Auctions

COMP6203: Intelligent Agents
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Learning Outcomes

By the end of this lecture, you should be able to

- **Define** four families of single-item auctions.
- **Describe** four desirable properties/objectives for auctions.
- **Explain** what the optimal behaviour is when you know the probablity distribution of opponent bids (as in a decision theoretic framework).
- **Define** strategy-proofness and **prove** that Vickrey auction is strategy-proof.
- **Describe** the connection between the optimal strategies in each of the four auctions discussed.

Conflict Resolution Mechanisms

- So far we have focused on *negotiation* as a way to resolve conflicts between self-interested agents.
- Negotiation is typically used in encounters between two agents.
- With more than two agents, other mechanisms may be more appropriate:
  - Auctions: used in order to allocate scarce resources.
  - Voting: used in order to make collective decisions.

Auction Design and Objectives

- There are many different auctions with different properties.
- Typical desirable objectives include:
  - The auction should maximise social welfare: allocate the resources to those who value them the most.
  - The auction should be individually rational: agents should not be worse off from participating.
  - The auction should not be manipulable: agents should be incentivised to behave *truthfully*.
  - In some cases, the aim could be to maximise revenue.

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  - In some cases, the aim could be to maximise revenue.
- Which auction is most appropriate and how do we analyse such auctions?

Some Canonical Single-item Auctions

Suppose we are selling a single indivisible item. There are a few famous families of single-item auctions.

Open-outcry auctions:

- English auction
- Dutch auction
- Japanese auction

Sealed-bid auctions:

- First-price auction
- Second-price (*aka* Vickrey) auction
- All-pay auction

Which one is better?

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- A seller with a single item, such as an old antique chair.
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- Our bidder utility model is quasilinear utility model.
  - If i loses, and has to pay p_i, her utility is $-p_i$.
    - In auctions where only winners pay, i's utility is 0.
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We need an allocation rule + a payment rule.

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- $\sum_{i=1}^{n} x_i \leq 1$ (feasibility constraint).

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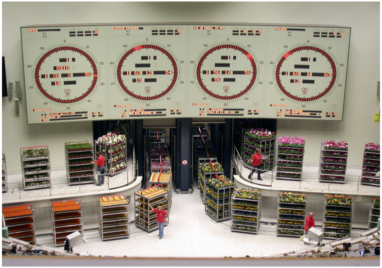
- In a single-item auction, maximizing social welfare means awarding the item to the bidder with the highest value.
- Let our allocation rule picks the bidder whose bid is the highest.

English auction

1. Auctioneer starts the bidding at some low "reservation price".
2. Bidders then shout out ascending prices.
   - usually by some minimum increment set by the auctioneer.
3. Auction ends once bidders stop shouting (i.e. no bidder is willing to bid higher).
4. Highest bidder wins the item and pays her bid.

Dutch auction

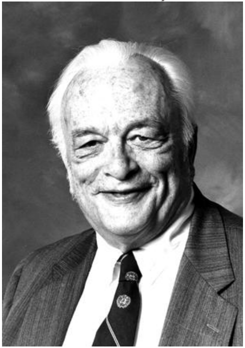
 

1. Auctioneer starts the bidding starts at a high price.
2. The auctioneer lowers the price until someone bids.
3. The item is allocated to the first bidder at that current price.

Sealed-bid auctions

1. Each bidder submits her bid in a sealed "envelope".
2. Bidders do not see each others' bids.
3. Bids are collected by the auctioneer.
4. The auctioneer determines the winner and the price to pay.
   - In the first-price sealed-bid auction, the highest bidder wins and pays her bid.

Vickrey (second-price sealed-bid) Auction



- The highest bidder wins but only pays the second highest bid (or the **reserve price**, whichever is higher).
- Named after William Vickrey.
  - Awarded Nobel Prize in Economics in 1996 along with James A. Mirrlees.
    "for their fundamental contributions to the economic theory of incentives under asymmetric informaiton".

Analysing Auctions

Every Auction is a Game

Consider the first-price auction.

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- Let $b = (b_1, b_2, \ldots, b_n)$ denote the bid profile of all bidders.

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  - virtually any non-negative real, unless there are rules in the auction, e.g. "only place integer bids", or "don't place a bid less than £3" (reserve price).

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- The utility of each bidder i depends on
  - her private valuation or type θ_i, as well as,
  - the outcome (allocation + payment), which in turn depends on b_i and the bids placed by the other bidders, hence on bid profile b.

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- The game induced by the first-price auction (in fact, any auction) is not a strategic-form game.
- This is an example of a Bayesian game.

Model (decision theoretic framework)

- A bidder does not know the private valuations or bids of other players (as assumed so far).
- However, in a **decision theoretic framework**, the bidder is assumed to have beliefs about the bid distribution.
  - Assume that in case of ties, the bidder loses (so the tie-breaking rule is not in favour of our bidder).
  - Let $F(b)$ denote the probability that all other bids are less than b. So $F(b)$ is the probability of winning given bid b.

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  - Let $F(b)$ denote the probability that all other bids are less than b. So $F(b)$ is the probability of winning given bid b.
- Goal is to find bidding strategy $s : \Theta \rightarrow A$, where A is the set of actions or bids, which maximises the expected utility for any valuation $\theta_i \in \Theta$

First-price auction

The utility is given by:

$$u_i(\theta_i) = \begin{cases} \theta_i - b_i & \text{if } \; b_i > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

The expected utility given bid b_i is then:

$$E[u_i(\theta_i)|b_i] = (\theta_i - b_i)F(b_i)$$

Which bid maximises the expected utility?

Exercise: Discrete Bids

$$u_i(\theta_i) = \begin{cases} \theta_i - b_i & \text{if } \; b_i > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

$$E[u_i(\theta_i)|b_i] = (\theta_i - b_i)F(b_i)$$

- Suppose bids are whole numbers, ranging from 0 to 5, and we have the following information:
  $F(0) = 0$, $F(1) = 0.2$, $F(2) = 0.7$, $F(3) = 0.8$, $F(4) = 0.9$, $F(5) = 1.0$.
- Calculate the expected utility when $\theta_i{=}4$ and the agent bids their true valuation.
- Calculate the expected utility when $\theta_i{=}4$ and the agent bids $b_i = 3$.
- What is the optimal bid?

Solution

$$E[u_i(\theta_i)|b_i] = (\theta_i - b_i)F(b_i)$$

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Expected utility when $\theta_i{=}4$ and the agent bids their true valuation:

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Expected utility when $\theta_i{=}4$ and the agent bids their true valuation:
$E[u_i(4)|4] = (4 - 4)F(4) = 0.$

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$E[u_i(4)|3] = (4 - 3)F(3) = 0.8.$

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Expected utility when θ_i=4 and the agent bids $b_i = 3$:
$E[u_i(4)|3] = (4-3)F(3) = 0.8.$

What is the optimal bid? To compute the optimal bid we need to compute the expected utility for all possible bids of agent i. It is clear that bidding above valuation at 5 will generate negative expected utility, and bidding at 0 will generate expected utility 0 since $F(0) = 0$.
$E[u_i(4)|2] = (4-2)F(2) = 2 \cdot 0.7 = 1.4.$
$E[u_i(4)|1] = (4-1)F(1) = 3 \cdot 0.2 = 0.6.$
It is therefore clear that bidding 2 generates the highest expected utility and thus 2 is the optimal bid.

Optimal Bid for Continuous Bid Distributions

$$u_i(\theta_i) = \begin{cases} \theta_i - b_i & \text{if } \ b_i > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

$$E[u_i(\theta_i) | b_i] = (\theta_i - b_i) F(b_i)$$

Let $f(b) = \frac{dF(b)}{db}$ denote the first derivative of F (f is called the probability density and F is the corresponding cumulative distribution) .

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For uniform distribution $F \sim U(0, 1)$ this gives (for $b_i \leq 1$):

$$b_i = \theta_i - b_i \leftrightarrow b_i = \frac{1}{2}\theta_i \leftrightarrow s(\theta_i) = \frac{1}{2}\theta_i$$

(no need to know continuous version for exam)

Vickrey Auction: Discrete Bids

The utility is given by:

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The expected utility given bid b_i is then:

$$E[u_i(\theta_i)|b_i] = \theta_i F(b_i) - \sum_{b=0}^{b_i-1} b \cdot f(b)$$

Since according to our tie-breaking rule i loses if tied with the highest opponent bid, to compute the expected payment we have to consider only cases where the highest opponent bid is at most $b_i - 1$.

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Expected utility when θ_i=4 and the agent bids their true valuation:

$E[u_i(4)|4] = 4 \cdot F(4) - \sum_{b=0}^{b=3} b \cdot (F(b+1) - F(b)) =$

$4 \cdot 0.9 - (0 \cdot 0.2 + 1 \cdot 0.5 + 2 \cdot 0.1 + 3 \cdot 0.1) = 3.6 - 1 = 2.6$.

Expected value is 3.6, expected payment is 1 and the expected utility is 2.6.

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What is the optimal bid? To compute the optimal bid we need to compute the expected utility for all possible bids of agent i. It is clear that bidding 0 will generate expected utility 0 since $F(0) = 0$.

$E[u_i(4)|5] = 4 \cdot 1 - (0.5 + 0.2 + 0.3 + 4(1 - 0.9)) = 4 - 1.4 = 2.6.$

$E[u_i(4)|2] = 4 \cdot 0.7 - (0.5) = 2.8 - 0.5 = 2.3.$

$E[u_i(4)|1] = 4 \cdot 0.2 - 0 = 0.8.$

Bidding 4 or 5 generates the highest expected utility and thus 4 and 5 are both optimal bids. So bidding her true valuation is an optimal strategy for bidder i.

Vickrey Auction: Continuous Bids

$$u_i(\theta_i) = \begin{cases} \theta_i - \max_{j \neq i} b_j & \text{if } \; b_i > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

For continuous distributions, f is the probability density as before, and we simply replace the sum by the integral. The expected utility given bid b_i then becomes:

$$E[u_i(\theta_i)|b_i] = \theta_i F(b_i) - \int_{b=0}^{b_i} b \cdot f(b) db$$

The integral is from 0 to b_i (and not $b_i - 1$). The upper bound of the integral should be the largest opponent bid at which i still wins, which would be $b_i - \epsilon$ for infinitely small ϵ. Therefore, under some conditions that we can assume holds for F, we can safely put b_i for the upper bound.

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Note that $s_i(\theta_i) = \theta_i$ is always an optimal outcome (easy to verify that second derivative is negative). What does this mean?

Strategy-proofness

Definition

A strategy is *weakly dominant* if, regardless of what any other players do, the strategy earns a player a utility at least as high as any other strategy.

- In the Vickrey auction, bidding $s_i(\theta_i) = \theta_i$ is a (weakly) dominant strategy
- Such an auction is called
  - truthful,
  - or, equivalently, strategy-proof,
  - or incentive-compatible in dominant strategies

Proof (1/2)

Theorem

Vickrey auction is truthful.

Proof.

Assume that the other bidders bid in some arbitrary way. We show that i maximises his/her utility by bidding truthfully. We break the proof into two cases:

1. By bidding honestly, i wins the auction.
2. By bidding honestly, i loses the auction.

□

Proof (2/2)

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- If i bids lower, s/he will still lose and pay nothing.
- Let b^{win} denote the winning bid. The only way to win the auction is by submitting a bid higher than b^{win}.
- Note that $\theta_i < b^{win}$ or else i would have won the auction in the first place.

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In either case, bidding truthfully maximises i's utility.

So what about the Dutch and the English auctions? How should one behave?

Dutch auction

- Here, strategy $s(\theta_i)$ is the point at which you bid when the clock reaches the value $s(\theta_i)$ (conditional on no-one else having bid earlier).
- What is the optimal strategy here?
- Would the fact that no-one has bid so far influence your optimal strategy during the auction?

First-price sealed-bid auction vs. Dutch auction

- In both auctions, if i is the winner she pays $s(\theta_i)$.
- The amount of available information in both cases is the same (nothing of value is learned until the winner is announced).
- Therefore, the optimal strategies in both Dutch and first-price sealed-bid auctions are identical. These auctions are said to be strategically equivalent.

English auction

- English auction has a more complicated strategy space: a bidder may place several bids throughout the auction, conditioning their new bid on the information revealed (an extensive form game).
- To simplify the representation of a bidder's strategy in English auction, let us interpret the strategy s_i of bidder i as the point at which i should stop bidding (and not bid higher). Note that s_i is not necessarily the same as the last bid placed by bidder i, b_i, but that $b_i \leq s_i$ (no bidder outbids herself).

Vickrey auction vs. English auction

- In English auction it is a dominant strategy for bidders to bid up to (and not beyond) their valuation; i.e. setting $s_i = \theta_i$ is a dominant strategy for each bidder i.
- In Vickrey auction it is a dominant strategy to bid truthfully, i.e. to set $b_i = \theta_i$.
- Bidding your true valuation (setting $b_i = \theta_i$) is not exactly the same as bidding up to, and not beyond, your true valuation (setting $s_i = \theta_i$) but it is closely similar. Therefore we say that these two auctions are **weakly strategically equivalent**.

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Note: This equivalence *only* holds for the independent private value model, and no longer holds if the valuation of an agent is affected by the valuation of other agents (since new information about the valuation of other bidders is learned in this case).

Revisiting Objectives

- Individual rationality: which auctions ensure that agents are not worse off from participating?
- Manipulation: which auctions incentivise truthful behaviour?
- Efficiency: does the item always go to the agent who values it the most?
- Revenue: which auction creates more revenue for the seller?

Decision-theoretic framework not suitable to answer all of these questions. Why not?

Model (Bayes-Nash framework)

- Recall that θ_i is the type of agent i. Let θ_{-i} denote the types of other agents.
- In the independent private value model, a player's utility depends on her type θ_i and on the **actions** (i.e. bids) of all the players (not on their types).
- Each player knows his own type θ_i, but does not know the type of other players θ_{-i}.
- Agents have a prior distribution $Pr(\theta_{-i})$ over other players' type profiles, which is common knowledge.
- Hence this is called a Bayesian game, and the solution concept a Bayes-Nash equilibrium (BNE).

Revenue equivalence

Assume that all bidders are risk-neutral and utility maximisers, and each has an independent private valuation for the single item, drawn from a common cumulative distribution $F(\theta)$ that is strictly increasing and atomless on $[L, H]$. Then any auction mechanism in which

- in equilibrium, the item will be allocated to the agent with the highest valuation, and
- any agent with valuation L has an expected utility of zero;

yields the same expected revenue, and hence results in any bidder with valuation θ making the same expected payment.

The four single-item auctions we discussed all satisfy the above. Hence they are **revenue equivalent**.

Trading Agents

Autonomous agents are used as bots to bid in auctions, such as:

- Online auctions such as eBay
- Financial exchanges (algorithmic trading)
- Betting exchanges
- Online advertising auctions

Advertising Auctions

Sponsored search: allocation of ad space alongside search results.

- Multiple slots on single page sold simultaneously.
- Uses an auction called Generalised Second Price (GSP) Auction.
  - For a while it was thought that GSP is a “proper” generalisation of Vickrey and is strategy-proof; but it has been shown that it is not!
- Bidding and auction fully automated.

Banner Advertising: advertising on regular webpages usually auctioned off.

- Auctions are run by ad exchanges such as DoubleClick and Facebook Exchange (FBX).
- Each *item* is a slot on a webpage.
- Auctioned off individually using the Vickrey auction.

Final Notes

This lecture only covers very simple simplest setting. Extensions include:

- Common value/interdependent types: when an agent's valuation for an item depends on how others value the item.
- Collusion: when bidders form a bidding ring to agree not to bid against one another.
- Combinatorial Auctions: what auction to use when you have multiple resources to allocate?
- Double auctions: settings with multiple sellers and buyers.
- Other issues: e.g. Risk averse bidders, corrupt auctioneers, the winner's curse.

Conclusion

- Auctions are used to allocate resources amongst self interested agents.
- We discussed the 4 main auctions for selling a single item.
- We discussed how to behave if you know the probability distribution of opponent bids.
- The Vickrey and English auction are strategyproof, meaning there is no incentive to manipulate.
- Game theory can be used to compare different auctions and their properties.
- Auctions become more complex when there are more items.