STRATEGIC-FORM GAMES (PURE STRATEGIES)

COMP6203 - Intelligent Agents

Enrico Marchioni
e.marchioni@soton.ac.uk
Building 32 - Room 4019

INTRODUCTION

- We are going to talk about strategic-form noncooperative games.
- This is the best-known class of games.
- In non-cooperative games, players act alone, do not make joint decision, and pursue their own goals.
- We always assume our games have finitely many players $P_1, \ldots, P_n$.
- In strategic-form games, each player P_i has different choices called strategies.
- Players choose simultaneously one among their strategies and their combined choices determine different outcomes.
- Each player will have their own preference over these outcomes which will be represented by a utility function.

- Two criminals (Prisoner 1 and Prisoner 2) are arrested.
- Each prisoner is in solitary confinement with no means of communicating with the other.
- The prosecutors lack sufficient evidence to convict the pair on the principal charge, but they have enough to convict both on a lesser charge.
- Simultaneously, the prosecutors offer each prisoner a bargain.
- Each prisoner is given the opportunity either to betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent.

- Each prisoner has 2 choice: betray their fellow criminal and confess (B), or cooperate with the other prisoner (C).
- If they both choose to betray each other, they serve 3 years in prison.
- If they cooperate and stay silent, both of them will only serve one year in prison (on the lesser charge).
- If one betrays the other, and the other stays silent, the traitor will be set free, and the other will serve 6 years.

		Prisoner 2	
		C	B
Prisoner 1	C	$1y$, $1y$	$6y$, $0y$
	B	$0y$, $6y$	$3y$, $3y$

- We can represent the Prisoner's Dilemma in matrix form.
- There are 4 possible outcomes, i.e.

$$\Omega = \{0y, 1y, 3y, 6y\}$$

- We can define a preference relation $\succ_i$ for each player P_i such that

$$0y \succeq_i 1y \succeq_i 3y \succeq_i 6y.$$

		Prisoner 2	
		C	B
Prisoner 1	C	8, 8	0, 10
	B	10, 0	5, 5

- Each $\succ_i$ can be represented in terms of a utility function u_i.
- For instance

$$u_i(0y) = 10, \quad u_i(1y) = 8, \quad u_i(3y) = 5, \quad u_i(6y) = 0$$

- We can represent the game in matrix form with the utility functions representing the players' preferences.

STRATEGIC-FORM GAMES

Definition

A **strategic-form game** is a tuple

$$\langle N, S_1, \dots, S_n, u_1, \dots, u_n \rangle$$

where

- $N = \{1, \dots, n\}$ is a finite set of players
- S_i is a finite set of strategies for each player i
- $u_i = S_1 \times \cdots \times S_n \to \mathbb{R}$ is a utility function for player i

- Each $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ is called a **strategy profile** or **strategy combination**.
- Strategy profiles are also denoted by

$$(s_i, s_{-i})$$

  to highlight the strategy of player i.
- s_{-i} denotes the strategy combination without player i,

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

- S_{-i} is the set of all strategy combinations of the form s_{-i}, i.e. excluding the strategies of player i.

- We assume that players are rational decision-makers and have complete and common knowledge about each other strategies, utilities and their rationality.
- We are not interested in how players play a game (i.e. descriptive, empirical interpretations).
- We are not interested in how players should play a game (i.e. normative interpretation)
- We are interested in trying to predict what will happen under the above assumptions (i.e. theoretical interpretation).
- So, if players act rationally, what outcome will they choose?
- We answer this question by defining solution concepts, i.e. criteria that will allow us to predict the solution of a game under the assumptions we make about the players' behaviour.

Dominated Strategies

Strictly Dominated Strategies

		Prisoner 2	
		C	B
Prisoner 1	C	8, 8	0, 10
	B	10, 0	5, 5

- The outcome for a player will always depend on the choice of others, but there are situations where one player can make independent choices that will always yield better outcomes.
- For instance, Prisoner 1 will always get a higher utility choosing B (betrayal) over C (cooperation), no matter what Prisoner 2 chooses.
- In this case we say that, for Prisoner 1, C is **strictly dominated** by B.

Definition

A strategy s_i of player i is **strictly dominated** if there exists another strategy s_i' of player i such that for each strategy vector $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}).$$

- In this case, we say that s_i is strictly dominated by s_i'.
- We have assumed that all players are rational and also know about each other's rationality.
- We can then assume that rational players will never play strictly dominated strategies, which can then be eliminated from the game.

		Prisoner 2	
		C	B
Prisoner 1	C	8, 8	0, 10
	B	10, 0	5, 5

- For Prisoner 1, B strictly dominates C
- Also, for Prisoner 2, B strictly dominates C
- So, under our assumptions we can conclude that the outcome of the game will be the strategy combination (B, B)
- This process is called **iterated elimination of strictly dominated strategies**.
- Whenever we can eliminate strictly dominated strategies, the result is aways independent of the order of elimination.

Player 2

		C	*D*
Player 1	*A*	1, 2	2, 3
	B	2, 2	2, 0

- The problem is that not all games have strictly dominated strategies, and so, we cannot always reach an outcome by elimination.
- The above game has no strictly dominated strategies.
- However, there are strategies that are at least as good as others for some players.
- If Player 1 selects B, the outcome will be as good as selecting A, if not better.

Definition

A strategy s_i of player i is **weakly dominated** if there exists another strategy s_i' of player i satisfying the following two conditions:

1. For every strategy vector $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})$$

2. There exists a strategy vector $t_{-i} \in S_{-i}$ of the other players such that

$$u_i(s_i, t_{-i}) < u_i(s_i', t_{-i})$$

- In this case, we say that s_i is weakly dominated by s_i'
- We can then assume that rational players will never play weakly dominated strategies, which can then be eliminated from the game.

Player 2

		D	E	F
	A	1, 2	2, 3	0, 3
Player 1	B	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

- Similar to strictly dominated strategies, elimination of weakly dominated strategies cannot always be performed.
- In addition, different from strictly dominated strategies, the order of elimination does matter and we can get different results.

		Player 2		
		D	*E*	*F*
Player 1	*A*	1, 2	2, 3	0, 3
	B	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Player 2

		D	*E*	*F*
Player 1	*A*	1, 2	2, 3	0, 3
	B	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Player 2

		D	*E*	*F*
Player 1	*B*	2 2	2 1	3 2
	C	2 1	0 0	1 0

		Player 2		
		D	*E*	*F*
Player 1	*B*	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Player 2

		D	*E*	
Player 1	*B*	2, 2	2, 1	
	C	2, 1	0, 0	

Player 2

Player 1		*D*	*E*	
	B	2, 2	2, 1	
	C	2, 1	0, 0	

Player 2

		D	*E*	
Player 1	*B*	2, 2	2, 1	

Player 2

		D	*E*	
Player 1	*B*	2, 2	2, 1	

Dominated Strategies

Player 2

		D		
Player 1	*B*	2, 2		

		Player 2		
		D	*E*	*F*
	A	1, 2	2, 3	0, 3
Player 1	*B*	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Player 2

Player 1		D	E	F
	A	1, 2	2, 3	0, 3
	B	2, 2	2, 1	3, 2
	C	2, 1	0, 0	1, 0

Player 2

		D	*E*	*F*
Player 1	*A*	1, 2	2, 3	0, 3
	B	2, 2	2, 1	3, 2

Dominated Strategies

Player 2

		D	*E*	*F*
	A	1, 2	2, 3	0, 3
Player 1	*B*	2, 2	2, 1	3, 2

Player 2

			E	*F*
	A		2 3	0 3
Player 1	*B*		2 1	3 2

Player 2

		E	*F*
	A	2, 3	0, 3
Player 1	*B*	2, 1	3, 2

Dominated Strategies

Player 2

				F
	A			0, 3
Player 1	B			3, 2

Player 2

				F
	A			0 3
Player 1	B			3 2

Player 2

				F
Player 1	*B*			3 2

Pure Nash Equilibria

Player 2

		D	*E*	*F*
Player 1	*A*	0, 6	6, 0	4, 3
	B	6, 0	0, 6	4, 3
	C	3, 3	3, 3	5, 5

- There are games where we cannot perform elimination of dominated strategies.
- We need different solution concepts.
- The most important one is the concept of stability: the Nash Equilibrium.

- To understand the concept of a Nash equilibrium we need to concept of a best response.
- A player's best response to a strategy profile is a choice that gives the player the highest utility.
- Clearly a best response does not have to be unique

Definition

Let s_{-i} be a strategy vector for all the players not including i. Player i's strategy s_i is called a **best response** to s_{-i} if

$$u_i(s_i, s_{-i}) = \max_{s_i' \in S_i} u_i(s_i', s_{-i}).$$

Player 2

Player 1	*D*	*E*	*F*
A	0, 6	6, 0	4, 3
B	6, 0	0, 6	4, 3
C	3, 3	3, 3	5, 5

- For player 1
  - B is a best response to D
  - A is a best response to E
  - C is a best response to F
- For player 2
  - D is a best response to A
  - E is a best response to B
  - F is a best response to C

Player 2

		D	E	F
	A	0, 6	6, 0	4, 3
Player 1	B	6, 0	0, 6	4, 3
	C	3, 3	3, 3	5, 5

- The strategy combination (C, F) is such that the strategies are best response to each other.
- If players select this combination, none of them will benefit from changing their choice, because they have chosen a best response.
- This is a situation of stability, in fact (C, F) is an example of a Nash equilibrium

Definition

A strategy combination $(s_1, \ldots, s_n)$ is a **Nash equilibrium** if s_i is a best response to s_{-i} for every player $i \in N$.

Determining the existence of a Nash equilibrium for a strategic form game is in logarithmic space
(G. Gottblob, G. Greco, F. Scarcello. Pure Nash equilibria: hard and easy games. *JAIR*, 2005.)

Computing Nash Equilibria

		Player 2		
		D	E	F
	A	0, 6	6, 0	4, 3
Player 1	B	6, 0	0, 6	4, 3
	C	3, 3	3, 3	5, 5

- For each player compute the strategy combination where their strategy is a best response.
- For player 1:
$$\{(B,D),(A,E),(C,F)\}$$
- For player 2:
$$\{(A,D),(B,E),(C,F)\}$$
- Take the intersection of the sets of all players.
- Their intersection (C,F) is a Nash equilibrium

Computing Nash Equilibria

		Player 2		
		D	*E*	*F*
	A	0, 6	6, 0	4, 3
Player 1	*B*	6, 0	0, 6	4, 3
	C	3, 3	3, 3	5, 5

- Alternatively, for each strategy combination, check if any player can increase their utility by deviating.
- If they can't, that's a Nash Equilibrium.

		Player 2	
		C	*D*
Player 1	*A*	10, 10	0, 8
	B	8, 0	7, 7

- Nash equilibria might not be unique.
- Coordination games are examples of games with multiple Nash equilibria
- Equilibria arise when players coordinate on the same strategy.

Matching Pennies

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

- Not all games have Nash equilibria.
- Matching pennies is one example of this.
- Two players toss a penny simultaneously.
- If the outcomes match, player 1 keeps both pennies.
- If the outcomes don't match, it is player 2 who gets to keep both coins.
- To be in a situation where equilibria always exist we will need the concept of a mixed strategy (more on next lectures!).

Prisoner 2

Prisoner 1	C	B
C	5, 5	0, 10
B	10, 0	8, 8

- Let $(s_1, \ldots, s_n)$ be a strategy profile obtained from iterated elimination of strictly dominated strategies. Then $(s_1, \ldots, s_n)$ is a Nash equilibrium.
- Moreover, $(s_1, \ldots, s_n)$ is the unique equilibrium of the game.
- Iterated elimination of strictly dominated strategies does not eliminate equilibria from the game.

P2

		L	R
P1	T	0, 0	2, 1
	B	3, 2	1, 2

- Given a game G, let $G^{\star}$ be the game obtained by iterated elimination of weakly dominated strategies.
- The set of equilibria of $G^{\star}$ is a subset of the set of equilibria of G
- This means that iterated elimination of weakly dominated strategies can result in the elimination of some (if not all!) the equilibria of the original game.

Iterated Elimination and Nash Equilibria

P2

P1		L	R
	T	0, 0	2, 1
	B	3, 2	1, 2

- Equilibria: (T, R) and (B, L).
- L is weakly dominated by R and, after eliminating L, B is weakly dominated by T.
- The result of eliminating weakly dominated strategies is (T, R) and equilibrium (B, L) is lost.

- M. Maschler, E. Solan, S. Zamir. *Game Theory*. Cambridge University Press, 2013. [Part of the material in these lectures is taken from Chapter 3 and Chapter 4]
- Y. Shoham, K. Leyton-Brown. *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*. Cambridge University Press, 2009.
- M. J. Osborne. *An Introduction to Game Theory*. Oxford University Press, 2003.