Algorithmic Game Theory
COMP6207

Lecture 9: Optimal Auctions

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Learning Outcomes

By the end of this session, you should be able to

- **Define** optimal auctions.
- **Describe** an optimal (single-item) auction, in terms of its choice rule and payment function.
- **Analyse** optimal auctions.

Optimal auctions
for selling a single item

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- A seller my be willing to risk failing to sell the item even when there is an interested buyer.
- A seller may be willing sometimes to sell to a buyer who didn't make the highest bid.
- Mechanisms which are designed to maximise seller's expected revenue are known as as optimal auctions

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As we have assumed so far in this module

- Independent private valuations (IPVs)
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Optimal auction: maximises seller's expected revenue subject to some form of individual rationality

Example

- 2 bidders, v_i uniformly distributed on $[0, 1]$
- Set reserve price R and then run a second price auction:
  - no sale if both bids below R
  - sale at price R if one bid above reserve price and other below
  - sale at second highest bid if both bids are above reserve

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- Expected revenue $= 2(1-R)R^2 + (1-R)^2 \frac{1+2R}{3}$
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- Maximising: $0 = 2R - 4R^2$, or $R = \frac{1}{2}$

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  - increase price when one bidder has low value other high: happens with probability $1/2$
- Like adding another bidder: increasing competition in the auction.

Designing optimal auctions

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Bidder i's bidder-specific reserve price r_i^* is the value for which $\psi_i(r_i^*) = 0$.

Myerson's theorem

Theorem (Myerson (1981))

The optimal (single-item) auction is a sealed-bid auction in which every agent is asked to declare his valuation. The good is sold to the agent $i = \operatorname{argmax}_i \psi_i(\hat{v}_i)$, *as long as* $\hat{v}_i \geq r_i^*$. *If the good is sold, the winning agent* i *is charged the smallest valuation that she could have declared while still remaining the winner; i.e.*

$$inf\{v_i^* : \psi_i(v_i^*) \geq 0 \; and \; \forall j \neq i, \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$$

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- Declarations are used to compute a virtual (declared) valuation $\psi_i(\hat{v}_i)$ for each agent i.
- The item is sold to the agent i whose virtual valuation is the highest, as long as this value is nonnegative; i.e. $\hat{v}_i$ is no less than her reserve price r_i^*.
- If every agent's virtual valuation is negative, the seller keeps the item and achieves a revenue of zero.
- If the item is sold, the winning agent i pays an amount equal to the smallest valuation that she could have declared while still remaining the winner:
  $inf\{v_i^* : \psi_i(v_i^*) \geq 0 \text{ and } \forall j \neq i, \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$

Analysing optimal auctions

Optimal auction:

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- How should bidders bid?
  - This is a second-price (Vickrey) auction with a reserve price, held in virtual valuation space.
  - Neither the reserve prices nor the virtual valuation transformation depends on the agent's declaration
  - Thus the proof that a Vickrey auction is dominant-strategy truthful applies here as well.

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- What happens in the general case?
  - The virtual valuations also increase weak bidders' bids, making them more competitive.
  - Low bidders can win, paying less.
  - However, bidders with higher expected valuations must bid more aggressively.

Using optimal auctions in practice

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- Trying to attract more bidders may be more important that trying to figure out bidders' valuation distributions in order to run an optimal auction.

Books

- Twenty Lectures on Algorithmic Game Theory, by Tim Roughgarden
- Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations by Yoav Shoham and Kevin Leyton-Brown
  - From now on we will refer to this book as **MAS**
- Algorithmic Game Theory, edited by Noam Nisan, Tim Roughgarden, Eva Tardos,Vijay V. Vazirani
  - From now on we will refer to this book as **AGT**

Further reading/watching

- Read MAS chapter 11.1.8
- Watch Game Theory II - Week 4 (Auctions): video 6

Acknowledgment

Some of the slides in this lecture were based on the slides by Kevin Leyton-Brown.