Algorithmic Game Theory
COMP6207

Recap and Q&A on Basics of Game Theory

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Game

- A set of players
- A set of strategies for each player
  - Countable
  - Uncountable
- A set of outcomes
- Game rule: mapping from strategy profiles to outcomes
- Payoff for each player: mapping from outcomes to some some set (we usually assume $\mathbb{R}$)

Two Standard Representations

- Normal Form (a.k.a. Matrix Form, Strategic Form)
  - Players move simultaneously
  - Example: Prisoner's dilemma

- Extensive Form (Sequential games)
  - Players move sequentially
  - Example: Chess

Normal Form (a.k.a. Strategic-form) Game

A tuple (N, A, u) where

- $N = \{1, \ldots, n\}$ is a finite set of agents.
- $A = A_1 \times \ldots \times A_n$, where A_i is a finite set of actions (i.e. pure strategies) available to agent i.
- $u = (u_1, \ldots, u_n)$, where $u_i : A \mapsto \mathbb{R}$ is the utility (a.k.a. payoff) function for player i.

Attention: In Enrico M recordings (COMP6203), S_i is used to denote the set of pure strategies available to player i. In this module, I use

- S_i to refer to the set of all strategies (pure and mixed) available to agent i, and use s_i to denote a (mixed) strategy of agent i, and
- A_i to denote the set of actions (or, pure strategies) available to agent i.

Traveler's Dilemma: Guess the Laptop Game

- Two of you each lost a laptop (identical) and claimed for compensation towards the insurance company.
- Insurance company does not know the actual value of the laptop, only you two know.
- To find out a reasonable compensation amount, the IC separates you two and asks you its value $\in [2, 100]$.
  - If you declare the same number x, each of you will be given x.
  - If you declare different numbers x and y ($x < y$), then you will be given $x + 2$ and $x - 2$ respectively

Normal Form representation of Traveler's Dilemma

	100	99	98	97	⋯	3	2
100	100, 100	97, 101	96, 100	95, 99	⋯	1, 5	0, 4
99	101, 97	99, 99	96, 100	95, 99	⋯	1, 5	0, 4
98	100, 96	100, 96	98, 98	95, 99	⋯	1, 5	0, 4
97	99, 95	99, 95	99, 95	97, 97	⋯	1, 5	0, 4
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮
3	5, 1	5, 1	5, 1	5, 1	⋯	3, 3	0, 4
2	4, 0	4, 0	4, 0	4, 0	⋯	4, 0	2, 2

Strict Dominance

Let s_i and s_i' be two strategies of agent i, and S_{-i} the set of all strategy profiles of the remaining agents.

- s_i strictly dominates s_i' if **for all** $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.
- In this case we say that s_i' is a stricly dominated strategy.

A strategy is stricly dominant for an agent if it strictly dominates any other strategy for that agent.

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Example. How many strictly dominated pure strategies does Player 1 have?

Player 2

Player 1		D	E	F
	A	1, 2	2, 3	0, 3
	B	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Weak Dominance

Let s_i and s'_i be two strategies of agent i, and S_{-i} the set of all strategy profiles of the remaining agents.

- s_i weakly dominates s'_i if **for all** $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$, and for at least one $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.
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Example. How many weakly dominated pure strategies does Player 2 have?

Player 2

		D	E	F
Player 1	A	1 2	2 3	0 3
	B	2 2	2 1	3 2
	C	2 1	0 0	1 0

Very Weak Dominance

Let s_i and s_i' be two strategies of agent i, and S_{-i} the set of all strategy profiles of the remaining agents.

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Very Weak Dominance

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- In this case we say that s_i' is a very weakly dominated strategy.

A strategy is very weakly dominant for an agent if it weakly dominates any other strategy for that agent.

Example. How many very weakly dominated pure strategies does player 2 have?

		Player 2		
		D	*E*	*F*
Player 1	*A*	1, 2	2, 3	0, 3
	B	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Summary: Dominated and dominant strategies

Let s_i and s'_i be two strategies of agent i, and S_{-i} the set of all strategy profiles of the remaining agents.

1. s_i strictly dominates s'_i if for all $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.
2. s_i weakly dominates s'_i if for all $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$, and for at least one $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.
3. s_i very weakly dominates s'_i if for all $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

In all cases above we say that s'_i is a strictly (resp. weakly; very weakly) dominated strategy.

A strategy is strictly (resp. weakly; very weakly) dominant for an agent if it strictly (resp. weakly; very weakly) dominates any other strategy for that agent.

Quiz question 1-5: dominated and dominant strategies

		Player 2		
		D	*E*	*F*
	A	1 2	2 3	0 3
Player 1	*B*	2 2	2 1	3 2
	C	2 1	0 0	1 0

Quiz question 1-5: dominated and dominant strategies

Player 2

		D	*E*	*F*
	A	1, 2	2, 3	0, 3
Player 1	*B*	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Quiz question 1: How many strictly dominated pure strategies does Player 2 have?

Quiz question 1-5: dominated and dominant strategies

		Player 2		
		D	*E*	*F*
	A	1, 2	2, 3	0, 3
Player 1	*B*	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Quiz question 2: How many weakly dominated pure strategies does Player 1 have?

Quiz question 1-5: dominated and dominant strategies

		Player 2		
		D	*E*	*F*
	A	1 2	2 3	0 3
Player 1	*B*	2 2	2 1	3 2
	C	2 1	0 0	1 0

Quiz question 3: How many very weakly dominated pure strategies does Player 1 have?

Quiz question 1-5: dominated and dominant strategies

Player 2

Player 1		*D*	*E*	*F*
	A	1, 2	2, 3	0, 3
	B	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Quiz question 4: How many weakly dominant pure strategies does Player 1 have?

Quiz question 1-5: dominated and dominant strategies

Player 2

Player 1		*D*	*E*	*F*
	A	1, 2	2, 3	0, 3
Player 1	*B*	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Quiz question 5: Write down a weakly dominant pure strategy profile in this game (if any exists).

On Rationality and Common Knowledge

In (traditional) Game Theory we assume that

- all agents are rational, and
- this fact F = "all agents are rational" is *common knowledge*.
  - All agents know F, they all know that they know F, they all know that they all know that they know F, and so on forever.

With this assumption, a rational player will never play strictly or weakly dominated strategies which can then be eliminated from the game.

Traveler's Dilemma: Elimination of Dominated Strategies

	100	99	98	97	⋯	3	2
100	100, 100	97, 101	96, 100	95, 99	⋯	1, 5	0, 4
99	101, 97	99, 99	96, 100	95, 99	⋯	1, 5	0, 4
98	100, 96	100, 96	98, 98	95, 99	⋯	1, 5	0, 4
97	99, 95	99, 95	99, 95	97, 97	⋯	1, 5	0, 4
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮
3	5, 1	5, 1	5, 1	5, 1	⋯	3, 3	0, 4
2	4, 0	4, 0	4, 0	4, 0	⋯	4, 0	2, 2

Traveler's Dilemma: Elimination of Weakly Dominated Strategies

	100	99	98	97	···	3	2
100	100, 100	97, 101	96, 100	95, 99	···	1, 5	0, 4
99	101, 97	99, 99	96, 100	95, 99	···	1, 5	0, 4
98	100, 96	100, 96	98, 98	95, 99	···	1, 5	0, 4
97	99, 95	99, 95	99, 95	97, 97	···	1, 5	0, 4
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮
3	5, 1	5, 1	5, 1	5, 1	···	3, 3	0, 4
2	4, 0	4, 0	4, 0	4, 0	···	4, 0	2, 2

Nash Equilibrium

Let s_{-i} be a strategy vector for all agents not including i. Agent i's strategy s_i is called a **best response** to s_{-i} if and only if

$$u_i(s_i, s_{-i}) = \max_{s_i' \in S_i} s_i', s_{-i}$$

Nash Equilibrium

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$$u_i(s_i, s_{-i}) = \max_{s_i' \in S_i} s_i', s_{-i}$$

A strategy profile $s = (s_1, \ldots, s_n)$ is a **Nash Equilibrium** if s_i is a best response to s_{-i} for every player $i \in N$.

Nash Equilibria and Dominated Strategies

Given a game G let G^* be the game obtained by iterated elimination of **strictly** dominated (pure) strategies.

- The order of elimination does not matter; We always get the same G^*.
- The set of NE of G^* is the same as the set of NE of G.
- So if we end up with a single strategy profile s, then s is the unique NE of G.

Nash Equilibria and Dominated Strategies

Given a game G let G^* be the game obtained by iterated elimination of **strictly** dominated (pure) strategies.

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Given a game G let G^* be the game obtained by iterated elimination of **weakly** dominated (pure) strategies.

- The order of elimination matters.
- The set of NE of G^* is a subset of the set of NE of G.
- The iterated elimination of weakly dominated strategies can result in the elimination of some (if not all!) the NE of the original game.

Guess 2/3 of the Average: Elimination of Weakly Dominated Strategies

Each of you write down a number between 0 and 100. Whoever is closest to 2/3 of the average wins.

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- If everyone writes down 100 the average is going to be 66.66.
- Guessing any number above 66.66 is weakly dominated for every player.

Guess 2/3 of the Average: Elimination of Weakly Dominated Strategies

Each of you write down a number between 0 and 100. Whoever is closest to 2/3 of the average wins.

- If everyone writes down 100 the average is going to be 66.66.
- Guessing any number above 66.66 is weakly dominated for every player.
- Once we eliminated these weakly dominated strategies, then guessing any number above 44.44 is weakly dominated for every player.

Guess 2/3 of the Average: Elimination of Weakly Dominated Strategies

Each of you write down a number between 0 and 100. Whoever is closest to 2/3 of the average wins.

- If everyone writes down 100 the average is going to be 66.66.
- Guessing any number above 66.66 is weakly dominated for every player.
- Once we eliminated these weakly dominated strategies, then guessing any number above 44.44 is weakly dominated for every player.
- By continuing this process, all numbers except 0 will be dominated.
- Writing down 0 is the unique pure strategy NE of this game.