COMP6207 ALGORITHMIC GAME THEORY, SPRING 2022

ARROW'S IMPOSSIBILITY THEOREM

Pavel Naumov
p.naumov@soton.ac.uk

KENNETH ARROW

1921 - 2017

Nobel Memorial Prize in Economic Sciences, 1972

PREFERENCE AGGREGATION FRAMEWORK

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- **Set A containing at least three <u>alternatives</u>**

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- Finite set N of individuals

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- Set A containing at least three <u>alternatives</u>
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n is the size of set N

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preference R is weak order relation on A

  - reflexive (aRa)
  - transitive (if aRb and bRc, then aRc)
  - complete (aRb or bRa)

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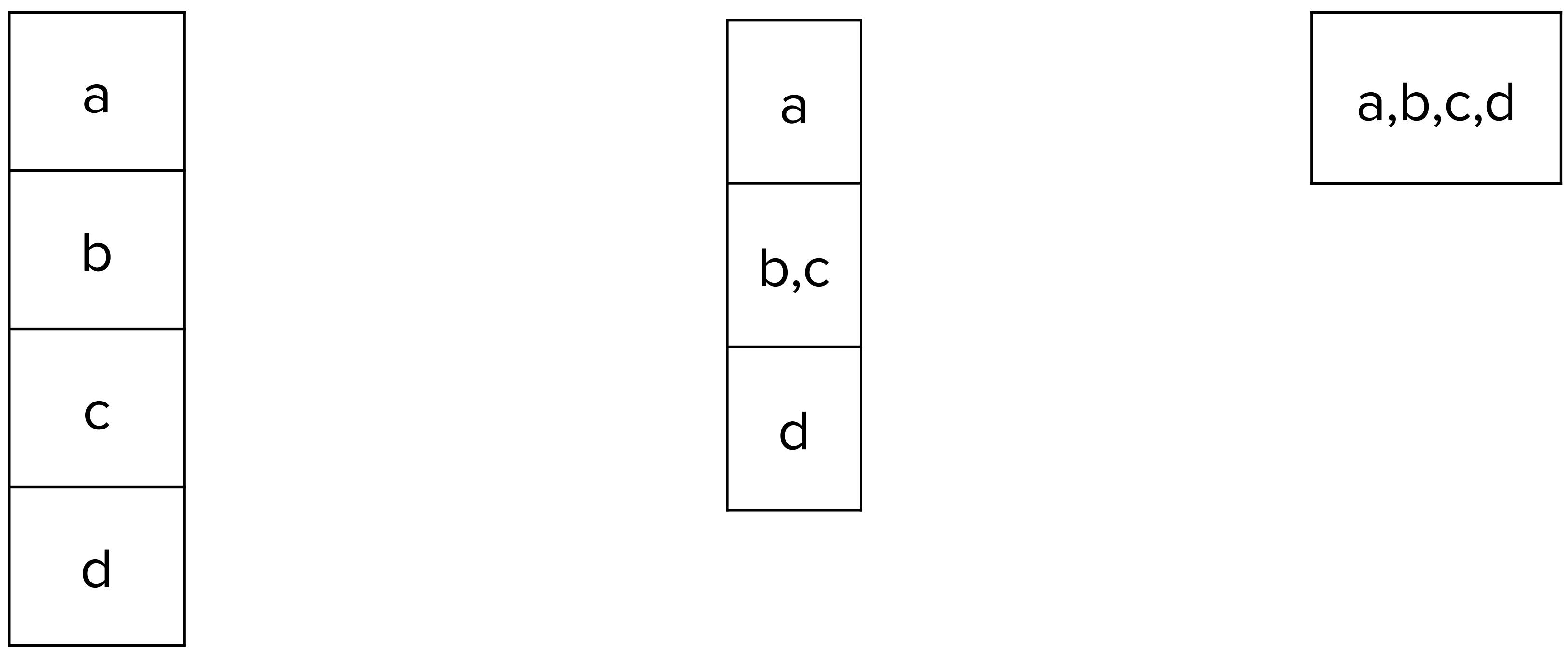
$\mathcal{R}$ is the set of all weak orders on A

preference profile is any element of $\mathcal{R}^n$

VISUAL NOTATION FOR PREFERENCES



“I care” **“I somewhat care”** **“I don’t care”**

AUXILIARY NOTATION 🤯

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PREFERENCE VS STRICT PREFERENCE

a
b,c
d

PREFERENCE VS STRICT PREFERENCE

a
b,c
d

aRb

bRc

cRb

bRd

PREFERENCE VS STRICT PREFERENCE

a
b,c
d

aRb aPb

bRc

cRb

bRd bPd

SOCIAL PREFERENCE (CHOICE) FUNCTION

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any function from $\mathcal{R}^n$ to $\mathcal{R}$

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input: $(R_1, \ldots , R_n)$ output: R

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any function from $\mathcal{R}^n$ to $\mathcal{R}$

input: $(R_1, \ldots, R_n)$ output: R

unanimity: if aP_ib for each i, then aPb

i is the dictator: if aP_ib, then aPb

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input: $(R_1, \dots , R_n)$ output: R

unanimity: if aP_ib for each i, then aPb

i is the dictator: if aP_ib, then aPb

Independence of Irrelevant Alternatives (IIA) :

for any $(R_1, \dots , R_n)$ and $(R'_1, \dots , R'_n)$ such that aR_ib iff aR'_ib for each i,

if R and R' are social preferences of profile $(R_1, \dots , R_n)$ and $(R'_1, \dots , R'_n)$, then aRb iff $aR'b$

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Equivalent version of IIA :

for any $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$ such that aP_ib iff aP'_ib for each i,

if R and R' are social preferences of profile $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$, then aPb iff $aP'b$

ARROW'S THEOREM

Arrow's Theorem. If a social preference function satisfies Unanimity and Independence of Irrelevant Alternatives (IIA), then some individual is a dictator.

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i is the dictator: if aP_ib, then aPb

i is decisive for a over b: if aP_ib, then aPb

STEP A

- Fix two distinct alternatives a and b.
- Consider profiles

R1	R2	…	Rn
a	a	…	a
b	b	…	b
…	…	…	…
…	…	…	…

aPb by Unanimity

R1	R2	…	Rn
b	b	…	b
a	a	…	a
…	…	…	…
…	…	…	…

bPa by Unanimity

STEP B

- **Pick i* such that**

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	a	a	…	a
a	…	a	b	b	…	b
…	…	…	…	…	…	…
…	…	…	…	…	…	…

still aPb

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	b	a	…	a
a	…	a	a	b	…	b
…	…	…	…	…	…	…
…	…	…	…	…	…	…

already not aPb
thus bRa

STEP C

from Stop B:

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

aPb

STEP C

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

aPb

CONSIDER:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

STEP C

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

aPb

CONSIDER:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPb by IIA from left profile

STEP C

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

aPb

CONSIDER:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPb by IIA from left profile

bPc by Unanimity

STEP C

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

aPb

CONSIDER:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPb by IIA from left profile

bPc by Unanimity

aPc because preference is transitive

STEP D

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

bRa

from Stop C:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPc

STEP D

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

bRa

from Stop C:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPc

CONSIDER
ANY PROFILE
OF THE FORM:

STEP D

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

bRa

from Stop C:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPc

CONSIDER ANY PROFILE OF THE FORM:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

STEP D

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

bRa

from Stop C:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPc

CONSIDER ANY PROFILE OF THE FORM:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bRa by IIA from upper left

STEP D

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

bRa

from Stop C:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPc

CONSIDER ANY PROFILE OF THE FORM:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bRa by IIA from upper left

aPc by IIA from upper right

STEP D

from Stop B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

bRa

from Stop C:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
c	...	c	b	b	...	b
a	...	a	c	c	...	c
...	...	...	...	...	...	...

aPc

CONSIDER ANY PROFILE OF THE FORM:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bRa by IIA from upper left

aPc by IIA from upper right

bPc by transitivity

STEP E

from Stop D:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc

STEP E

from Stop D:

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc

Note: any profile such that bPi*c ranks b and c the same way as one of the profiles on the left!

STEP E

from Stop D:

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc

Note: any profile such that $bP_{i^*}c$ ranks b and c the same way as one of the profiles on the left!

Thus, bPc any profile such that $bP_{i^*}c$

STEP E

from Stop D:

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc

Note: any profile such that bPi*c ranks b and c the same way as one of the profiles on the left!

Thus, bPc any profile such that $bP_{i^*}c$

Hence, i* is decisive for b over c by IIA

STEP E

from Stop D:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b/c	...	b/c	b	a	...	a
a	...	a	a	b/c	...	b/c
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc

Note: any profile such that bPi*c ranks b and c the same way as one of the profiles on the left!

Thus, bPc any profile such that $bP_{i^*}c$

Hence, i* is decisive for b over c by IIA

i* is decisive for b over c

STEP F

i* is decisive for b over c

STEP F

i* is decisive for b over c

consider:

R_1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	R_n
a/c	...	a/c	a	a/c	...	a/c
b	...	b	b	b	...	b
...	...	...	c	...	...	...
...	...	...	...	...	...	...

STEP F

i* is decisive for b over c

consider:

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
a/c	…	a/c	a	a/c	…	a/c
b	…	b	b	b	…	b
…	…	…	c	…	…	…
…	…	…	…	…	…	…

bPc because i* is decisive for b over c

STEP F

i* is decisive for b over c

consider:

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/c	...	a/c	a	a/c	...	a/c
b	...	b	b	b	...	b
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc because i* is decisive for b over c

aPb by Unanimity

STEP F

i* is decisive for b over c

consider:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/c	...	a/c	a	a/c	...	a/c
b	...	b	b	b	...	b
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc because i* is decisive for b over c

aPb by Unanimity

aPc by transitivity

STEP F

i* is decisive for b over c

consider:

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/c	...	a/c	a	a/c	...	a/c
b	...	b	b	b	...	b
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc because i* is decisive for b over c

aPb by Unanimity

aPc by transitivity

Note: any profile such that aPi*c ranks a and c the same way as one of the profiles on the left!

STEP F

i* is decisive for b over c

consider:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/c	...	a/c	a	a/c	...	a/c
b	...	b	b	b	...	b
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc because i* is decisive for b over c

aPb by Unanimity

aPc by transitivity

Note: any profile such that aPi*c ranks a and c the same way as one of the profiles on the left!

Hence, i* is decisive for a over c by IAA

STEP F

i* is decisive for b over c

consider:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/c	...	a/c	a	a/c	...	a/c
b	...	b	b	b	...	b
...	...	...	c	...	...	...
...	...	...	...	...	...	...

bPc because i* is decisive for b over c

aPb by Unanimity

aPc by transitivity

Note: any profile such that aPi*c ranks a and c the same way as one of the profiles on the left!

Hence, i* is decisive for a over c by IAA

i* is decisive for b over c
i* is decisive for a over c

STEP G

i* is decisive for b over c
i* is decisive for a over c

STEP G

i^* is decisive for b over c
i^* is decisive for a over c

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

still aPb

STEP G

i* is decisive for b over c
i* is decisive for a over c

STEP B:

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

still aPb

CONSIDER

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

STEP G

i* is decisive for b over c
i* is decisive for a over c

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

still aPb

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

aPb by IIA from left profile

STEP G

i^* is decisive for b over c
i^* is decisive for a over c

STEP B:

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	a	a	…	a
a	…	a	b	b	…	b
…	…	…	…	…	…	…
…	…	…	…	…	…	…

still aPb

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	c	c	…	c
c	…	c	a	a	…	a
a	…	a	b	b	…	b
…	…	…	…	…	…	…

aPb by IIA from left profile

cPa by Unanimity

STEP G

i* is decisive for b over c
i* is decisive for a over c

STEP B:

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	a	a	…	a
a	…	a	b	b	…	b
…	…	…	…	…	…	…
…	…	…	…	…	…	…

still aPb

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	c	c	…	c
c	…	c	a	a	…	a
a	…	a	b	b	…	b
…	…	…	…	…	…	…

aPb by IIA from left profile

cPa by Unanimity

cPb because preference is transitive

STEP H

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

already not aPb
thus bRa

STEP G:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

cPb

STEP H

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	a/c	...	a/c
a/c	...	a/c	b	b	...	b
...	...	...	a	...	...	...
...	...	...	...	...	...	...

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

already not aPb
thus bRa

STEP G:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

cPb

STEP H

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

already not aPb
thus bRa

STEP G:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

cPb

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	a/c	...	a/c
a/c	...	a/c	b	b	...	b
...	...	...	a	...	...	...
...	...	...	...	...	...	...

bRa by IIA from upper-left profile

STEP H

STEP B:

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	b	a	…	a
a	…	a	a	b	…	b
…	…	…	…	…	…	…
…	…	…	…	…	…	…

already not aPb
thus bRa

STEP G:

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	c	c	…	c
c	…	c	a	a	…	a
a	…	a	b	b	…	b
…	…	…	…	…	…	…

cPb

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
b	…	b	c	a/c	…	a/c
a/c	…	a/c	b	b	…	b
…	…	…	a	…	…	…
…	…	…	…	…	…	…

bRa by IIA from upper-left profile

cPb by IIA from lower-left profile

STEP H

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

already not aPb
thus bRa

STEP G:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

cPb

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	a/c	...	a/c
a/c	...	a/c	b	b	...	b
...	...	...	a	...	...	...
...	...	...	...	...	...	...

bRa by IIA from upper-left profile

cPb by IIA from lower-left profile

cPa by transitivity

STEP H

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	a/c	...	a/c
a/c	...	a/c	b	b	...	b
...	...	...	a	...	...	...
...	...	...	...	...	...	...

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

already not aPb
thus bRa

bRa by IIA from upper-left profile

cPb by IIA from lower-left profile

cPa by transitivity

Note: any profile such that cPi*a ranks a and c the same way as one of the profiles above

STEP G:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

cPb

STEP H

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

already not aPb
thus bRa

STEP G:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

cPb

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	a/c	...	a/c
a/c	...	a/c	b	b	...	b
...	...	...	a	...	...	...
...	...	...	...	...	...	...

bRa by IIA from upper-left profile

cPb by IIA from lower-left profile

cPa by transitivity

Note: any profile such that $cP_{i^*}a$ ranks a and c the same way as one of the profiles above

Hence, i* is decisive for c over a by IAA

STEP H

STEP B:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	b	a	...	a
a	...	a	a	b	...	b
...	...	...	...	...	...	...
...	...	...	...	...	...	...

already not aPb
thus bRa

STEP G:

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	c	...	c
c	...	c	a	a	...	a
a	...	a	b	b	...	b
...	...	...	...	...	...	...

cPb

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
b	...	b	c	a/c	...	a/c
a/c	...	a/c	b	b	...	b
...	...	...	a	...	...	...
...	...	...	...	...	...	...

bRa by IIA from upper-left profile

cPb by IIA from lower-left profile

cPa by transitivity

Note: any profile such that $cP_{i^*}a$ ranks a and c the same way as one of the profiles above

Hence, i* is decisive for c over a by IAA

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a

STEP I

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a

STEP I

i^* is decisive for b over c
i^* is decisive for a over c
i^* is decisive for c over a

CONSIDER

$R1$	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
a	…	a	c	a	…	a
b/c	…	b/c	a	b/c	…	b/c
…	…	…	b	…	…	…
…	…	…	…	…	…	…

STEP I

i^* is decisive for b over c
i^* is decisive for a over c
i^* is decisive for c over a

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a	...	a	c	a	...	a
b/c	...	b/c	a	b/c	...	b/c
...	...	...	b	...	...	...
...	...	...	...	...	...	...

cPa because i^* is decisive for c over a

STEP I

i^* is decisive for b over c
i^* is decisive for a over c
i^* is decisive for c over a

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
a	…	a	c	a	…	a
b/c	…	b/c	a	b/c	…	b/c
…	…	…	b	…	…	…
…	…	…	…	…	…	…

cPa because i^* is decisive for c over a

aPb by Unanimity

STEP I

i^* is decisive for b over c
i^* is decisive for a over c
i^* is decisive for c over a

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a	...	a	c	a	...	a
b/c	...	b/c	a	b/c	...	b/c
...	...	...	b	...	...	...
...	...	...	...	...	...	...

cPa because i^* is decisive for c over a

aPb by Unanimity

cPb by transitivity

STEP I

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
a	…	a	c	a	…	a
b/c	…	b/c	a	b/c	…	b/c
…	…	…	b	…	…	…
…	…	…	…	…	…	…

cPa because i* is decisive for c over a

aPb by Unanimity

cPb by transitivity

Note: any profile such that cPi*b ranks b and c the same way as one of the profiles on the left

STEP I

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a

CONSIDER

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a	...	a	c	a	...	a
b/c	...	b/c	a	b/c	...	b/c
...	...	...	b	...	...	...
...	...	...	...	...	...	...

cPa because i* is decisive for c over a

aPb by Unanimity

cPb by transitivity

Note: any profile such that cPi*b ranks b and c the same way as one of the profiles on the left

Hence, i* is decisive for c over b by IAA

STEP I

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a

CONSIDER

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a	...	a	c	a	...	a
b/c	...	b/c	a	b/c	...	b/c
...	...	...	b	...	...	...
...	...	...	...	...	...	...

cPa because i* is decisive for c over a

aPb by Unanimity

cPb by transitivity

Note: any profile such that cPi*b ranks b and c the same way as one of the profiles on the left

Hence, i* is decisive for c over b by IAA

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
a/b	…	a/b	a	a/b	…	a/b
…	…	…	c	…	…	…
…	…	…	b	…	…	…
…	…	…	…	…	…	…

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

$R1$	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
a/b	…	a/b	a	a/b	…	a/b
…	…	…	c	…	…	…
…	…	…	b	…	…	…
…	…	…	…	…	…	…

aPc because i* is decisive for a over c

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

$R1$	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
a/b	…	a/b	a	a/b	…	a/b
…	…	…	c	…	…	…
…	…	…	b	…	…	…
…	…	…	…	…	…	…

aPc because i* is decisive for a over c

cPb because i* is decisive for c over b

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/b	...	a/b	a	a/b	...	a/b
...	...	...	c	...	...	...
...	...	...	b	...	...	...
...	...	...	...	...	...	...

aPc because i* is decisive for a over c

cPb because i* is decisive for c over b

aPb by transitivity

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/b	...	a/b	a	a/b	...	a/b
...	...	...	c	...	...	...
...	...	...	b	...	...	...
...	...	...	...	...	...	...

aPc because i* is decisive for a over c

cPb because i* is decisive for c over b

aPb by transitivity

Note: any profile such that aPi*b ranks a and b the same way as one of the profiles on the left

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/b	...	a/b	a	a/b	...	a/b
...	...	...	c	...	...	...
...	...	...	b	...	...	...
...	...	...	...	...	...	...

aPc because i* is decisive for a over c

cPb because i* is decisive for c over b

aPb by transitivity

Note: any profile such that aPi*b ranks a and b the same way as one of the profiles on the left

Hence, i* is decisive for a over b by IAA

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

R1	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
a/b	...	a/b	a	a/b	...	a/b
...	...	...	c	...	...	...
...	...	...	b	...	...	...
...	...	...	...	...	...	...

aPc because i* is decisive for a over c

cPb because i* is decisive for c over b

aPb by transitivity

Note: any profile such that aPi*b ranks a and b the same way as one of the profiles on the left

Hence, i* is decisive for a over b by IAA

Similarly, i* is decisive for b over a by IAA

STEP J

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
a/b	…	a/b	a	a/b	…	a/b
…	…	…	c	…	…	…
…	…	…	b	…	…	…
…	…	…	…	…	…	…

aPc because i* is decisive for a over c

cPb because i* is decisive for c over b

aPb by transitivity

Note: any profile such that aPi*b ranks a and b the same way as one of the profiles on the left

Hence, i* is decisive for a over b by IAA

Similarly, i* is decisive for b over a by IAA

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

STEP K

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

STEP K

pick any x, y different from a, b

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

STEP K

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

pick any x, y different from a, b

CONSIDER

STEP K

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

pick any x, y different from a, b

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
x/y	…	x/y	x	x/y	…	x/y
a/b	…	a/b	a	a/b	…	a/b
…	…	…	b	…	…	…
…	…	…	y	…	…	…

STEP K

pick any x, y different from a, b

CONSIDER

$R1$	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
x/y	…	x/y	x	x/y	…	x/y
a/b	…	a/b	a	a/b	…	a/b
…	…	…	b	…	…	…
…	…	…	y	…	…	…

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

xPa because i* is decisive for x over y

STEP K

pick any x, y different from a, b

CONSIDER

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
x/y	...	x/y	x	x/y	...	x/y
a/b	...	a/b	a	a/b	...	a/b
...	...	...	b	...	...	...
...	...	...	y	...	...	...

i^* is decisive for b over c
i^* is decisive for a over c
i^* is decisive for c over a
i^* is decisive for c over b
i^* is decisive for a over b
i^* is decisive for b over a

xPa because i^* is decisive for x over y

aPb because i^* is decisive for a over b

STEP K

pick any x, y different from a, b

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
x/y	…	x/y	x	x/y	…	x/y
a/b	…	a/b	a	a/b	…	a/b
…	…	…	b	…	…	…
…	…	…	y	…	…	…

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

xPa because i* is decisive for x over y

aPb because i* is decisive for a over b

bPy because i* is decisive for b over y

STEP K

pick any x, y different from a, b

CONSIDER

$R1$	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
x/y	…	x/y	x	x/y	…	x/y
a/b	…	a/b	a	a/b	…	a/b
…	…	…	b	…	…	…
…	…	…	y	…	…	…

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

xPa because i* is decisive for x over y

aPb because i* is decisive for a over b

bPy because i* is decisive for b over y

xPy by transitivity

STEP K

pick any x, y different from a, b

CONSIDER

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
x/y	...	x/y	x	x/y	...	x/y
a/b	...	a/b	a	a/b	...	a/b
...	...	...	b	...	...	...
...	...	...	y	...	...	...

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

xPa because i* is decisive for x over y

aPb because i* is decisive for a over b

bPy because i* is decisive for b over y

xPy by transitivity

Note: any profile such that $xP_{i^*}y$ ranks x and y the same way as one of the profiles on the left

STEP K

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
i* is decisive for b over a

pick any x, y different from a, b

CONSIDER

$R1$	...	R_{i^*-1}	R_{i^*}	R_{i^*+1}	...	Rn
x/y	...	x/y	x	x/y	...	x/y
a/b	...	a/b	a	a/b	...	a/b
...	...	...	b	...	...	...
...	...	...	y	...	...	...

xPa because i* is decisive for x over y

aPb because i* is decisive for a over b

bPy because i* is decisive for b over y

xPy by transitivity

Note: any profile such that xPi*y ranks x and y the same way as one of the profiles on the left

Hence, i* is decisive for x over y by IAA

STEP K

pick any x, y different from a, b

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
x/y	…	x/y	x	x/y	…	x/y
a/b	…	a/b	a	a/b	…	a/b
…	…	…	b	…	…	…
…	…	…	y	…	…	…

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
i* is decisive for a over b
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xPa because i* is decisive for x over y

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Hence, i* is decisive for x over y by IAA

i* is the dictator

STEP K

pick any x, y different from a, b

CONSIDER

R1	…	R_{i^*-1}	R_{i^*}	R_{i^*+1}	…	Rn
x/y	…	x/y	x	x/y	…	x/y
a/b	…	a/b	a	a/b	…	a/b
…	…	…	b	…	…	…
…	…	…	y	…	…	…

i* is decisive for b over c
i* is decisive for a over c
i* is decisive for c over a
i* is decisive for c over b
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xPa because i* is decisive for x over y

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Note: any profile such that xPi*y ranks x and y the same way as one of the profiles on the left

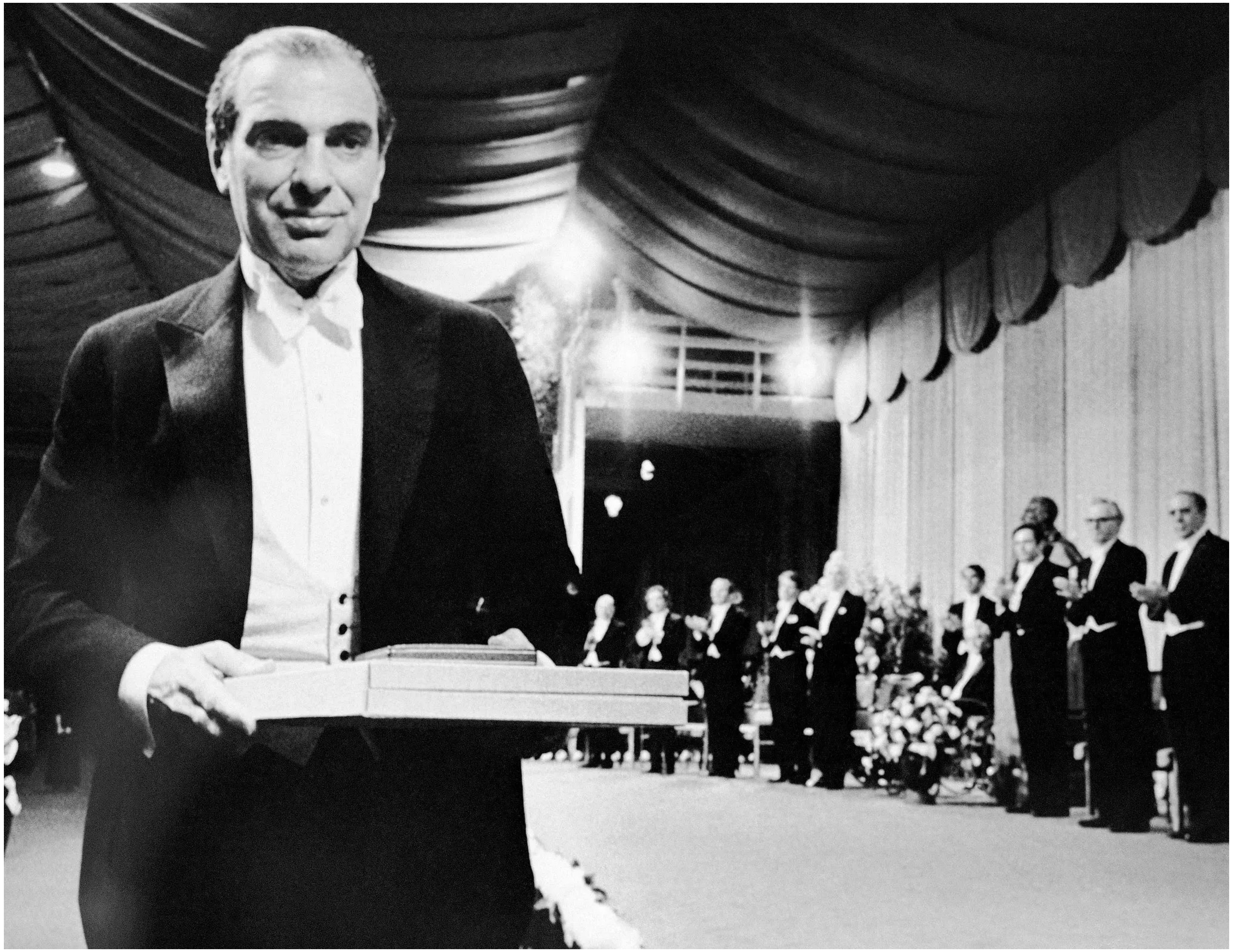
Hence, i* is decisive for x over y by IAA

😱 i* is the dictator 😱

ARROW'S THEOREM

Arrow's Theorem. If a social preference function satisfies Unanimity and Independence of Irrelevant Alternatives (IIA), then some individual is a dictator.

👏



Kenneth J. Arrow receiving the Nobel Memorial Prize in Economic Science in Stockholm in 1972.
Associated Press