EME5943
현대제어시스템

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Stability

- **Definitions**

Solution to a differential equation
with initial condition a

Stable?

if other solutions that start near a stay close to $x(t;a)$

if for all $\varepsilon > 0$ there exists a $\delta > 0$

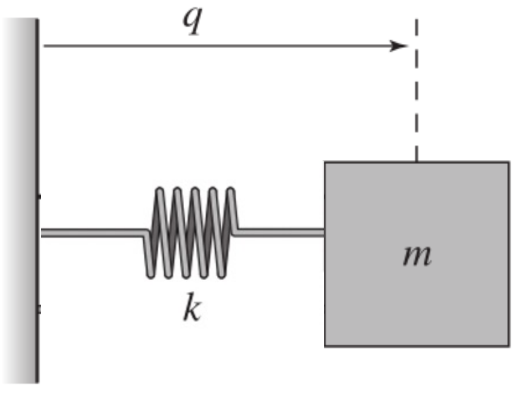
$$\|b-a\| < \delta \Rightarrow \|x(t;b)-x(t;a)\| < \varepsilon \quad \text{for all } t > 0$$

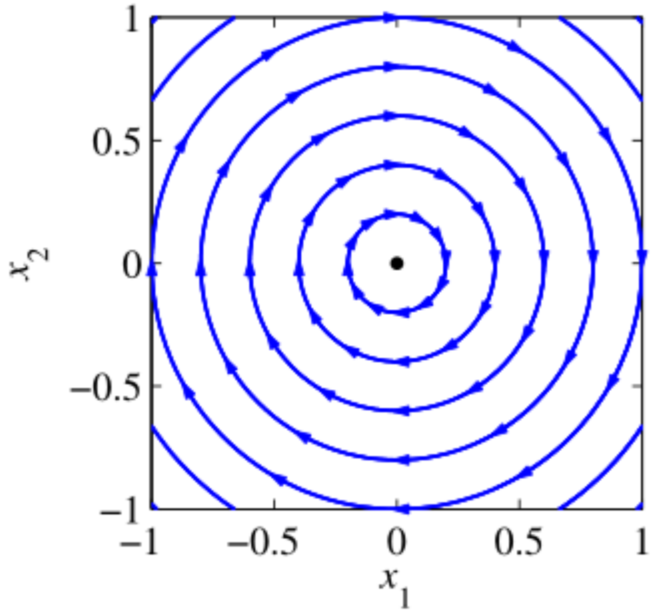
Stability

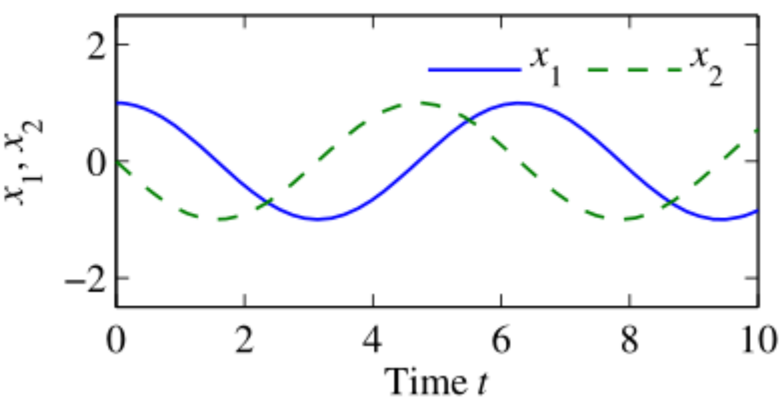
An important special case

$$x(t;a) = x_e$$

the equilibrium point is stable

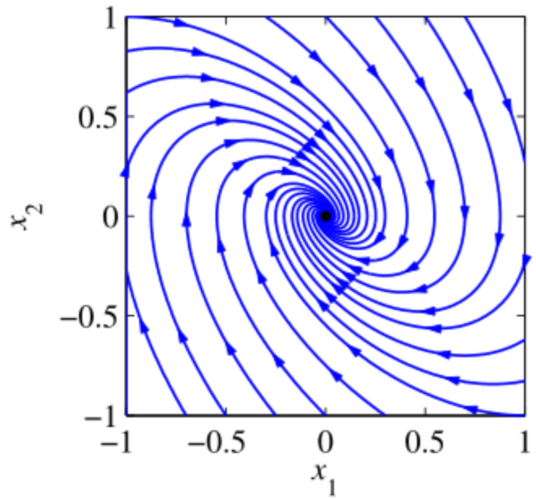




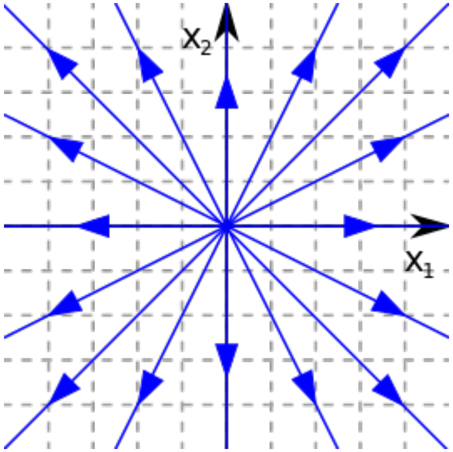


neutrally stable equilibrium point

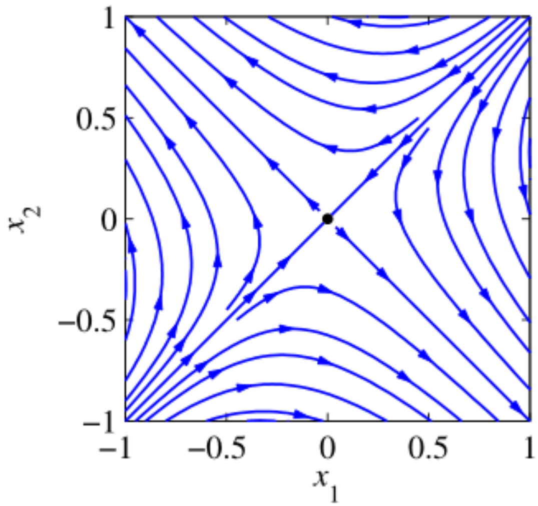
Stability



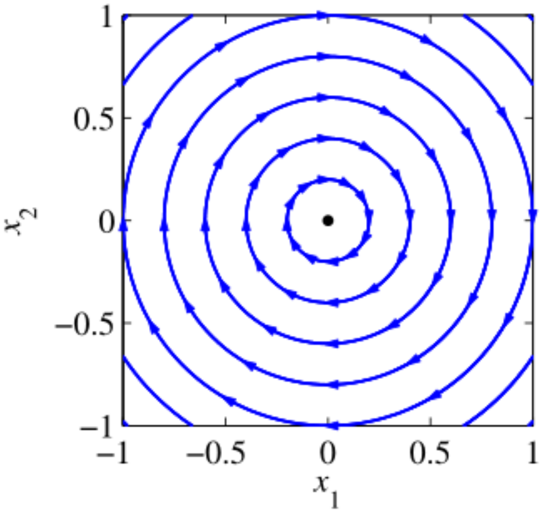
Sink
(or attractor)



Source



Saddle



Center

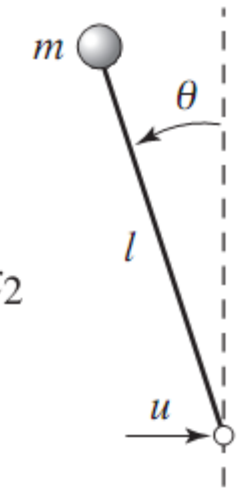
Stability

Ex) Inverted pendulum

$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ \sin x_1 - cx_2 + u\cos x_1 \end{pmatrix}$$

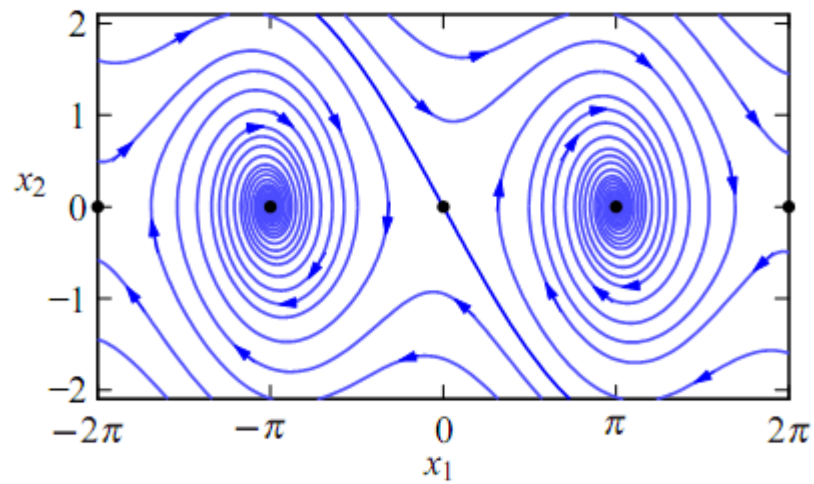
$$\theta = x_1$$

$$d\theta/dt = x_2$$



open loop dynamics

$$x_e = \begin{pmatrix} \pm n\pi \\ 0 \end{pmatrix}$$



Stability (Linear system)

Stability of linear systems

- Simple cases

Diagonal matrix

$$\frac{dx}{dt} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} x$$

$$\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}$$

$$\lambda_j \leq 0$$

$$\lambda_j < 0$$

Stability (Linear system)

- **Stability of linear systems**
  - Diagonalization

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \quad AP = P\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

$$P = (\vec{\alpha}_1 \quad \vec{\alpha}_2 \quad \cdots \quad \vec{\alpha}_n), \qquad A\vec{\alpha}_i = \lambda_i\vec{\alpha}_i \qquad (i = 1, 2, \cdots, n).$$

$$\mathrm{A}\mathrm{x} = s\mathrm{x}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}. \qquad \lambda_1 = 3, \quad \lambda_2 = 2, \quad \lambda_3 = 1.$$

$$v_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

$$P = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Stability (Linear system)

$$\frac{dx}{dt} = Ax$$

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

transformed system is stable ➡ original system has the same type of stability

Theorem 5.1 (Stability of a linear system). *The system*

$$\frac{dx}{dt} = Ax$$

is asymptotically stable if and only if all eigenvalues of A have a strictly negative real part and is unstable if any eigenvalue of A has a strictly positive real part.

현대제어시스템

Dynamic behavior (3)

Stability (Linear system)

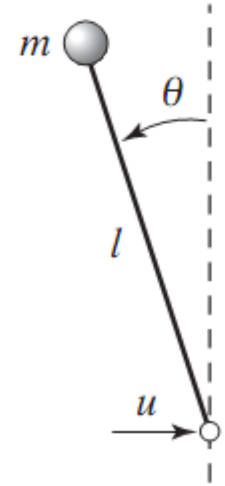
ex) A linear system

$$\frac{dx}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} x + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u. \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

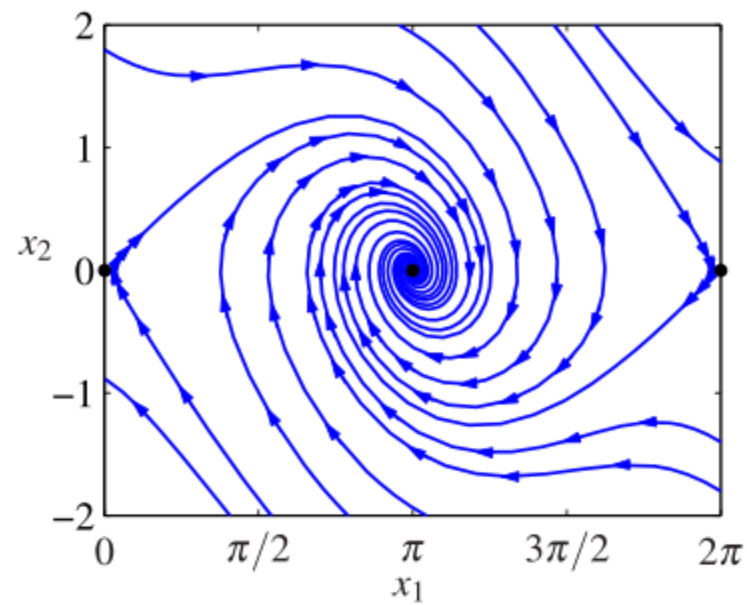
Stability (Linear system)

Stability (Linear system)

- **Stability analysis via linear approximation**
  - Approximating system to a linear system
    → Local stability of an equilibrium point
  - ex) Inverted pendulum

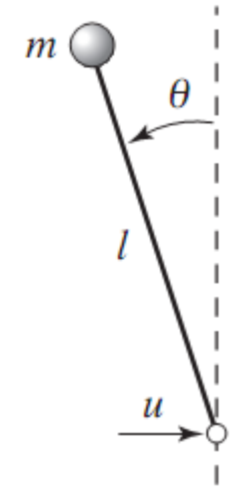


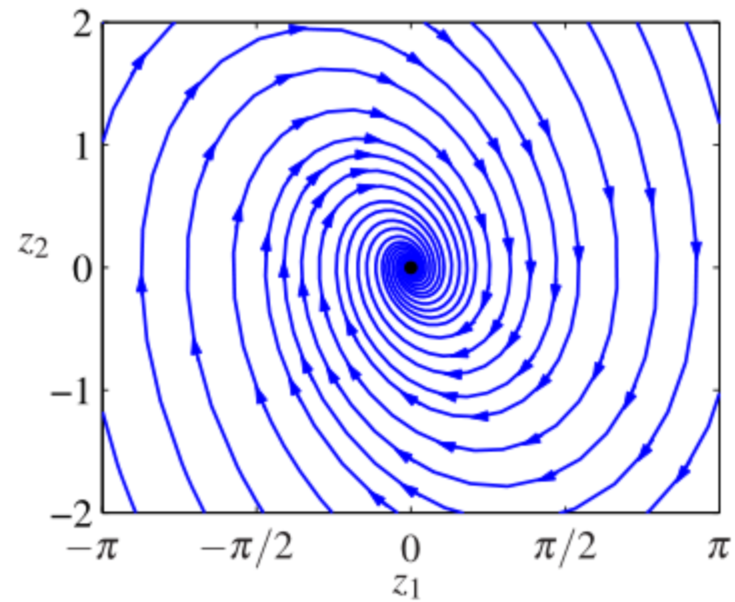
$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{pmatrix} \qquad x = (\theta, \dot{\theta})$$



Stability (Linear system)

at $x = (\pi, 0)$





Stability (Linear system)

- **Stability analysis via linear approximation**
  - Linear approximation (Linearization)

$$\frac{dx}{dt} = F(x) \qquad \text{an equilibrium point at } x_e$$

Tayler series expansion

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots.$$

$$\frac{dx}{dt} = F(x_e) + \left.\frac{\partial F}{\partial x}\right|_{x_e}(x-x_e) + \text{higher-order terms in } (x-x_e).$$

$$z = x - x_e \qquad \frac{dz}{dt} = Az, \qquad \text{where} \quad A = \left.\frac{\partial F}{\partial x}\right|_{x_e}$$

Stability (Linear system)

- **Stability analysis via linear approximation**
  - In nonlinear system ?

Nonlinear system → Linear approximation

→ Linear system at local area

→ Designing feedback control law
that keeps system near its equilibrium point

→ Stability of equilibrium point of nonlinear system

Lyapunov stability analysis

$$\frac{dx}{dt} = F(x), \quad x \in \mathbb{R}^n$$

stability of solutions for a nonlinear system

- **Lyapunov functions**
  - Energy-like function
    - ✓ Nonnegative, always decreased along trajectory
  - A few definitions

positive definite

positive semidefinite

negative definite

suppose that $x \in \mathbb{R}^2$

$$V_1(x) = x_1^2 \qquad V_2(x) = x_1^2 + x_2^2$$

Lyapunov stability analysis

- **Lyapunov functions**

Theorem 5.2 (Lyapunov stability theorem). *Let V be a function on $\mathbb{R}^n$ and let $\dot{V}$ represent the time derivative of V along trajectories of the system dynamics* (5.16)*:*

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}F(x).$$

If there exists $r > 0$ such that V is positive definite and $\dot{V}$ is negative semidefinite on B_r, then $x = 0$ is (locally) stable in the sense of Lyapunov. If V is positive definite and $\dot{V}$ is negative definite in B_r, then $x = 0$ is (locally) asymptotically stable.

If V satisfies one of the conditions above

V : Lyapunov function

If we don't know → candidate Lyapunov function

Lyapunov stability analysis

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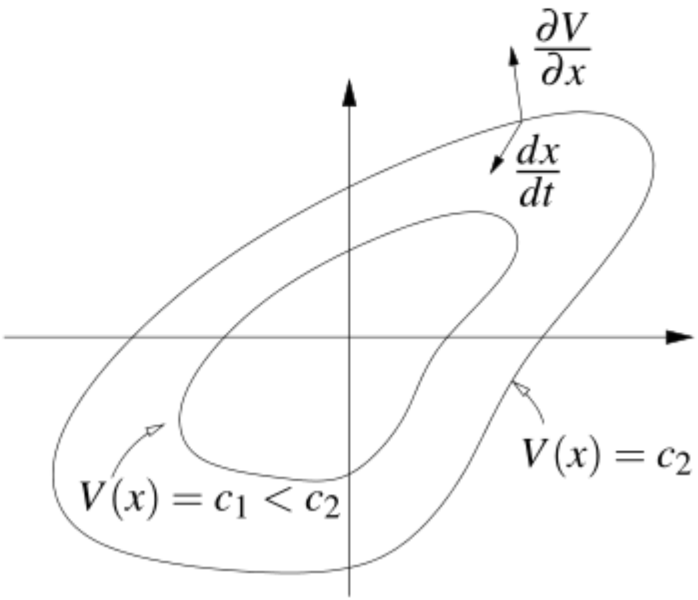

Lyapunov stability analysis

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$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}F(x).$$

$$\frac{dx}{dt} = F(x), \quad x \in \mathbb{R}^n$$

Chain rule

Lyapunov stability analysis

- **Lyapunov function**
  - Ex) Scalar nonlinear system

$$\frac{dx}{dt} = \frac{2}{1+x} - x$$

Lyapunov stability analysis

$$\dot{V}(z) = z\dot{z} = \frac{2z}{2+z} - z^2 - z$$

$$z \in B_r,\ r < 2$$

Lyapunov stability analysis

- **Lyapunov function**
  - Ex) Hanging pendulum

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = -\sin x_1$$

Tayler series expansion

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{for all } x$$

Lyapunov stability analysis

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt}$$

Lyapunov stability analysis

- **Finding Lyapunov functions**
  - not always easy to find
  - not unique
  - If a system is stable → a Lyapunov function exists

  - Sum-of-squares technique : a systematic approach
    - ✓ If need, see ref. in textbook

  - Systematic method for linear system?

$$\frac{dx}{dt} = Ax$$

$$V(x) = x^T P x$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix $(P = P^T)$

Lyapunov stability analysis

$$\frac{dx}{dt} = Ax$$

$$V(x) = x^T P x$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix $(P = P^T)$

The condition that V be positive definite $\Rightarrow$ $P > 0$

the condition that P be a *positive definite matrix*

$\Downarrow$

if P is symmetric if and only if all of its eigenvalues are real and positive

candidate Lyapunov function $V(x) = x^T P x$

Lyapunov stability analysis

candidate Lyapunov function $V(x) = x^T Px$

- Always has a solution if all of the eigenvalues of A are in the left half-plane
- $P > 0$ if $Q > 0$

Lyapunov stability analysis

▪ Finding Lyapunov function

- Stability of nonlinear system
  with finding Lyapunov function of linear system

$$\frac{dx}{dt} = F(x) =: Ax + \tilde{F}(x)$$

$$F(0) = 0$$

$\tilde{F}(x)$ contains terms that are second order and higher

Theorem 5.3. *Consider the dynamical system (5.18) with $F(0) = 0$ and $\tilde{F}$ such that $\lim \|\tilde{F}(x)\|/\|x\| \to 0$ as $\|x\| \to 0$. If the real parts of all eigenvalues of A are strictly less than zero, then $x_e = 0$ is a locally asymptotically stable equilibrium point of equation (5.18).*