Algorithmic Game Theory
COMP6207

Lectures 11 and 12:

Extensions of Stable Matching problem

Baharak Rastegari
b.rastegari@soton.ac.uk
Electronics and Computer Science
University of Southampton

Learning Outcomes

- By the end of this session, you should be able to
  - ***Describe*** the extensions of SM and implications of allowing ties and/or incomplete lists
  - ***Identify*** blocking pairs in instances of SMTI
  - ***Argue*** about the incentives facing agents participating in SMTI
  - ***Describe*** the relationship between leader-optimal stable matchings and follower-optimal stable matchings

Recap

(Classical) Stable Matching Problem

- **Participants**
  - A set of **n** leaders
  - A set of **n** followers
- **Preferences**
  - Each leader has strict preferences over all followers
  - Each follower has strict preferences over all leaders

Every leader ranks all followers (i.e. finds them **acceptable**) and vice versa.

- **Objective:** find a **one-to-one** **stable** matching
  - (one-to-one) Matching: each leader is paired with at most one follower and vice versa
  - Stable: no **blocking pair**
    - (l, f) is a **blocking pair** iff they both prefer each other to their assigned partners.

Gale-Shapley algorithm

Deferred-acceptance-leader-oriented (leaders, followers, preferences)

```
Assign all leaders and followers to be free; //initial state
While (some leader l is free and hasn’t proposed to every follower)
    f = first follower on l's list to whom l hasn't yet proposed;
     // next: l proposes to f
    If (f is free)
        assign l and f to be engaged; //tentatively matched
    else if (f prefers l to her fiancé l') { //f is engaged
        set l and f to be engaged;
        set l' to be free;
    else  f rejects l; //and l remains free
output the n engaged pairs, who form a stable matching;
```

Gale-Shapley returns a stable matching

Claim. Gale-Shapley algorithm always terminates and returns a stable matching.

Observation 1. Leaders propose to followers in decreasing order of preference

Observation 2. Once a follower is matched up, she never becomes unmatched; only ``trades up''.

Claim. Given an instance of SM, Gale-Shapley returns a **perfect** matching (i.e. everyone is matched).

Important note: all leaders rank all followers and vice versa.

Quiz

Go to vevox.com
Meeting ID: 110-844-851

- Do all executions of leader-oriented **GS** lead to the same stable matching?

A. No, because the algorithm is nondeterministic.
B. No, because an instance can have several stable matchings.
C. Yes, because each instance has a unique stable matching.
D. Yes, even though an instance can have several stable matchings and the algorithm is nondeterministic.

Quiz

- Do all executions of leader-oriented **GS** lead to the same stable matching?

  A. No, because the algorithm is nondeterministic.
  B. No, because an instance can have several stable matchings.
  C. Yes, because each instance has a unique stable matching.
  D. Yes, even though an instance can have several stable matchings and the algorithm is nondeterministic.

Concurrent proposals

- GS is **nondeterministic**: When choosing a free leader (who hasn't proposed to every follower) to propose, the algorithm can choose any such leader *arbitrarily.*

- We can consider a (parallelised) **variant** of GS in which all free leaders (who haven't proposed to every follower) **propose at the same time.**
  - If a follower receives several proposals, she compares them all with her current fiancé (if any), picks the one she most prefers and rejects the rest

All executions of leader-oriented (resp. follower-oriented) Gale-Shapley lead to the same stable matching.

Proof: Corollary 1 in a later slide

Extensions of Stable Matching problem

Reminder: some textbooks refer to the *Stable Matching* problem as the *Stable Marriage* problem.

Stable Matching with Incomplete Lists

Candidates: agents on the other side of the market.

Agents may declare some candidates unacceptable

→ **S**table **M**atching problem with **I**ncomplete lists (**SMI**)

l_1: f_1 f_2	f_1: l_2
l_2: f_2 f_1	f_2: l_1 l_2

An agent prefers to remain unmatched than to get matched to an unacceptable partner.

Matching: an individually rational pairing of leaders and followers.

1. Each leader is paired with at most one follower and vice versa.
2. Each agent finds his/her partner acceptable.

Stable matching: a matching with no blocking pair

Gale-Shapley for SMI

- Almost the same algorithm we have seen before works for the case with incomplete preference lists
- Only small changes required in lines 2 & 4 & 12

```
2  While (some leader l is free and hasn't proposed to every
follower he finds acceptable)

4  If (f is free and finds l acceptable)
5       assign l and f to be engaged;

12 output the engaged pairs, who form a stable matching;
```

Stable Matching with Ties

Agents may be indifferent among several candidates

→ Stable Matching problem with Ties (**SMT**)

$l_1: \ f_1 \ f_2$	$f_1: (l_2 \ l_1)$
$l_2: (f_2 \ f_1)$	$f_2: \ l_1 \ l_2$

- f_1 indifferent between l_1 and l_2
- l_2 indifferent between f_1 and f_2

Blocking pair: when ties are allowed, three different types of blocking pairs can be defined. The straightforward extension of the one you are already familiar with is the following:

- There is no leader-follower pair, each of whom would **strictly** prefer to match with each other rather than their assigned partner.

Stable matching: a matching with no blocking pair

SM with Ties and Incomplete Lists

Both incomplete lists and indifferences are allowed

→ **S**table **M**atching problem with **T**ies and **I**ncomplete lists (**SMTI**)

l_1: f_1	f_1: $(l_2\ l_1)$
l_2: $(f_2\ f_1)$	f_2: $l_1\ l_2$

- Almost the same algorithm for SMI works for SMTI (and thus SMT)
- Only change: break ties in preferences arbitrarily first

Optimal stable matchings

Achievable candidates

- For a given instance, there may be several stable matchings.

l_1:	f_2	f_1	f_3	f_1:	l_1	l_2	l_3
l_2:	f_1	f_2	f_3	f_2:	l_2	l_1	l_3
l_3:	f_1	f_2	f_3	f_3:	l_1	l_3	l_2

- This instance has two stable matchings: green and purple
- Both f_1 and f_2 are achievable to l_1
- Only l_3 is achievable to f_3

Definition. A leader l and a follower f are **achievable** to each other if there exists a stable matching in which l and f are matched.

Leader-optimal matching

Leader-optimal matching: a stable matching in which each leader receives his **best achievable** partner.

Follower-optimal matching: a stable matching in which each follower receives her **best achievable** partner.

Claim. When preferences are **strict**, all executions of leader-oriented GS return the leader-optimal matching.

Corollary 1. When preferences are strict, all executions of leader-oriented GS return the same stable matching.

Corollary 2. When preferences are strict, there exists a leader-optimal stable matching and a follower-optimal stable matching.

Claim. When preferences are **strict**, all executions of leader-oriented GS return the leader-optimal matching.

Proof. **[by contradiction]**

- Suppose a leader is matched with a follower other than best achievable partner
- Leaders propose in decreasing order of preferences

  => some leader is rejected by an achievable partner during GS
- Let l be first such leader and let **f** be the first (and hence best) achievable partner that rejects l
- Let μ be a stable matching where l and **f** are matched.
- When **f** rejects l in GS, she forms (or re-affirms) commitment to a leader, say l' $\Rightarrow$ $\boxed{\text{f prefers } l' \text{ to } l}$
- Let **f'** be the partner of l' in μ
- l' had not been rejected by any achievable partner (including **f'**) at the point when l is rejected by **f**.

  Because this is the first rejection by an achievable partner
- Thus, l' had not yet proposed to **f'** when l' proposed to **f** $\Rightarrow$ $\boxed{l' \text{ prefers f to f'}}$
- Thus **(l',f)** is a blocking pair in μ, a contradiction (to μ being stable)

Conflict of interests

Leader-pessimal matching: a stable matching in which each leader receives his **worst achievable** partner.

Follower-pessimal matching: a stable matching in which each follower receives her **worst achievable** partner.

Theorem. When preferences are strict, the leader-optimal matching is the follower-pessimal matching, and vice versa.

Note: the above theorem is an immediate corollary of a stronger claim by Knuth. For a proof, check the proof of Theorem 2.13 in **TSM** book.

The case with Ties

Corollary 2. When preferences are strict, there exists a leader-optimal stable matching and a follower -optimal stable matching.

What if ties are allowed?

l_1: (f_2 f_3) f_1 　　 f_1: l_1 l_2 l_3

l_2: f_2 f_1 　　 f_2: l_1 l_2

l_3: f_3 f_1 　　 f_3: l_1 l_3

Question 1. What are the stable matchings in this example?

Question 2. Which one is optimal for leaders?

Neither: l_2 prefers purple but l_3 prefers green.

Question 3. Which one is optimal for followers?

Neither: f_1 and f_2 prefer green but f_3 prefers purple.

The case with Ties

Claim. When ties are allowed, a leader-optimal stable matching and/or a follower-optimal stable matching may not exist.

l_1: (f_2 f_3) f_1	f_1: l_1 l_2 l_3
l_2: f_2 f_1	f_2: l_1 l_2
l_3: f_3 f_1	f_3: l_1 l_3

- This example has two stable matchings: green and purple.
- Neither is a leader-optimal stable matching: l_2 prefers purple but l_3 prefers green.
- Neither is a follower-optimal stable matching: f_1 and f_2 prefer green but f_3 prefers purple.

DS
truthfulness

Dominant-strategy truthfulness

- Agents (here leaders and followers) declare their preferences to the centralised system.
- Agent are strategic: they misreport if it is in their benefit
  - i.e. if providing a different ranking over candidates results in the system matching them with a better one

- A matching mechanism is *dominant-strategy truthful (DS truthful)*, if every agent finds it in his/her best interest to declare his/her true preference list, no matter what other agents choose to do.

Question: Is Gale-Shapley DS truthful?

Is Gale-Shapley DS truthful?

For now assume that **preferences are strict** =>
leader-oriented GS returns the leader-optimal stable matching.

- When the leader-oriented version of Gale-Shapley algorithm is executed, all leaders find it in their best interest to be truthful.
- Some followers, however, may benefit from misreporting their preferences.

In truth f_1 prefers l_2 to l_3

l_1: $\cancel{f_2}$ f_1 f_3	f_1: l_1 $\cancel{l_3}$ $\cancel{l_2}$
l_2: $\cancel{f_1}$ f_2 f_3	f_2: l_2 $\cancel{l_1}$ $\cancel{l_3}$
l_3: $\cancel{f_1}$ $\cancel{f_2}$ f_3	f_3: l_1 l_3 l_2

GS matching when f_1 misreports

Is Gale-Shapley DS truthful?

Will all agents reveal their preferences truthfully?

- When the leader-oriented version of Gale-Shapley algorithm is executed, all leaders find it in their best interest to be truthful.
- Some followers, however, may benefit from misreporting their preferences.

In truth f_1 prefers l_2 to l_3

l_1: f_2 f_1 f_3
l_2: f_1 f_2 f_3
l_3: f_1 f_2 f_3

f_1: l_1 l_3 l_2
f_2: l_2 l_1 l_3
f_3: l_1 l_3 l_2

GS matching when f_1 tells the truth

DS truthfulness

Theorem (Dubins and Freedleader; Roth)

The mechanism that yields the leader-optimal stable matching (in terms of stated preferences) makes it a dominant strategy for each leader to state his true preferences.

- Similarly, the mechanism that yields the follower-optimal stable matching makes it a dominant strategy for every follower to state her true preferences.
- If we have ties in preferences but a leader-optimal stable matching exists, then leader-oriented GS is DS truthful for leaders.

Impossibility of DS truthfulness

Theorem (Roth, 1982)

No stable matching mechanism exists for which truth-telling is a dominant strategy for every agent.

Even when preferences are strict

The next theorem can be viewed as even an stronger impossibility result.

Theorem (Roth and Sotomayor, 1990)

When any stable mechanism is applied to a marriage market in which preferences are strict and there is more than one stable matching, then at least one agent can profitably misrepresent his or her preferences, assuming the others tell the truth.

Impossibility of DS truthfulness

Theorem (Roth, 1982)

No stable matching mechanism exists for which truth-telling is a dominant strategy for every agent, even when preferences are strict.

Proof sketch.

1. Construct an instance of SMI.
2. Show that whatever stable matching the mechanism chooses, at least one agent can benefit by misreporting (i.e. if that agent declares a different preference, the mechanism matches the agent with a better partner).

Proof of the impossibility result

Consider the following instance with 2 leaders and 2 followers

l_1: f_1 f_2	f_1: l_2 l_1
l_2: f_2 f_1	f_2: l_1 l_2

1. This instance has two stable matchings: green and purple
2. Every stable matching mechanism must choose either green or purple
3. Suppose the mechanism chooses the green matching.
4. If f_2 now only declares l_1 as acceptable then the only stable matching is the purple matching.
5. Similarly, if the mechanism chooses the purple matching, then if l_2 declares only f_2 as acceptable then only the green matching is stable.
6. In either scenario, at least one agent benefits by not reporting truthfully.

Observation

- In the proof of the impossibility result, we allowed incomplete lists and agents could report by ``**truncating**'' their preferences and by falsely declaring an acceptable partner as unacceptable.

Exercise. What if agents cannot truncate? E.g. if

- every agent finds all candidates acceptable (SM),
- the mechanism knows this, and
- so the only way to misreport is to **permute** your preferences.

Hint: create a 3x3 setting and follow a similar approach as in Roth's proof.

Impossibility of truthful Nash Equilibrium

Even when preferences are strict

Theorem (Roth, 1982)

No stable matching mechanism exists for which truth-telling is a dominant strategy for every agent.

Corollary. No stable matching mechanism exists for which stating the true preferences is always a best response for every agent when all other agents state their true preferences.

Alternate statement of the above corollary:

Corollary. No stable matching mechanism exists for which it is always an equilibrium for every agent to state his or her true preferences.

Very few people can manipulate

- Roth and Peranson noted that, in practice, very few students and hospitals could have benefited by submitting false preferences.
  - E.g., in 1996, out of 24749 applicants, only 21 could have affected their match by submitting false preferences.
- They conjectured: the sheer size of the job market matters.

Theorem (Immorlica and Mahdian, ''Marriage, Honestly and Stability'', SODA 2005**)**

The number of students that have more than one achievable hospital vanishes. Therefore, with high probability, the truthful strategy a dominant strategy

- Another concept: approximate-truthfulness
  - The incentive to manipulate is smaller than epsilon

Sisterhood in GS

Question (by a student when I taught this lecture in 2018).

Assume a follower misreports and benefits by misreporting (receives a better partner). Can her misreport harm any of the other followers? i.e. could it result in some other follower receiving a worse partner?

Answer (**Gonczarowski and Friedgut, ''Sisterhood in the Gale-Shapley Matching Algorithm'', J. of Combinatorics, 2013**)

In an instance of SM (no ties or incomplete preferences), and assuming that lying is only possible by permuting one's preferences: if no lying follower is worse off then

- No follower is worse off
- No leader is better off

What was in this lecture

- Extensions of Stable Marriage problem by allowing ties and incomplete lists

- Leader-optimal (resp. follower-optimal) stable matchings

- DS truthfulness in GS

From Stable Marriage to the **Hospitals/Residents problem** and its variants

Next Week

***Match Day 2017**. Credit: Charles E. Schmidt College of Medicine, FAU.*
For more photos of this important day of medical students' life click here.

Books

- Algorithmics of Matching under Preferences by David F. Manlove.

- Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis (**TSM**) by Alvin E. Roth, Marilda A. Oliviera Sotomayor.

- Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations (**MAS**) by Yoav Shoham and Kevin Leyton-Brown

- Algorithmic Game Theory (**AGT**), edited by Noam Nisan, Tim Roughgarden, Eva Tardos, Vijay V. Vazirani

Acknowledgment

Some of the slides in this lecture were based on the slides by **Jie Zhang** and **Kevin Wayne**.