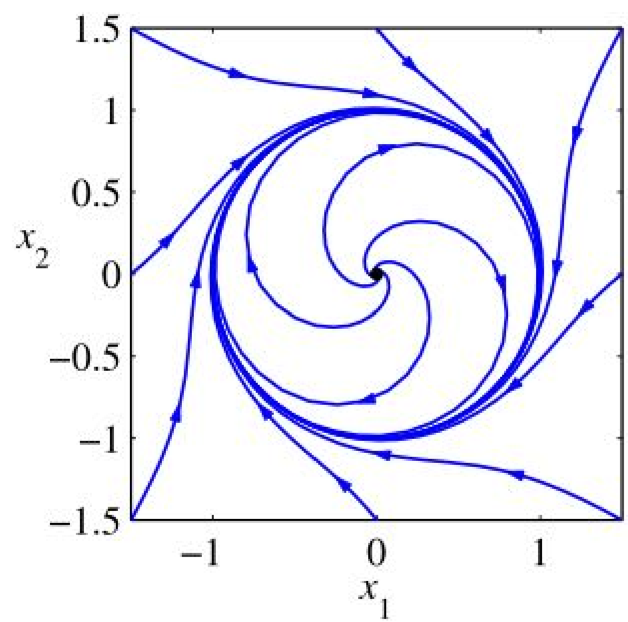


Qualitative analysis

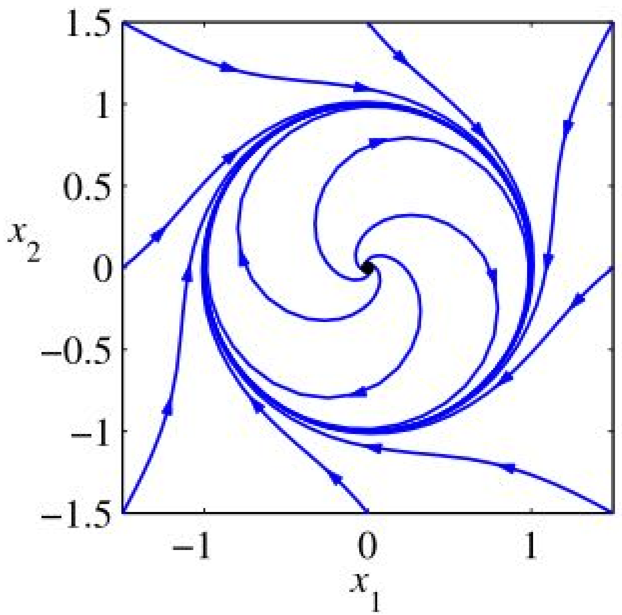
- **Equilibrium points & Limit cycles**
  - Limit cycles
    - ✓ Stationary periodic solutions
    - ✓ Ex) Electronic oscillator

$$\frac{dx_1}{dt} = x_2 + x_1(1 - x_1^2 - x_2^2), \quad \frac{dx_2}{dt} = -x_1 + x_2(1 - x_1^2 - x_2^2)$$





Qualitative analysis



Limit cycle

$T > 0$ if $x(t+T) = x(t)$ for all $t \in \mathbb{R}$

To determine limit cycle

- analytical methods for second-order system
- generally, computational analysis

Stability

- **Definitions**

Solution to a differential equation
with initial condition a

Stable?

if other solutions that start near a stay close to $x(t;a)$

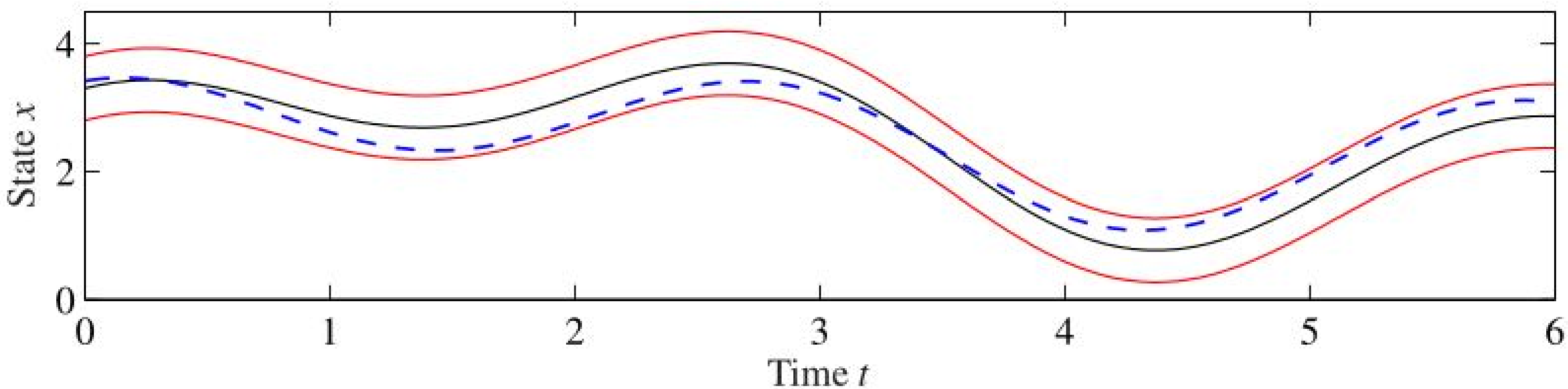
if for all $\varepsilon > 0$ there exists a $\delta > 0$

$\|b - a\| < \delta$ ⇨

Stability

if for all $\varepsilon > 0$ there exists a $\delta > 0$

$$\|b-a\| < \delta \Rightarrow \|x(t;b)-x(t;a)\| < \varepsilon \quad \text{for all } t > 0$$



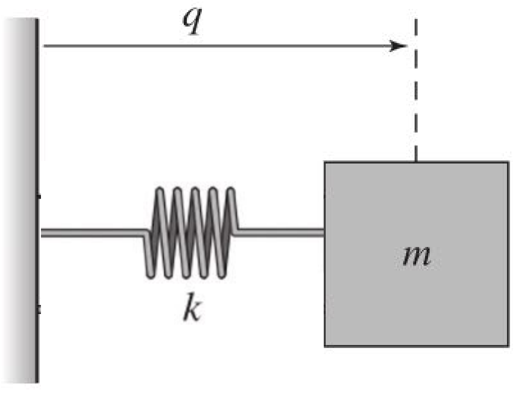
If a solution is stable in this sense and the trajectories do not converge

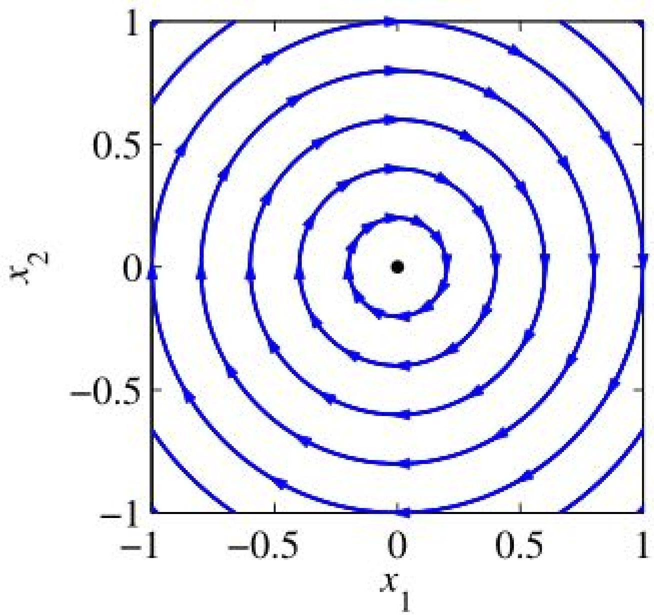
Stability

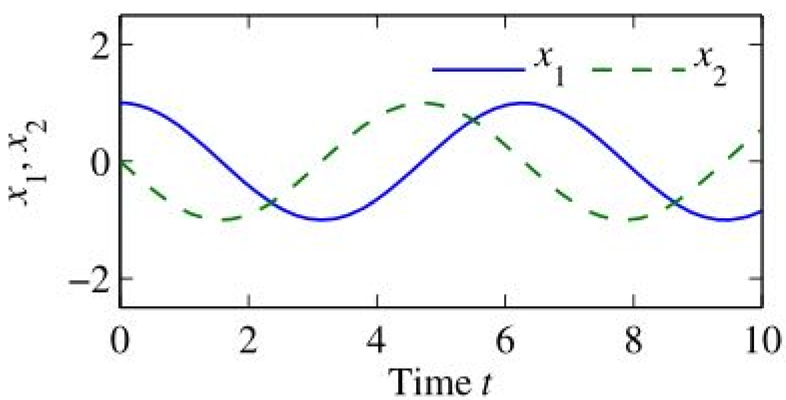
An important special case

$$x(t;a) = x_e$$

the equilibrium point is stable





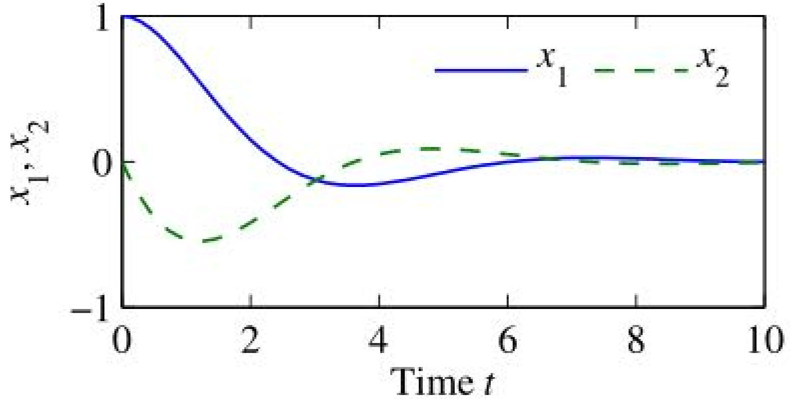


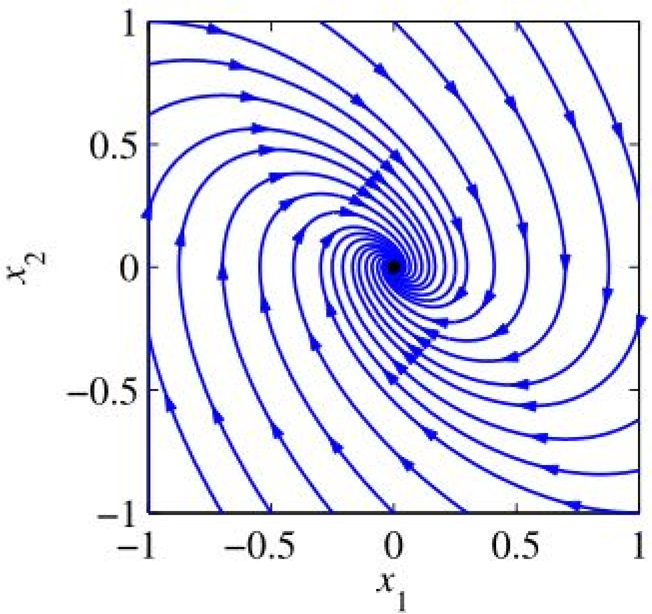
neutrally stable equilibrium point

Stability

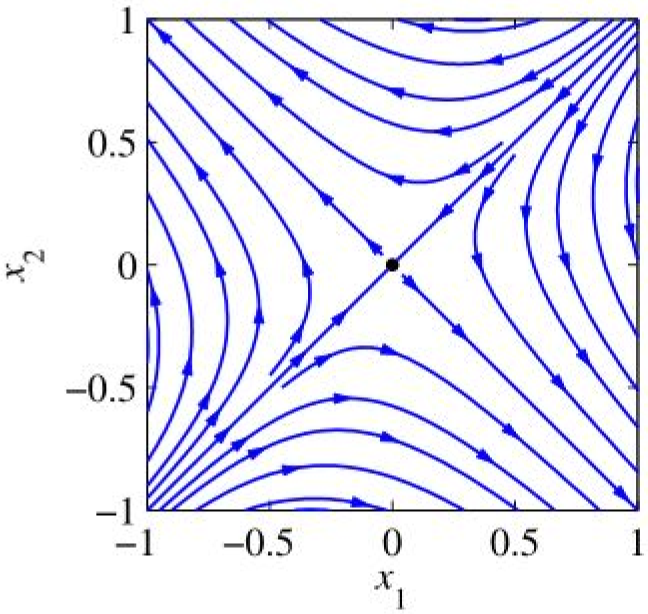
In stability in the sense of Lyapunov,

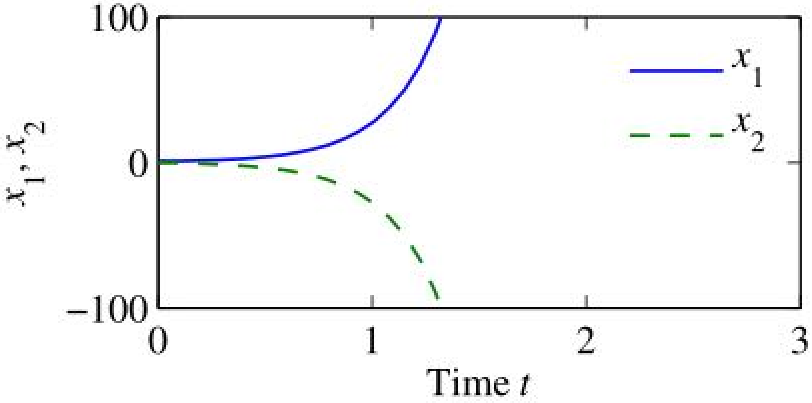
$$x(t;b) \to x(t;a) \qquad \text{as } t \to \infty \text{ for } b \text{ sufficiently close to } a$$





Stability

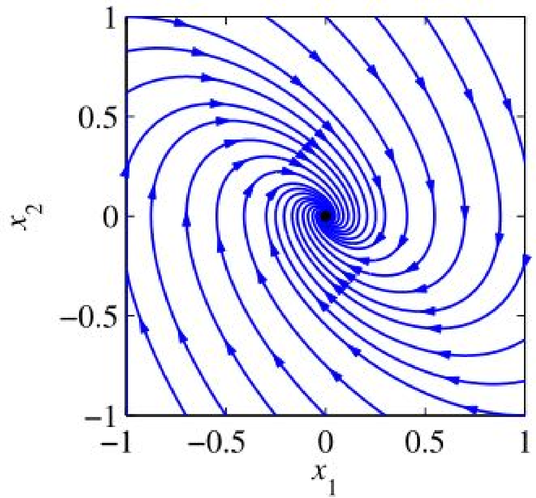




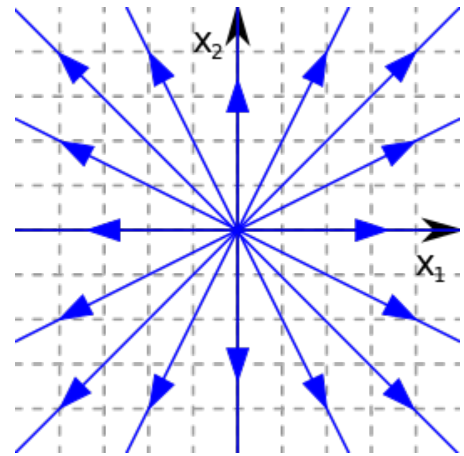
stable for all initial conditions $x \in B_r(a)$ $B_r(a) = \{x : \|x - a\| < r\}$

for all $r > 0$

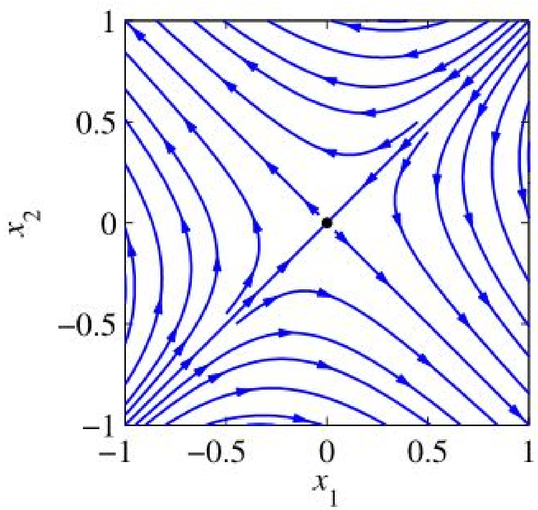
Stability



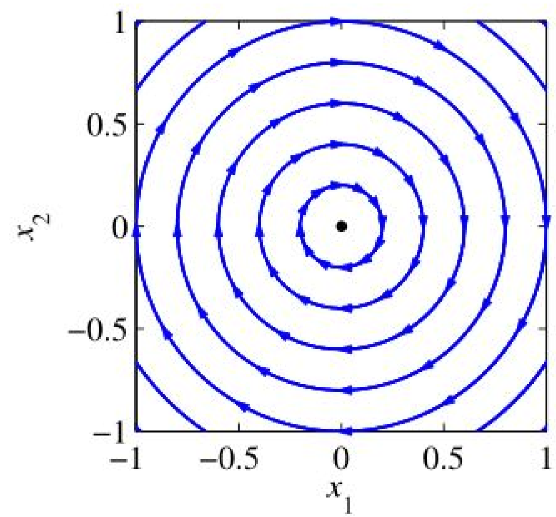
Sink
(or attractor)



Source



Saddle



Center

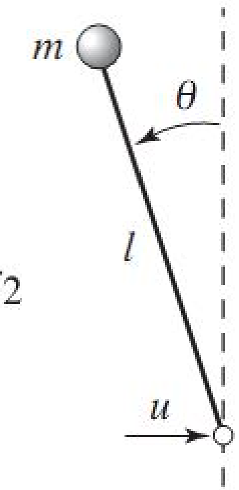
Stability

Ex) Inverted pendulum

$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ \sin x_1 - cx_2 + u\cos x_1 \end{pmatrix}$$

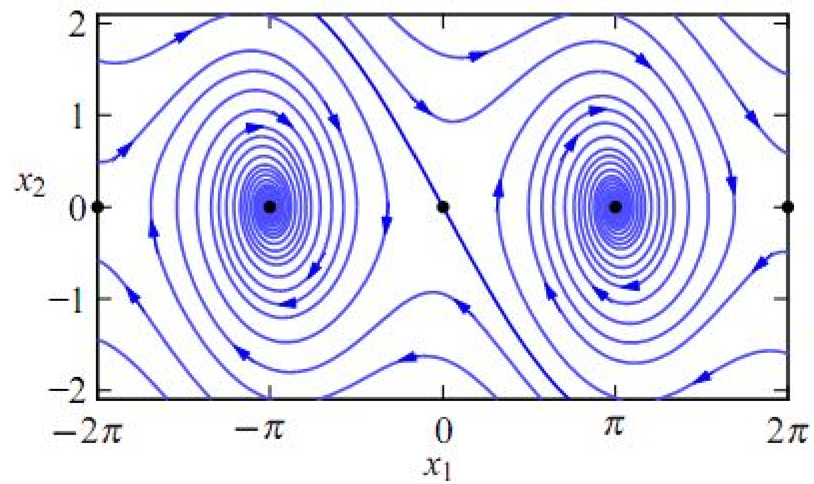
$$\theta = x_1$$

$$d\theta/dt = x_2$$



open loop dynamics

$$x_e = \begin{pmatrix} \pm n\pi \\ 0 \end{pmatrix}$$



Stability (Linear system)

- **Stability of linear systems**

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0$$

  - determined by eigenvalues of system matrix

$$\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}$$

characteristic polynomial

Ex) eigenvalue

$$A\mathrm{x} = s\mathrm{x} \qquad A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad \mathrm{x} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

Stability (Linear system)

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Eigenvalue equation

$$\mathrm{A}\mathrm{x} = s\mathrm{x}$$

$$\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}$$

- Only depending on A

→ Stability of the system

Stability (Linear system)

- **Stability of linear systems**
  - Simple cases

Diagonal matrix

$$\frac{dx}{dt} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} x$$

$$\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}$$

$$\lambda_j \leq 0$$

$$\lambda_j < 0$$

Stability (Linear system)

- **Stability of linear systems**
  - Diagonalization

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \quad AP = P\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

$$P = (\vec{\alpha}_1 \quad \vec{\alpha}_2 \quad \cdots \quad \vec{\alpha}_n), \qquad A\vec{\alpha}_i = \lambda_i \vec{\alpha}_i \qquad (i = 1, 2, \cdots, n).$$

$$\mathrm{A}x = sx$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}. \qquad \lambda_1 = 3, \quad \lambda_2 = 2, \quad \lambda_3 = 1.$$

$$v_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

$$P = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Stability (Linear system)

$$\frac{dx}{dt} = Ax$$

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

transformed system is stable $\Rightarrow$ original system has the same type of stability

Theorem 5.1 (Stability of a linear system). *The system*

$$\frac{dx}{dt} = Ax$$

is asymptotically stable if and only if all eigenvalues of A have a strictly negative real part and is unstable if any eigenvalue of A has a strictly positive real part.