EXTENSIVE-FORM GAMES

COMP6203 - Intelligent Agents

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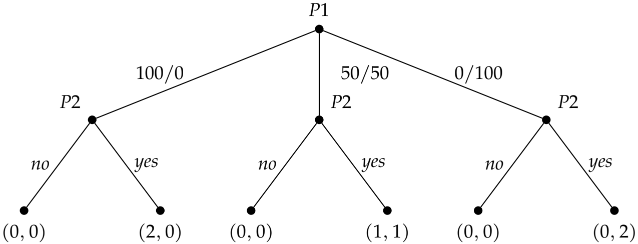
INTRODUCTION

INTRODUCTION

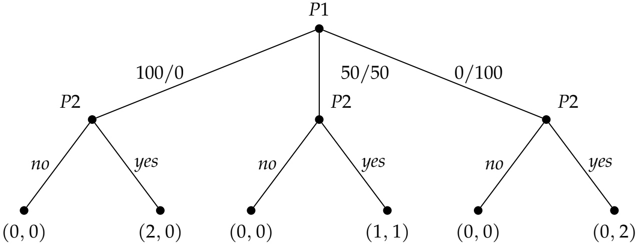
- In strategic-form games, players make a move simultaneously.
- We are now going to see games where players make choices sequentially.
- Extensive-form games make the dynamic aspect of strategic interaction explicit.
- Extensive-form games model different kinds of strategic sequential decision-making:
    - games with perfect information: players know exactly how the current state of the game was reached.
    - games with imperfect information: players may be uncertain about previous moves, may not know how the current state was reached.
    - games of perfect recall: no player forgets the action they previously chose.
- In this lecture we will only focus on games with perfect information.

EXAMPLE: ULTIMATUM GAME



- Two agents want to share a cake (see lectures on Negotiation).
- The first player suggests a split:
  - *P*1 keeps the whole cake.
  - *P*2 keeps the whole cake.
  - Both players get 50%.
- *P*2 can either accept the proposed split or reject it.

EXAMPLE: ULTIMATUM GAME



- The game is represented as a tree in the sense of graph theory.
- Players act sequentially following an order.
- At each (non-terminal) node, a choice is made by one of the players.
- Each edge represents a possible action.
- The leaves represent the final outcomes.

THIS LECTURE

- We will define a formal notion of extensive-form game with perfect information (and perfect recall).
- We will define a notion of strategy.
- We will explore the relation with strategic-form games.
- We will see what role pure strategy Nash equilibria have.
- We will refine this notion and define the concept of subgame perfect equilibrium.
- We will study an algorithm to compute subgame perfect equilibria.

EXTENSIVE-FORM GAMES

EXTENSIVE-FORM GAMES: DEFINITION

Definition

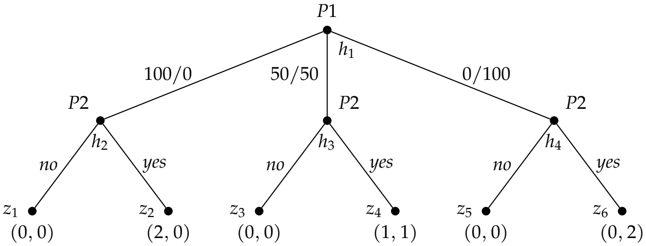
An finite **extensive-form game with perfect information** is a tuple

$$G = (N, A, H, Z, \alpha, \pi, \sigma, u),$$

where.

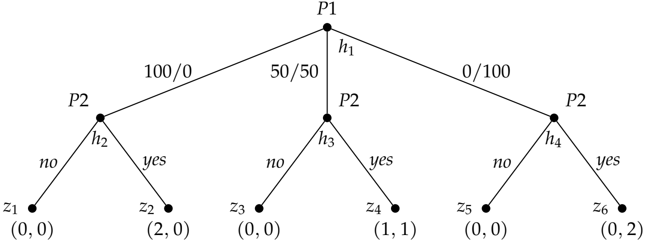
- N is a finite set of players.
- A is a (single) set of actions.
- H is a set of nonterminal choice nodes.
- Z is a set of terminal nodes, disjoint from H.
- $\alpha : H \to 2^A$ is the action function, which assigns to each choice node a set of possible actions.
- $\pi : H \to N$ is the player function, which assigns to each nonterminal node a player $i \in N$ who choses an action at that node.
- $\sigma : H \times A \to H \cup Z$ is the successor function, which maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$, then $h_1 = h_2$ and $a_1 = a_2$.
- $u = (u_1, \ldots, u_n)$, where $u_i : Z \to \mathbb{R}$ is a real-valued utility function for player i on the terminal nodes Z.

EXTENSIVE-FORM GAMES: DEFINITION



- Players: $N = \{P1, P2\}$
- Actions: $A = \{100/0, 50/50, 0/100, \mathit{yes}, \mathit{no}\}$
- Choice nodes: $H = \{h_1, \ldots, h_4\}$
- Terminal nodes: $Z = \{z_1, \ldots, z_6\}$

EXTENSIVE-FORM GAMES: DEFINITION



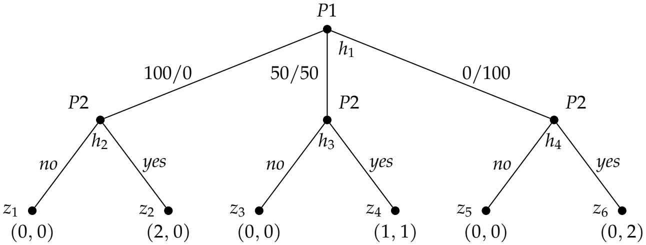
- Action function α:

$$h_1 \overset{\alpha}{\mapsto} \{100/0, 50/50, 0/100\} \qquad h_2 \overset{\alpha}{\mapsto} \{no, yes\} \qquad h_3 \overset{\alpha}{\mapsto} \{no, yes\} \qquad h_4 \overset{\alpha}{\mapsto} \{no, yes\}$$

- Player function π:

$$h_1 \overset{\pi}{\mapsto} P_1 \qquad h_2 \overset{\pi}{\mapsto} P_2 \qquad h_3 \overset{\pi}{\mapsto} P_2 \qquad h_4 \overset{\pi}{\mapsto} P_2$$

EXTENSIVE-FORM GAMES: DEFINITION



- Successor function σ:

$$(h_1, 100/0) \overset{\sigma}{\mapsto} h_2 \qquad (h_1, 50/50) \overset{\sigma}{\mapsto} h_3 \qquad (h_1, 0/100) \overset{\sigma}{\mapsto} h_4$$

$$(h_2, no) \overset{\sigma}{\mapsto} z_1 \qquad (h_2, yes) \overset{\sigma}{\mapsto} z_2 \qquad (h_3, no) \overset{\sigma}{\mapsto} z_3 \qquad \cdots\cdots\cdots$$

- Utility functions $u = (u_1, \ldots, u_n)$:

$$u_1(z_1) = 0 \quad \cdots \quad u_1(z_6) = 0 \qquad u_2(z_1) = 0 \quad \cdots \quad u_2(z_6) = 2$$

STRATEGIES

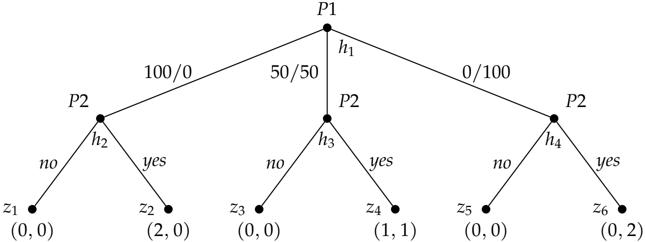
- A strategy for player *i* in an extensive-form game is a complete specification of the action taken at every note belonging to the player.
- An agent's strategy requires a decision at each choice node, regardless of whether or not it is possible to reach that node given the other choice nodes.

Definition

Let *G* be a perfect-information extensive-form game. The **pure strategies** of player *i* consist of the Cartesian product

$$\prod_{h \in H, \pi(h) = i} \alpha(h).$$

STRATEGIES



- $P1$ makes a decision only at h_1, so

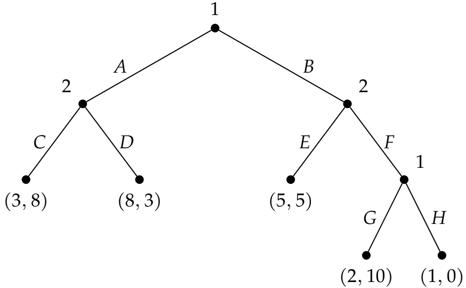
$$S_1 = \{100/0, 50/50, 0/100\}.$$

- $P2$ makes a decision at h_2, h_3, h_4, so

$$\begin{aligned} S_2 &= \{no, yes\} \times \{no, yes\} \times \{no, yes\} \\ &= \{(yes, yes, yes), (yes, yes, no), (yes, no, yes), (yes, no, no), (no, yes, yes), \\ &\quad (no, yes, no), (no, no, yes), (no, no, no)\} \end{aligned}$$

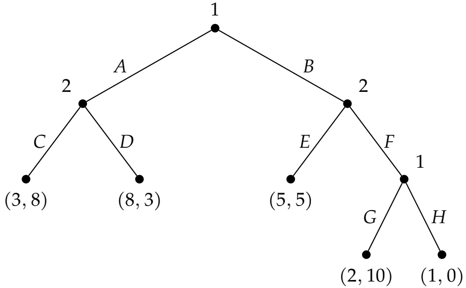
STRATEGIC-FORM TRANSFORMATION

STRATEGIC-FORM TRANSFORMATION



- We have defined a formal notion of strategy for extensive-form games, so now we can take strategy combinations.
- To each strategy combination we can assign a utility value for each player, following the outcomes of the extensive-form game.
- We then convert the extensive-form game into a strategic-form game.

STRATEGIC-FORM TRANSFORMATION



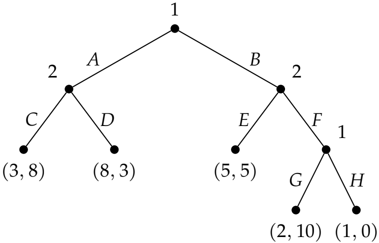
- Strategies for player 1:

$$S_1 = \{A, B\} \times \{G, H\} = \{(A, G), (A, H), (B, G), (B, H)\}.$$

- Strategies for player 2:

$$S_2 = \{C, D\} \times \{E, F\} = \{(C, E), (C, F), (D, E), (D, F)\}$$

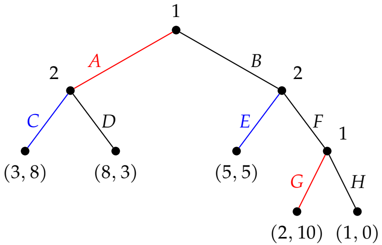
STRATEGIC-FORM TRANSFORMATION



2

		(C, E)	(C, F)	(D, E)	(D, F)
1	(A, G)	8 3	8 3	3 8	3 8
	(A, H)	8 3	8 3	3 8	3 8
	(B, G)	5 5	10 2	5 5	10 2
	(B, H)	5 5	0 1	5 5	0 1

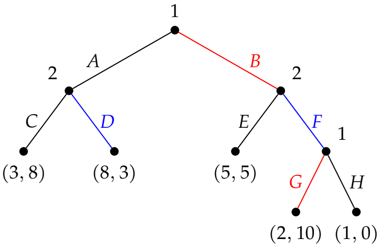
STRATEGIC-FORM TRANSFORMATION



2

		(C, E)	(C, F)	(D, E)	(D, F)
1	(A, G)	3 8	3 8	8 3	8 3
	(A, H)	3 8	3 8	8 3	8 3
	(B, G)	5 5	2 10	5 5	2 10
	(B, H)	5 5	1 0	5 5	1 0

STRATEGIC-FORM TRANSFORMATION



2

		(C, E)	(C, F)	(D, E)	(D, F)
1	(A, G)	3 8	3 8	8 3	8 3
	(A, H)	3 8	3 8	8 3	8 3
	(B, G)	5 5	2 10	5 5	2 10
	(B, H)	5 5	1 0	5 5	1 0

STRATEGIC-FORM TRANSFORMATION

- It is always possible to transform an extensive-form game into a strategic-form game.
- This transformation though negates the temporal nature of the game.
- We add redundant elements: in the original version of the previous game we have 5 outcomes, while its strategic form has 16.
- Notice that it is not always possible to transform a strategic-form game into an extensive-form game.

EQUILIBRIA

NASH EQUILIBRIA

- Now that we have shown how to transform an extensive-form game into a strategic-form game, we can compute the pure strategy Nash equilibria of the game.
- In the original game, players take turns, and everyone sees everything that happened at the point of making a move.
- For extensive-form games the existence of pure Nash equilibria is always guaranteed.

Theorem (Zermelo, 1913)

Given any finite perfect-information extensive-form game G, let $G^\star$ be its strategic-form. Then $G^\star$ has a pure strategy Nash equilibrium.

NASH EQUILIBRIA

2

		(C,E)	(C,F)	(D,E)	(D,F)
	(A,G)	8 3	8 3	3 8	3 8
1	(A,H)	8 3	8 3	3 8	3 8
	(B,G)	5 5	10 2	5 5	10 2
	(B,H)	5 5	0 1	5 5	0 1

- Let’s compute the equilibria of the above game.

NASH EQUILIBRIA

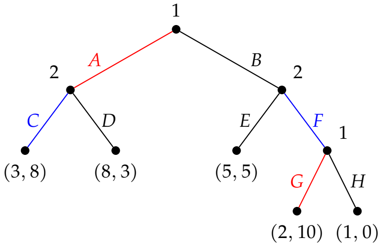
2

		(C,E)	(C,F)	(D,E)	(D,F)
1	(A,G)	3 8	3 8	8 3	8 3
	(A,H)	3 8	3 8	8 3	8 3
	(B,G)	5 5	2 10	5 5	2 10
	(B,H)	5 5	1 0	5 5	1 0

- The equilibria are:

$$((A,G),(C,F)) \qquad ((A,H),(C,F)) \qquad ((B,H),(C,E))$$

NASH EQUILIBRIA



		2			
		(C, E)	(C, F)	(D, E)	(D, F)
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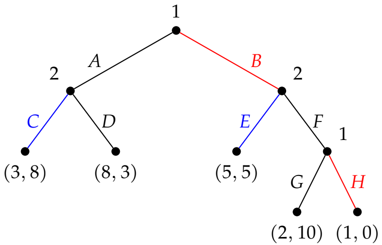
	(A, H)	3 8	3 8	8 3	8 3
	(B, G)	5 5	2 10	5 5	2 10
	(B, H)	5 5	1 0	5 5	1 0

- Consider the equilibrium $((A, G), (C, F))$
- Is this a satisfying outcome?
- Both players seem to be making rational choices.

NASH EQUILIBRIA



			2		
		(C,E)	(C,F)	(D,E)	(D,F)
1	(A,G)	3 8	3 8	8 3	8 3
	(A,H)	3 8	3 8	8 3	8 3
	(B,G)	5 5	2 10	5 5	2 10
	(B,H)	5 5	1 0	5 5	1 0

- Consider the equilibrium $((B,H),(C,E))$
- Is this a satisfying outcome?
- The choices made are questionable.
- This equilibrium does not seem to match the choices we expect rational players to make.

SUBGAMES

- To formally capture why some equilibria are unsatisfying, we introduce a new solution concept, called subgame perfect equilibrum.
- In order to understand it, we first need the notion of a subgame.

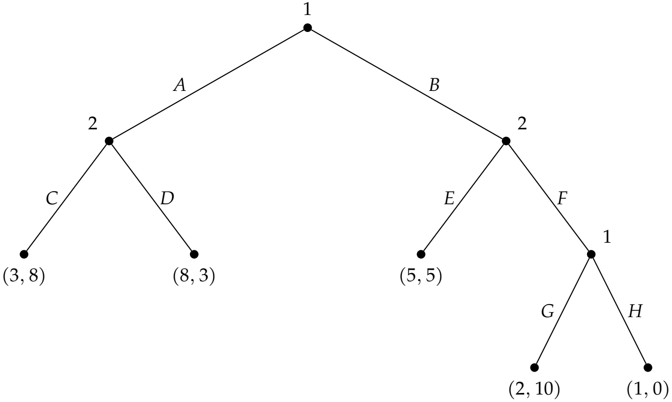
Definition

Given a perfect-information extensive-form game G, the **subgame** of G rooted at node h is the restriction of G to the descendants of h.
The set of subgames of G consists of all subgames rooted at some node in G

- To find the subgames of an extensive-form game, we look at each nonterminal node of the game and take the tree originating from the node.

SUBGAMES

The following game has 4 distinct subgames.



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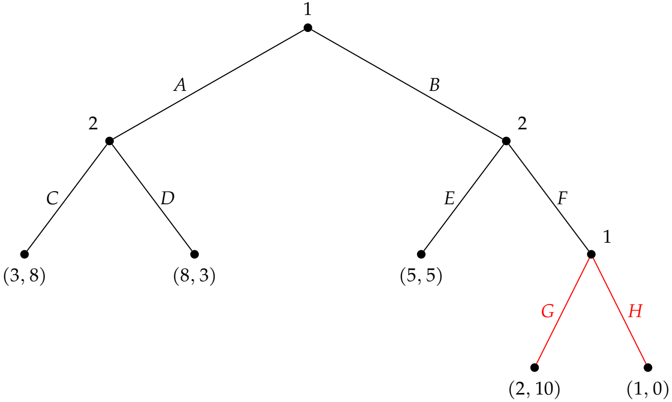

SUBGAMES

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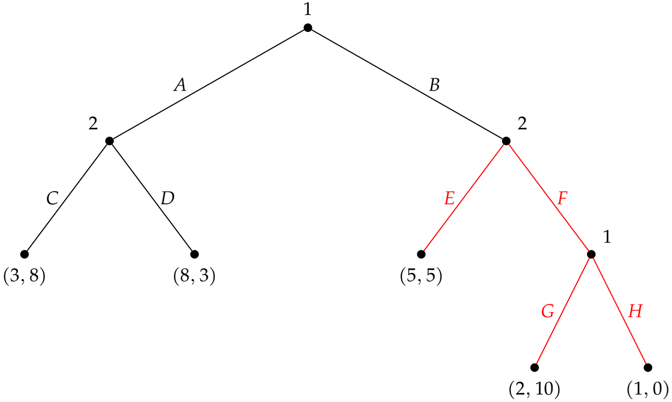

SUBGAMES

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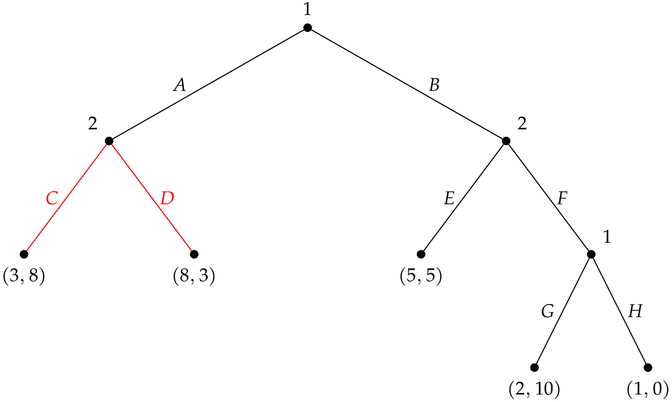

SUBGAMES

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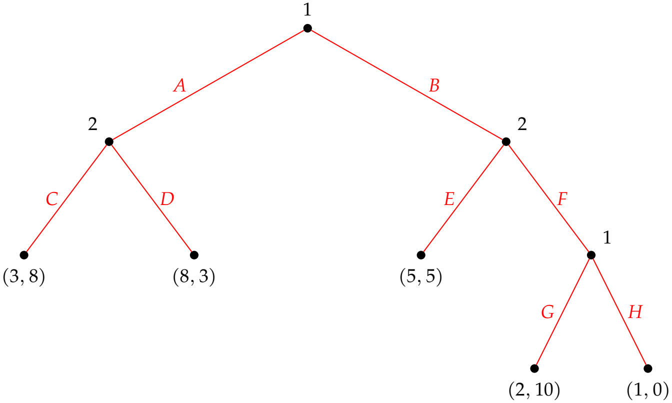


SUBGAME PERFECT EQUILIBRIA

Definition

A strategy profile s in an extensive-form game G is a **subgame perfect equilibrium** if, for each subgame G' of G, the restriction of s to G' is a pure strategy Nash equilibrium.

- The definition of a subgame perfect equilibrium requires that we look at rational behaviour at every node and respects the temporal dynamic of the game.
- Since every subgame perfect equilibrium is a Nash equilibrium, we have the following results.

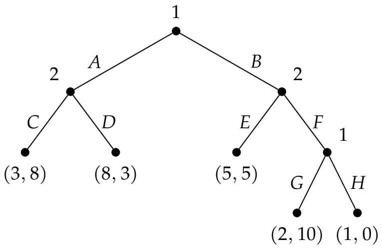
Theorem

Every extensive-form game has a subgame perfect equilibrum.

Theorem

For every extensive-form game the set of subgame perfect equilibria is a subset of the set of pure strategy Nash equilibria.

SUBGAME PERFECT EQUILIBRIA



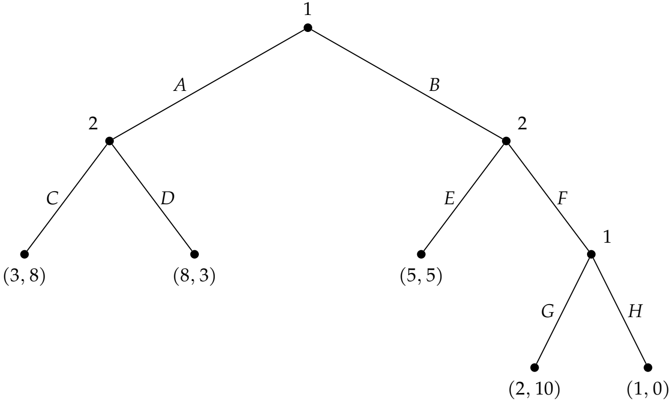
		2			
		(C, E)	(C, F)	(D, E)	(D, F)
1	(A, G)	3 8	3 8	8 3	8 3
	(A, H)	3 8	3 8	8 3	8 3
	(B, G)	5 5	2 10	5 5	2 10
	(B, H)	5 5	1 0	5 5	1 0

- Let's look at the game we defined above and see which of the Nash equilibria are also subgame perfect equilibria.
- The Nash equilibria equilibria are:

$$((A,G),(C,F)) \qquad ((A,H),(C,F)) \qquad ((B,H),(C,E))$$

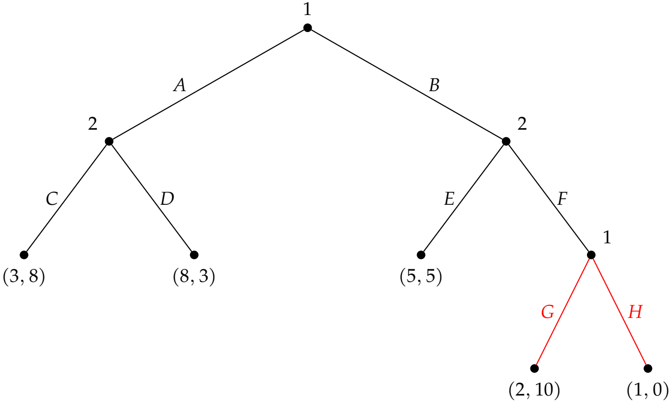
SUBGAME PERFECT EQUILIBRIA

Is $((B, H), (C, E))$ a subgame perfect equilibrium?

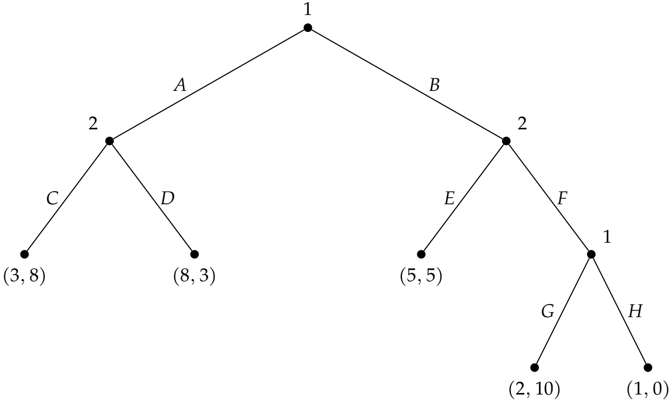


SUBGAME PERFECT EQUILIBRIA

$((B, H), (C, E))$ is not a Nash equilibrium in the subgame below.

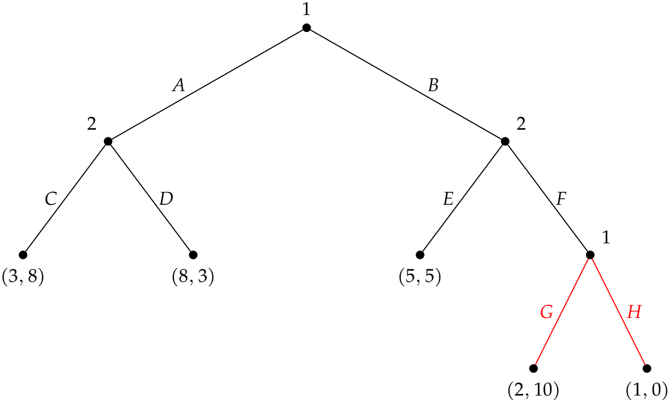


Is $((A, H), (C, F))$ a subgame perfect equilibrium?



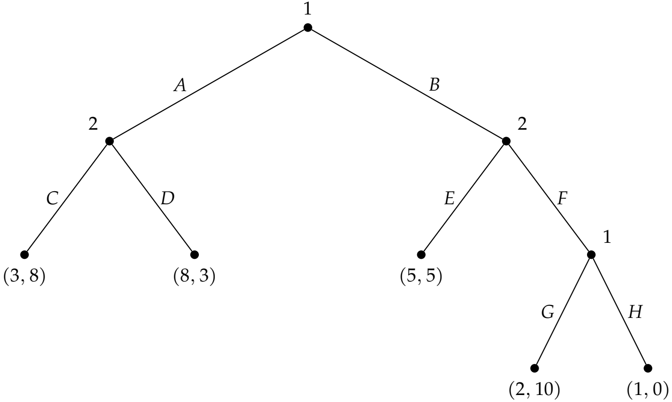
SUBGAME PERFECT EQUILIBRIA

$((A, H), (C, F))$ is not a Nash equilibrium in the subgame below.



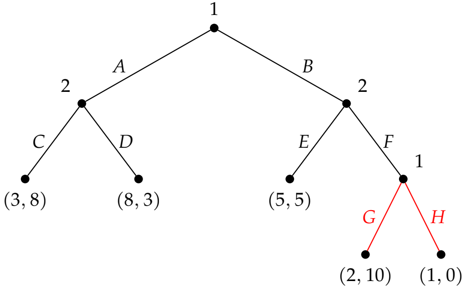
SUBGAME PERFECT EQUILIBRIA

Is $((A,G),(C,F))$ a subgame perfect equilibrium?



SUBGAME PERFECT EQUILIBRIA

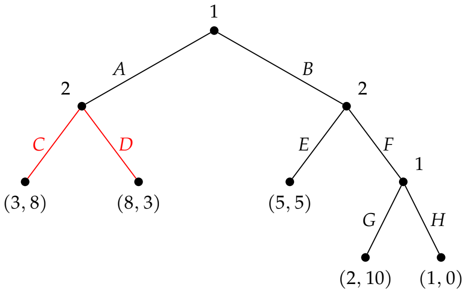
G is a Nash equilibrium.



		2
1	G	2, 10
	H	1, 0

SUBGAME PERFECT EQUILIBRIA

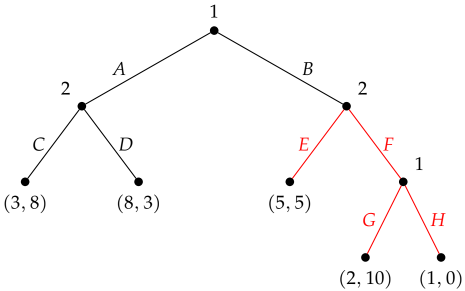
C is a Nash equilibrium.



	2	
	C	D
1	8 3	3 8

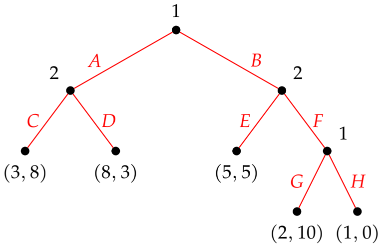
SUBGAME PERFECT EQUILIBRIA

(G, F) is a Nash equilibrium.



		2	
		E	F
1	G	5 5	10 2
	H	5 5	0 1

SUBGAME PERFECT EQUILIBRIA



			2		
		(C, E)	(C, F)	(D, E)	(D, F)
1	(A, G)	3, 8	3, 8	8, 3	8, 3
	(A, H)	3, 8	3, 8	8, 3	8, 3
	(B, G)	5, 5	2, 10	5, 5	2, 10
	(B, H)	5, 5	1, 0	5, 5	1, 0

- $((A, G), (C, F))$ is a Nash equilibrium in every subgame, and so is a subgame perfect equilibrium.

BACKWARD INDUCTION

BACKWARD INDUCTION

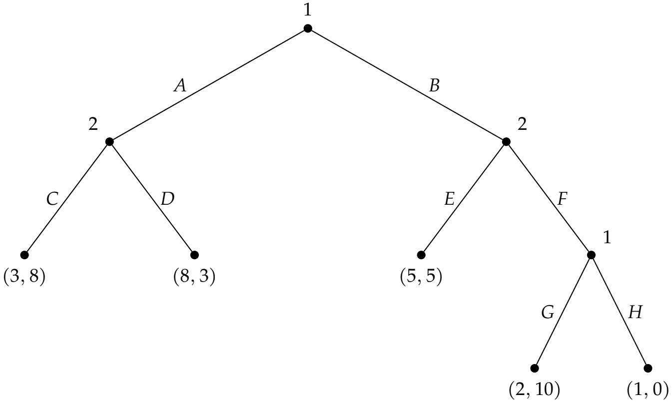
- We have reached the concept of a subgame perfect equilibrium trough the concept of a Nash equilibrium.
- We started from an extensive-form game.
- We have found the sets of strategies for each player.
- We have transformed the game into a strategic-form game.
- We have computed the set of pure Nash equilibria.
- We have checked which of those Nash equilibria are also subgame perfect.
- This process works but is extremely inefficient.
- We show an algorithm based on backward induction that reaches the same conclusion much more efficiently.

BACKWARD INDUCTION

- Take the tree representing an extensive-form game with n players.
- Starting from the smallest subgames (i.e. those rooted at nodes whose children are terminal nodes), compute the Nash equilibrium for the player making the choice.
- Replace the subgame with utility u obtained, i.e. make the node of the subgame a terminal node and label it with the utility of the equilibrium.
- Repeat the process for every subgame rooted in a node leading to terminal nodes, i.e.: compute the Nash equilibrium and replace the game with the result of its equilibrium.
- If at any step a subgame has multiple Nash equilibria (i.e. when the utilities of different choices are the same), select any of them.
- The algorithm terminates when we reach the root node of the whole game and runs in polynomial time in the size of the tree.
- For every player i, let s_i be the vector where each component is the choice made by the player in the process of selecting the equilibrium of a subgame.
- $(s_1, \ldots, s_n)$ is a subgame perfect equilibrium and the utility u at the root node is the value of the equilibrium.

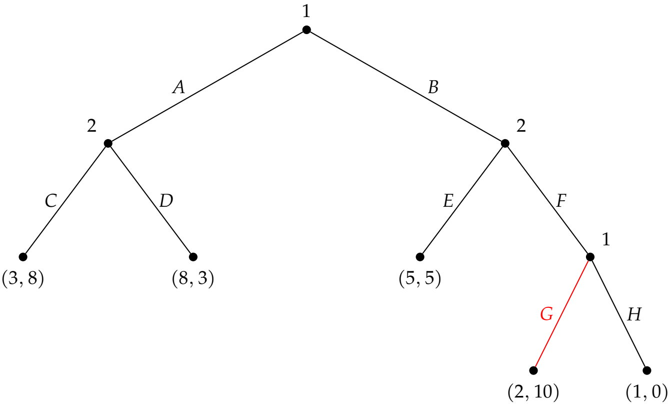
BACKWARD INDUCTION

Let's use backward induction to compute the subgame perfect equilibria of the game below.



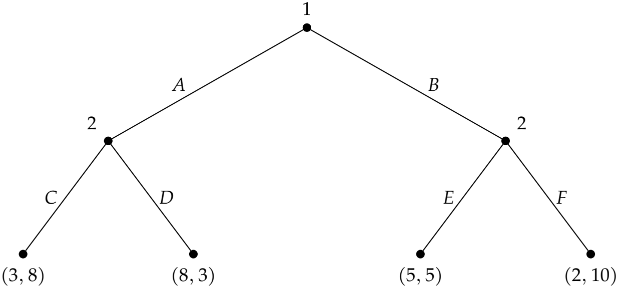
BACKWARD INDUCTION

We start with one of the smallest subgames. The choice is made by player 1. The equilibrium is G, with utility $(2, 10)$.



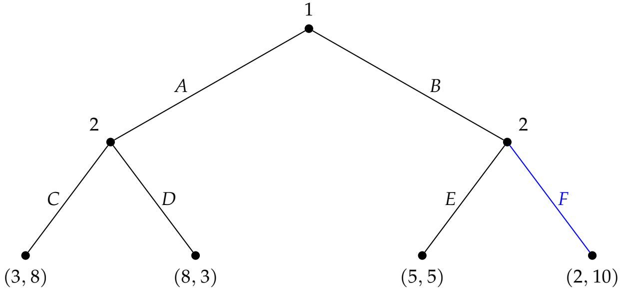
BACKWARD INDUCTION

We replace the subgame with the utility of the equilibrium.



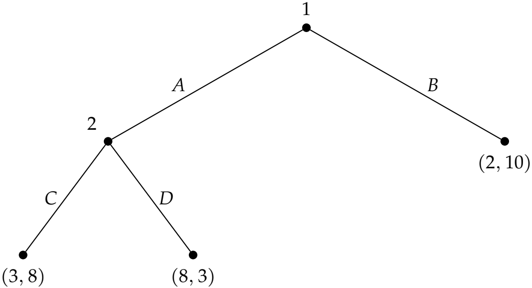
BACKWARD INDUCTION

Take the rightmost subgame. The choice is made by player 2. The equilibrium is *F* with utility $(2, 10)$.



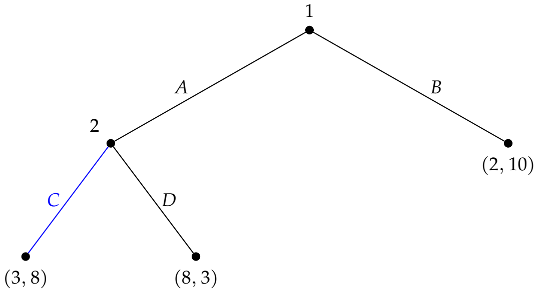
BACKWARD INDUCTION

We replace the subgame with the utility of the equilibrium.

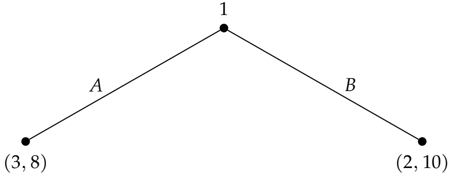


BACKWARD INDUCTION

Take the smallest subgame. The choice is made by player 2. The equilibrium is C with utility $(3, 8)$.

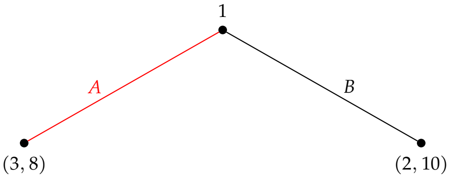


We replace the subgame with the utility of the equilibrium.

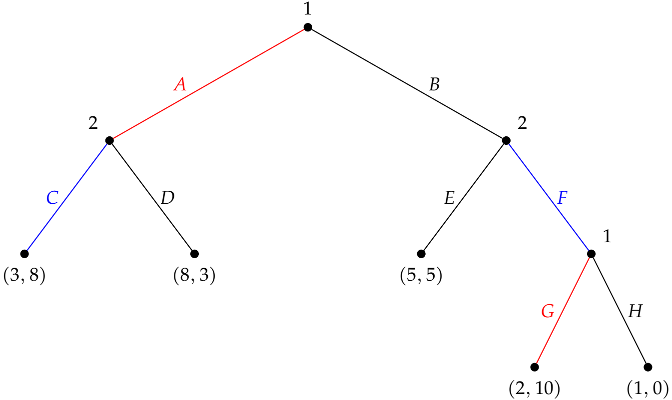


BACKWARD INDUCTION

This is the last step. The choice is made by player 1. The equilibrium is A with utility $(3, 8)$.

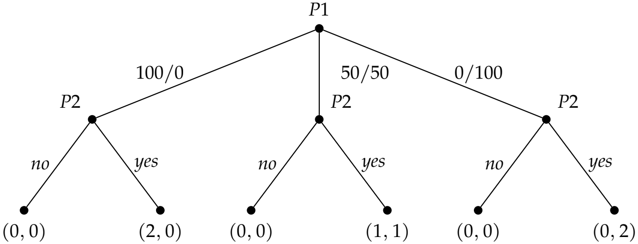


The subgame perfect equilibrium is $((A, G), (C, F))$ and the value is $(3, 8)$.



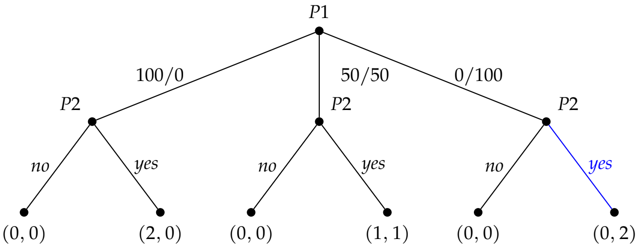
EXAMPLES

ULTIMATUM GAME

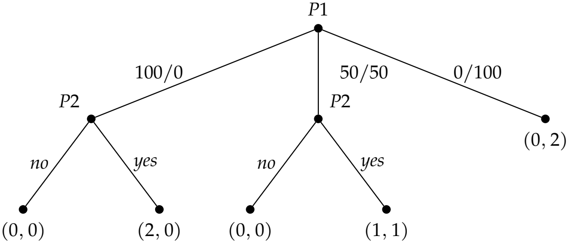


- Let’s go back to the ultimatum game and compute the subgame perfect equilibria

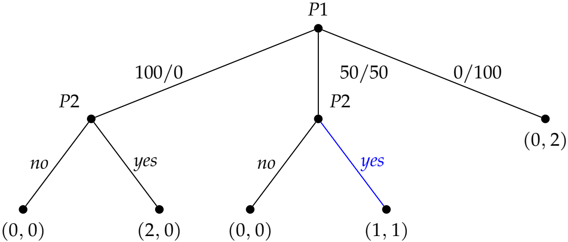
ULTIMATUM GAME



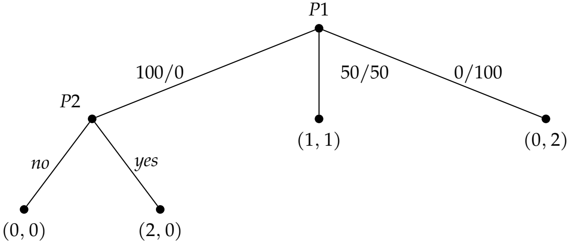
ULTIMATUM GAME



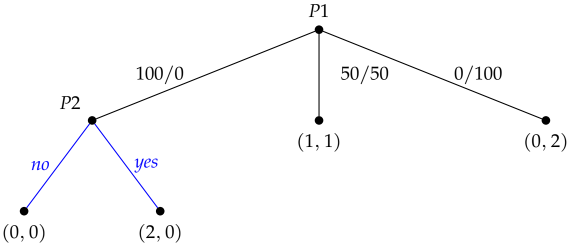
ULTIMATUM GAME



ULTIMATUM GAME

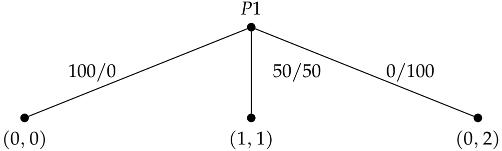


ULTIMATUM GAME

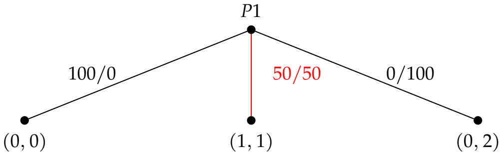


- Here both *yes* and *no* are equilibria for $P2$, so we can select either one of them.
- This means that this game will have (at least) 2 subgame perfect equilibria.

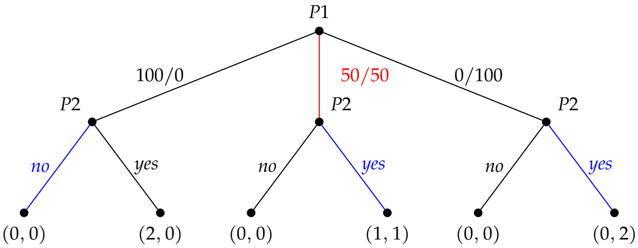
ULTIMATUM GAME



ULTIMATUM GAME

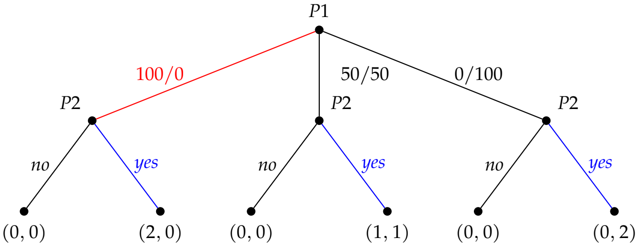


ULTIMATUM GAME



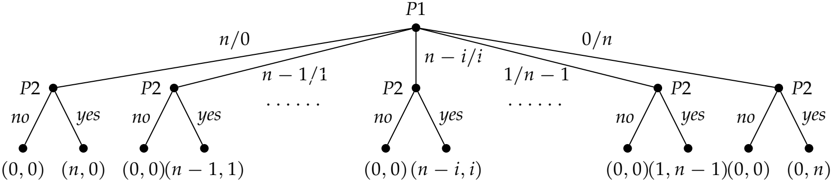
- One subgame perfect equilibrium is $(50/50, (no, yes, yes))$

ULTIMATUM GAME



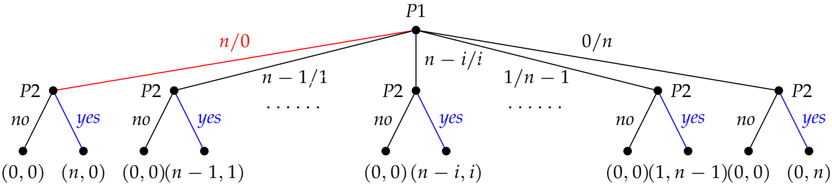
- The other subgame perfect equilibrium is (100/0, (*yes*, *yes*, *yes*))

ULTIMATUM GAME (GENERAL VERSION)



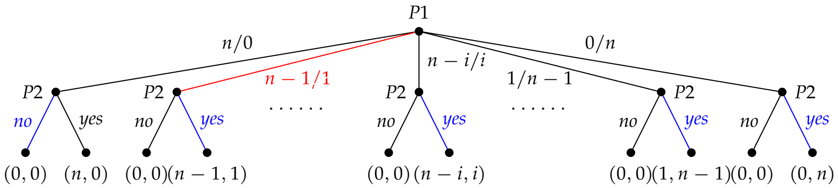
- We now look at a more general version of the ultimatum game, where player 1 can propose a division of the cake in n parts, for any natural number $n > 1$.
- Player 2 can either accept the subdivision or reject it, and in that case, nobody gets anything.
- For any game of this kind, no matter what n we choose, there always are exactly two subgame perfect equilibria.

ULTIMATUM GAME (GENERAL VERSION)



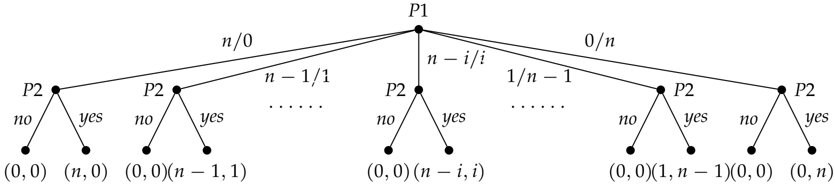
- In the first equilibrium, $P2$ chooses *yes* at every node, and so $P1$ chooses $n/0$.

ULTIMATUM GAME (GENERAL VERSION)



- In the second equilibrium, $P2$ chooses *yes* at every node except at the leftmost node. So $P1$ chooses $n - 1/1$.

ULTIMATUM GAME (GENERAL VERSION)

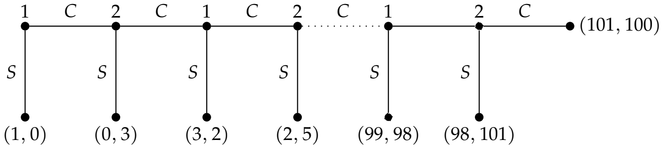


- While there are only two subgame perfect equilibria, there are many more Nash equilibria.
- For instance any strategy combination

  $$(n/0, (yes, x_1, \ldots, x_n)),$$

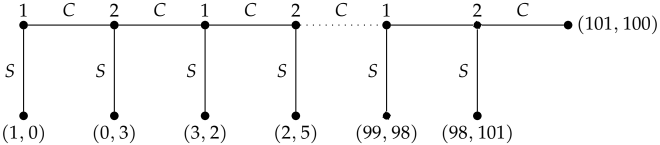
  where each $x_i \in \{no, yes\}$ is a choice by player 2, is a Nash equilibrium.
- $P1$ gets the best possible payoff by choosing $n/0$.
- When $P1$ chooses $n/0$, the payoff for player 2 will be the same no matter what choice is made.

CENTIPEDE GAME



- Two-player game with 100 stages.
- In odd stages, $t = 1, 3, \dots, 99$, player 1 can either:
  - stop the game S, with payoff of $(t, t-1)$,
  - continue the game C.
- In even stages, $t = 2, 4, \dots, 100$, player 2 can either:
  - stop the game S, with payoff of $(t-2, t+1)$,
  - continue the game C.
- The game ends after 100 stages if no player decides to stop before.

CENTIPEDE GAME



- Backward induction shows that players choose to stop the game at every stage.
- That is the only subgame perfect equilibrium.
- In particular, at this equilibrium, player 1 stops the game at the first stage, and the payoff is $(1, 0)$.
- This is unreasonable: no player will be satisfied with this outcome and they can do much better by continuing the game.

REFERENCES

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- Y. Shoham, K. Leyton-Brown. *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*. Cambridge University Press, 2009.
  [Part of the material in these lectures is taken from **Chapter 5**]
- M. J. Osborne. *An Introduction to Game Theory*. Oxford University Press, 2003.