COOPERATIVE GAMES
COMP6203 - Intelligent Agents

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Coalitional Games

- So far we have studied games where players act independently and strive to achieve their own private goals.
- We are now going to focus on **cooperative games**, which model situation where players may conclude binding agreements that impose a particular action or series of actions on each player.
- Cooperative game theory concentrates on questions such as:
    - Which sets of players (coalitions) will agree to conclude binding agreements?
    - Which agreements can reasonably be expected to be arrived at by players (and which are not reasonable)?

- Cooperative games are also called **coalitional games** to underscore the fact that they focus on the formation of coalitions by the players.

- Players can be a person, corporation, nation, and so on.

- The only requirement is that players be capable of arriving at decisions, and committing to those decisions.

Definition

A **coalitional game with transferable utility** is a pair (N, v) such that:

1. $N = \{1, 2, \ldots, n\}$ is a finite set of players. A subset S of N is called a **coalition**. The collection of all coalitions is denoted by 2^N.
2. $v : 2^N \to \mathbb{R}$ is a function associating every coalition S with a real number $v(S)$, satisfying $v(\emptyset) = 0$. This function is called the **coalitional function** of the game.

- The real number $v(S)$ is called the **worth of the coalition**.
- If the members of S agree to form the coalition, then, as a result, they can expect to receive utility $v(S)$.
- The utilities of all the players can be measured in a common unit (real number).
- Utility can be transferred between players.
- The coalition of all players is called the **grand coalition**.

- When players form a coalition, they all agree to join it and commit to it
- A coalition cannot form without the agreement of all its members.
- When two coalitions S and T are formed, we assume they are disjoint, i.e.

$$S \cap T = \emptyset$$

- In a coalitional game, the value of the coalition and what players can expect to get does not depend on the behaviour of players that are not in it, nor on other coalitions that may form.

Types of Coalitional Games

Definition

A coalitional game (N, v) is said to be **superadditive** if it satisfies

$$v(S) + v(T) \leq v(S \cup T)$$

for every pair of disjoint coalitions $S, T \subseteq N$.

- In superadditive games, players have no compelling reason to form separate coalitions.
- The agents can earn at least as much profit by working together within the grand coalition.
- For superadditive games it is usually assumed that the agents form the grand coalition.

- Three entrepreneurs: O, R, and W.
- O has ideas for inventions and patents and estimates profits of £170k per year.
- R wants to form a business consultancy, which can yield profits of £150k per year.
- W is a salesman and wants to form a sales company, which he estimates can yield profits of £180k per year.
- The entrepreneurs know that if they work together they can profit from their cooperation.

- R can advise O on which patents have greatest market demand: they can profit £350k per year.
- W can sell O's inventions, so they can profit £380k per year.
- R and W can form a business consulting and sales company and gain profits of £360k per year.
- If they work all together they can get £560k per year.
- The coalitional game corresponds to this situation.

$$v(\emptyset) = 0 \qquad v(W) = 180k \qquad v(O) = 170k \qquad v(R) = 150k$$

$$v(O, W) = 380k \qquad v(R, W) = 360k \qquad v(O, R) = 350k \qquad v(O, W, R) = 560k$$

- Three children, Charlie (C), Marcie (M), and Pattie (P), want to buy ice cream.
- Charlie has £3, Marcie has £4, and Pattie has £5.
- The ice cream tubs come in three different size: $500g$, which costs £7, $750g$, which costs £9, and $1000g$, which costs £11.
- The children value ice cream, and assign no utility to money.
- The value of each coalition is determined by how much ice cream they can buy.
- The function v can be defined by the following table.

Coalition	$\emptyset$	$\{C\}$	$\{M\}$	$\{P\}$	$\{C,M\}$	$\{C,P\}$	$\{M,P\}$	$\{C,M,P\}$
v	0	0	0	0	500	500	750	1000

Definition

A coalitional game (N, v) is called **simple** if for each coalition S, either $v(S) = 0$ or $v(S) = 1$.

- So a coalitional game is simple if the worth of any coalition is either 1 or 0.
- In a simple game, a coalition S is called **winning** if $v(S) = 1$, and is called **losing** if $v(S) = 0$.
- It is sometimes convenient to represent simple games by indicating the family of winning coalitions
$$\mathcal{W} = \{S \subseteq N \mid v(S) = 1\}$$

- Simple games can model committee votes.
- The United Nations Security Council has 5 permanent members, and 10 nonpermanent members.
- For any resolution it passes, it needs at least 9 votes.
- Every permanent member can cast a veto on the resolution.
- So (ignoring abstentions), for any resolution to be adopted it needs the support of all 5 permanent members and at least 4 nonpermanent members.
- The following function represents this as a game:

$$v(S) = \begin{cases} 1 & |S| \geq 9 \text{ and } S \text{ contains all permament members} \\ 0 & \text{for any other coalition } S \end{cases}$$

- Weighted majority games are a special case of simple games.
- The House of Commons in the British Parliament has 650 members.
- A coalition requires 326 members to form a government (i.e. the majority).
- Suppose that there are 3 parties: the first with 282 seats, the second with 260 seats and the third with 108 seats.
- Denote by 1 the worth of being the governing coalition and 0 the worth of being in the opposition.
- No single party reaches the majority, no party alone can form a government, therefore
$$v(1) = v(2) = v(3) = 0$$
So, no party alone makes a winning coalition.
- Each pair of parties can can reach the majority, so any coalition with at least two party is a winning coalition, i.e.
$$v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = 1.$$

Definition

A coalitional game is a **weighted majority game** if there exists a **quota** $q \geq 0$ and nonnegative real weights w_i, $i \in N$, one for each player, such that the worth of each nonempty coalition S is

$$v(S) = \begin{cases} 1 & \sum_{i \in S} w_i \geq q \\ 0 & \sum_{i \in S} w_i < q \end{cases}$$

- A weighted majority game can be given an **explicit representation** denoted by

$$[q; w_1, \ldots, w_n]$$

- The above example with the parties in the British House of Commons has the following explicit representation as a weighted majority game:

$$[326; 282, 260, 108]$$

OUTCOMES

Definition

Given a coalitional game (N, v), a **coalition structure** over N is a partition of N, i.e. a collection of nonempty subsets $CS = \{S_1, \dots, S_k\}$ such that

- $\bigcup_{j=1}^{k} S_j = N$, and
- $S_i \cap S_j = \emptyset$ for any $i, j \in \{1, \dots k\}$ such that $i \neq j$.

- We denote the space of coalition structures over N by $\mathcal{CS}_N$.
- Example. If we have three players $N = \{1, 2, 3\}$, there are seven possible non-empty coalitions:,

$$\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\},$$

  and five possible coalition structures

$$\mathcal{CS}_N = \{\{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}, \{\{1, 2, 3\}\}\}.$$

Definition

Given a coalitional game (N, v), a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that

$$x_i \geq 0$$

for all $i \in N$ is called **a payoff vector**.

Definition

An **outcome** of a coalitional game (N, v) is a pair $(CS, \mathbf{x})$, where CS is a coalition structure over N ad $\mathbf{x}$ is a payoff vector.

Given a payoff vector $\mathbf{x}$, we write

$$\mathbf{x}(S) = \sum_{i \in S} x_i$$

to denote the total payoff of a coalition $S \subseteq N$.

Definition

Given a coalitional game (N, v) and a coalition structure $CS = \{S_1, \ldots, S_k\}$, a payoff vector $\mathbf{x}$ is called **efficient** if

$$\mathbf{x}(S_j) = v(S_j)$$

for every $j \in \{1, \ldots k\}$.

- When players divide into coalitions, we can assume they divide its worth among themselves.
- They cannot assign to themselves more than their worth.
- It is unreasonable to assume they will assign less than the total, wasting part of their worth.
- A payoff vector is efficient for a coalition structure if every coalition gets exactly its worth.

Definition

Given a coalitional game (N, v), a payoff vector $\mathbf{x}$ is called **individually rational** if

$$x_i \geq v(\{i\})$$

for all $i \in N$

- Each player i can guarantee themselves $v(\{x_i\})$.
- It is reasonable to assume that a player will agree to be part of a coalition if they can at least get as much as they would get by staying alone.

Definition

Let (N, v) be a coalitional game and CS a coalition structure. A vector $\mathbf{x}$ is called an **imputation** if it is efficient for CS and is individually rational.

- It is reasonable to assume that players will look for an outcome where each coalition gets its worth and where they don't commit to join forces with others while they'd be better off by themselves.
- Then the only outcomes we will consider are those where the vector is an imputation.
- Not all such outcomes are ideal or desirable though.
- We will evaluate outcomes following different criteria:
- **Stability**: what are the incentive for a player to stay on the coalition structure.
- **Fairness**: how well each agent's payoff reflects their contribution.

THE CORE

- Consider a coalitional game (N, v) and an outcome $(CS, \mathbf{x})$ where $\mathbf{x}$ is an imputation.
- If $\mathbf{x}(S) < v(S)$ for some coalition S, the agents in S could do better by abandoning the coalition structure CS and forming a coalition of their own.
- For example, they could distribute the additional payoff earned by this coalition by setting

$$x_i' = x_i + \frac{v(S) - \mathbf{x}(S)}{|S|},$$

  i.e., share the extra profit equally among themselves.
- This outcome is unstable because some players will have an incentive to break the coalition structure.

Definition

The **core** of a coalitional game (N, v), denoted by **core**(N, v), is the collection of all outcomes $(CS, \mathbf{x})$ where $\mathbf{x}$ is an imputation such that

$$\mathbf{x}(S) \geq v(S)$$

for every $S \subseteq N$

- The outcomes in the core are **stable** in the sense that no set of players has any incentive to deviate as it cannot do better on its own.

For superadditive games, the core contains outcomes based on the grand coalition and is the set of all vectors $\mathbf{x}$ that satisfy:

- $x_i \geq 0$ for all $i \in N$.
- $\mathbf{x}(N) = v(N)$.
- $\mathbf{x}(S) \geq v(S)$ for all $S \subseteq N$

- Let's go back to the ice cream game.
- Three children, Charlie (C), Marcie (M), and Pattie (P), want to buy ice cream.
- Charlie has £3, Marcie has £4, and Pattie has £5.
- The ice cream tubs come in three different size: 500g, which costs £7, 750g, which costs £9, and 1000g, which costs £11.
- The function v can be defined by the following table.

Coalition	$\emptyset$	$\{C\}$	$\{M\}$	$\{P\}$	$\{C, M\}$	$\{C, P\}$	$\{M, P\}$	$\{C, M, P\}$
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- What can we say about its core?

- Notice that if any outcome $(CS, \mathbf{x})$ is in the core, then it must be the case that $CS = \{N\}$.
- Otherwise, we would have

$$\mathbf{x}(N) = \sum_{S \in CS} v(S) < v(N)$$

violating the core constraint for N.

- The values of x_C, x_P, x_M must satisfy the following constraints

$$\begin{array}{rcl} x_C \geq 0, \quad x_P \geq 0, & & x_M \geq 0 \\ x_C + x_M + x_P & = & 1000 \\ x_C + x_M & \geq & 500 \\ x_C + x_P & \geq & 500 \\ x_M + x_P & \geq & 750 \end{array}$$

- The above systems has several solutions, for instance both imputations below guarantee that the grand coalition is in the core

$$x_C = 0, x_M = x_P = 500 \qquad x_C = 100, x_M = 400, x_P = 500$$

- Consider the following three-player majority game (N, v) where

$$N = \{1, 2, 3\}$$

and

$$v(S) \begin{cases} 1 & |S| \geq 2 \\ 0 & |S| < 2 \end{cases}$$

for all $S \subseteq N$.

- A coalition S is winning if it includes at least two players.
- This is a superadditive game, so we assume the outcome is based on the grand coalition

- We show that the game has an empty core.
- For any outcome to be in the core it must be the case that

$$x_1 + x_2 \geq 1 \qquad x_1 + x_3 \geq 1 \qquad x_2 + x_3 \geq 1$$

- This means that

$$x_1 + x_2 + x_3 \geq \frac{3}{2}$$

  contradicting the efficiency requirement

$$x_1 + x_2 + x_3 = 1.$$

- So, the core is empty.

The Shapley Value

- The Shapley value is a solution concept that tries to capture the notion of fairness.
- It is usually formulated with respect to the grand coalition and it assigns to every coalition game an imputation.
- It is based on the intuition that the payment that each agent receives should be proportional to their contribution.

- A naive implementation of this idea would be to pay each agent according to how much they increase the value of the coalition of all other players when they join it, i.e., set the payoff of the player i to $v(N) = v(N \setminus \{i\})$.
- However, under this payoff scheme the total payoff assigned to the agents may differ from the value of the grand coalition.
- Take, for instance the 3-player majority game, we defined above.
- Each agent's payoff under this scheme would be 0, whereas the value of the grand coalition is 1.

- To avoid this problem, we can fix an ordering of the agents and pay each agent according to how much they contribute to the coalition formed by their predecessors in this ordering:
  - Player 1 receives $v(\{1\})$.
  - Player 2 receives $v(\{1,2\}) - v(\{1\})$
  - etc.
- This payoff scheme distributes the value of the grand coalition among the agents.
- However, it suffers from another problem: two agents that play symmetric roles in the game may receive very different payoffs.
- In the 3-player majority game, one agent (the one that is second in the ordering) will receive 1, whereas two other agents receive 0, even though all three agents are clearly identical in terms of their contribution.
- The payoffs in this scheme strongly depend on the selected ordering of the agents.

- The dependence on the ordering can be eliminated by averaging over all possible permutations of the players.
- Take a coalitional game (N, v) and let Π_N be the set of all permutations of N, i.e. one-to-one mappings of N onto itself.
- Given a permutation $\pi \in \Pi_N$, denote by $S_\pi(i)$ the set of all predecessors of i in π, i.e.

$$S_\pi(i) = \{j \in N \mid \pi(j) < \pi(i)\}.$$

- For example for $N = \{1, 2, 3\}$, we have

$$\Pi_N = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$$

- So, if $\pi = (3, 2, 1)$ then

$$S_\pi(3) = \emptyset, \qquad S_\pi(2) = \{3\} \qquad S_\pi(1) = \{2, 3\}.$$

- The **marginal contribution** of an agent i with respect to a permutation π in a coalitional game (N, v) is denoted by $\Delta_\pi^{(N,v)}(i)$ and given by

$$\Delta_\pi^{(N,v)}(i) = v(S_\pi(i) \cup \{i\}) - v(S_\pi(i)).$$

- This quantity measures by how much i increases the value of the coalition consisting of its predecessors when i joins the coalition.
- We can define the Shapley value of a player i as the average marginal contribution, where the average is taken over all permutations of N.

Definition

Given a coalitional game (N, v) with $|N| = n$, the **Shapley value** of a player $i \in N$ denoted by $Sh_i(N, v)$ is given by

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \Delta_\pi^{(N,v)}(i).$$

- The set of players is $N = \{C, M, P\}$ and the coalition function:

Coalition	$\emptyset$	$\{C\}$	$\{M\}$	$\{P\}$	$\{C, M\}$	$\{C, P\}$	$\{M, P\}$	$\{C, M, P\}$
v	0	0	0	0	500	500	750	1000

- Let's compute the Shapley value for C.
- There are 6 permutations of the players

$$\pi_1 = CMP \quad \pi_2 = CPM \qquad \pi_3 = MCP$$

$$\pi_4 = PCM \qquad \pi_5 = MPC \qquad \pi_6 = PMC$$

- We have:

$$\begin{array}{lcll}
\Delta_{\pi_1}^{(N,v)}(C) & = & v(\{C\}) - v(\emptyset) & = & 0 \\
\Delta_{\pi_2}^{(N,v)}(C) & = & v(\{C\}) - v(\emptyset) & = & 0 \\
\Delta_{\pi_3}^{(N,v)}(C) & = & v(\{C, M\}) - v(\{M\}) & = & 500 \\
\Delta_{\pi_4}^{(N,v)}(C) & = & v(\{C, P\}) - v(\{P\}) & = & 500 \\
\Delta_{\pi_5}^{(N,v)}(C) & = & v(\{N\}) - v(\{M, P\}) & = & 250 \\
\Delta_{\pi_6}^{(N,v)}(C) & = & v(\{N\}) - v(\{M, P\}) & = & 250
\end{array}$$

and so

$$Sh_C(N, v) = (500 + 500 + 250 + 250)/6 = 250.$$

- The Shapley value is efficient, i.e., it distributes the value of the grand coalition among all agents.

Proposition

For any coalitional game (N, v)

$$\sum_{i=1}^{n} Sh_i(N, v) = v(N)$$

- A player i is called a **dummy player** if for any coalition $S \subseteq N$, $v(S) = v(S \cup \{i\})$.
- The Shapley value does not allocate any payoff to dummy players, i.e. those players who do not contribute to any coalition.

Proposition

For any coalitional game (N, v)*, if a player i is a dummy player then*

$$Sh_i(N, v) = 0.$$

- Two players i, j are symmetric if they contribute equally to each coalition, i.e. for all $S \subseteq N$, $v(S \cup \{i\}) = v(S \cup \{j\})$.

Proposition

For any coalitional game (N, v), if $i, j \in N$ are symmetric then

$$Sh_i(N, v) = Sh_j(N, v).$$

- Consider players N involved in coalitional games (N, v) and (N, v')
- The sum of (N, v) and (N, v') is the game $(N, v + v')$.

Proposition

Given two coalitional games (N, v) and (N, v') and their sum $(N, v + v')$, we have

$$Sh_i(N, v + v') = Sh_i(N, v) + Sh_i(N, v'),$$

for all players $i \in N$.

- We have seen that the Shapley value has the following properties:
- **Efficiency**.
- **Dummy player**.
- **Symmetry**.
- **Additivity**.
- The Shapley value is the only payoff division scheme that satisfies those properties, i.e. they uniquely characterise the Shapley value.
- These properties can also be used to simplify the computation of the Shapley value.

- Take a coalitional game (N, v) with n players such that
  - $v(N) = 1$
  - $v(S) = 0$ for any $S \subset N$.
- We can compute $Sh_i(N, v)$ by observing that $\Delta_\pi^{(N,v)}(i) = 1$ if i appears in the last position in π and $\Delta_\pi^{(N,v)}(i) = 0$ otherwise.
- There are exactly $(n-1)!$ permutations with player i in the last position, then

$$Sh_i(N, v) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

- Alternatively, we can exploit the fact that all players are symmetric, and so

$$Sh_1(N, v) = \cdots = Sh_n(N, v).$$

- By the efficiency property, for all $i \in N$

$$Sh_i(N, v) = \frac{v(N)}{n} = \frac{1}{n}.$$

- The Banzhaf index is another solution concept motivated by fairness.
- It measures the players' expected marginal contributions.
- Instead of averaging over all permutations of players, it averages over all coalitions in the game.

Definition

Given a coalitional game (N, v) with $|N| = n$, the **Banzhaf index** of a player $i \in N$ denoted by $Ba_i(N, v)$ is given by

$$Ba_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)).$$

- The Banzhaf index satisfies the same properties as the Shapley value, except efficiency.

- The Banzhaf index and the Shapley value have an attractive interpretation in simple games.
- They measure the **power** of a player, i.e. the probability that the player can influence the outcome of the game.
- A player is said to be **pivotal** for a coalition $S \subseteq N$ in a game (N, v), if $v(S) = 1$ and $v(S \setminus \{i\}) = 0$.
- A player is pivotal for a permutation $\pi : N \to N$ if it is pivotal for the coalition $S_\pi(i) \cup \{i\}$ consisting of the player and the predecessor.

- It is easy to see that, in simple games, we can define the Shapley value as follows:

$$Sh_i(N, v) = \frac{1}{n!} \left|\{\pi \in \Pi_N \mid v(S_\pi(i)) = 0, v(S_\pi(i) \cup \{i\} = 1)\}\right|$$

- An agent's Shapley value counts the fraction of permutations that i is pivotal for.
- If agents join the coalition in random order, $Sh_i(N, v)$ is exactly the probability that player i turns a losing coalition into a winning one.
- Similarly, the Banzhaf index measures the probability that a given agent turns a losing coalition into a winning one if each of the other agents decides whether to join the coalition by independently tossing a fair coin.

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