EME5943

현대제어시스템

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Stability (Linear system)

- **Stability analysis via linear approximation**
  - Linear approximation (Linearization)

$$\frac{dx}{dt} = F(x) \qquad \text{an equilibrium point at } x_e$$

Tayler series expansion

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots.$$

$$\frac{dx}{dt} = F(x_e) + \left.\frac{\partial F}{\partial x}\right|_{x_e}(x-x_e) + \text{higher-order terms in } (x-x_e).$$

$$z = x - x_e \qquad \frac{dz}{dt} = Az, \qquad \text{where} \quad A = \left.\frac{\partial F}{\partial x}\right|_{x_e}$$

Lyapunov stability analysis

$$\frac{dx}{dt} = F(x), \quad x \in \mathbb{R}^n$$

stability of solutions for a nonlinear system

▪ Lyapunov functions

- Energy-like function
  - ✓ Nonnegative, always decreased along trajectory
- A few definitions

positive definite

positive semidefinite

negative definite

suppose that $x \in \mathbb{R}^2$

$$V_1(x) = x_1^2 \qquad\qquad V_2(x) = x_1^2 + x_2^2$$

Lyapunov stability analysis

▪ Lyapunov functions

Theorem 5.2 (Lyapunov stability theorem). *Let V be a function on $\mathbb{R}^n$ and let $\dot{V}$ represent the time derivative of V along trajectories of the system dynamics* (5.16)*:*

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}F(x).$$

If there exists $r > 0$ such that V is positive definite and $\dot{V}$ is negative semidefinite on B_r, then $x = 0$ is (locally) stable in the sense of Lyapunov. If V is positive definite and $\dot{V}$ is negative definite in B_r, then $x = 0$ is (locally) asymptotically stable.

If V satisfies one of the conditions above

V : Lyapunov function

If we don't know → candidate Lyapunov function

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Dynamic behavior (4)

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Lyapunov stability analysis

▪ Finding Lyapunov functions

- not always easy to find
- not unique
- If a system is stable → a Lyapunov function exists

- Sum-of-squares technique : a systematic approach
  - ✓ If need, see ref. in textbook

- Systematic method for linear system?

$$\frac{dx}{dt} = Ax$$

$$V(x) = x^T P x$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix ($P = P^T$)

Lyapunov stability analysis

$$\frac{dx}{dt} = Ax$$

$$V(x) = x^T P x$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix ($P = P^T$)

The condition that V be positive definite ➡ $P > 0$

the condition that P be a *positive definite matrix*

⇩

if P is symmetric   if and only if all of its eigenvalues are real and positive

candidate Lyapunov function $V(x) = x^T P x$

Lyapunov stability analysis

candidate Lyapunov function $V(x) = x^T P x$

- Always has a solution if all of the eigenvalues of A are in the left half-plane
- $P > 0$ if $Q > 0$

Lyapunov stability analysis

▪ Finding Lyapunov function

- Stability of nonlinear system
  with finding Lyapunov function of linear system

$$\frac{dx}{dt} = F(x) =: Ax + \tilde{F}(x)$$

$$F(0) = 0$$

$\tilde{F}(x)$ contains terms that are second order and higher

Theorem 5.3. *Consider the dynamical system (5.18) with $F(0) = 0$ and $\tilde{F}$ such that $\lim \|\tilde{F}(x)\|/\|x\| \to 0$ as $\|x\| \to 0$. If the real parts of all eigenvalues of A are strictly less than zero, then $x_e = 0$ is a locally asymptotically stable equilibrium point of equation (5.18).*