Algorithmic Game Theory
COMP6207

Lecture 5: Vickrey-Clarks-Grove (VCG) Mechanism

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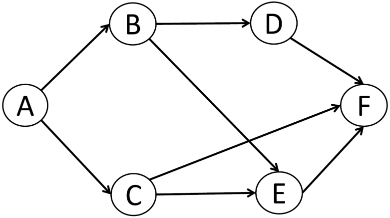
Learning Outcomes

By the end of this and next lecture, you should be able to

- **Define** *VCG mechanism* and the family of *Groves mechanisms*.
- **Apply** *VCG mechanism* to different settings and compute the outcome and agents' payments.
- **Describe** what properties does *VCG mechanism* have, and **prove** that VCG does have these properties.
- **Outline** the limitations a mechanism designer faces when requiring all of the properties defined in this lecture (except perhaps tractability). That is, describe the relevant theorems and explain what they mean and imply.

Fun Game

Fun game: Selfish routing



- A network with 6 vertices and 8 edges.
- Each edge has a cost and there is an agent associated with each edge.
- 8 students play as agents; others act as mediators.
- Agents' utility functions: $u_i =$ payment - cost if your edge is chosen; 0 otherwise.
- Mediators: find a path from A to F at the lowest cost you can.
- Agents: agree to be paid whatever you like; claim whatever you like; don't show your paper to anyone.

Quasilinear mechanism design

Direct quasilinear mechanism with IPVs

Setting:

- n strategic agents
- A finite set X of choices
- An agent's valuation for choice $x \in X$ is $v_i(x) = u_i(x, \theta_i)$
  - the maximum amount i is willing to pay for x to be chosen

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In a direct mechanism:

- Agents are asked to declare $v_i(x)$ for each $x \in X$
- Let $\hat{v}_i$ denote the valuation that agent i declares to the mechanism
  - $\hat{v}_i$ may be different from her true valuation v_i
  - Let $\hat{v}$ denote the declared valuation profile of all agents, and $\hat{v}_{-i}$ the declared valuation profile of all agents except i.
- The mechanism maps $\hat{v}$ to a choice $x \in X$ and a payment for each agent

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Often when it is understood we are in a quasilinear setting, $x \in X$ is referred to as an **outcome**.

Properties

Efficiency

Definition (Efficiency)

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- Efficiency is also called economic efficiency to distinguish from other (e.g. computational) notions.
- Note that efficiency is defined in terms of true (not declared) valuations.

(Dominant-strategy) Truthfulness

Definition (Dominant-strategy truthful mechanism)

A direct quasilinear mechanism is dominant-strategy truthful (truthful) if for each agent i, declaring $\hat{v}_i = v_i$ maximises i's utility, no matter what the other agents declare:

$$u_i(\chi(v_i, v_{-i}), p_i(v_i, v_{-i})) \geq u_i(\chi(\hat{v}_i, v_{-i}), p_i(\hat{v}_i, v_{-i})), \ \forall \hat{v}_i \ \forall v_{-i}$$

- Our definition before, adapted for the quasilinear setting.

Individual Rationality

Definition (Ex post individual rationality)

A mechanism is ex post individually rational if in equilibrium, the utility of each agent is at least 0.

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A mechanism is ex post individually rational if in equilibrium, the utility of each agent is at least 0.

- So no agent loses by participating in the mechanism.
- There is also the notion of *ex interim individual rationality*.

Budget Balance

Definition (Budget balance)

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- Regardless of the agents' types, the mechanism collects and disburses the same amount of money from and to the agents.
- There are also *weak* and *ex ante* variants.

Tractability

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- The mechanism is computationally feasible.

Some other Properties (that we will discuss in future)

- Revenue maximisation
- Fairness

Efficiency + Dominant-Strategy Truthfulness

- Recall that Vickrey (second-price) auction is both efficient (social-welfare maximising) and dominant-strategy truthful.
- Is there a mechanism that is both efficient and dominant-strategy truthful in general quasilinear settings with IPVs?

Efficiency + Dominant-Strategy Truthfulness

- Recall that Vickrey (second-price) auction is both efficient (social-welfare maximising) and dominant-strategy truthful.

- Is there a mechanism that is both efficient and dominant-strategy truthful in general quasilinear settings with IPVs? Yes.

- There is a general class of mechanism called **Groves mechanisms** that are both efficient and dominant-strategy truthful.
- In fact, in settings where agents may have unrestricted quasilinear utilities, Groves mechanisms are the only mechanisms that are both efficient and dominant-strategy truthful. [Theorem by Green-Laffont]

Groves mechanisms

Definition (Groves mechanisms)

Any direct quasilinear mechanism (χ, p) where

$$\chi(\hat{v}) = \operatorname*{argmax}_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

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- VCG is a specific mechanism within the class of Groves mechanism. (The most famous mechanism within this class.)
- Some people refer to Groves mechanisms as VCG mechanisms.
- Vickrey auction is a special case of VCG, and hence VCG is sometimes known as *generalised vickrely auction*.

Groves discussion

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- We will prove later in the lecture that Groves mechanisms are dominant-strategy truthful.

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- The choice rule should not come as a surprise.
  - Groves mechanisms are truthful and efficient.
- So what's going on with the payment rule?
  - agent i must pay some amount $h_i(\hat{v}_{-i})$ that doesn't depend on his own declared valuation.
  - agent i is paid the sum of others' declared valuations for the chosen choice.

Vickrey-Clarks-Groves (VCG) mechanism

The Clarke tax sets the h_i term in the definition of the Groves mechanism as:

$$h_i(\hat{v}_{-i}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i}))$$

Definition (Vickrey-Clarke-Groves (VCG) mechanism)

The VCG mechanism is a direct quasilinear mechanism (χ, p) where

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- VCG is also called pivotal mechanism.

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- **Question**: What is u_i in terms of social welfare?

VCG discussion

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Examples

VCG Example: combinatorial auction example

- two goods A and B
- n agents (here bidders)
- Set of outcomes X has $(n+1)^2$ elements: who gets A (if anyone) and who gets B (if anyone)

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- Therefore, we can write an agent i's valuation by specifying only 3 values: $v_i(A)$, $v_i(B)$ and $v_i(AB)$.

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- VCG chooses who gets what item(s) and how much each agent pays.

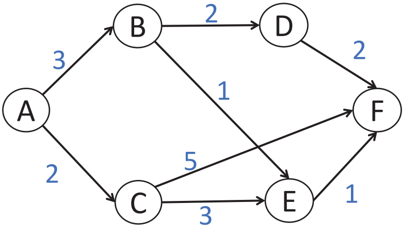
Combinatorial auction example contd.

What choice does VCG pick (i.e. who gets A and who gets B), and what is the payment for each agent?

Combinatorial auction setting with two agents

- $v_1(A) = 3$, $v_1(B) = 2$, $v_1(AB) = 6$
- $v_2(A) = 1$, $v_2(B) = 4$, $v_2(AB) = 4$

VCG Example: Selfish routing example



- The number on each edge is the cost of transporting along that edge.
- Each edge is owned by a different agent and the costs are private information of the agents.
- Goal: Find the shortest (least-cost) path from from A to F.
- The set of outcomes include all possible paths from from A to F.
- Note that numbers on edges are costs, not benefits.
  - If we select a path that crosses an edge of cost c, its owner is incurring a cost of c which means his value for this path is $-c$.
  - If our path doesn't cross agent i's edge, his value for the path is 0.

What path does VCG pick? break ties lexicographically

What is the payment for each agent?

Books

- Twenty Lectures on Algorithmic Game Theory, by Tim Roughgarden
- Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations by Yoav Shoham and Kevin Leyton-Brown
  - From now on we will refer to this book as **MAS**
- Algorithmic Game Theory, edited by Noam Nisan, Tim Roughgarden, Eva Tardos,Vijay V. Vazirani
  - From now on we will refer to this book as **AGT**

Further reading/watching

For further introduction to Mechanism Design, Groves mechanisms and VCG

- Read MAS chapters 10.1, 10.2, 10.3, 10.4.1-10.4.6 (we haven't covered some of the material in these sections, of which we will cover some in future lectures)
- Read AGT Chapters 9.1, 9.2, 9.3 (note that MAS and AGT sometimes use different notations and definitions for the same concepts)
- Watch Game Theory II - Week 3 (VCG): 6 videos

Acknowledgment

Some of the slides in this lecture were based on the slides by Jie Zhang and Kevin Leyton-Brown.