COMP6203 Intelligent Agents 2022/2023

Solutions to Exercises on Auctions

Read me: On Envy-Freeness

In the negotiation lectures you have seen the definition of envy-freeness which says "no agent prefers the resources allocated to other agents".
The notion of envy-freeness is not restricted to negotiation settings. In general, envy-freeness means that no agent prefers to receive the outcome received by another agent. In auctions, this means that no agent prefers to receive the allocation and payment of another agent; i.e. they don't prefer to receive the allocation of another agent and pay their payments.
Think of it this way: in the negotiation setting, an agent receives (a share) of one or several resources. In a single-item auction setting, an agent does not just receive an item/resource, s/he also receives a price tag for it.

Exercise 1: Full information game

Consider the following independent private value auction setting for a single item. There are 2 agents, April (A) and Ben (B), wanting to buy the item being auctioned. Assume that the agents' valuations are at least 0 pounds and at most 5 pounds and everyone (both agents and the auctioneer) knows this; therefore agents can only place bids between 0 and 5 pounds. Furthermore, assume that agents can only place bids in full pounds (so this is a discrete bid setting).
Agent A values the item at $\theta_A = 2$ pounds, and B values the item at $\theta_B = 4$ pounds. Assume a complete information setting (i.e. the agents know each others' valuations). Assume that the second-price sealed-bid auction (Vickrey) is used to allocate the item, and that ties are broken in lexicographic order.

1. Compute the utility of the agents when they both bid truthfully.
2. Is the resulting outcome (from truthful bidding) envy-free? Explain.
3. Identify all pure Nash equilibria of the strategic-form game induced by this auction.
4. Identify all pure strategy profiles that result in an envy-free outcome.
5. Does the set of pure Nash equilibria change if a different (deterministic) tie-breaking rule is used?
6. Does the set of envy-free outcomes change if a different (deterministic) tie-breaking rule is used?
7. What can you say about the set of all pure Nash equilibria of this game if the agents' valuations are between 0 and 50 (i.e., can you describe and characterise this set succinctly without identifying each member one by one)?

Solution.

1. When both agents bid truthfully, agent B wins (since her bid is higher) and pays A's valuation. Hence $u_B = \theta_B - \theta_A = 4 - 2 = 2$ and $u_A = 0$.

2. Yes it is. To check for envy-freeness, we need to check whether either of the agents prefers the other agent's outcome (i.e. allocation and payment). If agent B receives A's allocation and payment, she will receive no item and pays 0, so her utility will be $0 < 2 = u_B$. If agent A receives B's allocation and payment, he will receive the item and pay 2, so his utility will be $2 - 2 = 0 = u_A$. Hence neither of the agents prefers the other agent's outcome.

3. We can present the strategic-form game induced by this auction as a 6×6 matrix. Table 1 depicts this game where the first number in each cell corresponds to A's utility and the second number to B's utility. A pure strategy profile is a NE if neither of the players can gain a higher utility by choosing a different pure strategy. The outcomes corresponding to pure NE are shown in red. From this table it is easy to see that pure strategy profile (i, j) is a pure NE if and only if

- $0 \leq i \leq 4 \wedge 2 \leq j \leq 5 \wedge i < j$, or
- $4 \leq i \leq 5 \wedge 0 \leq j \leq 2$.

(Note: $\wedge$ is logical "and". You can just use 'and' or ',' or ' & ' instead. I decided to use $\wedge$ as this notation will become useful in part 7 where I will also use $\vee$, logical "or".)

		Agent B					
		0	1	2	3	4	5
	0	$2,0$	$0,4$	$0,4$	$0,4$	$0,4$	$0,4$
	1	$2,0$	$1,0$	$0,3$	$0,3$	$0,3$	$0,3$
	2	$2,0$	$1,0$	$0,0$	$0,2$	$0,2$	$0,2$
Agent A	3	$2,0$	$1,0$	$0,0$	$-1,0$	$0,1$	$0,1$
	4	$2,0$	$1,0$	$0,0$	$-1,0$	$-2,0$	$0,0$
	5	$2,0$	$1,0$	$0,0$	$-1,0$	$-2,0$	$-3,0$

Table 1: Strategic-form game induced by the Auction in Exercise 1. Outcomes corresponding to pure NE are shown in red.

4. One way to answer this question is to check each bid profile individually: calculate the outcome (i.e the allocation/winner and the payments, and hence the utilities of both agents) and see if either of the agents prefers the outcome received by the other agent. This can be done easily using the strategic-form game table. The envy-free outcomes are shown in green in Table 2.

		Agent B					
		0	1	2	3	4	5
	0	$2,0$	$0,4$	$0,4$	$0,4$	$0,4$	$0,4$
	1	$2,0$	$1,0$	$0,3$	$0,3$	$0,3$	$0,3$
	2	$2,0$	$1,0$	$0,0$	$0,2$	$0,2$	$0,2$
Agent A	3	$2,0$	$1,0$	$0,0$	$-1,0$	$0,1$	$0,1$
	4	$2,0$	$1,0$	$0,0$	$-1,0$	$-2,0$	$0,0$
	5	$2,0$	$1,0$	$0,0$	$-1,0$	$-2,0$	$-3,0$

Table 2: Envy-free outcomes are shown in green.

A more general approach is to observe that an outcome (i.e. allocation and payments) is envy-free if and only if the winner receives non-negative utility [or otherwise the winner prefers to receive the outcome of the loser: lose, pay zero, and receive zero utility] and the payment of the winner is no less than the valuation of the loser [or else the loser prefers the outcome of the winner: win the item, pay less than her valuation and hence receive positive utility].

5. Yes. If B wins in case of ties then e.g. pure strategy profile $(2, 2)$ is also a Nash equilibrium.

6. Yes. If B wins in case of ties then e.g. the outcome resulting from pure strategy profile $(2, 2)$ is also envy-free.

7. With a bit of thinking and mulling over the answer to part 3 (where bids are between 0 and 5) it is not too hard to see that pure strategy profile (i, j) is a pure NE if and only if

- $0 \leq i \leq 4 \wedge 2 \leq j \leq 50 \wedge i < j$, or
- $4 \leq i \leq 50 \wedge 0 \leq j \leq 2$.

In the first bullet point, bidder B wins and pays at most his valuation (hence non-negative utility for B), and bidder A does not have a profitable deviation (if she wants to win the auction, she has to outbid B and pay $j \geq 2$ hence receiving at most zero utility). In the second bullet point, bidder A wins and pays at most her valuation (hence non-negative utility for A, and bidder B does not have a profitable deviation (if he wants to win the auction, he has to outbid A and pay $i \geq 4$ hence receiving at most zero utility).

Next I'll elaborate on a more general approach for answering such questions. You can answer this question by directly characterising the set of pure strategy profiles that are pure NE. Let us refer to pure strategy profiles as bid profiles. I find it easier to first think about the bid profiles that are NOT NE. A bid profile (i, j) is NOT a NE if either of the agents can increase her/his utility by placing a different bid. So a bid profile (i, j) is NOT a NE if any of the following cases hold:

- Bidder A wins and
  - she is bidding less than 4 (the value of bidder B) [in which case bidder B can increase his utility by placing a bid higher than i], or
  - her utility is negative (bidder B is bidding above 2) [in which case bidder A can increase her utility by placing a bid lower than j].
- Bidder B wins and
  - is bidding less than 2 (bidder A's value) [in which case bidder A can increase her utility by placing a bid higher than j], or
  - his utility is negative (Bidder A is bidding above 4) [in which case bidder B can increase his utility by placing a bid lower than i].

Converting the above to math, we get that a bid profile (i, j) is NOT a NE if and only if

$$(0 \leq j \leq i < 4) \vee (2 < j \leq i \leq 50) \vee (0 \leq i < j < 2) \vee (4 < i < j \leq 50).$$

By negating the above expression, we arrive at the following characterisation of bid profiles (i, j) that are NE. A strategy profile (i, j) is a NE if and only if

- $0 \leq i \leq 4 \wedge 2 \leq j \leq 50 \wedge i < j$, or
- $4 \leq i \leq 50 \wedge 0 \leq j \leq 2$.

Exercise 2: Optimal bids

Consider the following independent private value auction setting for a single item. There are n agents wanting to buy the item being auctioned. Assume that agents can only place bid in full pounds (e.g. they can bid 15 pounds bot not 25.5 pounds). Assume that agent i values the item at 7 pounds and that she has beliefs about the bid distribution. Let $F(b)$ denote the probability that all other bids are less than b pounds, and suppose that agent i has the following information:

$$F(2) = 0, F(3) = 0.1, F(4) = 0.25, F(5) = 0.35, F(6) = 0.40, F(7) = 0.65,$$
$$F(8) = 0.85, F(9) = 0.95, F(10) = 1$$

8. Suppose that we are running a first-price sealed-bid auction and that ties are broken in favour of the opponents of i. Compute all optimal bids for bidder i and her expected utility if she places an optimal bid.

9. Suppose that we are running a second-price sealed-bid (a.k.a. Vickrey) and that ties are broken in favour of the opponents of i. Compute all optimal bids for bidder i and her expected utility if she places an optimal bid..

Solution.

8. Expected utility of bidder i is $E(u_i(7)|b_i) = (7 - b_i)F(b_i)$. Therefore,

- when $b_i \leq 2$, $E(u_i(7)|b_i) = 0$ since $F(2) = 0$ and hence $F(1) = F(0) = 0$,
- when $b_i = 3$, $E(u_i(7)|3) = 4 \cdot 0.1 = 0.4$,
- when $b_i = 4$, $E(u_i(7)|4) = 3 \cdot 0.25 = 0.75$,
- when $b_i = 5$, $E(u_i(7)|5) = 2 \cdot 0.35 = 0.7$,
- when $b_i = 6$, $E(u_i(7)|6) = 1 \cdot 0.4 = 0.4$,
- when $b_i \geq 7$, $E(u_i(7)i|b_i) \leq 0$.

Optimal bid is $b_i = 4$ that generates 0.75 pounds utility.

9. Expected utility of bidder i is $E[u_i(\theta_i)|b_i] = \theta_i F(b_i) - \sum_{b=0}^{b_i-1}(b \cdot (F(b+1) - F(b)))$. We know that Vickrey is trufhul therefore bidding her valuation is an optimal bid for bidder i. By bidding truthfully, bidder i obtains utility 1.75 pounds as calculated below:

$$E[u_i(7)|7] = 7 \cdot F(7) - \sum_{b=0}^{b=6}(b(F(b+1) - F(b))) =$$

$$7 \cdot 0.65 - (0 + 0 + 2 \times 0.1 + 3 \times 0.15 + 4 \times 0.1 + 5 \times 0.05 + 6 \times 0.25) = 4.55 - 2.8 = 1.75$$

As the tie-breaking rule is against i, bidding one pound above valuation at 8 is also optimal and produces the same expected utility. To see this, note that by bidding 1 pound more, her expected valuation is increased by $F(8) - F(7) = 0.2$ and her expected payment is also increased by $F(8) - F(7) = 0.2$.
No bid less than 7 or more than 8 is optimal. To see this, consider bidding less than 7 at e.g. 6: expected valuation is reduced by $7(F(7) - F(6)) = 7 \times 0.25 = 1.75$; expected payment is reduced by $6(F(7) - F(6)) = 1.5$. Now also consider bidding more than 8, e.g. at 9: expected valuation is increased by $7(F(9) - F(8)) = 0.7$ and expected payment is increased by $8(F(9) - F(8)) = 0.8$.