Algorithmic Game Theory
COMP6207

Lecture 16: **Size vs Stability**

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Learning Outcomes

- By the end of this session, you should be able to
  - ***Describe*** MAX SMTI
  - ***Describe*** *Kiraly's* algorithm as an extension of Gale-Shapley
  - ***Compute*** the stable matching produced by executing *Kiraly's* algorithm on an instance of SMTI

Maximum size Stable Matching

Stable matchings of different sizes

- All stable matchings in a given instance of **SM**, or **SMT**, or **SMI**, are of the same size.
- When both ties and incomplete lists are allowed (i.e. we have an instance of **SMTI**), stable matchings can have different sizes

m_1 : (w_1 w_2) w_1 : m_1 m_2

m_2 : w_1 w_2 : m_1

- A maximum (cardinality) stable matching can be (at most) twice the size of a minimum stable matching [Manlove et al, 2002]

Maximum stable matchings

- Problem of finding a maximum stable matching in an instance of SMTI (MAX SMTI) is NP-hard [Iwama, Manlove et al, 1999], even if (simultaneously):
  - the ties occur on one side only
  - each preference list is either strictly ordered or is a single tie
  - *and*
    - *either* each tie is of length **2** [Manlove et al, 2002]
    - *or* each preference list is of length **≤3** [Irving, Manlove, O'Malley, 2009]

This result implies that MAX HRT is also NP-hard.

- Minimisation problem is NP-hard too, for similar restrictions! [Manlove et al, 2002]

Reminder: computational complexity

- Given two functions f and g, we say $f(n)=O(g(n))$ if there are positive constants c and N such that $f(n) \le c.g(n)$ for all $n \ge N$
- An algorithm for a problem has *time complexity* $O(g(n))$ if its running time f satisfies $f(n)=O(g(n))$ where n is the input size
- An algorithm runs in *polynomial time* if its time complexity is $O(n^k)$ for some constant k, where n is the input size
- A *decision problem* is a problem whose solution is yes or no for any input
- A decision problem belongs to the class **P** if it can be solved by a *polynomial-time algorithm*
- A decision problem belongs to the class **NP** if it can be verified in *polynomial time*
- A decision problem A is **NP**-hard if every other problem in **NP** reduces to A.
- A decision problem A is **NP**-complete if it **NP**-hard and it belongs to **NP**.
- If a decision problem is **NP**-complete it has no polynomial-time algorithm unless **P=NP**

Reminder: approximation algorithms

- An *optimisation problem* is a problem that involves maximising or minimising (subject to a suitable measure) over a set of feasible solutions for a given instance
  - e.g., colour a graph using as few colours as possible
- If an optimisation problem is **NP**-hard it has no polynomial-time algorithm unless **P=NP**
- An *approximation algorithm* A for an optimisation problem is a polynomial-time algorithm that produces a feasible solution $A(I)$ for any instance I.
- A has *performance guarantee* c, for some $c>1$ if
  - $|A(I)| \leq c.opt(I)$ for any instance I (in the case of a minimisation problem)
  - $|A(I)| \geq (1/c).opt(I)$ for any instance I (in the case of a maximisation problem)

  where $opt(I)$ is the measure of an optimal solution and $|A(I)|$ the size of the solution produced by A.

  ➢ We say that A is a ***c-approximation*** algorithm for this problem.

MAX HRT: approximability

- MAX HRT is not approximable within **33/29** unless P=NP, even if each hospital has capacity **1** [Yanagisawa, 2007]
- MAX HRT is not approximable within **4/3-ε** assuming the *Unique Games Conjecture* (UGC) [Yanagisawa, 2007]
- Trivial **2**-approximation algorithm for MAX HRT
- Succession of papers gave improvements, culminating in:
  - **MAX HRT is approximable within 3/2** [McDermid, 2009; Király, 2012; Paluch 2012]
- Experimental comparison of approximation algorithms and heuristics for MAX HRT and MAX SMTI [Irving and Manlove, 2009; Podhradský 2010]

Kiraly's $\frac{3}{2}$-approximation for MAX SMTI (man-oriented version)

- An extension of Gale-Shapely
- When a man is rejected by all women in his list, he is given a *second chance*

- For a man **m**, and for two women $\mathbf{w_i}$ and $\mathbf{w_j}$, we say that **m** prefers $\mathbf{w_i}$ to $\mathbf{w_j}$ if
  1. either he prefers $\mathbf{w_i}$ in the usual sense
  2. or he is indifferent between the two, $\mathbf{w_j}$ is engaged and $\mathbf{w_i}$ is free.

- For a woman **w**, and for two men $\mathbf{m_i}$ and $\mathbf{m_j}$, we say that **w** prefers $\mathbf{m_i}$ to $\mathbf{m_j}$ if
  1. either she prefers $\mathbf{m_i}$ in the usual sense
  2. or she is indifferent between the two, $\mathbf{m_i}$ has a second chance (he is proposing to the women in his list for the 2nd time) and $\mathbf{m_j}$ does not (he is proposing to the women in his list for the 1st time).

Kiraly's $\frac{3}{2}$-approximation for MAX SMTI (man-oriented version) contd.

- An unassigned man proposes to his most-preferred woman on his list, according to his new definition of *prefers*

- An unassigned woman always accepts a proposal (as was the case in GS)

- An assigned woman **w** accepts a new proposal from a man **m**, and rejects her current partner $\mathbf{m_k}$, if
  1. either she prefers **m** to her current partner, according to her new definition of *prefers*
  2. or her current partner prefers some woman to **w**, again according to his new definition of *prefers*. (In this case we call **w** *precarious*.)

- When a woman **w** rejects a man **m**, and she is not precarious, **m** and **w** are deleted from each others' lists

SMTI: stable matching (1)

m_1 : (w_1 w_2)

m_2 : w_1

m_3 : w_3 w_4

m_4 : w_3

w_1 : (m_1 m_2)

w_2 : m_1

w_3 : (m_3 m_4)

w_4 : m_3

$M = \{(m_1, w_1), (m_3, w_3)\}$ (size **2**)

SMTI: stable matching (2)

m_1 : (w_1 w_2)

m_2 : w_1

m_3 : w_3 w_4

m_4 : w_3

w_1 : (m_1 m_2)

w_2 : m_1

w_3 : (m_3 m_4)

w_4 : m_3

$M = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}$ (size **4**)

Example: Kiraly's algorithm

m_1 : ($\cancel{w_1}$ w_2)

m_2 : w_1

m_3 : $\cancel{w_3}$ w_4

m_4 : $\cancel{w_3}$

w_1 : ($\cancel{m_1}$ m_2)

w_2 : m_1

w_3 : ($\cancel{m_3}$ $\cancel{m_4}$)

w_4 : m_3

- w_1 is precarious: her current partner m_1 prefers another woman, w_2, according to his new definition of prefers.
- w_3 is not precarious and is indifferent between m_3 and m_4, even according to her new definition of prefers.
- m_4 is given a second chance.
- w_3 prefers m_4 to m_3, according to her new definition of prefers.
  - w_3 and w_3 are deleted from each others' lists

Example: Kiraly's algorithm

m_1 : (w_1 w_2)

m_2 : w_1

m_3 : w_3 w_4

m_4 : w_3

w_1 : (m_1 m_2)

w_2 : m_1

w_3 : (m_3 m_4)

w_4 : m_3

$M = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}$ (size **4**)

Quiz: (man-oriented) Kiraly

m_1 : (w_2 w_3)

m_2 : (w_1 w_2 w_4)

m_3 : w_1

m_4 : w_2

w_1 : m_2 m_3

w_2 : (m_1 m_2 m_4)

w_3 : m_1

w_4 : m_2

Kiraly's short summary

Preferences

- For a man m, and for two women w_i and w_j , we say that m prefers w_i to w_j if
  1. either he prefers w_i in the usual sense
  2. or he is indifferent between the two, w_j is engaged and w_i is free.

- For a woman w, and for two men m_i and m_j , we say that w prefers m_i to m_j if
  1. either she prefers m_i in the usual sense
  2. or she is indifferent between the two, m_i has a second chance (he is proposing to the women in his list for the 2nd time) and m_j does not (he is proposing to the women in his list for the 1st time).

Proposals and rejections

- An assigned woman w accepts a new proposal from a man m, and rejects her current partner m_k , if
  1. either she prefers m to her current partner, according to her new definition of *prefers*
  2. or her current partner prefers some woman to w, again according to his new definition of *prefers*. (In this case we call w *precarious*.)

- When a woman w rejects a man m, and she is not precarious, m and w are deleted from each others' lists

DS truthfulness

- Is Kiraly's algorithm DS truthful?
  - No. (Recall Roth's impossibility theorem)
- Is the man-oriented Kiraly DS truthful for men?
  - No. (Exercise: prove this; a simple example works)
- If not, can we achieve 3/2 approximation ratio with another mechanism that is DS for men?
  - No
- If not, can we achieve 3/2 approximation ratio with another mechanism that is DS for men and ties are only on one side of the market?
  - No if ties are on women's side.
  - Yes if ties are on men's side.

Strategy-Proof Approximation Algorithms for the Stable Marriage Problem with Ties and Incomplete Lists, by K. Hamada, S. Miyazaki, H. Yanagisawa, **2019**

Other important interesting problems

``Almost stable'' matchings

- Sometimes matching more people is very import.
- A small number of blockings pairs could be tolerated if it is possible to find a larger matching.

MAX SIZE MIN BP SMI is the problem of finding a matching, out of all maximum cardinality matchings, which has the minimum number of blocking pairs, given an instance of **SMI**.

[Biro, Manlove and Mittal, 2010]

- is NP-hard even if every preference list is of length ≤ 3
- not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P=NP
- Solvable in polynomial time if each woman's list is of length ≤ 2

And more problems

- Stable Marriage problem with Forbidden pairs and/or Forced pairs
- Balanced stable matchings
- Stronger forms of stability when ties are allowed
  - Strong stability
  - Super stability
- Social stability
- …..

Acknowledgement

Some of the slides in this lecture were based on the slides by **David Manlove**.

Book

- Algorithmics of Matching under Preferences
  by David F. Manlove.