Algorithmic Game Theory
COMP6207

Lecture 17: **Size vs Stability**

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Learning Outcomes

- By the end of this session, you should be able to
  - ***Describe*** MAX SMTI
  - ***Describe*** *Kiraly's* algorithm as an extension of Gale-Shapley
  - ***Compute*** the stable matching produced by executing *Kiraly's* algorithm on an instance of SMTI

Anything in yellow boxes are not examinable

Maximum size
Stable Matching

Stable matchings of different sizes

- All stable matchings in a given instance of **SM**, or **SMT**, or **SMI**, are of the same size.
- When both ties and incomplete lists are allowed (i.e. we have an instance of **SMTI**), stable matchings can have different sizes

$$
\begin{array}{llll}
l_1: & (f_1\ f_2) & \qquad f_1: & l_1\ l_2 \\
l_2: & f_1 & \qquad f_2: & l_1
\end{array}
$$

- A maximum (cardinality) stable matching can be (at most) twice the size of a minimum stable matching [leaderlove et al, 2002]

Maximum stable matchings

- Problem of finding a maximum stable matching in an instance of SMTI (MAX SMTI) is NP-hard [Iwama, leaderlove et al, 1999], even if (simultaneously):
  - the ties occur on one side only
  - each preference list is either strictly ordered or is a single tie
  - *and*
    - *either* each tie is of length **2** [leaderlove et al, 2002]
    - *or* each preference list is of length **≤3** [Irving, leaderlove, O'Malley, 2009]

This result implies that MAX HRT is also NP-hard.

- Minimisation problem is NP-hard too, for similar restrictions! [leaderlove et al, 2002]

Reminder: computational complexity

- Given two functions f and g, we say $f(n)=O(g(n))$ if there are positive constants c and N such that $f(n) \leq c.g(n)$ for all $n \geq N$
- An algorithm for a problem has *time complexity* $O(g(n))$ if its running time f satisfies $f(n)=O(g(n))$ where n is the input size
- An algorithm runs in *polynomial time* if its time complexity is $O(n^k)$ for some constant k, where n is the input size
- A *decision problem* is a problem whose solution is yes or no for any input
- A decision problem belongs to the class **P** if it can be solved by a *polynomial-time algorithm*
- A decision problem belongs to the class **NP** if it can be verified in *polynomial time*
- A decision problem *A* is **NP**-hard if every other problem in **NP** reduces to *A*.
- A decision problem *A* is **NP**-complete if it **NP**-hard and it belongs to **NP**.
- If a decision problem is **NP**-complete it has no polynomial-time algorithm unless **P=NP**

Reminder: approximation algorithms

- An *optimisation problem* is a problem that involves maximising or minimising (subject to a suitable measure) over a set of feasible solutions for a given instance
  - e.g., colour a graph using as few colours as possible
- If an optimisation problem is **NP**-hard it has no polynomial-time algorithm unless **P=NP**
- An *approximation algorithm* A for an optimisation problem is a polynomial-time algorithm that produces a feasible solution $A(I)$ for any instance I.
- A has *perforleaderce guarantee* c, for some $c>1$ if
  - $|A(I)| \leq c.opt(I)$ for any instance I (in the case of a minimisation problem)
  - $|A(I)| \geq (1/c).opt(I)$ for any instance I (in the case of a maximisation problem)

  where ***opt*(*I*)** is the measure of an optimal solution and **|*A*(*I*)|** the size of the solution produced by ***A*.**

  ➢ We say that ***A*** is a ***c-approximation*** algorithm for this problem.

MAX HRT: approximability

- MAX HRT is not approximable within **33/29** unless P=NP, even if each hospital has capacity **1** [Yanagisawa, 2007]

- MAX HRT is not approximable within **4/3-ε** assuming the *Unique Games Conjecture* (UGC) [Yanagisawa, 2007]

- Trivial **2**-approximation algorithm for MAX HRT

- Succession of papers gave improveleadersts, culminating in:

  - **MAX HRT is approximable within 3/2** [McDermid, 2009; Király, 2012; Paluch 2012]

- Experileaderstal comparison of approximation algorithms and heuristics for MAX HRT and MAX SMTI [Irving and leaderlove, 2009; Podhradský 2010]

Kiraly's $\frac{3}{2}$-approximation for MAX SMTI (leader-oriented version)

- An extension of Gale-Shapely
- When a leader is rejected by all followers in his list, he is given a *second chance*

- For a leader l, and for two followers f_i and f_j , we say that l prefers f_i to f_j if
  1. either he prefers f_i in the usual sense
  2. or he is indifferent between the two, f_j is engaged and f_i is free.

- For a follower f, and for two leaders l_i and l_j , we say that f prefers l_i to l_j if
  1. either she prefers l_i in the usual sense
  2. or she is indifferent between the two, l_i has a second chance (he is proposing to the followers in his list for the 2^{nd} time) and l_j does not (he is proposing to the followers in his list for the 1^{st} time).

Kiraly's $\frac{3}{2}$-approximation for MAX SMTI (leader-oriented version) contd.

- An unassigned leader proposes to his most-preferred follower on his list, according to his new definition of *prefers*

- An unassigned follower always accepts a proposal (as was the case in GS)

- An assigned follower **f** accepts a new proposal from a leader $\boldsymbol{l}$, and rejects her current partner $\boldsymbol{l}_k$, if
  1. either she prefers $\boldsymbol{l}$ to her current partner, according to her new definition of *prefers*
  2. or her current partner prefers some follower to **f**, again according to his new definition of *prefers*. (In this case we call **f** *precarious*.)

- When a follower **f** rejects a leader $\boldsymbol{l}$, and she is not precarious, $\boldsymbol{l}$ and **f** are deleted from each others' lists

SMTI: stable matching (1)

l_1:	(f_1 f_2)	f_1:	(l_1 l_2)
l_2:	f_1	f_2:	l_1
l_3:	f_3 f_4	f_3:	(l_3 l_4)
l_4:	f_3	f_4:	l_3

$M = \{(l_1, l_1), (l_3, f_3)\}$ (size **2**)

SMTI: stable matching (2)

l_1: (f_1 f_2)

l_2: f_1

l_3: f_3 f_4

l_4: f_3

f_1: (l_1 l_2)

f_2: l_1

f_3: (l_3 l_4)

f_4: l_3

$M = \{(l_1, f_2), (l_2, f_1), (l_3, f_4), (l_4, f_3)\}$ (size **4**)

Example: Kiraly’s algorithm

l_1 : ($\cancel{f_1}$ f_2)

l_2 : f_1

l_3 : $\cancel{f_3}$ f_4

l_4 : $\cancel{f_3}$

f_1 : ($\cancel{l_1}$ l_2)

f_2 : l_1

f_3 : ($\cancel{l_3}$ $\cancel{l_4}$)

f_4 : l_3

- f_1 is precarious: her current partner l_1 prefers another follower, f_2, according to his new definition of prefers.
- f_3 is not precarious and is indifferent between l_3 and l_4, even according to her new definition of prefers.
- l_4 is given a second chance.
- f_3 prefers l_4 to l_3, according to her new definition of prefers.
  - f_3 and f_3 are deleted from each others’ lists

Example: Kiraly's algorithm

l_1: (f_1 f_2)		f_1: (l_1 l_2)
l_2: f_1		f_2: l_1
l_3: f_3 f_4		f_3: (l_3 l_4)
l_4: f_3		f_4: l_3

$M = \{(l_1, f_2), (l_2, f_1), (l_3, f_4), (l_4, f_3)\}$ (size **4**)

Quiz: (leader-oriented) Kiraly

l_1: (f_2 f_3)	f_1: l_2 l_3
l_2: (f_1 f_2 f_4)	f_2: (l_1 l_2 l_4)
l_3: f_1	f_3: l_1
l_4: f_2	f_4: l_2

Kiraly's short summary

Preferences

- For a leader l, and for two followers f_i and f_j , we say that l prefers f_i to f_j if
  1. either he prefers f_i in the usual sense
  2. or he is indifferent between the two, f_j is engaged and f_i is free.

- For a follower f, and for two leaders l_i and l_j , we say that f prefers l_i to l_j if
  1. either she prefers l_i in the usual sense
  2. or she is indifferent between the two, l_i has a second chance (he is proposing to the followers in his list for the 2nd time) and l_j does not (he is proposing to the followers in his list for the 1st time).

Proposals and rejections

- An assigned follower f accepts a new proposal from a leader l, and rejects her current partner l_k , if
  1. either she prefers l to her current partner, according to her new definition of *prefers*
  2. or her current partner prefers some follower to f, again according to his new definition of *prefers*. (In this case we call f *precarious*.)

- When a follower f rejects a leader l, and she is not precarious, l and f are deleted from each others' lists

DS truthfulness

- Is Kiraly's algorithm DS truthful?
  - No. (Recall Roth's impossibility theorem)
- Is the leader-oriented Kiraly DS truthful for leaders?
  - No. (Exercise: prove this; a simple example works)
- If not, can we achieve 3/2 approximation ratio with another mechanism that is DS for leaders?
  - No
- If not, can we achieve 3/2 approximation ratio with another mechanism that is DS for leaders and ties are only on one side of the market?
  - No if ties are on followers' side.
  - Yes if ties are on leaders' side.

Strategy-Proof Approximation Algorithms for the Stable Marriage Problem with Ties and Incomplete Lists, by K. Hamada, S. Miyazaki, H. Yanagisawa, **2019**

Other important interesting problems

``Almost stable'' matchings

- Sometimes matching more people is very import.
- A small number of blockings pairs could be tolerated if it is possible to find a larger matching.

MAX SIZE MIN BP SMI is the problem of finding a matching, out of all maximum cardinality matchings, which has the minimum number of blocking pairs, given an instance of **SMI**.

[Biro, leaderlove and Mittal, 2010]

- is NP-hard even if every preference list is of length ≤ 3
- not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P=NP
- Solvable in polynomial time if each follower's list is of length ≤ 2

And more problems

- Stable Marriage problem with Forbidden pairs and/or Forced pairs

- Balanced stable matchings

- Stronger forms of stability when ties are allowed
  - Strong stability
  - Super stability

- Social stability

- …..

Acknowledgeleaderst

Some of the slides in this lecture were based on the slides by **David leaderlove**.

Book

- Algorithmics of Matching under Preferences
  by David F. leaderlove.