EME5943
현대제어시스템

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Input/output response

$$\frac{dx}{dt} = Ax + Bu$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\frac{dx}{dt} = Ae^{At}x(0) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)$$

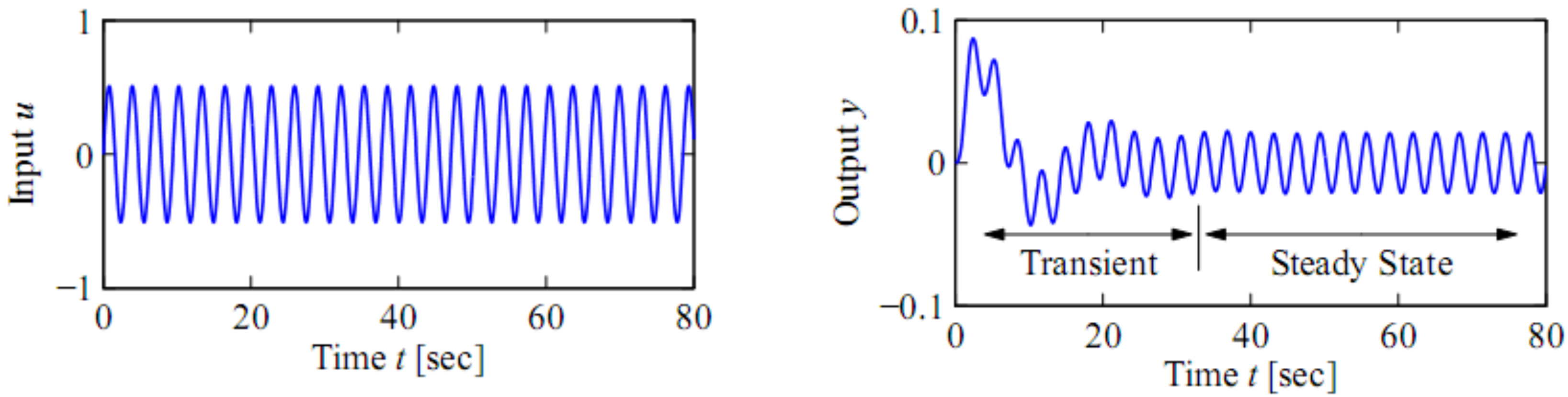
Theorem 6.4. *The solution to the linear differential equation* (6.13) *is given by*

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{6.14}$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Input/output response

- **Steady-state response**



- Transient response
  - ✓ Reflecting mismatch between initial condition and steady-state solution

- Steady-state response
  - ✓ Reflecting long-term behavior of system under given input

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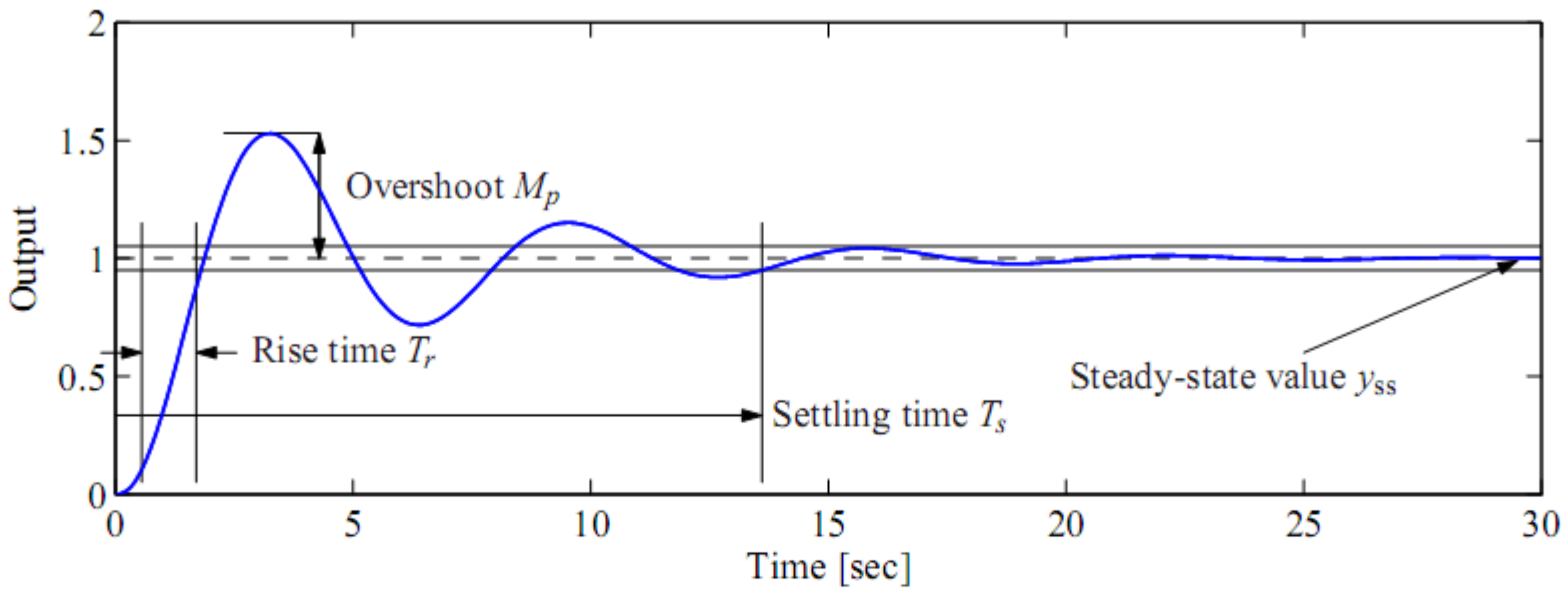
Input/output response

- **Steady-state response**
  - Step response : under unit step

unit step $$u = S(t) = \begin{cases} 0 & t = 0 \\ 1 & t > 0 \end{cases}$$

$$\begin{aligned} y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \\ &= C\int_0^t e^{A(t-\tau)}Bd\tau + D \\ &= C\int_0^t e^{A\sigma}Bd\sigma + D = C\left(A^{-1}e^{A\sigma}B\right)\Big|_{\sigma=0}^{\sigma=t} + D \\ &= CA^{-1}e^{At}B - CA^{-1}B + D \end{aligned}$$

Input/output response



- Steady-state value
- Rise time
  - ✓ Amount of time required for signal to go from 10% to 90% of its final
- Overshoot
  - ✓ Percentage of final value by which signal initially rises above final value
- Settling time
  - ✓ Amount of time required for signal to stay within 2% of its final value for all future

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Linear systems (3)

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Input/output response

- **Steady-state response**
  - Frequency response : under sinusoidal excitation

$$u = \cos \omega t$$

$$\Downarrow$$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Be^{s\tau}d\tau + De^{st}$$

$$= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau}Bd\tau + De^{st}$$

$$= Ce^{At}x(0) + Ce^{At}\left((sI-A)^{-1}e^{(sI-A)\tau}B\right)\Big|_0^t + De^{st}$$

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Input/output response

$$y(t) = Ce^{At}x(0) + Ce^{At}\left(\left(sI-A\right)^{-1}e^{(sI-A)\tau}B\right)\Big|_0^t + De^{st}$$

$$= Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I\right)B + De^{st}$$

$$= Ce^{At}x(0) + C(sI-A)^{-1}e^{st}B - Ce^{At}(sI-A)^{-1}B + De^{st}$$

$$= Ce^{At}\left(x(0) - (sI-A)^{-1}B\right) + \left(C(sI-A)^{-1}B + D\right)e^{st}$$

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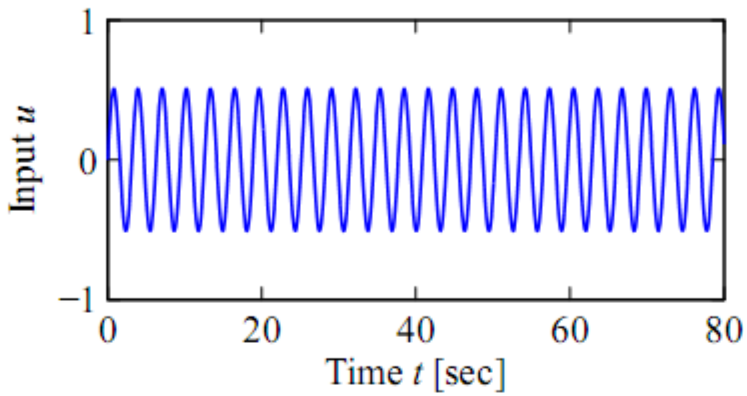
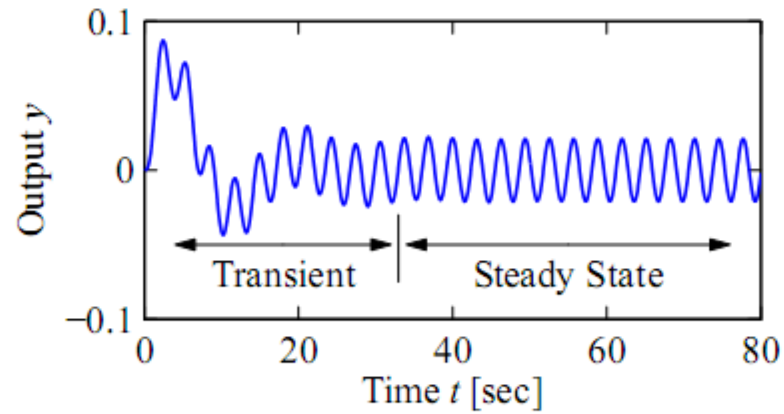
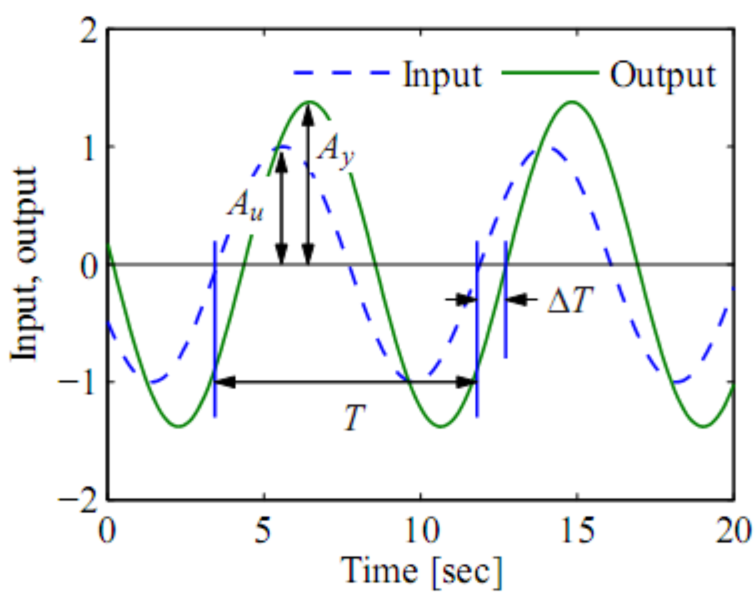
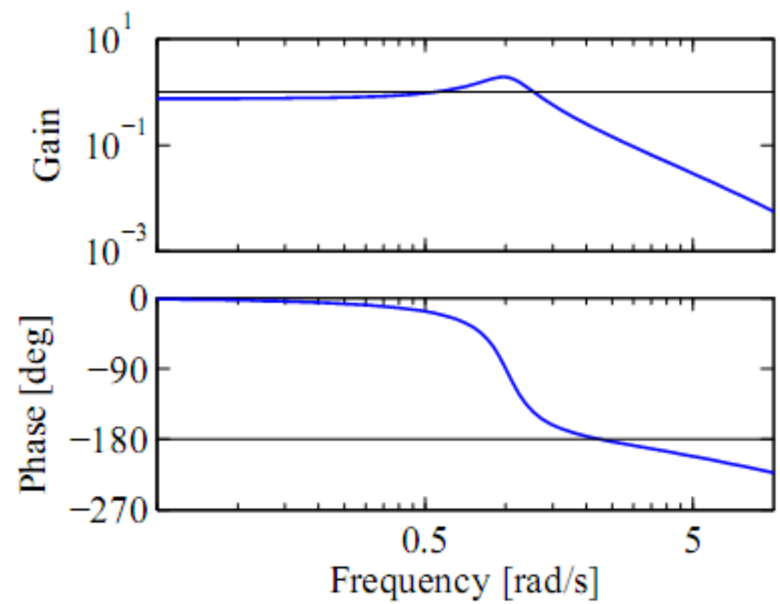
Input/output response

$$u = \cos \omega t \quad \Rightarrow \quad y_{ss}(t) = M \cos(\omega t + \theta)$$









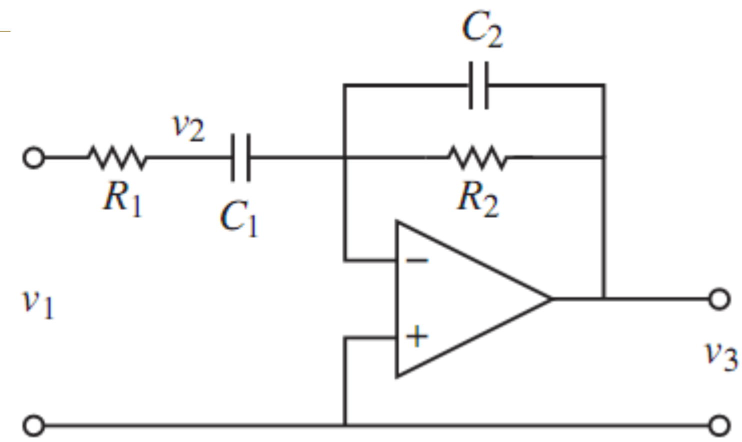
Input/output response

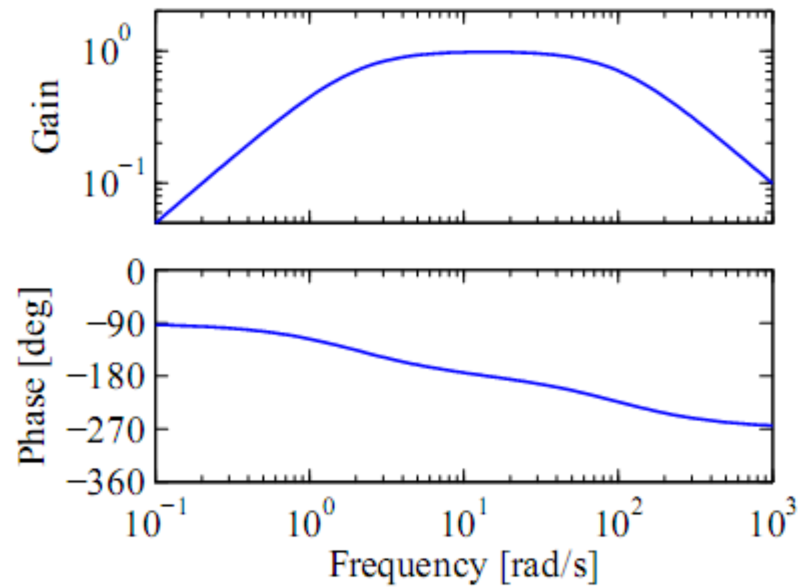
Ex) Active band-pass filter

$$\frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1C_1} \qquad \frac{dv_3}{dt} = \frac{-v_3}{R_2C_2} - \frac{v_1 - v_2}{R_1C_2}$$

$$\frac{dx}{dt} = \begin{pmatrix} -\dfrac{1}{R_1C_1} & 0 \\ \dfrac{1}{R_1C_2} & -\dfrac{1}{R_2C_2} \end{pmatrix} x + \begin{pmatrix} \dfrac{1}{R_1C_1} \\ \dfrac{-1}{R_1C_2} \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x \qquad x = (v_2, v_3) \qquad \begin{aligned} u &= v_1 \\ y &= v_3 \end{aligned}$$





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Input/output response

- **Steady-state response**
  - Standard properties of frequency response
    - ✓ Zero-frequency gain (DC gain)
      - ratio between a constant input and the steady output
    - ✓ Bandwidth
      - frequency range over which the gain has decreased by no more than a factor of $1/\sqrt{2}$ from its reference value
      - c.f. high-pass filter: reference gain is taken as the high-frequency gain.
    - ✓ Resonance peak, peak frequency
      - frequency of the sinusoidal input that produces the largest possible output and the gain at the frequency

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Linearization

- **Jacobian linearization**

$$\frac{dx}{dt} = f(x,u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R} \qquad\qquad y = h(x,u), \qquad y \in \mathbb{R}$$

$$z = x - x_e \qquad v = u - u_e \qquad w = y - h(x_e, u_e)$$

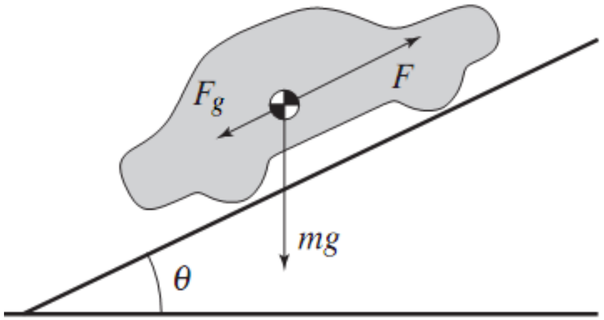
Tayler series expansion

$$\frac{dx}{dt} = F(x_e) + \left.\frac{\partial F}{\partial x}\right|_{x_e} (x - x_e) + \text{higher-order terms in } (x - x_e).$$

$$\frac{dz}{dt} = Az + Bv. \qquad\qquad w = Cz + Dv.$$

Linearization

Ex) Cruise control system



$$m\frac{dv}{dt} = \alpha_n u T(\alpha_n v) - mgC_r \operatorname{sgn}(v) - \frac{1}{2}\rho C_v A v^2 - mg\sin\theta$$

$$\frac{d(v-v_e)}{dt} = a(v-v_e) - b_g(\theta-\theta_e) + b(u-u_e) \quad + \text{higher order terms}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad a = \frac{u_e\alpha_n^2 T'(\alpha_n v_e) - \rho C_v A v_e}{m} \qquad b_g = g\cos\theta_e$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)} \qquad b = \frac{\alpha_n T(\alpha_n v_e)}{m}$$

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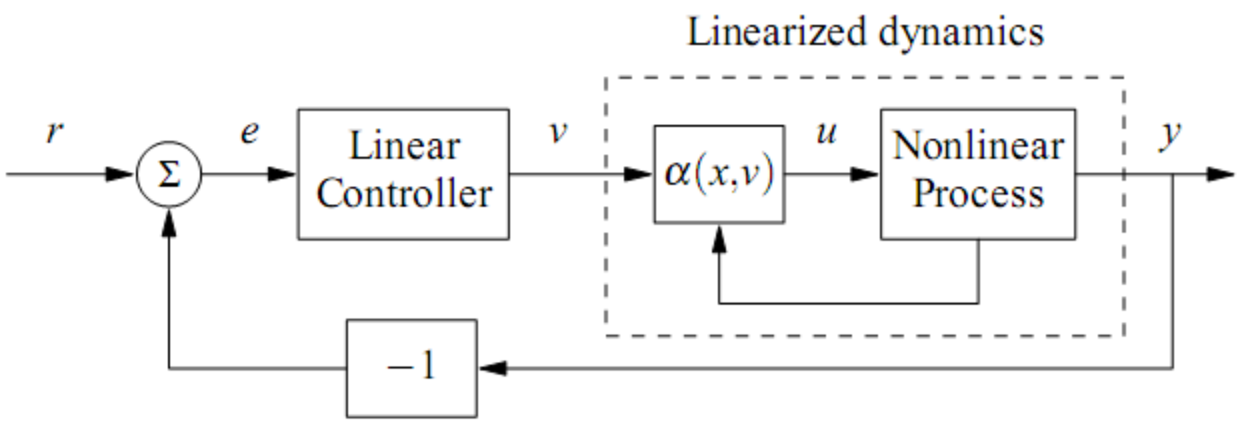
Linearization

- **Feedback linearization**
  - Use of feedback to convert the dynamics of a nonlinear system into those of a linear one



$$m\frac{dv}{dt} = \alpha_n u T(\alpha_n v) - mgC_r\,\mathrm{sgn}(v) - \frac{1}{2}\rho C_d A v^2 - mg\sin\theta$$

$$u = \frac{1}{\alpha_n T(\alpha_n v)}\left(u' + mgC_r\,\mathrm{sgn}(v) + \frac{1}{2}\rho C_v A v^2\right)$$

$$m\frac{dv}{dt} = u' - mg\sin\theta$$

Linearization

$$\frac{dx}{dt} = f(x, u), \qquad y = h(x)$$

feedback linearizable if we can find a control law $u = \alpha(x, v)$ such that the resulting closed loop system is input/output linear with input v and output y

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State feedback (1)

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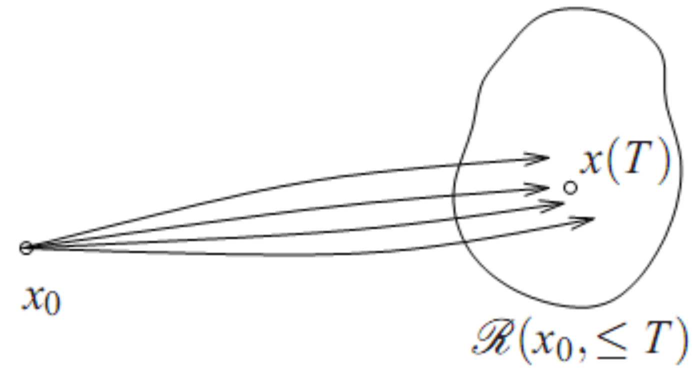
Reachability

- **Definition of reachability**

$$\frac{dx}{dt} = Ax + Bu$$



Definition 7.1 (Reachability). A linear system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u\colon [0,T] \to \mathbb{R}$ such that if $x(0) = x_0$ then the corresponding solution satisfies $x(T) = x_f$.

✓ Possibility to reach all points in the state space in a transient fashion

✓ The set of points that we are most interested in reaching

→ The set of equilibrium points of the system

Reachability

- **Testing for reachability**
  - Ex) an input consisting of a sum of impulse and their derivative

$$u(t) = \alpha_1 \delta(t) + \alpha_2 \dot{\delta}(t) + \alpha_3 \ddot{\delta}(t) + \cdots + \alpha_n \delta^{(n-1)}(t)$$

$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$x_\delta = \int_0^t e^{A(t-\tau)} B \delta(\tau)\, d\tau \;= e^{At} B$$

impulse response

$$h(t) = \int_0^t C e^{A(t-\tau)} B \delta(\tau)\, d\tau \;= C e^{At} B$$

$$\frac{dx_\delta}{dt} = A e^{At} B$$

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = f * \frac{dg}{dx}$$

$$u(t) = \alpha_1 \delta(t) + \alpha_2 \dot{\delta}(t) + \alpha_3 \ddot{\delta}(t) + \cdots + \alpha_n \delta^{(n-1)}(t)$$

$$x(t) = \alpha_1 e^{At} B + \alpha_2 A e^{At} B + \alpha_3 A^2 e^{At} B + \cdots + \alpha_n A^{n-1} e^{At} B$$

Reachability

$$x(t) = \alpha_1 e^{At} B + \alpha_2 A e^{At} B + \alpha_3 A^2 e^{At} B + \cdots + \alpha_n A^{n-1} e^{At} B$$

$$\lim_{t \to 0+} x(t) = \alpha_1 B + \alpha_2 AB + \alpha_3 A^2 B + \cdots + \alpha_n A^{n-1} B$$

$$W_r = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

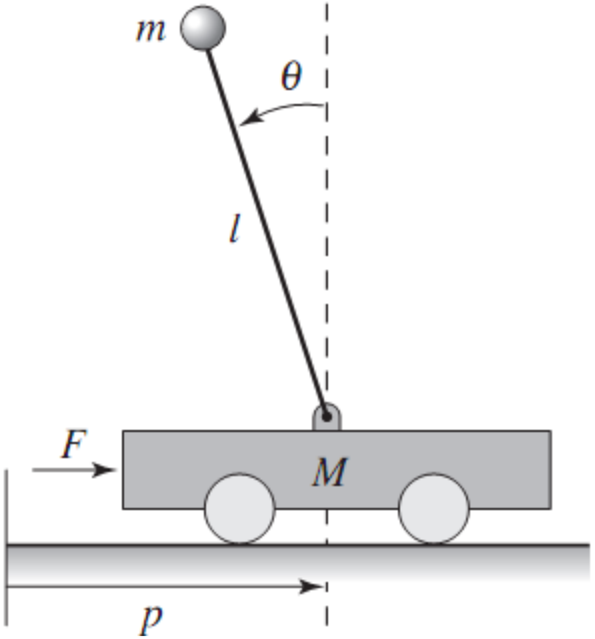
Reachability matrix

→ n linear independent columns

→ invertible

Theorem 7.1 (Reachability rank condition). *A linear system of the form* (7.1) *is reachable if and only if the reachability matrix* W_r *is invertible (full column rank).*

Reachability



Ex) Balance system

$$(M+m)\ddot{p} - ml\cos\theta\,\ddot{\theta} = -c\dot{p} - ml\sin\theta\,\dot{\theta}^2 + F$$
$$(J+ml^2)\ddot{\theta} - ml\cos\theta\,\ddot{p} = -\gamma\dot{\theta} + mgl\sin\theta.$$

$c = \gamma = 0$ Linearization around $(p,0,0,0)$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2l^2g/\mu & 0 & 0 \\ 0 & M_t mgl/\mu & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ J_t/\mu \\ lm/\mu \end{pmatrix}$$

$$W_r = \begin{pmatrix} 0 & J_t/\mu & 0 & gl^3m^3/\mu^2 \\ 0 & lm/\mu & 0 & gl^2m^2(m+M)/\mu^2 \\ J_t/\mu & 0 & gl^3m^3/\mu^2 & 0 \\ lm/\mu & 0 & gl^2m^2(m+M)/\mu^2 & 0 \end{pmatrix}$$

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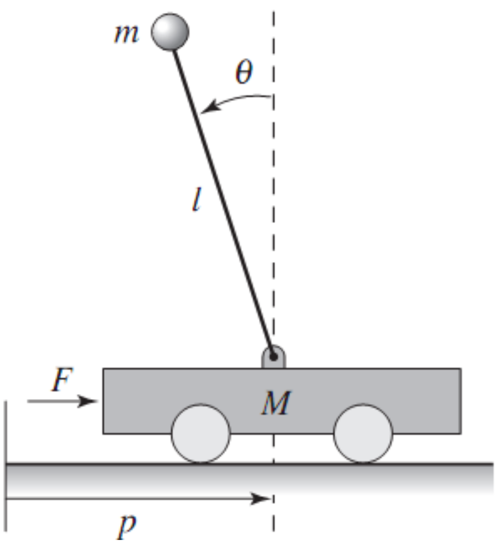
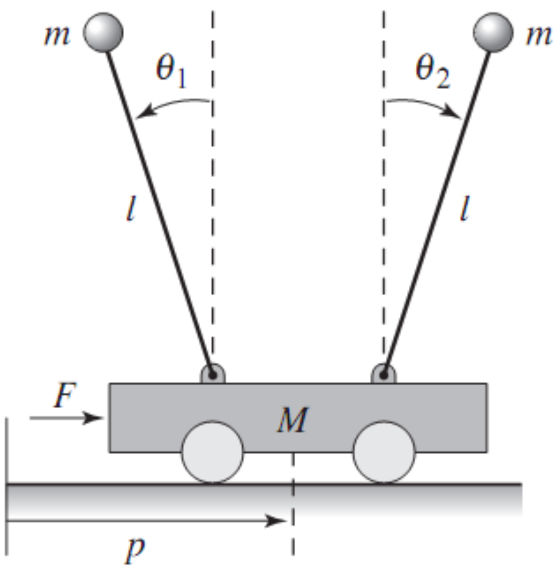
Reachability

$$W_r = \begin{pmatrix} 0 & J_t/\mu & 0 & gl^3m^3/\mu^2 \\ 0 & lm/\mu & 0 & gl^2m^2(m+M)/\mu^2 \\ J_t/\mu & 0 & gl^3m^3/\mu^2 & 0 \\ lm/\mu & 0 & gl^2m^2(m+M)/\mu^2 & 0 \end{pmatrix}$$

$$\det(W_r) = \frac{g^2 l^4 m^4}{(\mu)^4} \neq 0 \quad \Rightarrow$$ Reachable





Non-reachable

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