STRATEGIC-FORM GAMES (MIXED STRATEGIES)

COMP6203 - Intelligent Agents

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Mixed Nash Equilibria

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

- Let's now go back to the concept of a Nash equilibrium.
- Again, we are assuming that players are rational decision-makers and have common knowledge of the game and of their rationality.
- Again, we are trying to predict, under those assumption, what the outcome of the game will be.
- We have seen that not all strategic-form games have Nash equilibria (see Matching Pennies above).

In order to guarantee the existence of equilibria, we extend the concept of a strategic-form game with the notion of **mixed strategy**.

Definition

In a strategic-form game, a **mixed strategy** σ_i for player i is a probability distribution over the set of strategies S_i, i.e. a function $\sigma_i : S_i \to [0, 1]$ such that

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

From now on, to distinguish them from mixed strategies, we will call each strategy $s_i \in S_i$, for all players i, a **pure strategy**.

- Mixed strategies can be interpreted in different ways.
- A mixed strategy can be seen as a sort of randomised choice.
- If you were to repeat the game and play every time the same way you would become predictable.
- So randomising your choice could keep the other players guessing.
- If we assume this, is there a way to randomise your choice and maximise your outcome?

Player 2

		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

- When we deal with mixed strategies, we are in the realm of expected utilities.
- Suppose we are playing Matching Pennies and
  - Player 1 plays Heads with probability 0.4, i.e.

    $$\sigma_1(H) = 0.4, \qquad \sigma_1(T) = 0.6$$

    [because $\sigma_1(T) = 1 - \sigma_1(H)$]
  - Player 2 plays Heads with probability 0.3, i.e.

    $$\sigma_2(H) = 0.3, \qquad \sigma_2(T) = 0.7$$

		Player 2: H	Player 2: T
Player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

- The expected utility for Player 1, given the mixed strategies σ_1, σ_2 is

$$\begin{aligned} EU_1(\sigma_1, \sigma_2) &= \sigma_1(H)\sigma_2(H)u_1(H,H) + \sigma_1(H)\sigma_2(T)u_1(H,T) + \\ &\quad \sigma_1(T)\sigma_2(H)u_1(T,H) + \sigma_1(T)\sigma_2(T)u_1(T,T) \\ &= (0.4 \cdot 0.3 \cdot 1) + (0.4 \cdot 0.7 \cdot (-1)) + \\ &\quad (0.6 \cdot 0.3 \cdot (-1)) + (0.6 \cdot 0.7 \cdot 1) \\ &= 0.08 \end{aligned}$$

- The expected utility for Player 2 is computed in a similar way

- Given a strategic-form game, for each player i, let Σ_i be the set of all mixed strategies over S_i, i.e.

$$\Sigma_i = \left\{ \sigma_i \mid \sigma_i : S_i \to [0,1], \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}$$

- Let Σ

$$\Sigma = \Sigma_1 \times \cdots \times \Sigma_n.$$

- We call every element $(\sigma_1, \ldots, \sigma_n) \in \Sigma$ a **mixed strategy profile** or **mixed strategy combination**

- Notice that, similar to pure strategies, we use the notation (σ_i, σ_{-i}).

- Given a mixed strategy profile $(\sigma_1, \ldots, \sigma_n)$, the expected utility of player i is given by

$$EU_i(\sigma_1, \ldots, \sigma_n) = \sum_{(s_1,\ldots,s_n) \in S_1 \times \cdots \times S_n} \left(\left(\prod_{j \in N} \sigma_j(s_j) \right) u_i(s_1, \ldots, s_n) \right).$$

Definition

Let

$$G = \langle N, S_1, \ldots, S_n, u_1, \ldots, u_n \rangle$$

be a strategic-form game. The **mixed extension** of G is the game

$$\Gamma = \langle N, \Sigma_1, \ldots, \Sigma_n, U_1, \ldots, U_n \rangle$$

where

- Each Σ_i is the set of mixed strategies of player i over S_i.
- Each $U_i : \Sigma \to \mathbb{R}$ is a payoff function that associates with each mixed strategy combination $(\sigma_1, \ldots, \sigma_n) \in \Sigma$ its expected utility

$$U_i(\sigma_1, \ldots, \sigma_n) = EU_i(\sigma_1, \ldots, \sigma_n) = \sum_{(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n} \left(\left(\prod_{j \in N} \sigma_j(s_j) \right) u_i(s_1, \ldots, s_n) \right).$$

- A mixed extension is a “new” game built on top of a strategic-form game with only pure strategies.
- Each player now has a new set of strategies, i.e. all the mixed strategies defined over their original strategy set.
- Each player also has a new utility function, which, for each mixed strategy combination, outputs the expected utility determined by the players selecting those mixed strategies.
- Notice that this generalises the concept of a strategic-form game, because pure strategies are special cases of mixed strategies, i.e. pure strategies correspond to those cases where a probability distribution assigns 1 to one pure strategy.

- We want to generalise the concept of a Nash equilibrium to mixed strategies.
- In order to do so, we first generalise the concept of a best response.

Definition

Let G be a strategic-form game and let Γ be its mixed extension. Let σ_{-i} be a mixed strategy vector for all the players not including i. Player i's mixed strategy σ_i is called a **best response** to σ_{-i} if

$$EU_i(\sigma_i, \sigma_{-i}) = \max_{\sigma_i' \in \Sigma_i} EU_i(\sigma_i', \sigma_{-i}).$$

We are now ready to generalise the notion of a Nash equilibirum

Definition

A mixed strategy combination $(\sigma_1, \ldots, \sigma_n)$ is a **mixed strategy Nash equilibrium** if σ_i is a best response to σ_{-i} for every player $i \in N$.

From now on, we will use the term "mixed strategy Nash equilibrium" (or "mixed Nash equilibrium") to refer to this general concept. Whenever we talk only about pure strategies, we will use the term "pure strategy Nash equilibrium" (or "pure Nash equilibrium").

Clearly, since pure strategies are a special form of mixed strategies, a pure Nash equilibrium is a special form of mixed Nash equilibrium.

We saw before that pure Nash equilibria do not always exist. That is not the case for mixed Nash equilibria.

Theorem (Nash, 1951)

Every strategic-form game has mixed extension that has mixed strategy Nash equilibria.

We know mixed Nash equilibria always exist, but how hard is it to find a Nash equilibrium for a strategic-form game?

Theorem (Daskalakis, Goldberg, Papadimitriou, 2009)

Finding mixed Nash equilibria in a strategic-form game is PPAD-complete.

Mixed Nash Equilibria

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

- In Matching Pennies there is a unique mixed Nash equilibrium given by the combination σ_1, σ_2 where

$$\sigma_1(H) = 0.5 \qquad \text{and} \qquad \sigma_2(H) = 0.5.$$

Computing Mixed Nash Equilibria

		Player 2	
		L	R
Player 1	T	$u_1(T,L)$, $u_2(T,L)$	$u_1(T,R)$, $u_2(T,R)$
	B	$u_1(B,L)$, $u_2(B,L)$	$u_1(B,R)$, $u_2(B,R)$

- Finding mixed Nash equilibria is not easy.
- We are going to see that for generic 2×2 games (see above) we have relatively simple ways to do it.
- We represent a mixed strategy for player 1 as $x \in [0,1]$:
  - Play T with probability x
  - Play B with probability $1 - x$
- We represent a mixed strategy for player 2 as $y \in [0,1]$:
  - Play L with probability y
  - Play R with probability $1 - y$

- The first approach is to compute the best response functions for mixed strategies, and draw a graphical representation of these functions.
- The best response function $br_1(y)$ for player 1 is

$$br_1(y) = \underset{x \in [0,1]}{\operatorname{argmax}} EU_1(x, y)$$

- The best response function $br_2(x)$ for player 2 is

$$br_2(x) = \underset{y \in [0,1]}{\operatorname{argmax}} EU_2(x, y)$$

- The intersection between the best response functions will give the mixed Nash equilibria of the game.

Best Response Method

		Player 2	
		H	*T*
Player 1	*H*	2, 1	0, 0
	T	0, 0	1, 2

- Consider the game above.
- This game has 2 pure strategy Nash equilibria (H, H) and (T, T).
- We want to compute its set of mixed Nash equilibria (if there are any), which will include the pure equilibria we already know of.

		Player 2	
		H	T
Player 1	H	2, 1	0, 0
	T	0, 0	1, 2

- Let's compute the best response functions.
- The best response function $br_1(y)$ for player 1 is

$$br_1(y) = \underset{x \in [0,1]}{\operatorname{argmax}} EU_1(x, y)$$

- The expected utility for player 1 is the following

$$EU_1(x, y) = 2xy + 0x(1 - y) + 0(1 - x)y + 1(1 - x)(1 - y) = x(3y - 1) - y + 1$$

- For each value of y we need to find all the values of x that maximise $x(3y - 1) - y + 1$.
- To derive the values of x that are best response, we just need to check when $(3y - 1)$ changes sign and when it equals 0.

		Player 2	
		H	T
Player 1	H	2, 1	0, 0
	T	0, 0	1, 2

- The best response function $br_2(x)$ for player 2, i.e.

$$br_2(x) = \underset{y \in [0,1]}{\operatorname{argmax}} EU_2(x, y)$$

- The expected utility for player 1 is the following

$$EU_2(x, y) = 1xy + 0x(1 - y) + 0(1 - x)y + 2(1 - x)(1 - y) = y(3x - 2) - 2x + 2$$

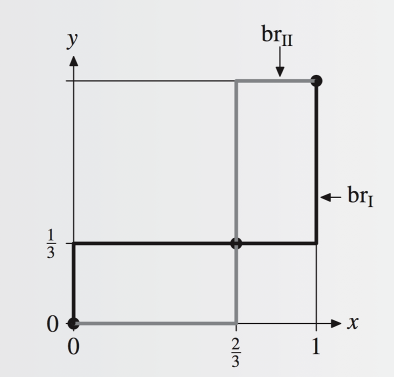
- For each value of x we need to find all the values of y that maximise $y(3x - 2) - 2x + 2$.
- To derive the values of y that are best response, we just need to check when $(3x - 2)$ changes sign and when it equals 0.

		Player 2	
		H	T
Player 1	H	2, 1	0, 0
	T	0, 0	1, 2

- It is easy to compute the following response functions

$$br_1(y) = \begin{cases} 0 & y < \frac{1}{3} \\ [0,1] & y = \frac{1}{3} \\ 1 & y > \frac{1}{3} \end{cases} \qquad br_2(x) = \begin{cases} 0 & x < \frac{2}{3} \\ [0,1] & x = \frac{2}{3} \\ 1 & x > \frac{2}{3} \end{cases}$$

The intersection of the graphs of $br_1(y)$ and $br_2(x)$ gives us the set of all equilibria of the game.
Notice that the pure strategy equilibria are included.



- There is another way to compute mixed Nash equilibria in 2×2 games.
- We use the indifference principle

Theorem (Indifference Principle)

Let $(\sigma_1, \ldots, \sigma_n)$ be a mixed Nash equilibrium of a strategic-form game, and let s_i and s_i' be two pure strategies of player i. If

$$\sigma_i(s_i) > 0 \qquad \textit{and} \qquad \sigma_i(s_i') > 0,$$

then

$$EU_i(s_i, \sigma_{-i}) = EU_i(s_i', \sigma_{-i})$$

- The indifference principle says that if in a mixed Nash equilibrium a player assigns positive probabilities to two different pure strategies, then the expected payoff of playing those strategies is the same.

Definition

We call a mixed strategy σ_i of player i a **completely mixed strategy** (or **fully mixed**) if

$$\sigma_i(s_i) > 0$$

for every pure strategy $s_i \in S_i$.

We call a mixed Nash equilibrium $(\sigma_1, \ldots, \sigma_n)$ a **completely mixed Nash equilibrium** (or **fully mixed**) if for every player i, the strategy σ_i is completely mixed.

- What the indifference principle tells us is that if a completely mixed Nash equilibrium exists, then for each player the expected utility of playing any pure strategies (under the given equilibrium) is the same.
- Each player is indifferent to playing a pure strategy over another.

		Player 2 L	Player 2 R
Player 1	T	$u_1(T,L)$, $u_2(T,L)$	$u_1(T,R)$, $u_2(T,R)$
	B	$u_1(B,L)$, $u_2(B,L)$	$u_1(B,R)$, $u_2(B,R)$

- Let $(x,y) \in [0,1]^2$ be a pair of mixed strategies in the generic 2×2 game.
- Define

$$\begin{aligned} EU_1(T,y) &= u_1(T,L)y + u_1(T,R)(1-y) \\ EU_1(B,y) &= u_1(B,L)y + u_1(B,R)(1-y) \\ EU_2(L,x) &= u_2(T,L)x + u_2(B,L)(1-x) \\ EU_2(R,x) &= u_2(T,R)x + u_2(B,R)(1-x) \end{aligned}$$

- $EU_1(T,y)$ is the expected utility for player 1 when choosing T.
- $EU_1(B,y)$ is the expected utility for player 1 when choosing B, etc.

Theorem

A pair of probability distributions (x, y) is a completely mixed Nash Equilibrium in the generic 2×2 game if and only if

$$EU_1(T, y) = EU_1(B, y) \quad \textit{and} \quad EU_2(L, x) = EU_2(R, x).$$

Proof.

Note: The proof is here for your information. You do not have to study this proof for the exam. Knowing the above theorem is enough.

Suppose

$$EU_1(T, y) = EU_1(B, y),$$

and let $x' \in (0, 1)$ be an arbitrary mixed strategy. We have

$$\begin{aligned} EU_1(x, y) &= xEU_1(T, y) + (1 - x)EU_1(B, y) \\ &= xEU_1(T, y) + (1 - x)EU_1(B, y) \\ &= EU_1(T, y) \\ &= x'EU_1(T, y) + (1 - x')EU_1(T, y) \\ &= x'EU_1(T, y) + (1 - x')EU_1(B, y) \\ &= EU_1(x', y) \end{aligned}$$

□

Proof.

So, for any x',

$$EU_1(x, y) \geq EU_1(x', y).$$

The argument for the second player is analogous. So, (x, y) is a mixed Nash equilibrium.
To prove the other direction, suppose

$$EU_1(T, y) = EU_1(B, y),$$

and, without loss of generality,

$$EU_1(T, y) > EU_1(B, y).$$

We have

$$\begin{aligned} EU_1(x, y) &= xEU_1(T, y) + (1 - x)EU_1(B, y) \\ &< xEU_1(T, y) + (1 - x)EU_1(T, y) \\ &= EU_1(T, y) \end{aligned}$$

and so (x, y) is not a mixed Nash equilibrium. The same argument can be made for player 2. □

- This gives us a method for finding completely mixed Nash equilibria.
- First check the equality
$$EU_1(T, y) = EU_1(B, y)$$
and find solutions for y.
- Then check the equality
$$EU_2(L, x) = EU_2(R, x)$$
and find solutions for x.
- Any pair of solutions (x, y) is a mixed Nash equilibrium.
- We are dealing with linear inequalities, so we can solve them in polynomial time.

Player 2

		L	R
Player 1	T	$1, -1$	$0, 2$
	B	$0, 1$	$2, 0$

- This is a game with no pure Nash equilibria.
- Let's check

$$EU_1(T, y) = EU_1(B, y):$$

$$\text{we have} \quad y = 2(1 - y) \quad \text{and so} \quad y = \frac{2}{3}$$

- Then check

$$EU_2(L, x) = EU_2(R, x)$$

$$\text{we have} \quad -x + 1 - x = 2x \quad \text{and so} \quad x = \frac{1}{4}$$

- $(\frac{1}{4}, \frac{2}{3})$ is a completely mixed Nash equilibrium.

Player 2

		L	R
Player 1	T	4, 4	2, 8
	B	8, 2	1, 1

- (T, R) and (B, L) are both pure Nash equilibria. Let's check if there are any other equilibria.
- For $EU_1(T, y) = EU_1(B, y)$:

$$\text{we have} \quad 4y + 2(1-y) = 8y + (1-y) \quad \text{and so} \quad y = \frac{1}{5}$$

- For $EU_2(L, x) = EU_2(R, x)$

$$\text{we have} \quad 4x + 2(1-x) = 8x + (1-x) \quad \text{and so} \quad x = \frac{1}{5}$$

- $\left(\frac{1}{5}, \frac{1}{5}\right)$ is a fully mixed Nash equilibrium.

		Player 2	
		L	R
Player 1	T	1, 1	4, 0
	B	2, 10	3, 5

- (B, L) is a pure Nash equilibrium. Let's check if there are any other equilibria.
- For $EU_1(T, y) = EU_1(B, y)$:

$$\text{we have} \quad y + 4(1-y) = 2y + 3(1-y) \quad \text{and so} \quad y = \frac{1}{2}$$

- For $EU_2(L, x) = EU_2(R, x)$

$$\text{we have} \quad x + 10(1-x) = +5(1-x) \quad \text{and so} \quad x = \frac{5}{4}$$

- $\frac{5}{4}$ is a not a mixed strategy, so the game has no completely mixed equilibria.

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

- Given a pair of mixed strategies (x, y) we can use the indifference principle to check if (x, y) is an equilibrium.
- Consider Matching Pennies, with mixed profile $\left(\frac{1}{3}, \frac{3}{4}\right)$
- We must have $EU_1(H, \frac{3}{4}) = EU_1(T, \frac{3}{4})$, but:
  - $EU_1(H, \frac{3}{4}) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$
  - $EU_1(T, \frac{3}{4}) = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$
- So $\left(\frac{1}{3}, \frac{3}{4}\right)$ is not an equilibrium.

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- Y. Shoham, K. Leyton-Brown. *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*. Cambridge University Press, 2009.
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