Algorithmic Game Theory
COMP6207

Lecture 6: VCG contd.

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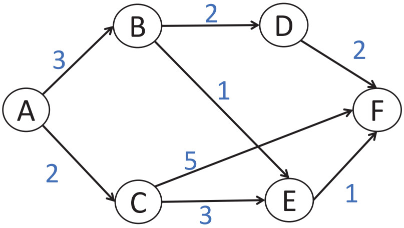
Learning Outcomes

By the end of this lecture, you should be able to

- **Define** *VCG mechanism* and the family of *Groves mechanisms*.
- **Apply** *VCG mechanism* to different settings and compute the outcome and agents' payments.
- **Describe** what properties does *VCG mechanism* have, and **prove** that VCG does have these properties.
- **Outline** the limitations a mechanism designer faces when requiring all of the properties defined in this lecture (except perhaps tractability). That is, describe the relevant theorems and explain what they mean and imply.

VCG Example: Selfish routing example



- The number on each edge is the cost of transporting along that edge.
- Each edge is owned by a different agent and the costs are private information of the agents.
- Goal: Find the shortest (least-cost) path from from A to F.
- The set of outcomes include all possible paths from from A to F.
- Note that numbers on edges are costs, not benefits.
  - If we select a path that crosses an edge of cost c, its owner is incurring a cost of c which means his value for this path is $-c$.
  - If our path doesn't cross agent i's edge, his value for the path is 0.

What path does VCG pick? break ties lexicographically

What is the payment for each agent?

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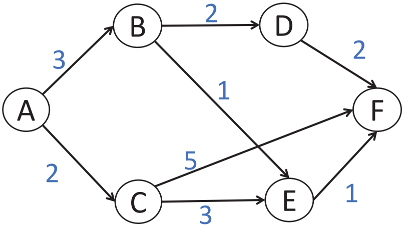
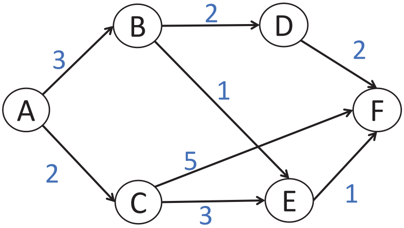


- What choice will be selected by χ?

VCG Example: Selfish routing example



- What choice will be selected by χ? *ABEF*

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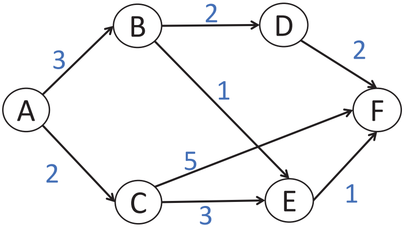

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- How much will *AC* have to pay?

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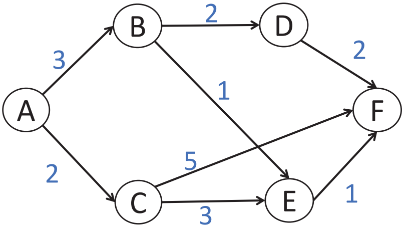

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  - The shortest path when *AC* is **not** present has length 5 and imposes cost of -5 on the remaining agents.

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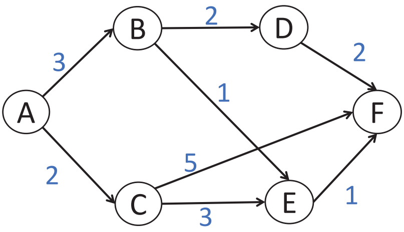

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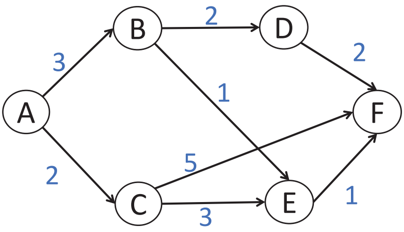

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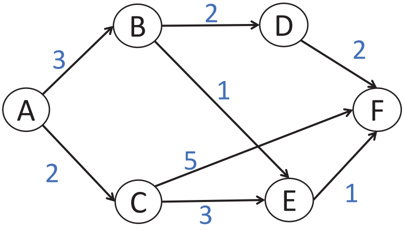

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- Who else pays zero?

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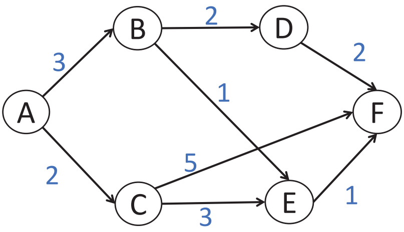

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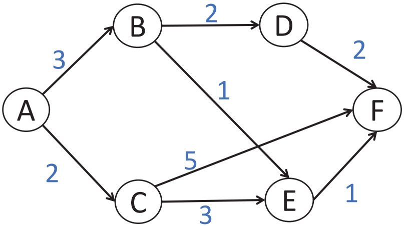
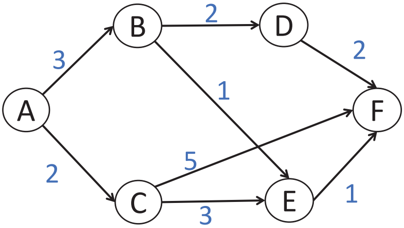
  - The shortest path when *AC* is present has length 5, and imposes cost of -5 on other agents.
  - Thus, $P_{AC} = (-5) - (-5) = 0$.
- Who else pays zero? *BD*, *CE*, *CF* and *DF*.

Selfish routing example contd.



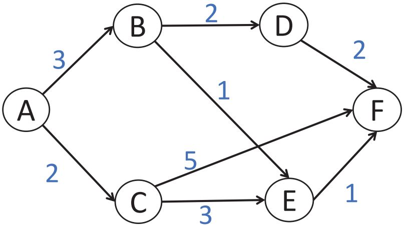
- Quiz: How much will AB have to pay?

Selfish routing example contd.



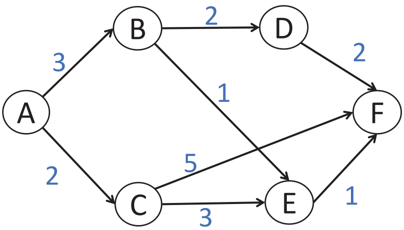
- How much will *BE* have to pay?

Selfish routing example contd.



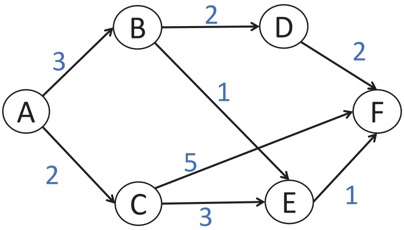
- How much will BE have to pay? $p_{BE} = (-6) - (-4) = -2$

Selfish routing example contd.



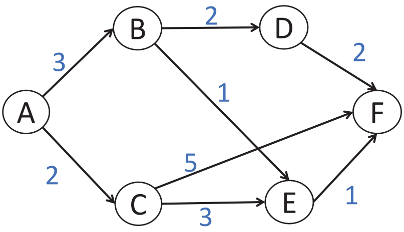
- How much will BE have to pay? $p_{BE} = (-6) - (-4) = -2$
- How much will EF have to pay?

Selfish routing example contd.



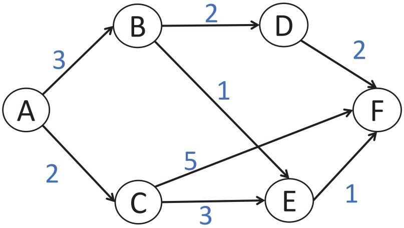
- How much will BE have to pay? $p_{BE} = (-6) - (-4) = -2$
- How much will EF have to pay? $p_{EF} = (-7) - (-4) = -3$

Selfish routing example contd.



- How much will BE have to pay? $p_{BE} = (-6) - (-4) = -2$
- How much will EF have to pay? $p_{EF} = (-7) - (-4) = -3$
- EF and BE have the same costs but are paid different amounts. Why?

Selfish routing example contd.



- How much will *BE* have to pay? $p_{BE} = (-6) - (-4) = -2$
- How much will *EF* have to pay? $p_{EF} = (-7) - (-4) = -3$
- *EF* and *BE* have the same costs but are paid different amounts. Why?
  - *EF* has more market power: for other agents, the situation without *EF* is worse than the situation without *BE*.

Truthfulness

VCG and Groves Mechanism: Truthfulness

Theorem

Truth-telling is a dominant strategy under Groves mechanisms.

Proof.

Consider a situation where every agent j other than i follows some arbitrary strategy $\hat{v}_j$. Consider agent i's problem of choosing the best strategy $\hat{v}_i$. The best strategy for i is one that solves

$$\max_{\hat{v}_i}(v_i(\chi(\hat{v})) - p_i(\hat{v}))$$

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Substituting in the payment function from the Groves mechanism, we have

$$\max_{\hat{v}_i}\left(v_i(\chi(\hat{v})) - h_i(\hat{v}_{-i}) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))\right)$$

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$$\max_{\hat{v}_i}\left(v_i(\chi(\hat{v})) - h_i(\hat{v}_{-i}) + \sum_{j\neq i}\hat{v}_j(\chi(\hat{v}))\right)$$

Since $h_i(\hat{v}_{-i})$ does not depend on $\hat{v}_i$, it is sufficient to solve

$$\max_{\hat{v}_i}\left(v_i(\chi(\hat{v})) + \sum_{j\neq i}\hat{v}_j(\chi(\hat{v}))\right)$$

Proof contd (Groves mechanisms are DS truthful)

$$\max_{\hat{v}_i}\left(v_i(\chi(\hat{v}))+\sum_{j\neq i}\hat{v}_j(\chi(\hat{v}))\right)$$

The only way the declaration $\hat{v}_i$ influences this maximisation is through the choice of χ. If possible, i would like to pick a declaration $\hat{v}_i$ that will lead the mechanism to pick an $x \in X$ which solves

$$\max_{x}\left(v_i(x)+\sum_{j\neq i}\hat{v}_j(x)\right) \tag{1}$$

Proof contd (Groves mechanisms are DS truthful)

$$\max_{\hat{v}_i} \left(v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right)$$

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$$\max_{x} \left(v_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right) \tag{1}$$

Under Groves mechanisms,

$$\chi(\hat{v}) = \operatorname*{argmax}_{x} \left(\sum_{i} \hat{v}_i(x) \right) = \operatorname*{argmax}_{x} \left(\hat{v}_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right)$$

A Groves mechanism will choose x in a way that solves the maximisation problem in Equation (1) when i declares $\hat{v}_i = v_i$. Because this argument does not depend in any way on the declarations of the other agents, truth-telling is a dominant strategy for agent i.

Proof Intuition

- Externalities are internalized
  - agents may be able to change the choice of the mechanism to another one that they prefer, by changing their declaration
  - however, their utility doesn't just depend on the choice—it also depends on their payment.
  - since they get paid the (reported) valuation of all the other agents under the chosen alternative/choice, they now have an interest in maximising everyone's utility rather than just their own

- Individual's incentives are aligned with the society's incentives.

- In general, DS truthful mechanisms have the property that an agent's payment doesn't (directly) depend on the amount of his declaration, but only on the other agents' declarations
  - the agent's declaration is used only to choose the choice, and to set other agents' payments

Groves Uniqueness

Theorem by **Green-Laffont** (informal statement): in settings where agents may have unrestricted quasilinear utilities, Groves mechanisms are the only mechanisms that are both efficient and dominant-strategy truthful.

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And the formal statement if you are interested:

Theorem (Green-Laffont)

Suppose that for all agents any $v_i : X \mapsto \mathbb{R}$ is a feasible preference. Then an efficient quasilinear mechanism has truthful reporting as a dominant strategy for all agents and preferences only if it is a Groves mechanism.

VCG in Summary, so far

- VCG, and Groves mechanisms in general, are efficient and dominant-strategy truthful.

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- Despite having these two fantastic properties, VCG is rarely used in practice.
    - It has been called "Lovely but Lonely".
- Why? because it also has a long list of weaknesses.
    - One of them you will encounter in coursework 1.

VCG in Summary, so far

- VCG, and Groves mechanisms in general, are efficient and dominant-strategy truthful.
- Despite having these two fantastic properties, VCG is rarely used in practice.
  - It has been called "Lovely but Lonely".
- Why? because it also has a long list of weaknesses.
  - One of them you will encounter in coursework 1.
- An important weakness of VCG is that the finding the social-welfare maximising choice might be hard (NP-hard).
  - This is true e.g. in combinatorial auctions.
  - So VCG is not tractable in general combinatorial auctions.

Limitations of VCG

Optional: watch this 11 minutes 22 seconds video
Game Theory II - Week 3 - Video 4: Limitations of VCG

Individual Rationality in VCG

VCG and Individual Rationality

Definition (Ex post individual rationality)

A mechanism is ex post individually rational if $\forall v$, in equilibrium the utility of each agent is at least 0.

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- VCG in general does not give rise to individual rationality.
- VCG is individual rational if two mild constraints (next slide) are satisfied.

Two Assumptions

Definition (Choice-set monotonicity)

A setting exhibits choice-set monotonicity if $\forall i, X_{-i} \subseteq X$.

- In other words: removing any agent weakly decreases—-that is, never increases—the mechanism's set of possible choices X.

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Definition (No negative externalities)

A setting exhibits no negative externalities if $\forall i, \forall x \in X_{-i}, v_i(x) \geq 0$.

- In other words: every agent has zero or positive utility for any choice that can be made without her participation.

Example: road referendum

Consider the problem of holding a referendum to decide whether or not to build a road.

- The set of choices is independent of the number of agents, satisfying choice-set monotonicity.
- No agent negatively values the project, though some might value the situation in which the project is not undertaken more highly than the situation in which it is.

VCG and Individual Rationality

Theorem

The VCG mechanism is ex-post individual rational when the **choice set monotonicity** *and* **no negative externalities** *properties hold.*

Proof.

All agents truthfully declare their valuations in equilibrium. Then

$$
\begin{aligned}
u_i &= v_i(\chi(v)) - \left(\sum_{j \neq i} v_j(\chi(v_{-i})) - \sum_{j \neq i} v_j(\chi(v)) \right) \\
&= \sum_{j} v_j(\chi(v)) - \sum_{j \neq i} v_j(\chi(v_{-i})) \qquad (2)
\end{aligned}
$$

$\chi(v)$ is the choice that maximises social welfare, and so the optimisation could have picked $\chi(v_{-i})$ instead (by choice set monotonicity). Thus,

$$\sum_{j} v_j(\chi(v)) \geq \sum_{j} v_j(\chi(v_{-i}))$$

VCG and Individual Rationality

Theorem

The VCG mechanism is ex-post individual rational when the **choice set monotonicity** *and* **no negative externalities** *properties hold.*

Proof contd.

$$\sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i}))$$

Furthermore, from no negative externalities,

$$v_i(\chi(v_{-i})) \geq 0$$

Therefore,

$$\sum_j v_j(\chi(v)) \geq \sum_{j \neq i} v_j(\chi(v_{-i}))$$

And thus Equation (2) is non-negative.

$\square$

Budget Balance and VCG

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Definition (Budget balance)

A quasilinear mechanism is budget balanced when $\forall v, \sum_{i=1}^{n} p_i(s(v)) = 0$, where s is the equilibrium strategy profile.

- VCG is not budget balanced (even in a simple single-item auction).

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Definition (Weak Budget balance)

A quasilinear mechanism is weakly budget balanced when
$\forall v, \sum_{i=1}^{n} p_i(s(v)) \geq 0$, where s is the equilibrium strategy profile.

- Weak budget balance requires that the mechanism does not lose money.
- VCG is not weakly budget balanced (e.g. in the selfish routing example it had to pay the agents and didn't collect any money)

Another property

Definition (No single-agent effect)

A setting exhibits no single-agent effect if $\forall i, \forall v_{-i}, \forall x \in \text{argmax}_y \sum_j v_j(y)$ there exists a choice x' that is feasible without i and that has $\sum_{j \neq i} v_j(x') \geq \sum_{j \neq i} v_j(x)$.

- In other words, welfare of agents other than i is weakly increased by dropping i.

Example

Consider a single-item auction. Dropping an agent reduces the amount of competition, making the others better off.

Good News

Theorem

The VCG mechanism is weakly budget-balanced when the **no single-agent effect** *property holds.*

Proof.

All agents truthfully declare their valuations in equilibrium. We must show that the sum of transfers (i.e. payments) from agents to the center is greater than or equal to zero.

Good News

Theorem

The VCG mechanism is weakly budget-balanced when the **no single-agent effect** *property holds.*

Proof.

All agents truthfully declare their valuations in equilibrium. We must show that the sum of transfers (i.e. payments) from agents to the center is greater than or equal to zero.

$$\sum_i p_i(v) = \sum_i \left(\sum_{j \neq i} v_j(\chi(v_{-i})) - \sum_{j \neq i} v_j(\chi(v)) \right)$$

From no single-agent effect condition we have that for all i

$$\sum_{j \neq i} v_j(\chi(v_{-i})) \geq \sum_{j \neq i} v_j(\chi(v))$$

The result thus directly follows. □

More Good News

Theorem (Krishna & Perry, 1998)

In any Bayesian game setting in which VCG is ex post individually rational, VCG collects at least as much revenue as any other efficient and ex interim individually-rational mechanism.

- Ex interim individual rationality is a weaker condition than ex post individual rationality
- This result somewhat surprising: does not require dominant strategies, and hence compares VCG to all Bayes-Nash mechanisms.
- A useful corollary: VCG is as budget balanced as any efficient mechanism can be
  - It satisfies weak budget balance in every case where any dominant strategy, efficient and ex interim individual rational mechanism would be able to do so.

Bad News

Theorem (Green-Laffont; Hurwics)

No dominant-strategy truthful mechanism is always both efficient and weakly budget balanced, even if agents are restricted to the simple exchange setting.

- *simple exchange* is an environment consisting of buyers and sellers with quasilinear utility functions, all interested in trading a single identical unit of some good.

Bad News

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- *simple exchange* is an environment consisting of buyers and sellers with quasilinear utility functions, all interested in trading a single identical unit of some good.

Theorem (Myerson-Satterthwaite)

No Bayes-Nash incentive-compatible mechanism is always simultaneously efficient, weakly budget balanced and ex interim individual rational, even if agents are restricted to quasilinear utility functions.

- Class of Bayes-Nash incentive-compatible mechanisms includes the class of dominant-strategy truthful mechanisms
- ex interim IR is weaker than ex post IR

Summary of the two VCG lectures

- Groves mechanisms are efficient and truthful.
- VCG is ex post individual rational if it satisfies the two conditions of choice-set monotonicity and no negative externalities.
- VCG is weakly budget balanced if it it satisfies the condition of no single-agent effect.
- VCG is is as budget balanced as any efficient mechanism can be.
- Limits of designing a mechanism that satisfies all our desired properties (Theorem by Green-Laffont;Hurwics and Theorem by Myerson-Satterthwaite)

Books

- Twenty Lectures on Algorithmic Game Theory, by Tim Roughgarden
- Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations by Yoav Shoham and Kevin Leyton-Brown
  - From now on we will refer to this book as **MAS**
- Algorithmic Game Theory, edited by Noam Nisan, Tim Roughgarden, Eva Tardos,Vijay V. Vazirani
  - From now on we will refer to this book as **AGT**

Further reading/watching

For further introduction to Mechanism Design, Groves mechanisms and VCG

- Read MAS chapters 10.1, 10.2, 10.3, 10.4.1-10.4.6 (we haven't covered some of the material in these sections, of which we will cover some in future lectures)
- Read AGT Chapters 9.1, 9.2, 9.3 (note that MAS and AGT sometimes use different notations and definitions for the same concepts)
- Watch Game Theory II - Week 3 (VCG): 6 videos

Acknowledgment

Some of the slides in this lecture were based on the slides by Jie Zhang and Kevin Leyton-Brown.