EME5943
현대제어시스템

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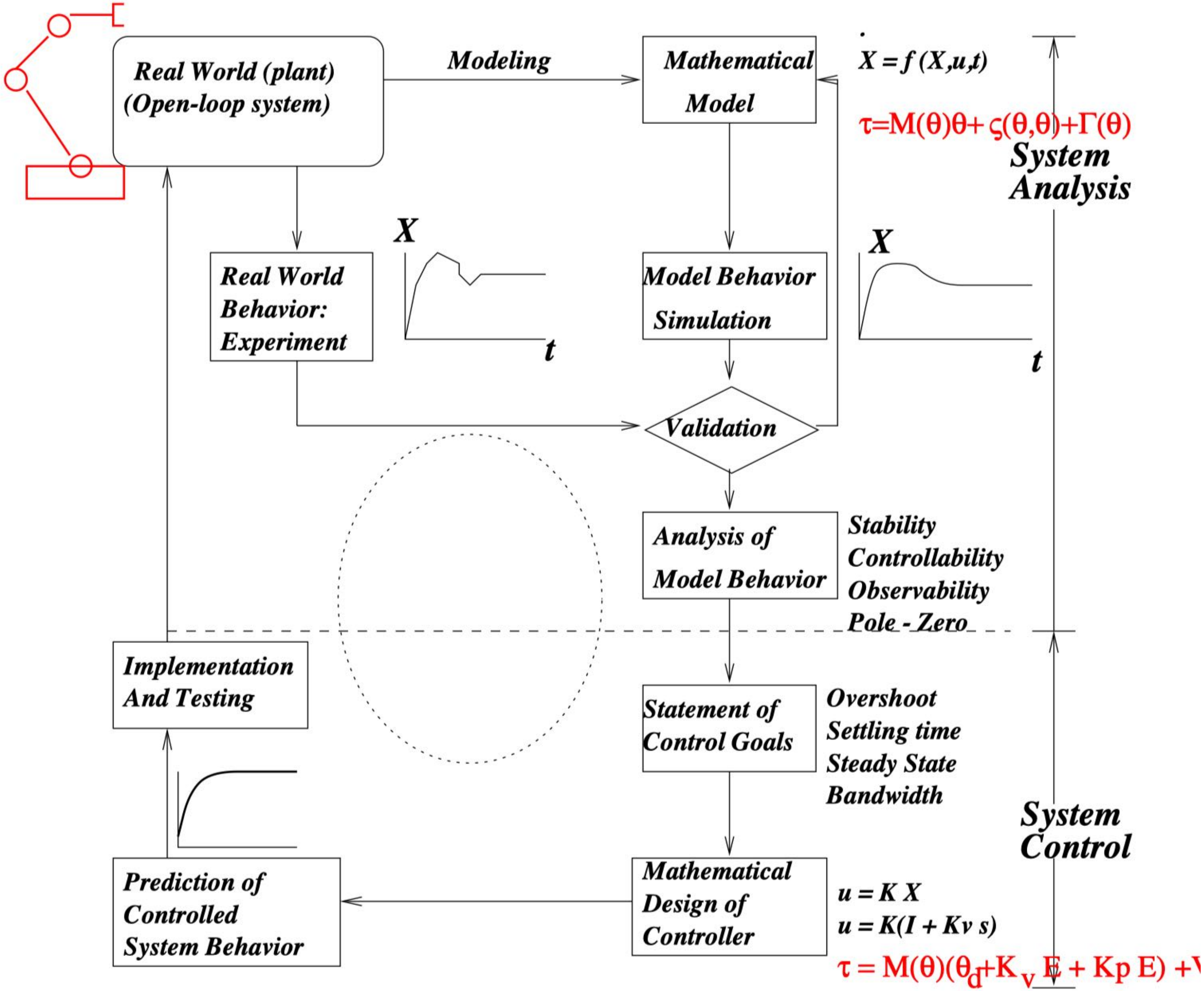
What is control?



What is feedforward?

- **Feedback: reactive**
  - There must be an error before corrective actions are taken.

- **Feedforward**
  - Possible to measure a disturbance before it enters the system
  - Taking corrective action before the disturbance has influenced
  - Reducing the effect of the disturbance

Feedback	Feedforward
Closed loop	Open loop
Acts on deviations	Acts on plans
Robust to model uncertainty	Sensitive to model uncertainty
Risk for instability	No risk for instability

Simple forms of feedback

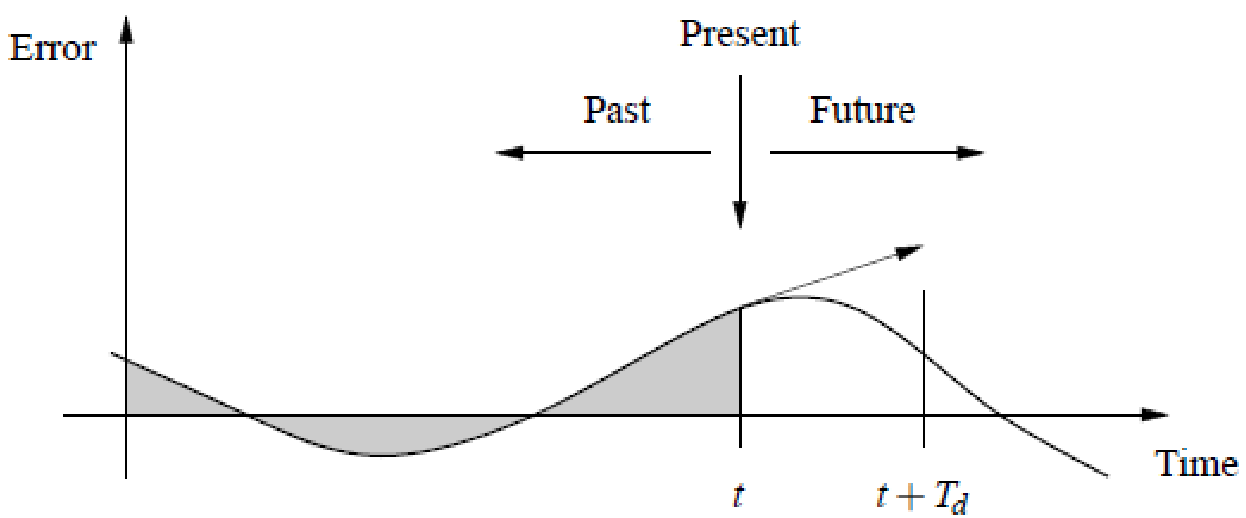
- **PID control**

$$u = \begin{cases} u_{\max} & \text{if } e \geq e_{\max} \\ k_p e & \text{if } e_{\min} < e < e_{\max} \\ u_{\min} & \text{if } e \leq e_{\min}, \end{cases}$$

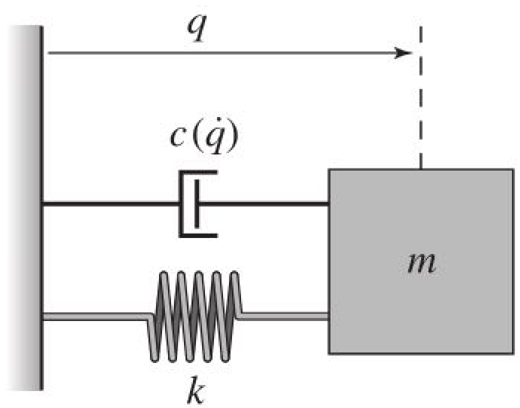
$$u(t) = k_i \int_0^t e(\tau)d\tau.$$

$$e(t+T_d) \approx e(t) + T_d \frac{de(t)}{dt}$$

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau)\,d\tau + k_d \frac{de(t)}{dt}$$



Modeling concept



$$m\ddot{q} + c(\dot{q}) + kq = 0$$

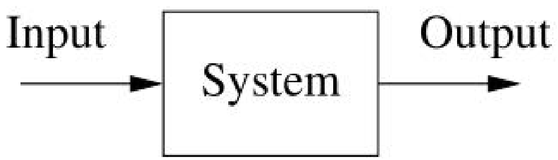
- **Input**
  - A system variable that is independently prescribed, or defined by the environment
- **Output**
  - Any system variable of interest
- **State variables**
  - A minimum set of system variable that completely characterizes the motion of a system for the purpose of predicting future motion

Modeling concept

- **State vector**
  - A vector whose elements consists of state variables
- **State equation**
  - The relationship among the change of state, present state, and input
- **State-space model (representation)**
  - A mathematical model of a system using a state equation
  - It determines the system behavior for all time, given
    - ✓ The initial values of state variables
    - ✓ The specification of the inputs to the system for all times

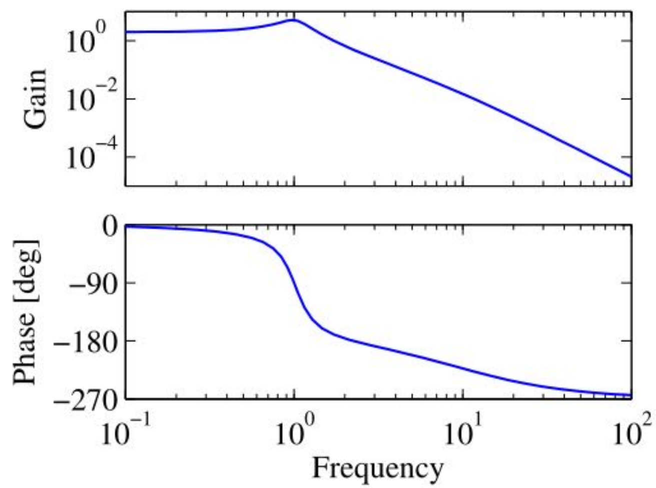
Modeling concept

- **Input-output view**
  - Heritage of electrical engineering
  - Useful for
    Linear Time-Invariant (LTI) system



- **Transfer function**
  - Transfer ratio of the Laplace transform of the output to the Laplace transform of the input

$$\frac{Y(s)}{U(s)} = H(s) \qquad \dot{y}(t) + ky(t) = u(t)$$



State-space model

▪ Ordinary differential equations

- State space model: a form of differential equation

$$\frac{dx}{dt} = f(x,u) \qquad\qquad y = h(x,u)$$

$x \in \mathbb{R}^n$ state vector $u \in \mathbb{R}^p$ input (vector)

$y \in \mathbb{R}^q$ output (vector)

- Linear state space system

$$\frac{dx}{dt} = Ax + Bu \qquad\qquad y = Cx + Du$$

dynamic matrix control matrix sensor matrix direct term

State-space model

- **Ordinary differential equations**
  - Another form of linear differential equations

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = u$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} d^{n-1}y/dt^{n-1} \\ d^{n-2}y/dt^{n-2} \\ \vdots \\ dy/dt \\ y \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = x_n$$

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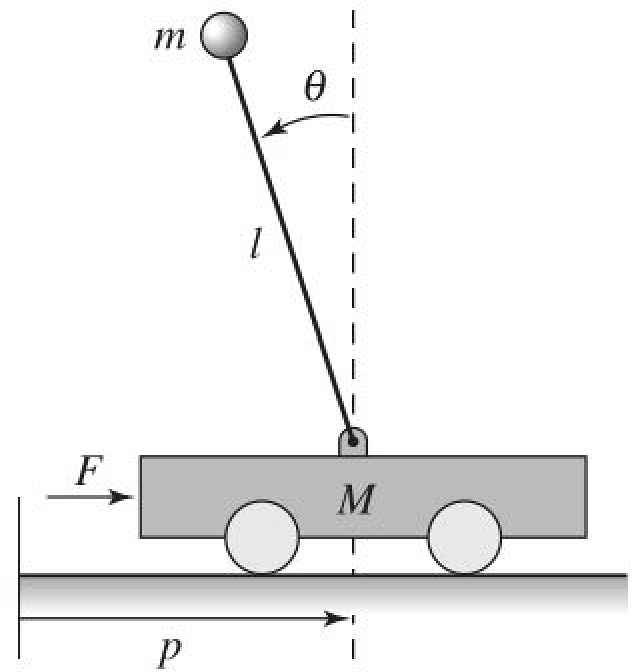
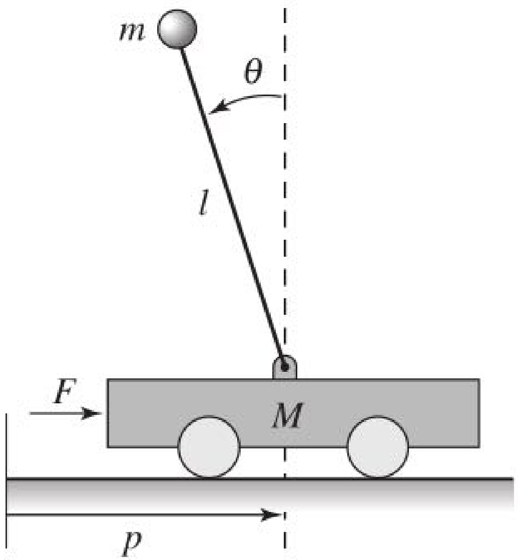
System modeling (2)

State-space model

- **Ordinary differential equations**
  - Example: Balance systems



State-space model



Dynamics of system

$$\begin{pmatrix} (M+m) & -ml\cos\theta \\ -ml\cos\theta & (J+ml^2) \end{pmatrix} \begin{pmatrix} \ddot{p} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} c\dot{p} + ml\sin\theta\,\dot{\theta}^2 \\ \gamma\dot{\theta} - mgl\sin\theta \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$M_t = M + m \qquad J_t = J + ml^2$$
$$c_\theta = \cos\theta \qquad s_\theta = \sin\theta$$

$$\frac{d}{dt}\begin{pmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{p} \\ \dot{\theta} \\ \dfrac{-mls_\theta\dot{\theta}^2 + mg(ml^2/J_t)s_\theta c_\theta - c\dot{p} - (\gamma/J_t)mlc_\theta\dot{\theta} + u}{M_t - m(ml^2/J_t)c_\theta^2} \\ \dfrac{-ml^2 s_\theta c_\theta\dot{\theta}^2 + M_t gls_\theta - clc_\theta\dot{p} - \gamma(M_t/m)\dot{\theta} + lc_\theta u}{J_t(M_t/m) - m(lc_\theta)^2} \end{pmatrix} \qquad y = \begin{pmatrix} p \\ \theta \end{pmatrix}$$

State-space model

$$\frac{d}{dt}\begin{pmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{p} \\ \dot{\theta} \\ \dfrac{-mls_\theta\dot{\theta}^2 + mg(ml^2/J_t)s_\theta c_\theta - c\dot{p} - (\gamma/J_t)mlc_\theta\dot{\theta} + u}{M_t - m(ml^2/J_t)c_\theta^2} \\ \dfrac{-ml^2 s_\theta c_\theta\dot{\theta}^2 + M_t gls_\theta - clc_\theta\dot{p} - \gamma(M_t/m)\dot{\theta} + lc_\theta u}{J_t(M_t/m) - m(lc_\theta)^2} \end{pmatrix}$$

$$\Downarrow$$

$$\frac{d}{dt}\begin{pmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2l^2g/\mu & -cJ_t/\mu & -\gamma lm/\mu \\ 0 & M_tmgl/\mu & -clm/\mu & -\gamma M_t/\mu \end{pmatrix}\begin{pmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ J_t/\mu \\ lm/\mu \end{pmatrix} u$$

State-space model

- **Difference equations**
  - Not continuously in time, but discrete instants of time

    → Discrete-time system

$$\frac{dx}{dt} = f(x,u) \quad \Rightarrow \quad x[k+1] = f(x[k], u[k])$$

$$y = h(x,u) \quad \Rightarrow \quad y[k] = h(x[k], u[k])$$

Linear cases

State-space model

$$x[k+1] = Ax[k] + Bu[k] \qquad y[k] = Cx[k] + Du[k]$$

Solution ?

$$x[k] = A^k x[0] + \sum_{j=0}^{k-1} A^{k-j-1} Bu[j],$$

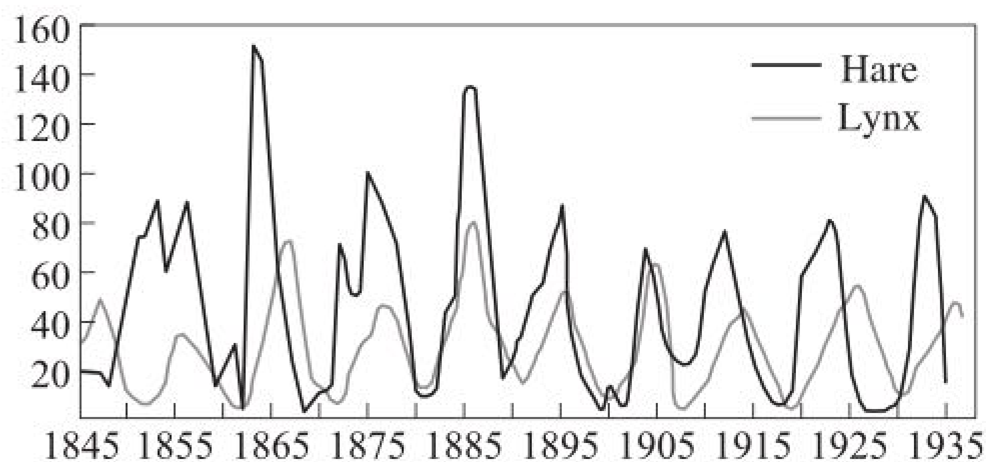
$$k > 0.$$

$$y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1} Bu[j] + Du[k],$$

State-space model

- **Difference equations**
  - Example: Predator-prey
    - ✓ Ecological system with two species, one of which feeds on the other

Canadian lynx *vs* Snowshoe hare



A discrete data

State-space model

Model as a difference equation
by keeping track of rate of births and deaths of each species

$$H[k+1] = H[k] + b_r(u)H[k] - aL[k]H[k]$$
$$L[k+1] = L[k] + cL[k]H[k] - d_f L[k],$$

H population of hares

L population of lynxes

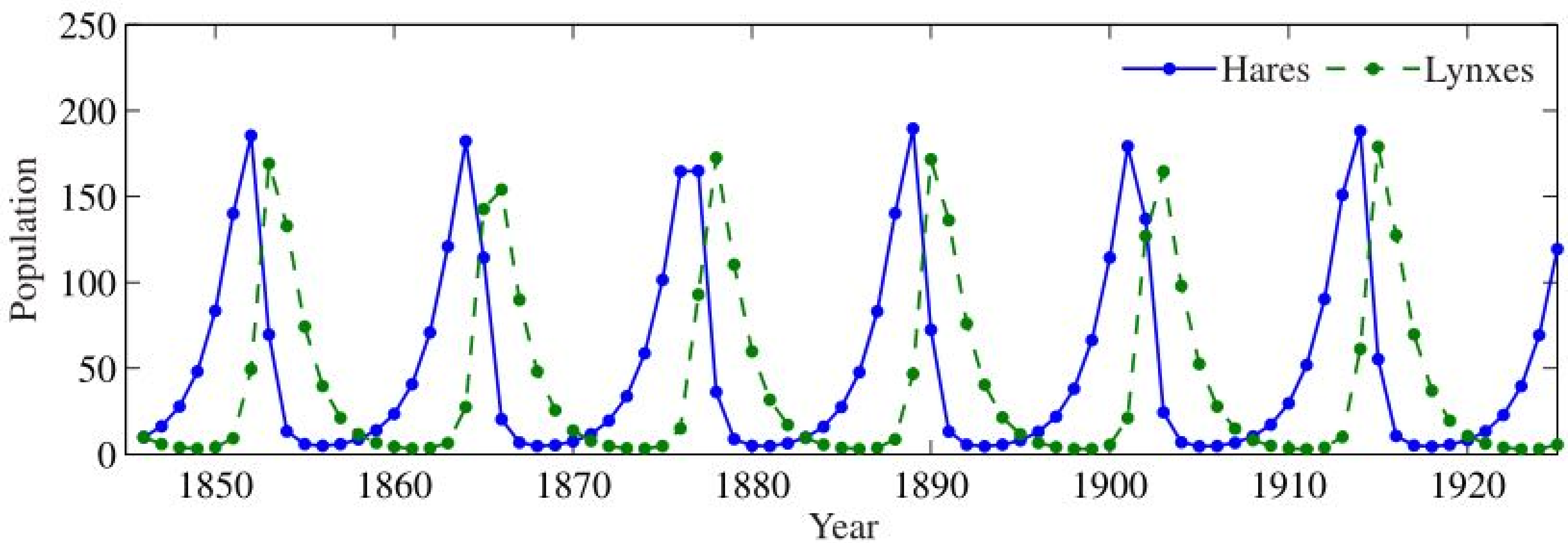
$b_r(u)$ hare birth rate

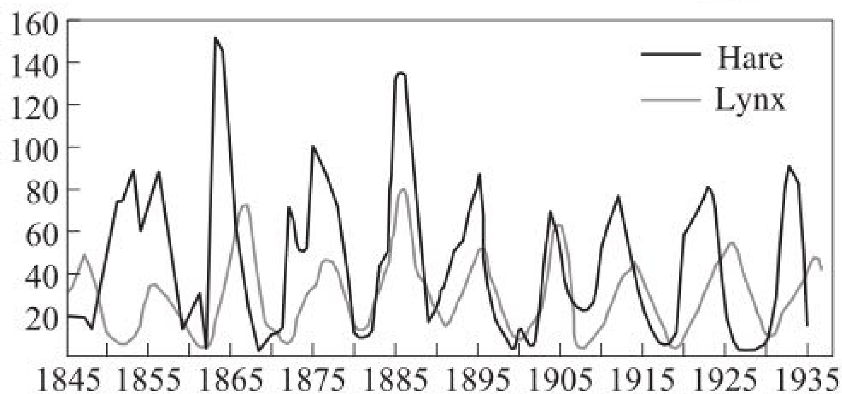
d_f lynx mortality rate

with many simplifying assumptions....

State-space model

$$H[k+1] = H[k] + b_r(u)H[k] - aL[k]H[k]$$
$$L[k+1] = L[k] + cL[k]H[k] - d_f L[k],$$

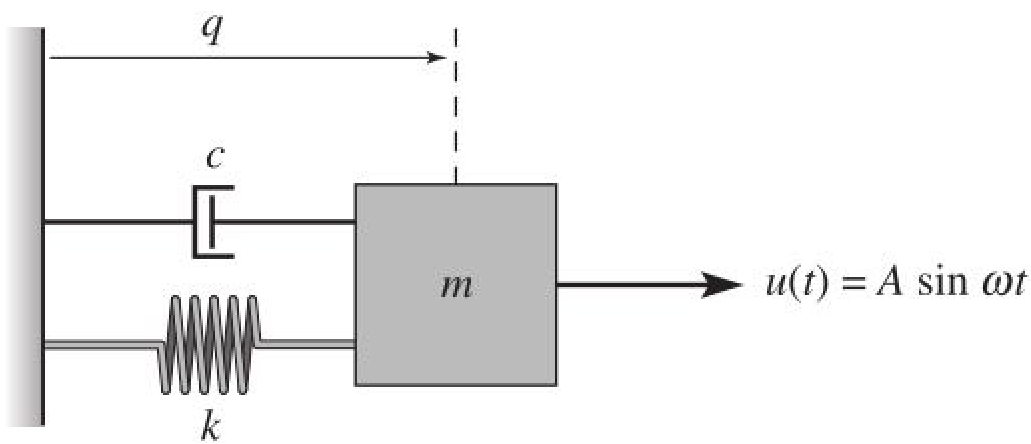




State-space models

- **Simulation & Analysis**
  - Predicting evolution of system state from an initial condition
    - ✓ in closed form
    - ✓ through computer simulation
  - Analyzing overall behavior of system without simulation

ex) mass-spring-damper system



$$m\ddot{q} + c\dot{q} + kq = u$$

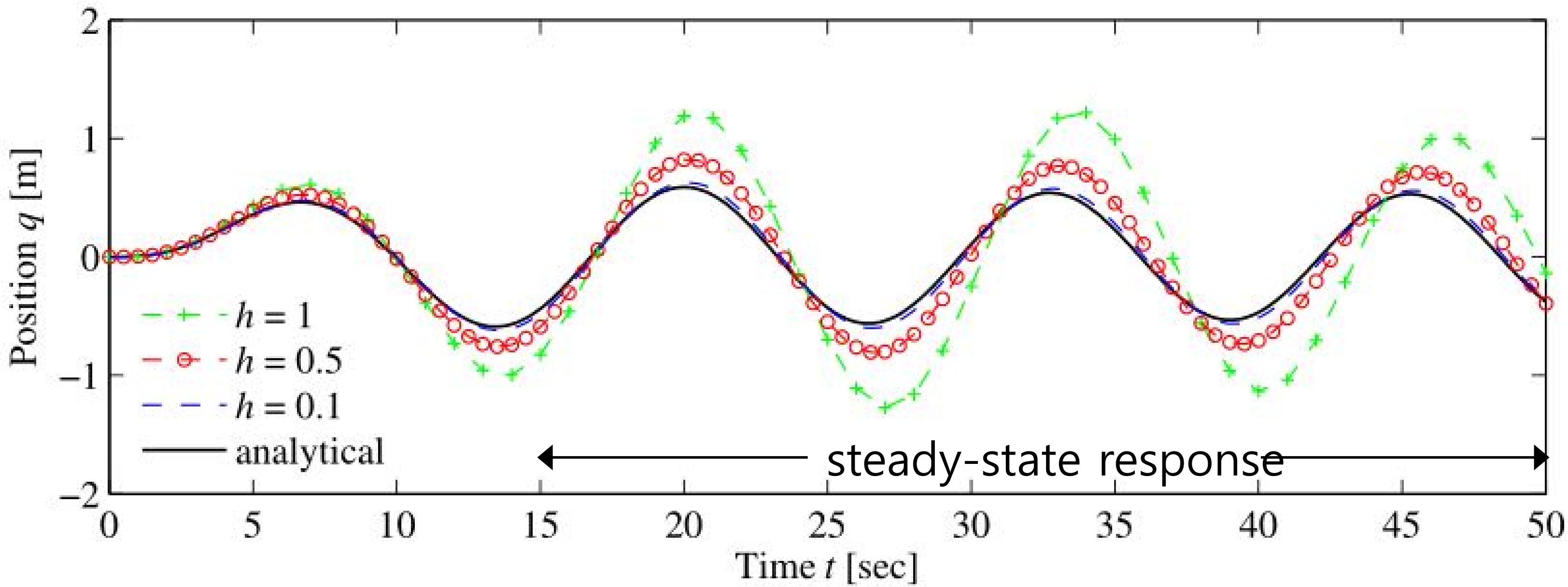
State-space models

$$m\ddot{q} + c\dot{q} + kq = u \quad \Rightarrow \quad x = (q, \dot{q}) \quad \frac{dx}{dt} = \begin{pmatrix} x_2 \\ -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{u}{m} \end{pmatrix}$$

$$y = x_1$$

How can we predict?

$$\frac{dx}{dt} = f(x, u)$$



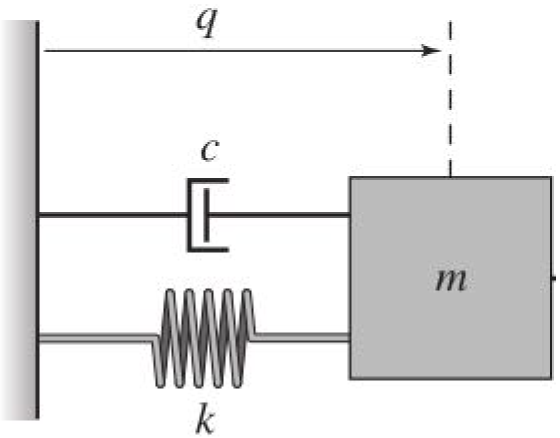
State-space models

▪ Simulation & Analysis

- Stability of an equilibrium point

ex) Equation of motion of
mass-spring-damper system with no input

$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ -\frac{c}{m}x_2 - \frac{k}{m}x_1 \end{pmatrix}$$



If the initial state of the system is away from the rest position, the system will return to the rest position eventually…

the rest position is *asymptotically stable*

Modeling methodology

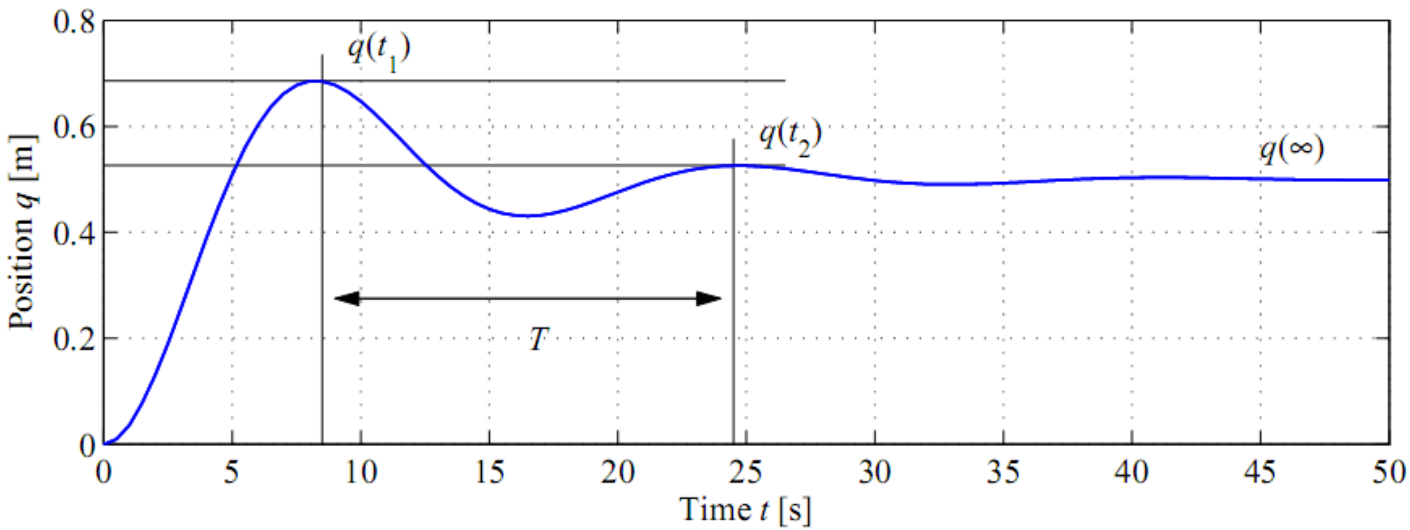
- **Modeling from experiments**

$$m\ddot{q} + c\dot{q} + kq = u$$

step response

$$q(t) = \frac{F_0}{k}\left(1 - \frac{1}{\omega_d}\sqrt{\frac{k}{m}}\exp\left(-\frac{ct}{2m}\right)\sin(\omega_d t + \varphi)\right)$$

$$\omega_d = \frac{\sqrt{4km - c^2}}{2m} \qquad \varphi = \tan^{-1}\left(\frac{\sqrt{4km - c^2}}{c}\right)$$



Modeling methodology

▪ Normalization & Scaling

- Scaling by dimension-free variable
- Purpose
  - ✓ Simplifying equations for system by reducing # of parameters
  - ✓ Revealing interesting properties of model

ex) Spring-mass system

$$m\ddot{q} + kq = u$$

$$x = q/l$$

$$\tau = \omega_0 t$$

$$v = u/(ml\omega_0^2)$$

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Dynamic behavior

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Solving differential equations

- **ODEs**

$$\frac{dx}{dt} = f(x,u) \qquad y = h(x,u)$$

$$u \in \mathbb{R}^p \qquad y \in \mathbb{R}^q$$

$p = q = 1$ ➡ single-input, single-output (SISO) systems

Solution?

$$\frac{dx(t)}{dt} = F(x(t)) \quad \text{for all } t_0 < t < t_f$$

Many solutions

Initial value problem

A unique solution

Solving differential equations

- **ODEs**
  - ex) damped oscillator

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = 0 \qquad \begin{aligned} x_1 &= q \\ x_2 &= \dot{q}/\omega_0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{dx_1}{dt} &= \omega_0 x_2 \\ \frac{dx_2}{dt} &= -\omega_0 x_1 - 2\zeta\omega_0 x_2 \end{aligned}$$

$$x_1(t) = e^{-\zeta\omega_0 t}\left(x_{10}\cos\omega_d t + \frac{1}{\omega_d}(\omega_0\zeta x_{10} + x_{20})\sin\omega_d t\right)$$

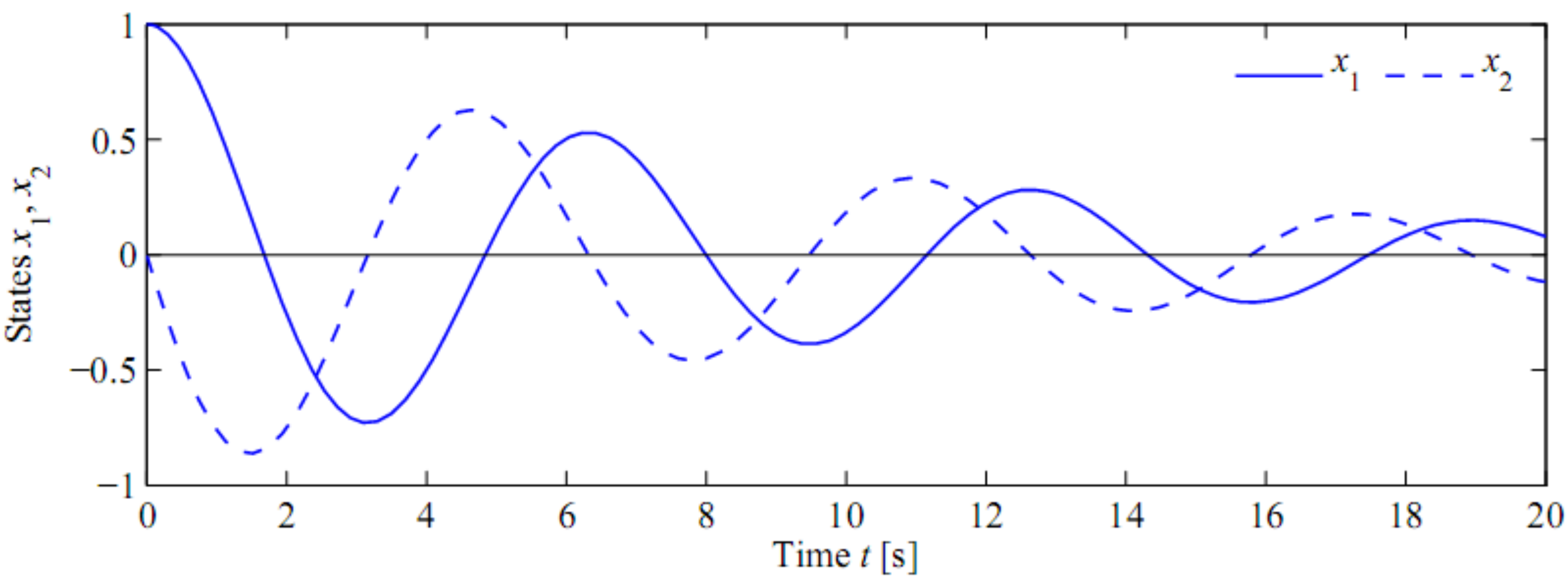
$$x_2(t) = e^{-\zeta\omega_0 t}\left(x_{20}\cos\omega_d t - \frac{1}{\omega_d}(\omega_0^2 x_{10} + \omega_0\zeta x_{20})\sin\omega_d t\right)$$

where $x_0 = (x_{10}, x_{20}) \quad \omega_d = \omega_0\sqrt{1-\zeta^2}$

Solving differential equations

$$x_1(t) = e^{-\zeta\omega_0 t}\left(x_{10}\cos\omega_d t + \frac{1}{\omega_d}(\omega_0\zeta x_{10} + x_{20})\sin\omega_d t\right)$$
$$x_2(t) = e^{-\zeta\omega_0 t}\left(x_{20}\cos\omega_d t - \frac{1}{\omega_d}(\omega_0^2 x_{10} + \omega_0\zeta x_{20})\sin\omega_d t\right)$$

where $x_0 = (x_{10}, x_{20})$ $\quad \omega_d = \omega_0\sqrt{1-\zeta^2}$

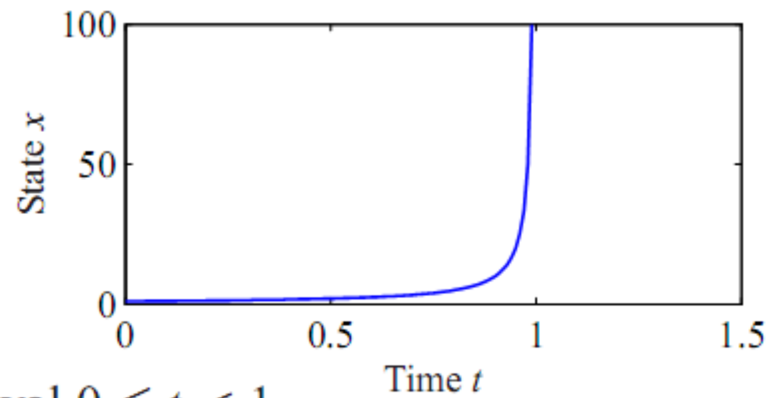


solution holds only for $0 < \zeta < 1$

Solving differential equations

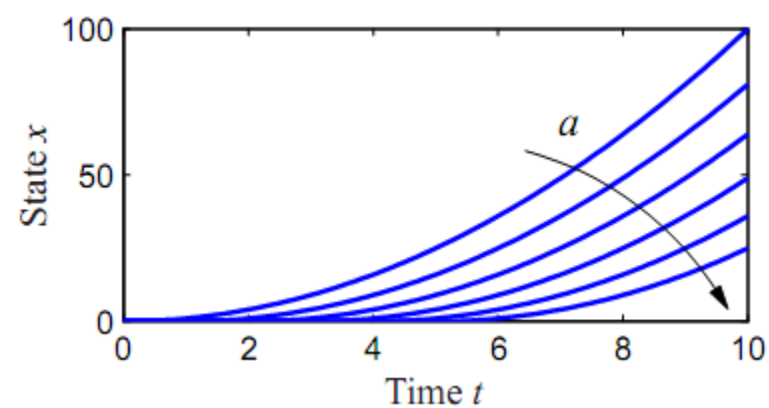
- **ODEs**
  - ex) No solution, Non-unique solution

$$\frac{dx}{dt} = x^2 \qquad x(0) = 1$$



only in the time interval $0 \leq t < 1$

$$\frac{dx}{dt} = 2\sqrt{x} \qquad x(0) = 0$$



many solutions

Solving differential equations

- **Lipschitz continuity**
  - For guaranteeing existence & uniqueness

$$\frac{dx}{dt} = F(x)$$

$$\|F(x) - F(y)\| < c\|x - y\| \quad \text{for all } x, y.$$

  - Sufficient condition

$$\partial F / \partial x \quad \text{uniformly bounded for all } x$$

$$\frac{dx}{dt} = x^2 \quad \Rightarrow$$

$$\frac{dx}{dt} = 2\sqrt{x} \quad \Rightarrow$$

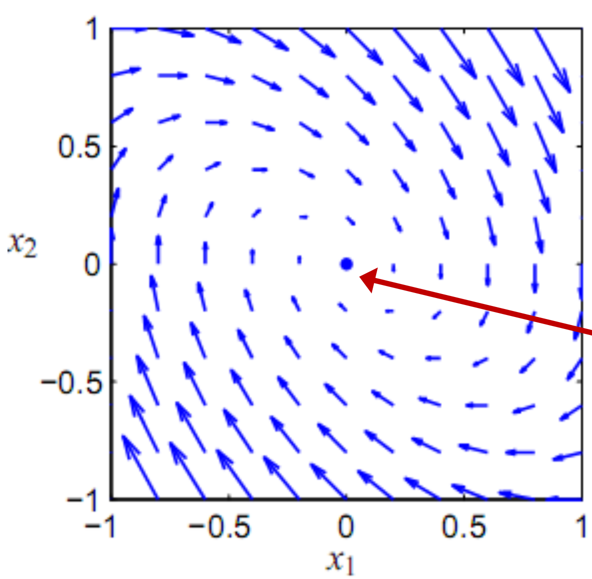
Qualitative analysis

- **Phase portraits**
  - To understand behavior of dynamical systems

  - Vector field

$$\frac{dx}{dt} = F(x)$$

vector representing the velocity of that state



stationary points of the flow

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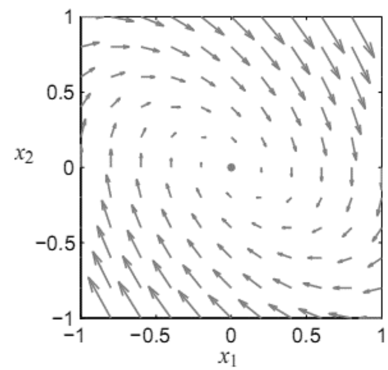
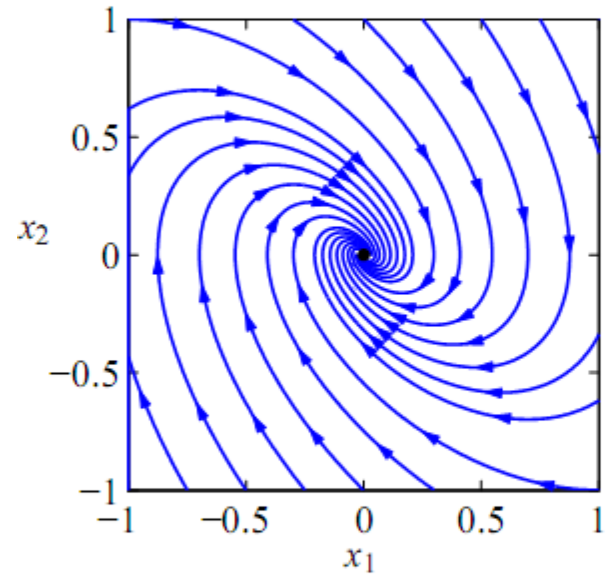
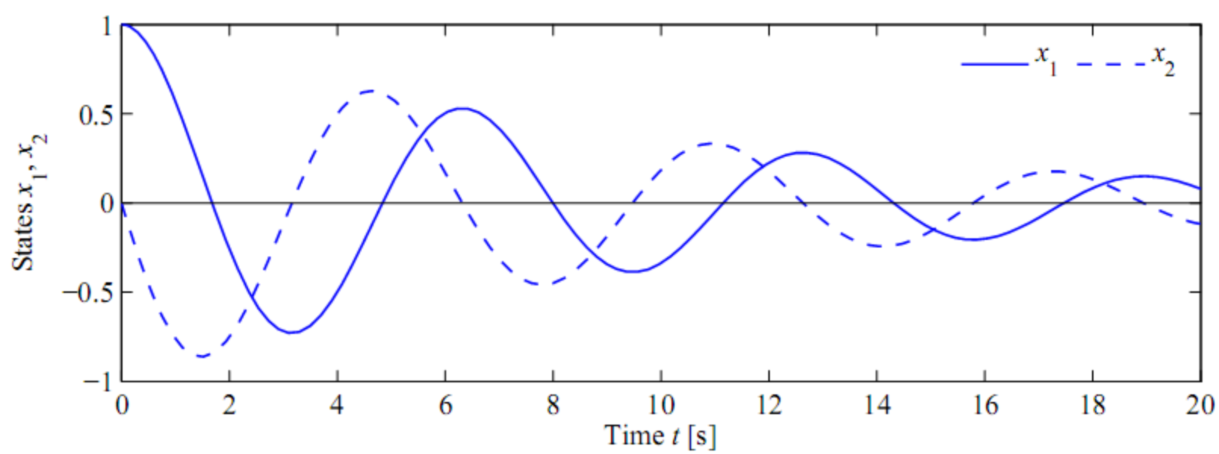
Qualitative analysis

- **Phase portraits**
  - Plotting solution of differential equation
  - Solutions (streamlines) from different initial conditions







Qualitative analysis

- **Equilibrium points & Limit cycles**
  - Equilibrium points
    - ✓ Stationary conditions for the dynamics

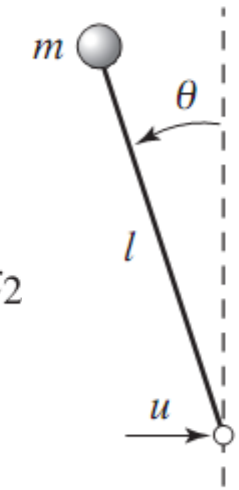
state x_e $$\frac{dx}{dt} = F(x)$$

    - ✓ How many?

Qualitative analysis

▪ Equilibrium points & Limit cycles

- Ex) Inverted pendulum

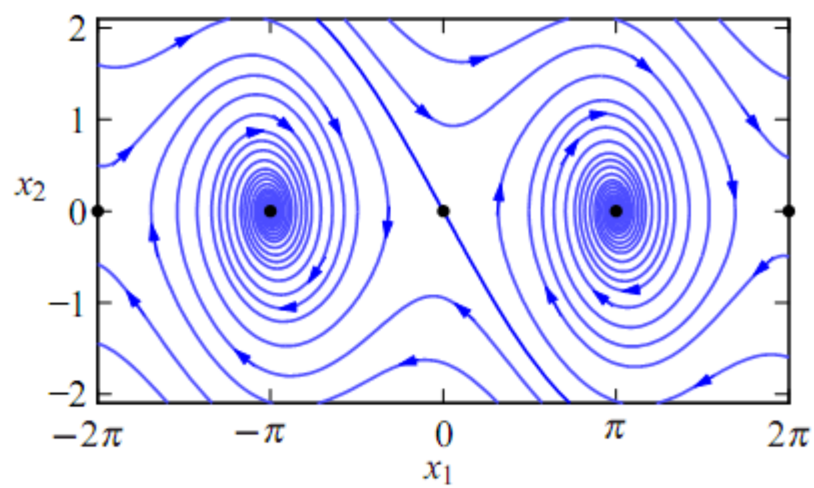


$$\theta = x_1$$

$$d\theta/dt = x_2$$

$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ \sin x_1 - c x_2 + u \cos x_1 \end{pmatrix}$$

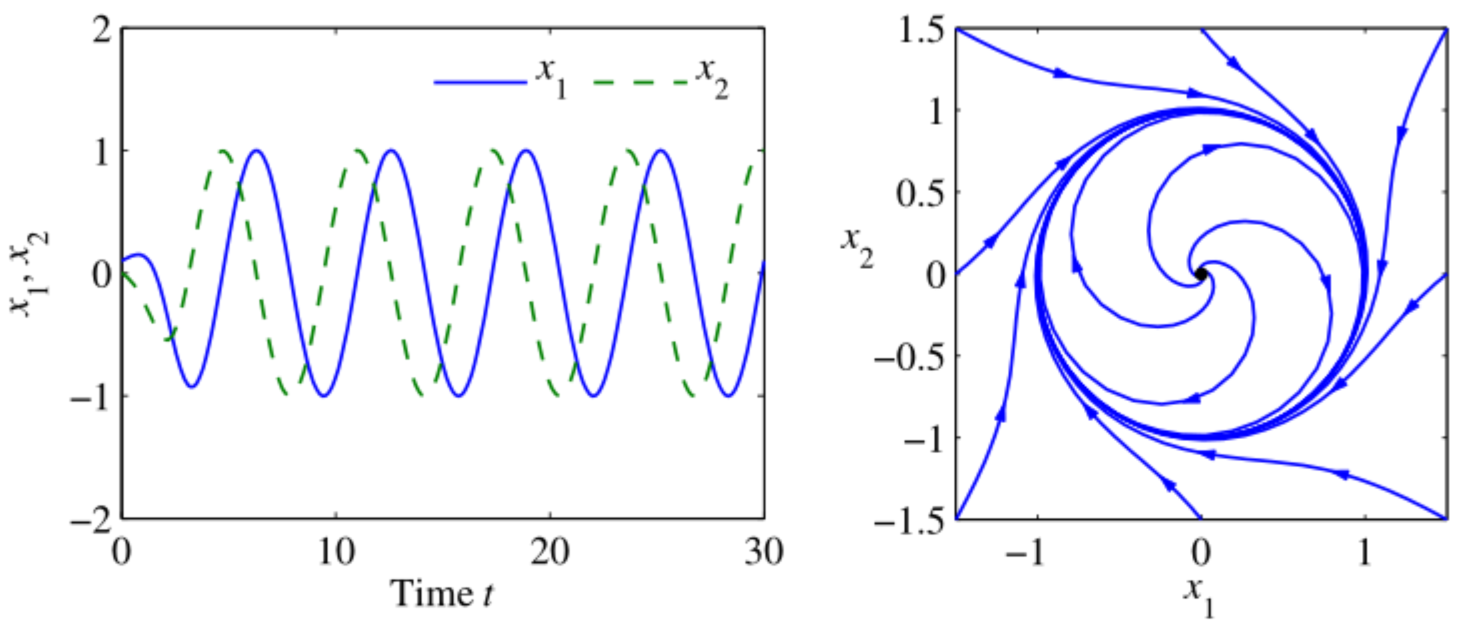
open loop dynamics



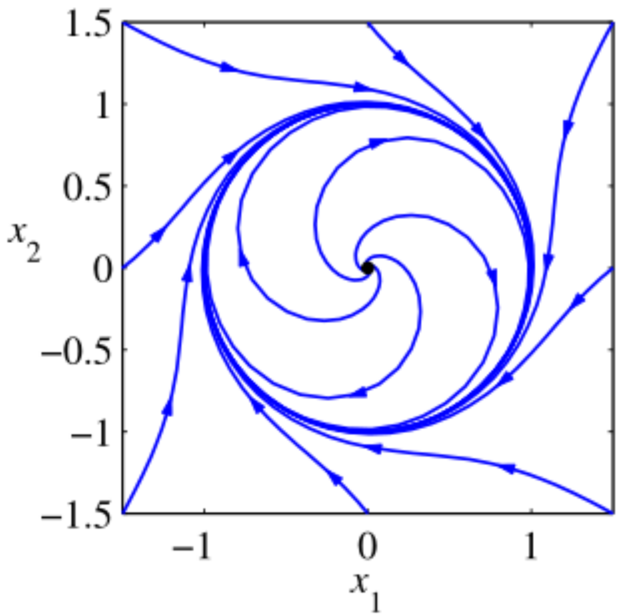
Qualitative analysis

- **Equilibrium points & Limit cycles**
  - Limit cycles
    - ✓ Stationary periodic solutions
    - ✓ Ex) Electronic oscillator

$$\frac{dx_1}{dt} = x_2 + x_1(1 - x_1^2 - x_2^2), \quad \frac{dx_2}{dt} = -x_1 + x_2(1 - x_1^2 - x_2^2)$$



Qualitative analysis



Limit cycle

$$T > 0 \text{ if } x(t+T) = x(t) \text{ for all } t \in \mathbb{R}$$

To determine limit cycle
- analytical methods for second-order system
- generally, computational analysis