Algorithmic Game Theory COMP6207

Lecture 16: Size vs Stability

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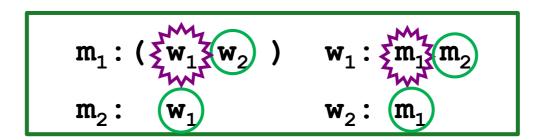
Learning Outcomes

- By the end of this session, you should be able to
 - Describe MAX SMTI
 - Describe Kiraly's algorithm as an extension of Gale-Shapley
 - Compute the stable matching produced by executing Kiraly's algorithm on an instance of SMTI

Maximum size Stable Matching

Stable matchings of different sizes

- All stable matchings in a given instance of SM, or SMT, or SMI, are of the same size.
- When both ties and incomplete lists are allowed (i.e. we have an instance of **SMTI**), stable matchings can have different sizes



 A maximum (cardinality) stable matching can be (at most) twice the size of a minimum stable matching [Manlove et al, 2002]

Maximum stable matchings

- Problem of finding a maximum stable matching in an instance of SMTI (MAX SMTI) is NP-hard [Iwama, Manlove et al, 1999], even if (simultaneously):
 - the ties occur on one side only
 - each preference list is either strictly ordered or is a single tie
 - and
 - either each tie is of length 2 [Manlove et al, 2002]
 - or each preference list is of length ≤3 [Irving, Manlove, O'Malley,
 2009]

This result implies that MAX HRT is also NP-hard.

Minimisation problem is NP-hard too, for similar restrictions!
 [Manlove et al, 2002]

Reminder: computational complexity

- Given two functions f and g, we say f(n) = O(g(n)) if there are positive constants c and N such that $f(n) \le c \cdot g(n)$ for all $n \ge N$
- An algorithm for a problem has time complexity O(g(n)) if its running time f satisfies f(n)=O(g(n)) where n is the input size
- An algorithm runs in *polynomial time* if its time complexity is $O(n^k)$ for some constant k, where n is the input size
- A decision problem is a problem whose solution is yes or no for any input
- A decision problem belongs to the class \mathbf{P} if it can be solved by a *polynomial-time algorithm*
- A decision problem belongs to the class NP if it can be verified in polynomial time
- A decision problem A is NP-hard if every other problem in NP reduces to A.
- A decision problem A is NP-complete if it NP-hard and it belongs to NP.
- If a decision problem is NP-complete it has no polynomial-time algorithm unless P=NP

Reminder: approximation algorithms

- An optimisation problem is a problem that involves maximising or minimising (subject to a suitable measure) over a set of feasible solutions for a given instance
 - e.g., colour a graph using as few colours as possible
- If an optimisation problem is NP-hard it has no polynomial-time algorithm unless P=NP
- An approximation algorithm A for an optimisation problem is a polynomial-time algorithm that produces a feasible solution A(I) for any instance I.
- A has performance guarantee c, for some c>1 if
 - $-|A(I)| \le c.opt(I)$ for any instance I (in the case of a minimisation problem)
 - $-|A(I)| \ge (1/c).opt(I)$ for any instance I (in the case of a maximisation problem)

where opt(I) is the measure of an optimal solution and |A(I)| the size of the solution produced by A.

 \triangleright We say that A is a c-approximation algorithm for this problem.

MAX HRT: approximability

- MAX HRT is not approximable within 33/29 unless P=NP, even if each hospital has capacity 1 [Yanagisawa, 2007]
- MAX HRT is not approximable within 4/3- ϵ assuming the *Unique Games Conjecture* (UGC) [Yanagisawa, 2007]
- Trivial 2-approximation algorithm for MAX HRT
- Succession of papers gave improvements, culminating in:
 - MAX HRT is approximable within 3/2 [McDermid, 2009; Király, 2012;
 Paluch 2012]
- Experimental comparison of approximation algorithms and heuristics for MAX HRT and MAX SMTI [Irving and Manlove, 2009; Podhradský 2010]

Kiraly's $\frac{3}{2}$ -approximation for MAX SMTI (man-oriented version)

- An extension of Gale-Shapely
- When a man is rejected by all women in his list, he is given a second chance
- For a man m, and for two women w_i and w_j , we say that m prefers w_i to w_i if
 - 1. either he prefers w_i in the usual sense
 - 2. or he is indifferent between the two, $\mathbf{w_i}$ is engaged and $\mathbf{w_i}$ is free.
- For a woman w, and for two men m_i and m_j , we say that w prefers m_i to m_i if
 - 1. either she prefers m_i in the usual sense
 - 2. or she is indifferent between the two, m_i has a second chance (he is proposing to the women in his list for the 2^{nd} time) and m_j does not (he is proposing to the women in his list for the 1^{st} time).

Kiraly's $\frac{3}{2}$ -approximation for MAX SMTI (man-oriented version) contd.

- An unassigned man proposes to his most-preferred woman on his list, according to his new definition of prefers
- An unassigned woman always accepts a proposal (as was the case in GS)
- An assigned woman \mathbf{w} accepts a new proposal from a man \mathbf{m} , and rejects her current partner $\mathbf{m}_{\mathbf{k}}$, if
 - 1. either she prefers m to her current partner, according to her new definition of *prefers*
 - 2. or her current partner prefers some woman to w, again according to his new definition of *prefers*. (In this case we call w *precarious*.)
- When a woman w rejects a man m, and she is not precarious,
 m and w are deleted from each others' lists

SMTI: stable matching (1)

```
m_1: (w_1) w_2 ) w_1: (m_1) m_2 ) m_2: w_1  w_2: m_1 m_3: w_3  w_4  w_3: (m_3) m_4 ) m_4: w_3  w_4: m_3
```

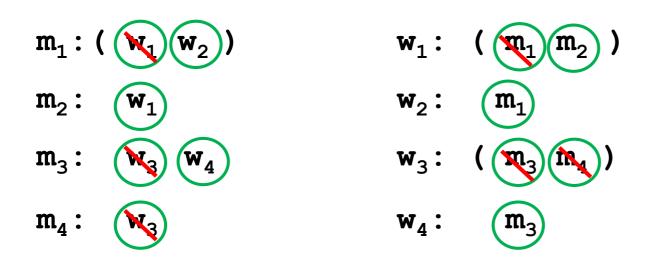
 $M = \{(m_1, w_1), (m_3, w_3)\}$ (size 2)

SMTI: stable matching (2)

```
m_1: (w_1 w_2) w_1: (m_1 m_2) m_2: w_1 w_2: m_1 w_3: (m_3 m_4) m_4: w_3 w_4: m_3
```

$$M = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}$$
 (size 4)

Example: Kiraly's algorithm



- $\mathbf{w_1}$ is precarious: her current partner $\mathbf{m_1}$ prefers another woman, $\mathbf{w_2}$, according to his new definition of prefers.
- w_3 is not precarious and is indifferent between m_3 and m_4 , even according to her new definition of prefers.
- m₄ is given a second chance.
- w_3 prefers m_4 to m_3 , according to her new definition of prefers.
 - w₃ and w₃ are deleted from each others' lists

Example: Kiraly's algorithm

```
m_1: (w_1 w_2) w_1: (m_1 m_2) m_2: w_1 w_2: m_1 m_3: w_3 w_4 w_3: (m_3 m_4) m_4: w_3 w_4: m_3
```

$$M = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}$$
 (size 4)

Quiz: (man-oriented) Kiraly

```
m_1: (w_2 w_3) w_1: m_2 m_3 m_2: (w_1 w_2 w_4) w_2: (m_1 m_2 m_4) m_3: w_1 w_3: m_1 m_4: w_2 w_4: m_2
```

Kiraly's short summary

Preferences

- For a man m, and for two women w_i and w_i , we say that m prefers w_i to w_i if
 - 1. either he prefers w_i in the usual sense
 - 2. or he is indifferent between the two, $\mathbf{w_i}$ is engaged and $\mathbf{w_i}$ is free.
- For a woman w, and for two men m; and m; , we say that w prefers m; to m; if
 - 1. either she prefers m_i in the usual sense
 - 2. or she is indifferent between the two, m_i has a second chance (he is proposing to the women in his list for the 2^{nd} time) and m_j does not (he is proposing to the women in his list for the 1^{st} time).

Proposals and rejections

- An assigned woman w accepts a new proposal from a man m, and rejects her current partner m_{k} , if
 - 1. either she prefers m to her current partner, according to her new definition of *prefers*
 - 2. or her current partner prefers some woman to \mathbf{w} , again according to his new definition of *prefers*. (In this case we call \mathbf{w} *precarious*.)
- When a woman w rejects a man m, and she is not precarious, m and w are deleted from each others' lists

DS truthfulness

- Is Kiraly's algorithm DS truthful?
 - No. (Recall Roth's impossibility theorem)
- Is the man-oriented Kiraly DS truthful for men?
 - No. (Exercise: prove this; a simple example works)
- If not, can we achieve 3/2 approximation ratio with another mechanism that is DS for men?
 - No
- If not, can we achieve 3/2 approximation ratio with another mechanism that is DS for men and ties are only on one side of the market?
 - No if ties are on women's side.
 - Yes if ties are on men's side.

Strategy-Proof Approximation Algorithms for the Stable Marriage Problem with Ties and Incomplete Lists, by K. Hamada, S. Miyazaki, H. Yanagisawa, 2019

Other important interesting problems

"Almost stable" matchings

- Sometimes matching more people is very import.
- A small number of blockings pairs could be tolerated if it is possible to find a larger matching.

MAX SIZE MIN BP SMI is the problem of finding a matching, out of all maximum cardinality matchings, which has the minimum number of blocking pairs, given an instance of SMI.

[Biro, Manlove and Mittal, 2010]

- is NP-hard even if every preference list is of length ≤3
- not approximable within $n^{1-\epsilon}$, for any $\epsilon > 0$, unless P=NP
- Solvable in polynomial time if each woman's list is of length ≤2

And more problems

- Stable Marriage problem with Forbidden pairs and/or Forced pairs
- Balanced stable matchings
- Stronger forms of stability when ties are allowed
 - Strong stability
 - Super stability
- Social stability
-

Acknowledgement

Some of the slides in this lecture were based on the slides by **David Manlove**.

Book

• Algorithmics of Matching under Preferences by David F. Manlove.

