COMP6207
Algorithmic Game Theory

Lecture 13 One-sided Matching Without Initial Ownership

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Learning Outcomes

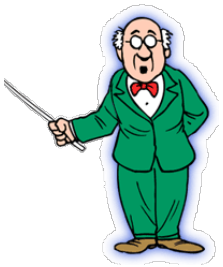
- By the end of this session, the students should be able to
  - ***Describe*** the one-sided matching problem
  - ***Compute*** the outcome of Probabilistic Serial mechanism (PS) and Random Priority (RP)
  - ***Understand*** the properties of these mechanisms

One-sided Matching without initial ownership

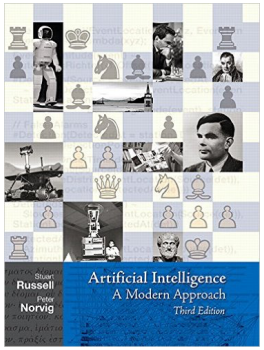
- Random Serial Dictatorship (RSD)
  - a.k.a., Random Priority (RP)
  - Randomized generalization of a deterministic mechanism Serial Dictatorship (SD)
- Probabilistic Serial (PS)
  - Defined by a Simultaneous Eating Algorithm
- Their properties

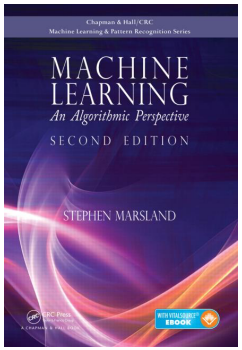
Teaching allocation

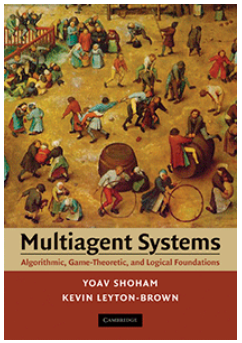
- Assigning teaching load to lecturers
  - Mechanism: {*preferences*} → {*assignment*}

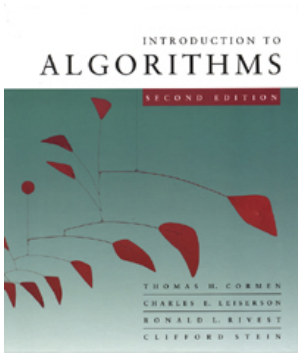


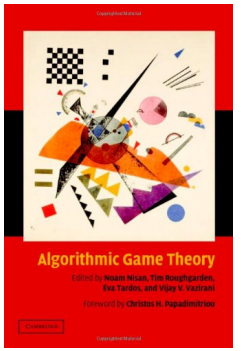






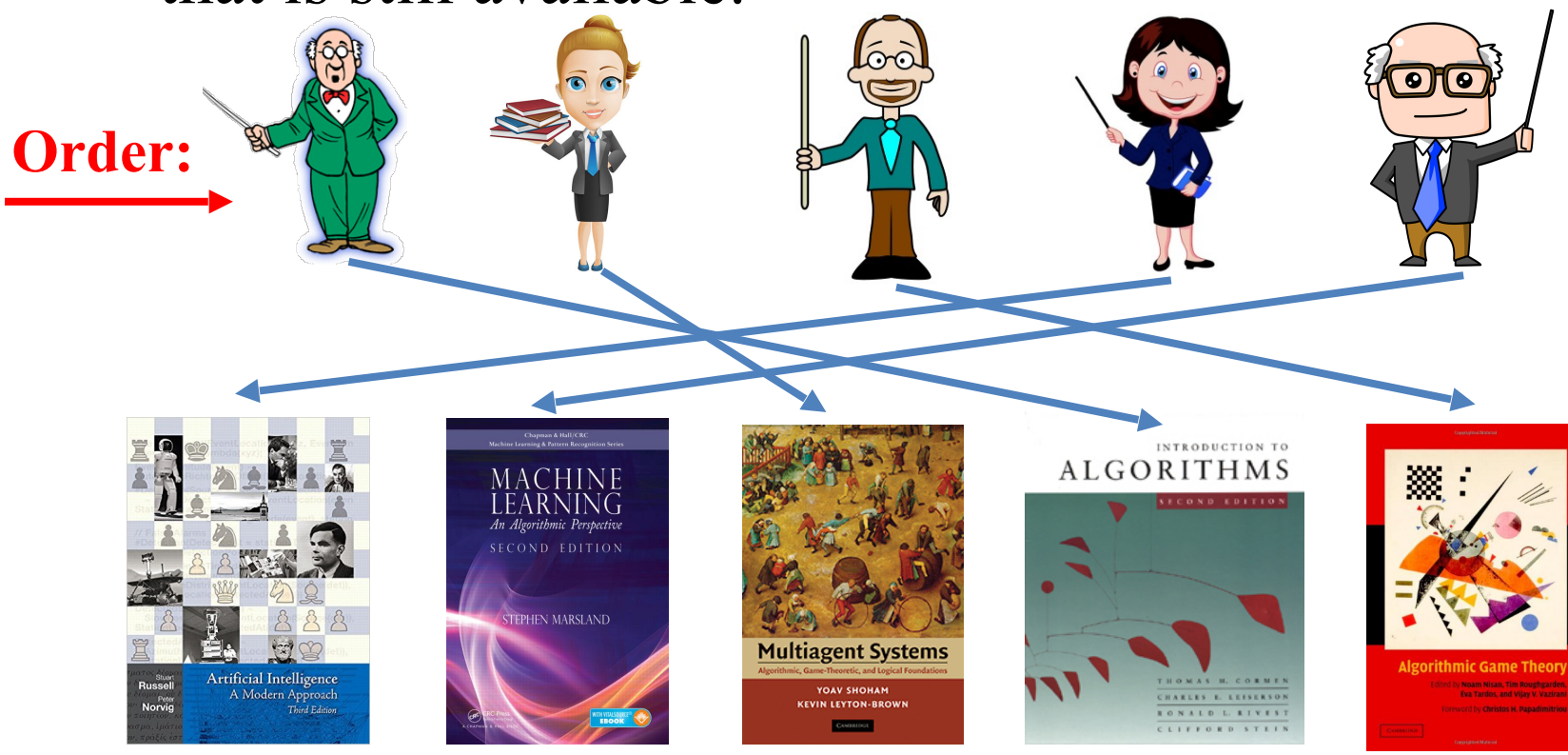






Serial Dictatorship

- Fix an ordering of the agents $a_1, a_2, \ldots, a_n$
  - It is a deterministic mechanism.
- Let them take turns to pick their most preferred item that is still available.



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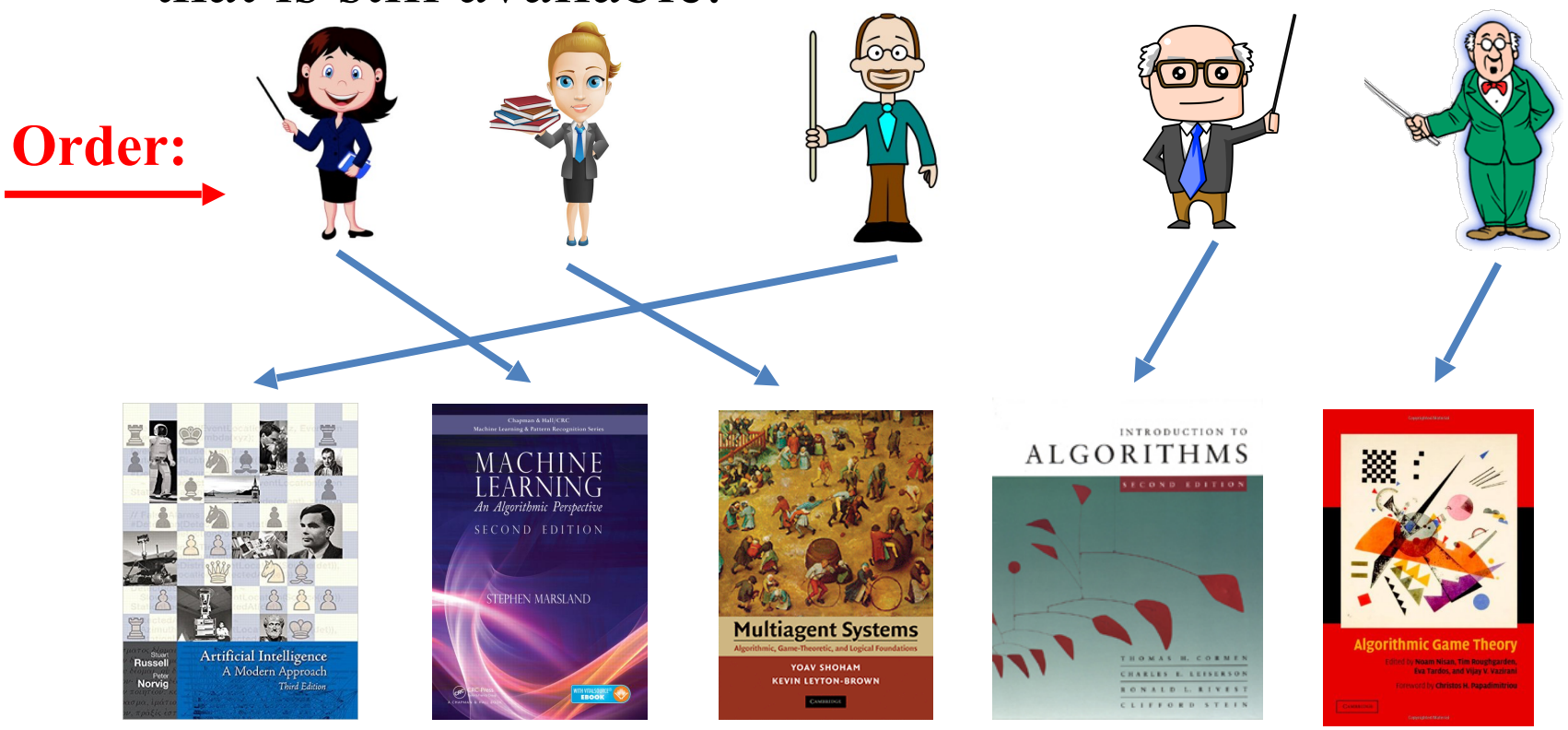
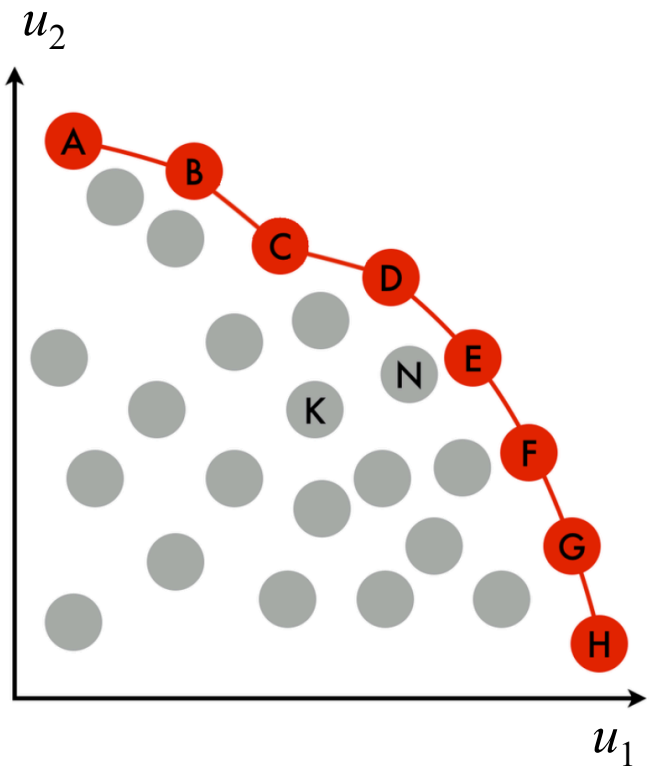


Pareto-optimality (PO)

- Given an initial situation (allocation), a Pareto improvement is a new situation where some agents will gain, and no agents will lose.
- A situation is called Pareto dominated if there exists a possible Pareto improvement.
- A situation is called Pareto optimal or Pareto efficient if no other situation could lead to an improvement for some agent without some other agent losing or if there's no scope for further Pareto improvement.
- The Pareto frontier is the set of all Pareto efficient allocations



Properties

- Serial Dictatorship is Pareto-optimal
  - by induction:
    - The agent with priority one (appearing first in the order) cannot improve her assignment (she is already getting her most favourite house)
    - Fixing her assignment, the agent with priority two (appearing second in the order) cannot improve her assignment (she is already getting her most favourite among the remaining houses)
    - And so on.

Properties

- Serial Dictatorship is truthful
  - by induction:
    - The agent with priority one receives her most preferred house, so clearly no incentive to lie
    - The agent with priority two receives her most preferred among remaining houses, so again no incentive to lie
    - And so on.

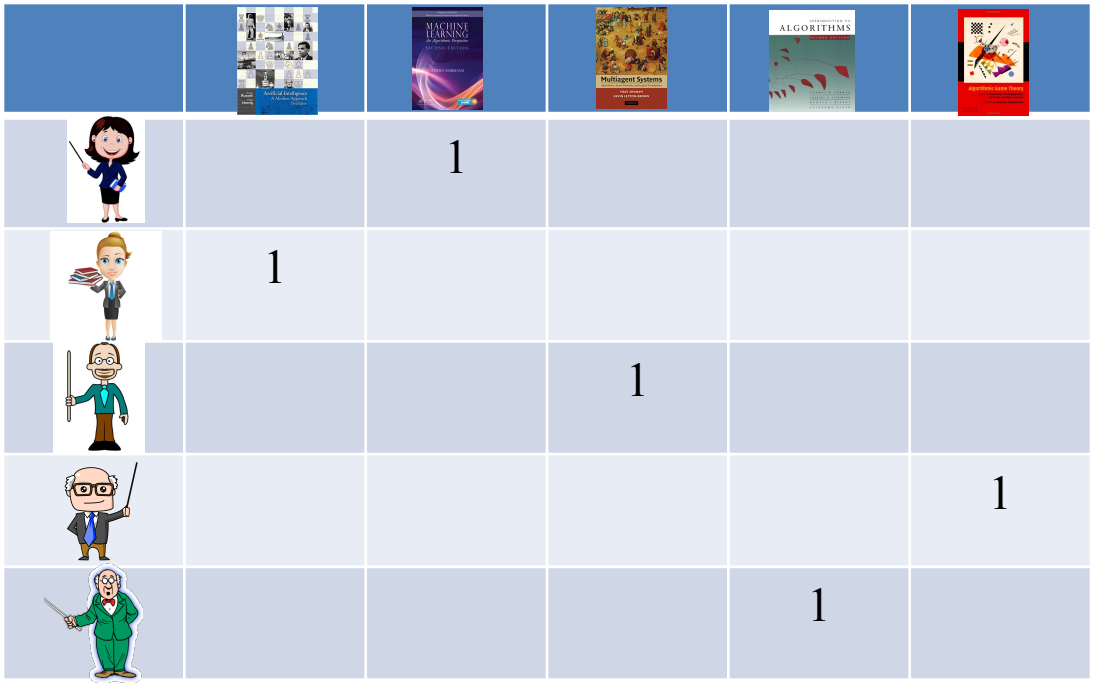
Random Serial Dictatorship (RSD)

- A.k.a., Random Priority (RP)
  - Draw a permutation of the agents uniformly at random. Then, let them successively choose an item in that order (so the first agent in the ordering gets first pick and so on).
  - It is a randomized mechanism.

Random Serial Dictatorship (RSD)

- 5 agents => 5! = 120 permutations => 120 orders
- For each order of the agents, run SD, denote the allocation matrix as A_i ; for example,

$A_i =$



- Then the randomized assignment returned by RSD is $\frac{1}{120}\sum_{i=1}^{120} A_i$

Ex-post Pareto-optimality

- We said that Serial Dictatorship is Pareto-optimal because it is a deterministic mechanism.

- Random Serial Dictatorship is
  - Ex-post Pareto-optimal, but it is not ex-ante PO
  - When the decision process is random, there is a difference between *ex-post* (after) and *ex-ante* (before) Pareto-efficiency
  - Ex-post Pareto-efficiency means that any outcome of the random process is Pareto efficient.
  - Ex-ante Pareto-efficiency means that the lottery (randomized allocation) determined by the process is Pareto-efficient with respect to the expected utilities. That is: no other lottery gives a higher expected utility to one agent and at least as high expected utility to all agents.

Random serial dictatorship (RSD) is not ex-ante Pareto efficient

			
	1	0.8	0
	1	0.2	0
	1	0.2	0

RSD gives a 1/3 chance of every object to each agent (because the preferences are the same). Expected utilities are (0.6, 0.4, 0.4).

Alternative lottery gives SECOND item to  the other two 50/50

Expected utilities are (0.8, 0.5, 0.5). Thus, RSD is not ex-ante Pareto efficient!

Universal Truthfulness

- Random Serial Dictatorship is
  - Universal truthful
  - For randomised mechanisms, there is a difference between universal truthfulness and truthful-in-expectation.
  - Universal Truthfulness: A universally-truthful mechanism is a probability distribution over deterministic truthful mechanisms. The mechanism is truthful for any realisation of the set of mechanisms.
  - Truthfulness-in-Expectation: A mechanism is truthful in expectation if an agent always maximises their expected utility by acting truthfully. The expectation is taken over the randomisation of the allocations.

Probabilistic Serial Mechanism

- Defined by a Simultaneous Eating procedure
  - Agents simultaneously "eat" their most preferred items at a uniform speed, moving onto their next most preferred item whenever an item is fully eaten

Simultaneous Eating (uniform speed)

Simultaneous Eating (uniform speed)

0.5

0.5

0.5

Simultaneous Eating (uniform speed)

0.5

0.5

0.5

Simultaneous Eating (uniform speed)

0.5

0.25

0.5

0.25

0.5+0.25

Simultaneous Eating (uniform speed)

0.5

0.25

0.5

0.25

0.5+0.25

Simultaneous Eating (uniform speed)

0.5

0.25

0.25

0.5

0.25+0.25

0.5+0.25

0.25

Simultaneous Eating (uniform speed)

- So, the final assignment is

			
	0.5	0.25	0.25
	0.5	0	0.5
	0	0.75	0.25

Quiz

- Given the following instance, what is the allocation returned by the PS mechanism?

$$1: b \succ a \succ c$$
$$2: a \succ b \succ c$$
$$3: a \succ c \succ b$$

$$X = \begin{matrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \end{matrix}$$

Truthfulness

- Is the PS mechanism truthful?
  - No (given some utilities).
  - Counterexample

True preference:
$u_1 = (0.9, 1, 0)$

$$1: b \succ a \succ c$$
$$2: a \succ b \succ c$$
$$3: a \succ c \succ b$$

$$X = \begin{matrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \end{matrix}$$

Report

$$1: a \succ b \succ c$$
$$2: a \succ b \succ c$$
$$3: a \succ c \succ b$$

$$X = \begin{matrix} 1/3 & 1/2 & 1/6 \\ 1/3 & 1/2 & 1/6 \\ 1/3 & 0 & 2/3 \end{matrix}$$

What is the intuition behind this counterexample? Under what circumstances an agent may misreport?

Bounded Incentives

- Bounded Incentives in Manipulating the Probabilistic Serial Rule
  - Wang, Wei, Zhang, 34th AAAI Conference on Artificial Intelligence (AAAI-2020)
- Theorem: No agent can misreport its preferences such that its utility becomes more than 1.5 times of what it is when reports truthfully. This ratio is a worst-case guarantee by allowing an agent to have complete information about other agents' reports
  - Shed some light on the robustness of Probabilistic Serial against strategic manipulation, which is one step further than knowing that it is not incentive-compatible.

Summary

- One-sided matching problem
  - When agents own items, i.e., so-called House Allocation problem
    - Top Trading Cycle mechanism (TTC)
    - You request my house, I get your turn (YRMH-IGYT)
  - When agents do not own items
    - Probabilistic Serial mechanism (PS)
    - Random Priority (RP)

Extra reading

- Random Serial Dictatorship / Random Priority
  - Atila Abdulkadiroglu and Tayfun So¨nmez. Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica*, 66(3):689–702, 1998.
  - Lars-Gunnar Svensson. Strategy-proof allocation of indivisible goods. Social Choice and Welfare, 16(4):557–567, 1999.

- Probabilistic serial mechanism
  - Bogomolnaia, Anna; Moulin, Hervé (2001). "A New Solution to the Random Assignment Problem". Journal of Economic Theory. 100 (2): 295.

Preview

- So far, we have learned various matching problems, but all are offline problems.
- Plan for the upcoming lectures
  - Online matching
  - Auctions + online matching

    => Search engine computational advertising
  - Social choice theory (voting rules)