# COMP6207
# Algorithmic Game Theory

## Lecture 21 Fair Division

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# Fair division

- Allocate resource in a "fair" way.
- Recourses: Land, time, computer memory, etc.
- E.g., cake-cutting: one cake, $n$ agents
- Different agent prefers different part of the cake
- Objective: cut the cake into $n$ pieces, give every person a piece of cake, in some *fair* manner.
  - Connected pieces
  - Fragile pieces

# Outline

- How to represent agents' utilities?
  - Different agent prefers different part of the cake
- What is fair?
- How to achieve that?
- Does a fair allocation always exist?
- How to find a fair allocation?
- How many cuts do we need?
- Game theory in fair allocation

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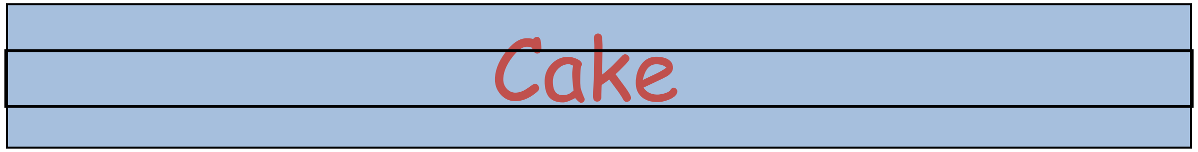
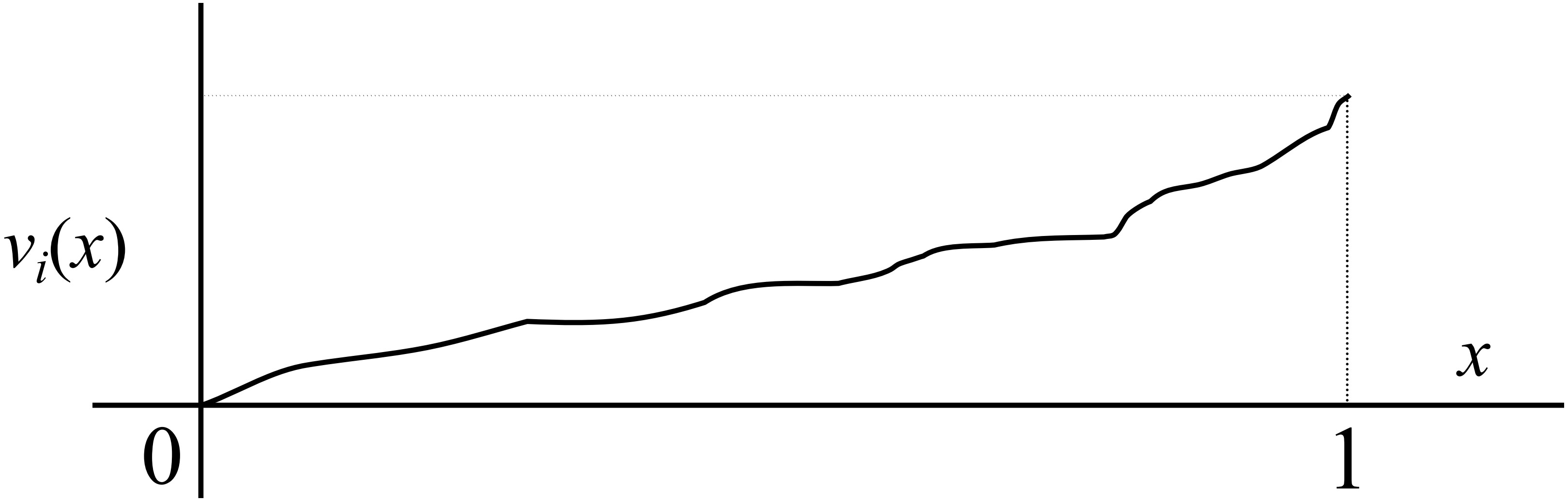
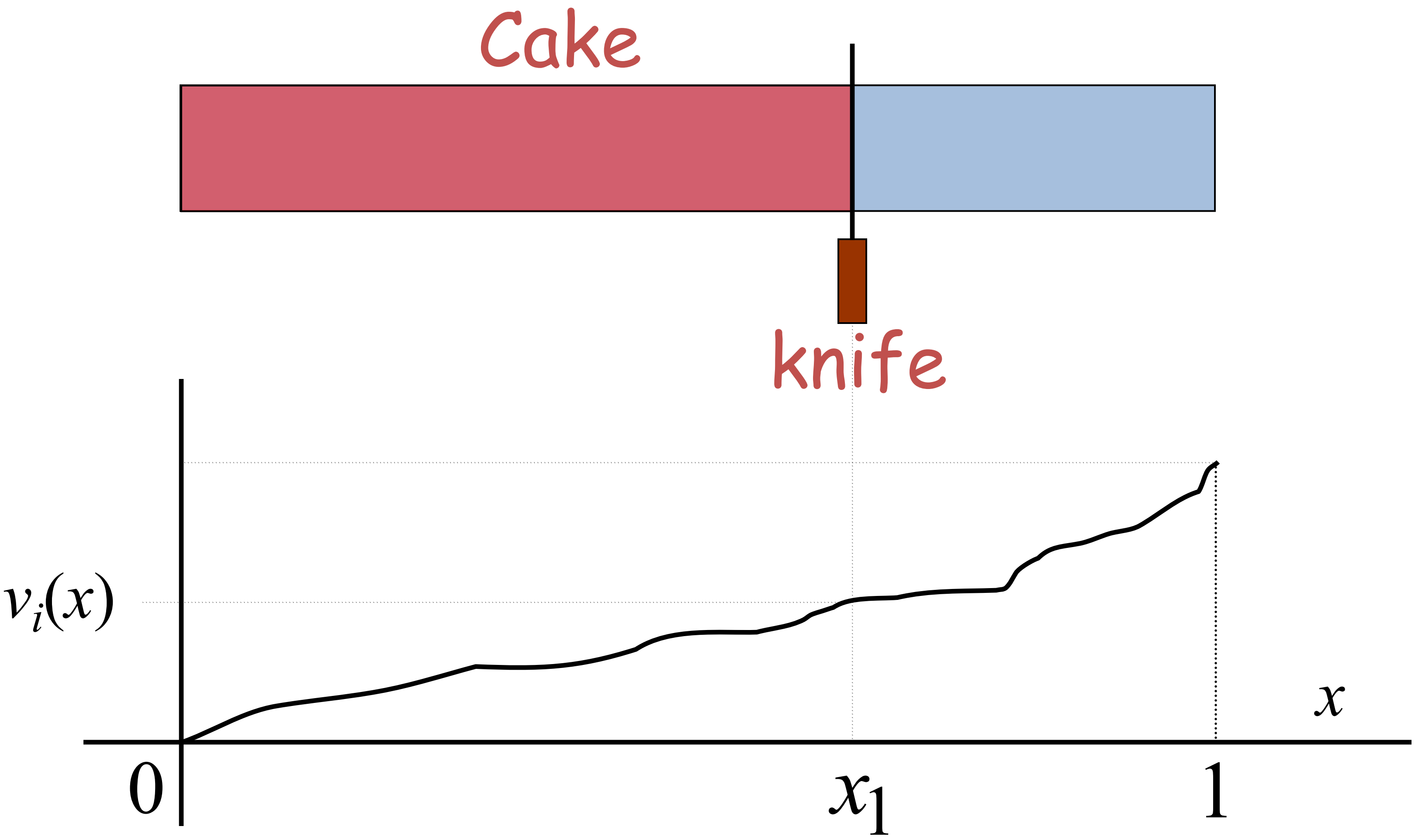


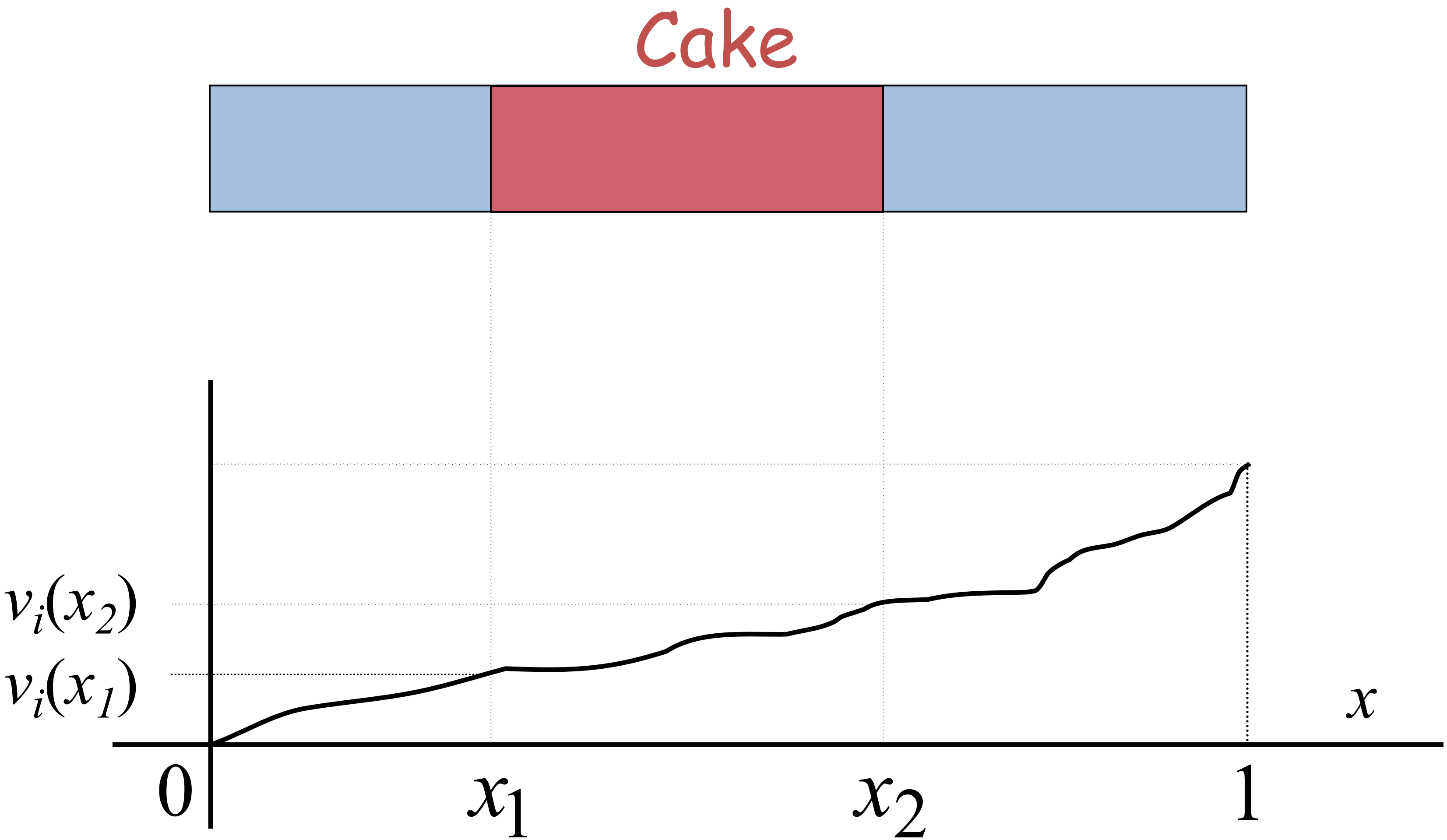

$u_i(S_i)$: utility of agent $i$ over segment $S_i$




$$u_i([0,1]) = \int_0^1 v_i(x)\,dx = 1$$




Value of piece $[0, x_1]$: $u_i = \int_0^{x_1} v_i(x)\, dx$




Value of piece $[x_1, x_2]$: $u_i(x_2) - u_i(x_1) = \int_{x_1}^{x_2} v_i(x)\, dx$

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# Fairness criteria

- Proportionality
  - everyone gets at least $1/n$ of the entire cake, according to its own opinion.
  - $u_i(S_i) \geq 1/n$ for any agent $i$
- Envy-freeness
  - Everyone gets the largest piece of cake amongst $n$ pieces;
  - Or equivalently, nobody wants to switch its own cake with others' cake.
  - $u_i(S_i) \geq u_i(S_j)$ for any agent $i, j$
- Equitability:
  - Everyone gets exactly the same utility as others, according to their own opinion.
  - $u_i(S_i) = u_j(S_j)$ for any agent $i, j$

# Envy-freeness implies proportionality

- For any number of agents, envy-freeness implies proportionality when all of the cake is allocated.
  - *proof*: *exercise*.
- When there are only two agents, proportionality is equivalent to envy-freeness
  - *proof*: *exercise*.

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# Envy-free protocol for $n$=2

- I Cut, You Choose

Claim: The protocol guarantees an envy-free division (and also proportional)

- How many cuts do we need?
  - In Cut-and-choose: just one cut → optimal
- Are portions continuous?
  - Yes

# Division protocol for $n \geq 3$

- Discrete protocols:
  - Steinhaus 1943
  - Selfridge-Conway 1960, 1993
  - Brams-Taylor, 1995 (unbounded cuts)
  - Aziz-Mackenzie 2015 (extendable to any *n,* bounded cuts)

- Continuous protocols (moving-knife)
  - Stromquist 1980

- Steinhaus is proportional only, all other are envy-free
- We care about #cuts

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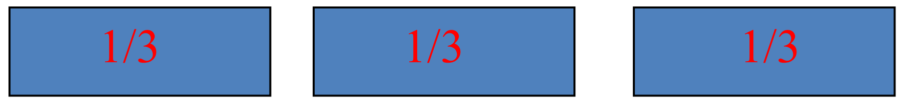
# Proportional protocol for $n = 3$ Steinhaus, 1943

- Agent 1 cuts the cake to three even pieces




- If there are two parts that worth at least ⅓ to agent 2, then:
    - Agents select pieces in the order 3,2,1
    - Otherwise, agent 2 marks the two small pieces as “bad”

# Proportional protocol for $n = 3$ Steinhaus, 1943

- (similarly) If there are two pieces that worth at least ⅓ to agent 3, then
  - Agents select pieces in the order 2,3,1
  - Otherwise agent 3 marks two small pieces as “bad”

- (if players 2 & 3 marked bad pieces) There is one piece marked twice
  - Agent 1 takes that piece
  - Other two pieces are merged to one portion, and agents 2,3 split it using cut-and-choose.

# Envy-free protocol for $n = 3$
# Selfridge-Conway protocol

- Alice cuts the cake into three equal pieces
- Bob trims the largest piece to be equal to the 2nd largest (according to his own value)
- Charlie chooses one piece first, then Bob, then Alice
  - If Charlie does not choose the trimmed piece, then Bob has to choose it (to make sure Alice does not get the trimmed piece)
- Call the person (must be either Bob or Charlie) who choose the trimmed piece T; call the other one NT
- Ask NT to cut the remaining piece into three equal pieces
- T, Alice, and NT choose in order.
- Q1: What is the high-level idea of this protocol?
- Q2: Prove envy-freeness.

# Strategic aspect of cake-cutting

- Design deterministic and randomized cake cutting mechanisms that are *truthful* and *fair* under different assumptions with respect to the valuation functions of the agents.
  – Chen, Lai, Parkes, Procaccia, Truth, justice, and cake cutting. Games and Economic Behavior 77(1): 284-297 (2013)

# Strategic aspect of cake-cutting

- Assumption: additive utilities
- Deterministic algorithms:
  - Theorem: assume that the agents have piecewise uniform valuations, then there is a deterministic algo that is truthful, proportional, envy-free, and polynomial-time.
    - Relates to Probabilistic Serial
- Randomized algorithms:
  - Theorem: assume that the agents have piecewise uniform valuations, then there is a randomized algo that is *truthful-in-expectation*, *universally EF* and *universally proportional*.