Preference Elicitation

Part 1/3

COMP6203 Intelligent Agents

Dr Baharak Rastegari
b.rastegari@soton.ac.uk
Electronics and Computer Science
University of Southampton

Part 1

Learning Outcomes

By the end of this lecture (part 1), you should be able to

- ***Define*** a preference ordering and the conditions usually assumed.
- ***Describe*** reasons for preference uncertainty and the two preference elicitation methods explained (pairwise comparisons and interviews).
- ***Describe*** the differences between pairwise comparisons and interviews.
- ***Define*** an elicitation plan.

Recap on Preferences

Preference Orderings

- Finite outcome set O

- Preference ordering $\succcurlyeq$ over O:
  - $y \succcurlyeq z$ means``**weakly prefer** y to z''
  - $y \succ z$ means``**strictly prefer** y to z''.
    It is true iff $y \succcurlyeq z$ and $z \not\succcurlyeq y$
  - $y \sim z$ means ``are **indifferent** between y and z''.
    It is true iff $y \succcurlyeq z$ and $z \succcurlyeq y$

Conditions

- Reflexive:

$$x \succcurlyeq x \text{ for all } x \in O$$

- Transitive:

for all $x, y, z \in O$, if $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$

- Total (or connected):

$y \succcurlyeq z$ or $z \succcurlyeq y$ for all $y, z \in O$

(connectivity implies reflexivity so in fact we don't need to explicitly require reflexivity)

In mathematics, $\succcurlyeq$ that satisfies the above conditions is referred to as a total preorder.

Transitivity

- If $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$
- If $x \sim y$ and $y \sim z$ then?
- If $x \succ y$ and $y \succ z$ then ?
- If $x \succcurlyeq y$ and $y \succ z$ then ?
- If $x \succ y$ and $y \succcurlyeq z$ then ?

How do Preference Orderings look like?

Reflexivity, transitivity and totality imply that preference orderings look like this: (an example)

$$x \succ y \succ \underbrace{z \sim a \sim b}_{\text{Tie}} \succ d \succ \underbrace{c \sim e}_{\text{Tie}}$$

- A tie (or an equivalence class) consists of outcomes that we are indifferent between.

Revisiting Transitivity

- Transitivity implies that I cannot prefer apple to orange, orange to pineapple and then pineapple to apple!

- Sometimes we have several preferences that need to be aggregated. E.g.
  - a hiring committee is ranking applicants who have been shortlisted for a job;
  - each member of the committee independently provides a ranking over the candidates; then these individual preferences need to be aggregated.

(We will discuss the aggregation of preferences in 2 weeks in the voting lecture.)

Example

- Committee with 3 members: Enrico, Bahar, Danesh
- Set of 3 candidates: Alan, Grace and Nicole

Enrico: Alan $\succ$ Grace $\succ$ Nicole

Bahar: Grace $\succ$ Nicole $\succ$ Alan

Danesh: Nicole $\succ$ Alan $\succ$ Grace

How can we aggregate these 3 preferences?

But for the rest of this lecture we will assume transitivity!

Preference Uncertainty

But do we always know our preferences?

- Learning our preferences is costly
  - Time
  - Money
  - Cognitive cost

- Often there are too many outcomes and fully ranking them is too costly and we cannot afford that.
- Instead, we invest our budget (could be time, money, cognitive power, … or a combination of them) on partially learning our preferences.
- What should our learning strategy be? How do we decide where to spend our elicitation budget?

Preference Elicitation

- We start with having some partial information (or none at all) about our preferences.
- We learn more about preferences by making some queries.
- These queries could be anything, e.g.
  - What cuisine do you prefer, Indian or Italian?
  - Which restaurant do you prefer, Lakaz Maman or Pho Vietnam?
  - Do you prefer strawberries to apples at least as much as you prefer chocolate cake to carrot cake?

Pairwise queries: compare two given outcomes

Interviews: learn as much as possible about the value of an outcome

Pairwise Queries

- Two given outcomes x and y are compared against each other and the result is one of the followings:
  - $x \succ y$
  - $y \succ x$
  - $x \sim y$

Question: given n outcomes and starting with no information about our preference ordering, how many pairwise queries are needed to fully learn our preference ordering

1. in the worst case scenario?
2. In the best case scenario?

Interviews

- A given outcome x is thoroughly investigated (i.e. interviewed).
  - e.g. interviewing candidates for a job

- After interviewing ℓ outcomes, we fully learn our preference ordering over these ℓ outcomes
  - e.g. after interviewing Alex, Bob, Camille, Diana and Edward we learn that Bob $\succ$ Camille $\succ$ Diana $\sim$ Edward $\succ$ Alex

Question: given n outcomes and starting with no information about our preference ordering, how many interviews are needed to fully learn our preference ordering

1. in the worst case scenario?
2. In the best case scenario?

Pairwise Queries VS Interviews

- Interviews are generally assumed to be way more costly than pairwise comparisons.
- In a pairwise queries, we only spend enough effort/time or cognitive cost to compare the given two outcomes.
- The choice (of pairwise query or interview) will depend on the application, and sometimes a combination of two is used. Usually,
  - pairwise comparisons are useful if the two outcomes are sufficiently distinct (or very much the same);
  - interviews are needed when the outcomes are similar and more information is needed in order to rank them.

Elicitation Scheme/Strategy/Plan

- What sort of queries or questions to use (perhaps a combination of few) and in what order?
- When to perform elicitation?

- An elicitation plan describes when and what to ask.
- The plan may depend on the result of the previous questions/queries or other changes to the state of the universe/game.

Example

- Five destinations to choose from for a hiking trip: Cornwall, Lake District, Snowdonia, West Highland Way, Dartmoor
- No preference information to start with
- Eliciting using pairwise comparison
  1. Cornwall ? Lake District
  2. Cornwall ? Snowdonia

Probabilistic Preferences

- We assumed so far that the result of a query is always deterministic
  - Comparing two outcomes we learn whether we prefer one to another or are indifferent between them.
  - Interviewing ℓ outcomes, we learn our preference ordering over them.

- In practice this might not be the case.
  - Comparing x and y you might only learn that $\mathrm{p}(x \succ y) = 0.3$ and thus $\mathrm{p}(y \succ x) = 0.7$
  - Interviewing 3 candidates x, y and z you might only learn that

$$\mathrm{p}(x \succ y \succ z) = 0.2$$
$$p(y \succ z \succ x) = 0.5$$
$$p(z \succ y \succ x) = 0.3$$

In the remainder of this lecture (next parts)

- We assume that the result of queries are always deterministic.
- We only use pairwise queries.

- We look at preference elicitation in Multiple Criteria Ranking using Multi-Attribute Additive Utility functions

Preference Elicitation

Part 2/3

COMP6203 Intelligent Agents

Dr Baharak Rastegari
b.rastegari@soton.ac.uk
Electronics and Computer Science
University of Southampton

Part 2

Learning Outcomes

By the end of this lecture (part 2 and part 3), you should be able to

- ***Define*** Multi-Criteria Decision Analysis.
- ***Describe*** ordinal regression.
- ***Describe*** UTA and UTA^{GMS} methods and their differences.
- ***Compute*** a compatible value function using UTA method (i.e. write the linear program).
- ***Compute*** necessary and possible weak preference relations using UTA^{GMS} method (i.e. write the linear programs).

Multi Criteria Ranking
using
Multi-Attribute Additive Functions

Multi-Criteria Decision Analysis (MCDA)

- A finite set of outcomes or alternatives $O = \{a, b, c, d, \dots\}$ is evaluated on a family of n criteria or issues where $g_i(o) \in \mathbb{R}$ is the evaluation of issue i in outcome o, for all $i \in \{1, \dots, n\}$ and $o \in O$

- The greater $g_i(o), o \in O$, the better outcome o on issue i

  \`\`a is at least as good as b'' w.r.t. issue $i \Leftrightarrow g_i(a) \geq g_i(b)$

- The agent (decision maker) is willing to rank the outcomes from the best to worst, according to his/her preferences.

Example

Outcomes	Education g_1	Work Experience g_2	Interview Assessment g_3
Alice	100	10	70
Grace	95	12	85
Nicole	75	15	85

- Outcomes: Alice, Grace, Nicole
- Criteria: i=1 Education, i=2 Work Experience, i=3 Interview Assessment

Multi-Attribute Additive Value Functions (MAAVF)

- The agent's utility is in the form of an additive value function, such that

$$U(o) = \sum_{i=1}^{n} u_i(g_i(o))$$

where $u_i: \mathbb{R} \to \mathbb{R}$ is a non-decreasing marginal value function for issue i, $i \in \{1, \dots, n\}$.

For simplicity I will sometimes write $\boldsymbol{u_i(o_i)}$ instead of $\boldsymbol{u_i(g_i(o))}$.

N.B. In the rest of the lecture I'm going to refer to U's as (additive) value functions, rather than utility, to be consistent with the terminology used in the related work.

- Value functions are increasing with respect to preference ordering
  - $a \succ b \Leftrightarrow U(a) > U(b)$
  - $a \sim b \Leftrightarrow U(a) = U(b)$

Weighted Additive Value Functions

Weighted additive value/utility functions (that you have seen in the negotiation lecture) is a particular case of the multi-attribute additive value function where $u_i = w_i \cdot g_i(o)$

N.B. Since we are only talking about one agent here, we do not indices to differentiate different agents (as in the negotiation lecture). Hence the notation looks slightly different.

Example

Outcomes	Education g_1	Work Experience g_2	Interview Assessment g_3
Alice	100	10	70
Grace	75	12	85
Nicole	85	10	85

- Assume weighted additive value function where $w_1 = 1, w_2 = 1.2, w_3 = 1.5$
- U(Alice) = ?
- U(Grace) =?
- U(Nicole) = ?

Ordinal Regression

- The agent has some partial information about her preference ordering.
- u_i's are unknown.
- Usually assumed that g_i's are known.

- Goal is to find the agent's preference ordering.
- Using the partial preference ordering, a set of parameters (i.e. u_i's) that respect the partial information are found, and then a complete preference ordering is generated.
  - Chosen parameters must be compatible with the given partial information

In the Rest of the Lecture

- **UTA**: the first and well-known **additive ordinal regression** method
  - Assumes that the agent knows her complete preference ordering over a set of *reference outcomes (or alternatives)* $O^R \subseteq O$.
  - Among possibly many compatible additive value functions that are consistent with the partial preference information, only one is selected and used to generate a complete preference ordering.
  - Assumes that marginal value functions u_i's are piecewise-linear.

- **UTA^{GMS}**: the first method of **robust additive ordinal regression**
  - The agent's ranking of reference outcomes does not need to be complete.
  - Takes into consideration the whole set of compatible additive value functions.
  - Marginal value functions are general non-decreasing functions.

Ordinal Regression via Linear Programming – principle of the UTA method

Assumptions

- Agent knows her complete preference ordering over a set of *reference outcomes* $O^R \subseteq O$.
- The range of g_i is $[\alpha_i, \beta_i]$, $\alpha_i < \beta_i$.
  - finite bound
  - α_i is the worst finite evaluation for issue i
  - β_i is the best finite evaluation for issue i
- u_i's are piecewise-linear, so that the interval $[\alpha_i, \beta_i]$ is divided into $\gamma_i \geq 1$ equal sub-intervals.
- u_i's are normalised to bound $U(o)$ in the interval [0,1].
  - $u_i(\alpha_i) = 0, \forall i \in \{1, \dots, n\}$
  - $\sum_{i=1}^{n} u_i(\beta_i) = 1$

Reference Outcomes

- Complete preference ordering over reference outcomes $O^R \subseteq O$, $|O^R| = m$, such that for all $c, d \in O^R$
  - ``c is at least as good as d'' $\Leftrightarrow$ $c \succcurlyeq d$
  - ``c is prefrred to d '' $\Leftrightarrow$ $[c \succcurlyeq d$ but $not(d \succcurlyeq c)]$ $\Leftrightarrow$ $c \succ d$
  - ``c is indifferent to d'' $\Leftrightarrow$ $[c \succcurlyeq d$ and $d \succcurlyeq c]$ $\Leftrightarrow$ $c \sim d$

- A value function U is compatible if and only if for each $\mathrm{c}, \mathrm{d} \in O^R$
  - $U(c) \geq U(d) \Leftrightarrow c \succcurlyeq d$
  - $U(c) > U(d) \Leftrightarrow c \succ d$
  - $U(c) = U(d) \Leftrightarrow c \sim d$

Piece-Wise Linear Marginal Value Functions

- For each $i \in \{1, \dots, n\}$, the range of g_i is $[\alpha_i, \beta_i]$, $\alpha_i < \beta_i$
- This interval is divided into $\gamma_i \geq 1$ equal sub-intervals $[x_i^0, x_i^1], [x_i^1, x_i^2], \dots, \left[x_i^{\gamma_i - 1}, x_i^{\gamma_i}\right]$ where $x_i^j = \alpha_i + \frac{j}{\gamma_i}(\beta_i - \alpha_i), j = 1 \dots \gamma_j$

For example: Assume that issue i=1 (education) is evaluated in the range [0,100], i.e.

- $0 \leq g_1(o) \leq 100$ for all outcomes o
- $\alpha_1 = 0, \beta_1 = 100$

Assume $\gamma_1 = 4$, then we have sub-intervals [0, 25], [25, 50], [50, 75], [75, 100]

- $x_1^0 = \alpha_1 = 0,\ x_1^1 = 25,\ x_1^2 = 50,\ x_1^3 = 75,\ x_1^4 = 100$

Piece-Wise Linear Marginal Value Functions

- The marginal value of an outcome $o \in O$ on issue i is obtained by linear interpolation

$$u_i(\mathrm{o}_i) = u_i(\mathrm{x}_\mathrm{i}^\mathrm{j}) + \frac{o_i - x_i^j}{x_i^{\mathrm{j+1}} - x_i^j}\left(u_i\left(x_i^{j+1}\right) - u_i\left(x_i^j\right)\right),\ o_i \in \left[x_i^j, x_i^{j+1}\right]$$

Continuing on the example of previous slide, assume we want to compute $u_1(65)$.

- 65 sits between breakpoints $50 = x_1^2$ and $75 = x_1^3$, therefore we get that

$$u_1(65) = u_1(50) + \frac{65 - 50}{75 - 50}(u_1(75) - u_1(50))$$

- The piecewise-linear additive model is completely defined by the marginal values at the breakpoints, i.e.

$$u_i\left(x_i^0\right) = u_i(\alpha_i), u_i\left(x_i^1\right), \dots, u_i\left(x_i^{\gamma_i}\right) = u_i(\beta_i).$$

Piece-Wise Linear Marginal Value Functions

- For each $i \in \{1, \dots, n\}$, the range of g_i is $[\alpha_i, \beta_i]$, $\alpha_i < \beta_i$
- This interval is divided into $\gamma_i \geq 1$ equal sub-intervals $[x_i^0, x_i^1], [x_i^1, x_i^2], \dots, \left[x_i^{\gamma_i - 1}, x_i^{\gamma_i}\right]$ where $x_i^j = \alpha_i + \frac{j}{\gamma_i}(\beta_i - \alpha_i), j = 1 \dots \gamma_j$
- The marginal value of an outcome $o \in O$ on issue i is obtained by linear interpolation

$$u_i(\mathrm{o}_i) = u_i(\mathrm{x}_\mathrm{i}^\mathrm{j}) + \frac{o_i - x_i^j}{x_i^{\mathrm{j}+1} - x_i^j}\left(u_i\left(x_i^{j+1}\right) - u_i\left(x_i^j\right)\right), o_i \in [x_i^j, x_i^{j+1}]$$

- The piecewise-linear additive model is completely defined by the marginal values at the breakpoints, i.e.

$$u_i\left(x_i^0\right) = u_i(\alpha_i), u_i\left(x_i^1\right), \dots, u_i\left(x_i^{\gamma_i}\right) = u_i(\beta_i).$$

Constraints for a Compatible Value Function

A value function $U(o) = \sum_{i=1}^{n} u_i(o_i)$ is compatible if it satisfies the following set of constraints:

$$\left.\begin{array}{l} U(c) > U(d) \Leftrightarrow c \succ d, \\ U(c) = U(d) \Leftrightarrow c \sim d, \end{array}\right\} \quad \forall c, d \in O^R,$$

$$u_i\left(x_i^{j+1}\right) - u_i\left(x_i^{j}\right) \geq 0, \quad i = 1, \ldots, n, j = 1, \ldots, \gamma_i - 1,$$

$$u_i(\alpha_i) = 0, \qquad i = 1, \ldots, n,$$

$$\sum_{i=1}^{n} u_i(\beta_i) = 1.$$

Linear Program

Minimise $F = \sum_{c \in O^R} (\sigma^+(c) + \sigma^-(c))$

Subject to:

$$\left.\begin{array}{l} U(c) + \sigma^+(c) - \sigma^-(c) \\ \geq\ U(d) + \sigma^+(d) - \sigma^-(d) + \epsilon \Leftrightarrow c \succ d, \\ \\ U(c) + \sigma^+(c) - \sigma^-(c) \\ =\ U(d) + \sigma^+(d) - \sigma^-(d) \Leftrightarrow c \sim d, \end{array}\right\} \quad \forall c, d \in O^R,$$

$$u_i\left(x_i^{j+1}\right) - u_i\left(x_i^j\right) \geq 0\,, i = 1, \dots, n, j = 1, \dots, \gamma_i - 1,$$

$$u_{i(\alpha_i)} = 0, i = 1, \dots, n,$$

$$\sum_{i=1}^{n} u_i(\beta_i) = 1\,,$$

$$\sigma^+(c) + \sigma^-(c) \geq 0, \quad \forall c \in O^R.$$

- $u_i(x_i^j)$'s are unknowns
- $\sigma^+(c)$'s and $\sigma^-(c)$'s are auxiliary variables
- ϵ an arbitrarily small value

Linear Program without σ's

Minimise

Subject to:

$$\left.\begin{array}{ll} U(c) \geq U(d) + \epsilon \Leftrightarrow c \succ d, \\ U(c) = U(d) \Leftrightarrow c \sim d, \end{array}\right\} \quad \forall c, d \in O^R,$$

$$u_i\left(x_i^{j+1}\right) - u_i\left(x_i^j\right) \geq 0\,, i = 1, \dots, n, j = 1, \dots, \gamma_i - 1,$$

$$u_{i(\alpha_i)} = 0, i = 1, \dots, n,$$

$$\sum_{i=1}^{n} u_i(\beta_i) = 1\,,$$

- $u_i(x_i^j)$'s are unknowns
- ϵ an arbitrarily small value

Solution or No Solution

- If the optimal value of the objective function is equal to zero (i.e. if all $\sigma^+(c)$'s and $\sigma^-(c)$'s are set to zero) then there exists at least one value function $U(o) = \sum_{i=1}^{n} u_i(o_i)$ compatible with the preference ordering on O^R.

- If the optimal value of the objective function is greater than zero, then there is no compatible value function. In this case, one might consider:
  - Increasing γ_i for one or several marginal values.
  - Revising the preference ordering on O^R

Preference Elicitation

Part 3/3

COMP6203 Intelligent Agents

Dr Baharak Rastegari
b.rastegari@soton.ac.uk
Electronics and Computer Science
University of Southampton

Part 3

UTA^{GMS}
– Robust Ordinal Regression

Limitations of UTA

- If the linear program is feasible, then the choice of a compatible value function is arbitrary.
- Marginal value functions are limited to piecewise linear functions. To specify the number of breakpoints (γ_i's) is arbitrary and restrictive.
- Complete preference ordering on reference outcomes is needed.

UTA^{GMS}

- Takes into consideration the whole set of compatible additive value functions.
- Marginal value functions are general non-decreasing functions.
- The agent’s ranking of reference outcomes does not need to be complete.

Assumptions

- Agent knows her partial preference ordering over a set of *reference outcomes* $O^R \subseteq O, |O^R| = m$.

As in UTA:

- u_i's are normalised to bound $U(o)$ in the interval [0,1].
  - $u_i(\alpha_i) = 0, \forall i \in \{1, \dots, n\}$
  - $\sum_{i=1}^{n} u_i(\beta_i) = 1$

- g_i's are known, and their range is bounded.

Necessary and Possible weak preference relations

For any $a, b \in O$ we can ask

- Are a and b ranked in the same way by *all* compatible value functions?
- Is there *at least one* compatible value function ranking a at last as good as b (or b at least as good as a)?

Answering these questions for all pairs of $(a, b) \in O \times O$ we get

- A necessary weak preference relation $\succcurlyeq^N$ where $a \succcurlyeq^N b \;\Leftrightarrow U(a) \geq U(b)$ for *all* compatible value functions
- A possible weak preference relation $\succcurlyeq^P$ where $a \succcurlyeq^P b \;\Leftrightarrow U(a) \geq U(b)$ for *at least one* compatible value function

From Partial Preference Ordering

For any $a, b \in O^R$

- If $a \succcurlyeq b \;\Rightarrow a \succcurlyeq^N b$
- If $a \succ b \;\Rightarrow not(b \succcurlyeq^p a)$

Constraints for a Compatible Value Function

A value function $U(o) = \sum_{i=1}^{n} u_i(o_i)$ is compatible if it satisfies the following set of constraints (that we will refer to as E^{A^R}):

$$\left.\begin{array}{l} U(c) > U(d) \Leftrightarrow c \succ d, \\ U(c) = U(d) \Leftrightarrow c \sim d, \end{array}\right\} \quad \forall c, d \in O^R$$

$$u_i(g_i(a_{\tau_{i(j)}})) - u_i(g_i(a_{\tau_{i(j-1)}})) \geq 0, \quad i = 1, \dots, n, j = 2, \dots, m,$$

$$u_i(g_i(a_{\tau_{i(1)}})) \geq 0, \; u_i(g_i(a_{\tau_{i(m)}})) \leq u_i(\beta_i), \quad i = 1, \dots, n,$$

$$u_i(\alpha_i) = 0, \qquad i = 1, \dots, n,$$

$$\sum_{i=1}^{n} u_i(\beta_i) = 1.$$

$\boldsymbol{\tau_i}$ is the permutation on the set of indices of outcomes from O^R that reorders them according to the increasing evaluation on attribute i, i.e.

$$g_i\left(a_{\tau_i(1)}\right) \leq g_i\left(a_{\tau_i(2)}\right) \leq \cdots \leq g_i\left(a_{\tau_i(m)}\right)$$

Turning E^{A^R} into a Linear Program

- Using the same trick as used for UTA, we can rewrite the first set of constraints as

$$U(c) \geq U(d) + \epsilon \Leftrightarrow c \succ d$$

for an arbitrary small ϵ

- If the LP is infeasible then no compatible value function exists. This could happen e.g. if
  – agent's preferences do not match the additive model, or
  – the agent have made an error in his/her statements.

Computation of $\succcurlyeq^N$ and $\succcurlyeq^P$

- For all pair of outcomes $(a, b) \in O \times O$, let π_i be a permutation of the indices of outcomes from set $O^R \cup \{a, b\}$ that reorders them according to increasing evaluation on attribute i, i.e.

$$g_i\left(a_{\pi_i(1)}\right) \leq g_i\left(a_{\pi_i(2)}\right) \leq \cdots \leq g_i\left(a_{\pi_i(w)}\right)$$

  where $w = |\, O^R \cup \{a, b\}\, |$.

- Fix the characteristic points of $u_i, i = 1, \ldots, n$, in $g_i^0 = \alpha_i, \quad g_i^j = g_i(a_{\pi_i(j)})$ for $j = 1, \ldots, w, \;\; g_i^{w+1} = \beta_i$

Ordinal Regression Constraints E(a,b)

For a given pair of outcomes $(a, b) \in O \times O$, we get the following set of constraints (that we will refer to as $E(a, b)$):

$$\left.\begin{array}{l} U(c) \geq U(d) + \epsilon \Leftrightarrow c \succ d, \\ U(c) = U(d) \Leftrightarrow c \sim d, \end{array}\right\} \quad \forall c, d \in O^R$$

$$u_i(g_i^j) - u_i(g_i^{j-1}) \geq 0, \quad i = 1, \dots, n, j = 1, \dots, \omega + 1,$$

$$u_i(g_i^0) = 0, \quad i = 1, \dots, n,$$

$$\sum_i^n u_i(g_i^{\omega+1}) = 1.$$

Characteristic points of u_i:

- $g_i^0 = \alpha_i, g_i^{w+1} = \beta_i$
- $g_i^j = g_i(a_{\pi_i(j)})$ for $j = 1, \dots, w$

Note that $E(a, b) = E(b, a)$.

Linear Programs to compute $\succcurlyeq^N$

For a given pair of outcomes $(a, b) \in O \times O$, $a \succcurlyeq^N b$ if and only if the optimal solution $d(a, b)$ to the following linear program (where constraints are $E(a, b)$) is nonnegative; i.e. $d(a, b) \geq 0$.

Minimise $\boldsymbol{U}(\boldsymbol{a}) - \boldsymbol{U}(\boldsymbol{b})$

Subject to:

$$\left.\begin{array}{l} U(c) \geq U(d) + \epsilon \Leftrightarrow c \succ d, \\ U(c) = U(d) \Leftrightarrow c \sim d \,, \end{array}\right\} \quad \forall c, d \in O^R$$

$$u_i(g_i^j) - u_i\,(g_i^{j-1}) \geq 0, \qquad i = 1, \ldots, n, j = 1, \ldots, \omega + 1\,,$$

$$u_i\left(g_i^0\right) = 0, \qquad i = 1, \ldots, n,$$

$$\sum_i^n u_i(g_i^{\omega+1}) = 1.$$

Characteristic points of u_i

- $g_i^0 = \alpha_i, g_i^{w+1} = \beta_i$
- $g_i^j = g_i(a_{\pi_i(j)})$ for $j = 1, \ldots, w$

Linear Programs to compute $\succcurlyeq^P$

For a given pair of outcomes $(a, b) \in O \times O$, $a \succcurlyeq^p b$ if and only if the optimal solution $\mathrm{D}(a, b)$ to the following linear program (where constraints are $E(a, b)$) is nonnegative; i.e. $\mathrm{D}(a, b) \geq 0$.

Maximise $\boldsymbol{U(a) - U(b)}$

Subject to:

$$\left.\begin{array}{l} U(c) \geq U(d) + \epsilon \Leftrightarrow c \succ d, \\ U(c) = U(d) \Leftrightarrow c \sim d\ , \end{array}\right\} \forall c, d \in O^R$$

$$u_i(g_i^j) - u_i\,(g_i^{j-1}) \geq 0, \quad i = 1, \ldots, n, j = 1, \ldots, \omega + 1\ ,$$

$$u_i\left(g_i^0\right) = 0, \quad i = 1, \ldots, n,$$

$$\textstyle\sum_i^n u_i(g_i^{\omega+1}) = 1.$$

Characteristic points of u_i

- $g_i^0 = \alpha_i, g_i^{w+1} = \beta_i$
- $g_i^j = g_i(a_{\pi_i(j)})$ for $j = 1, \ldots, w$

Summary of UTA$^{\mathrm{GMS}}$

- For each pair of outcomes $(a, b) \in O \times O$ write a pair of linear programs with the same constraints but different objective functions.
  - One for necessary weak preference relation and another for possibly weak preference relation.
  - So total of 4 LPs for every a and b in order to establish whether each of the following is true: $a \succcurlyeq^N b, b \succcurlyeq^N a, a \succcurlyeq^P b, b \succcurlyeq^P a$.

- By solving these linear programs we compute $\succcurlyeq^N$ and $\succcurlyeq^P$.
- Provide $\succcurlyeq^N$ and $\succcurlyeq^P$ to the agent, using which s/he can
  - extend their preference information, or
  - decide on next elicitation steps.

Assignment for you

1. What are $\succcurlyeq^N$ and $\succcurlyeq^P$ in the case where no partial preference ordering exists; i.e. no pairwise comparison of reference outcomes O^R is provided by the agent.
2. Do we need to solve all linear programs for each pair of $a, b \in O$ (that is, the LPs corresponding to $a \succcurlyeq^N b$, $a \succcurlyeq^P b$, $b \succcurlyeq^N a$ and $b \succcurlyeq^P a$) or can we deduce the outcome of one (i.e. whether the optimal solution is nonnegative or not) by knowing the outcome to the other one, at least in some cases?

Assignment for you: 1 of 2

Question: What are $\succcurlyeq^N$ and $\succcurlyeq^P$ in the case where no partial preference ordering exists; i.e. no pairwise comparison of reference outcomes O^R is provided by the agent ?

Answer: In this case,

- $\succcurlyeq^N$ boils down to the (very) weak dominance relation Δ in O where $a\Delta$b iff $g_i(a) \geq \ g_i(b), \forall\, i = 1, \ldots, n$
- $\succcurlyeq^P$ is a complete preference relation such that for any pair $(a, b) \in O \times O$:
  - $a\sim^P b$ (i.e. $a \succcurlyeq^p\ b$ and $b \succcurlyeq^p\ a$) iff [(not($a\Delta$b) and not(bΔa)) or ($a\Delta$b and bΔa)]
  - $a \succ^P\ b$ (i.e. $a \succcurlyeq^p\ b$ and $\mathrm{not}(b \succcurlyeq^p\ a)$) iff [$a\Delta$b and not(b$\Delta a$)]

Assignment for you: 2 of 2

Question: Do we need to solve all linear programs for each pair of $a, b \in O$ (that is, the LPs corresponding to $\mathrm{a} \succcurlyeq^N b$, $a \succcurlyeq^P b$, $b \succcurlyeq^N a$ and $\mathrm{b} \succcurlyeq^P a$) or can we deduce the outcome of one (i.e. whether the optimal solution is nonnegative or not) by knowing the outcome to the other one, at least in some cases?

Answer: We don't always need to solve both pair of linear programs for each pair of outcomes. The following equivalences hold:

- $d(a, b) \geq 0 \Leftrightarrow D(b, a) \leq 0,$
- $\mathrm{D}(a, b) \geq 0 \Leftrightarrow d(b, a) \leq 0,$
- $d(a, b) = 0 \Leftrightarrow D(b, a) = 0.$

Further Reading:

- **Robust Ordinal Regression,** by S. Greco, R. Słowinski, J. R. Figueira, and V. Mousseau. In ``*Trends in Multiple Criteria Decision Analysis*'', Chapter 9, 2010. [a summary of related work until 2010] https://https://link.springer.com/chapter/10.1007/978-1-4419-5904-1_9

- **UTA Method,** by Y. Siskos, E. Grigoroudis, and N. F. Matsatsinis. In ``*Multiple Criteria Decision Analysis: State of the Art Surveys''*, Chapter 7, 2005. [on variants of UTA methods] https://link.springer.com/chapter/10.1007/0-387-23081-5_8

- **Ordinal Regression Revisited: Multiple Criteria Ranking Using a Set of Additive Value Functions,** S. Greco, V. Mousseau, and R. Słowinski. *European Journal of Operation Research*, 2008 [UTA^{GMS} paper] https://www.sciencedirect.com/science/article/pii/S0377221707008752

And even further reading:

- **Preference Disaggregation in Multiple Criteria Decision Analysis,** Essays in Honor of Yannis Siskos, Editors: N. F. Matsatsinis and E. Grigoroudis, 2018 https://www.springer.com/gp/book/9783319905983

- **Automated Negotiations Under User Preference Uncertainty: A Linear Programming Approach,** by D. Tsimpoukis, T. Baarslag, M. Kaisers, and N. G. Paterakis. In ``Agreement Technologies'', 2018. [extension to categorical data] https://link.springer.com/chapter/10.1007/978-3-030-17294-7_9

And more further reading (on elicitation in other settings):

- **Learning to Rank Using Gradient Descent,** by C. Burges and A. Lazier, *In the proceedings of the 22nd international conference on Machine Learning (ICML),* 2005. [on presenting ranked order of results, such as news feed, ads, … to users] https://icml.cc/2015/wp-content/uploads/2015/06/icml_ranking.pdf