INTRODUCTION TO UTILITY THEORY

COMP6203 - Intelligent Agents

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- A game is a mathematical model of interactive decision making.
- Different agents interact with the goal to achieve the best possible outcome.
- "Best outcome" might mean different things: winning, maximising payoff, etc.
- In order to understand what this means, we need to define a formal model of decision making.

- For an agent, making a decision means selecting one among multiple choices.
- Let $S = \{s_1, \ldots, s_m\}$ denote the choices available to an agent.
- Making a choice leads to a certain outcome.
- Let $\Omega = \{\omega_1, \ldots, \omega_t\}$ denote a set of outcomes.
- Ω may be:
  - all the possible outcomes of a game of chess
  - the possible outcomes of negotiations between nations
  - the possible outcomes of an eBay auction
  - ... and so on.
- In these lectures we always assume the sets S, Ω to be **finite**.

- An outcome function
$$g : S \to \Omega$$
maps each choice to a an outcome.
- g specifies the consequence of making a certain choice.
- How does an agent select the best outcome?
- We assume that, given any pair of outcomes (ω, ω'), an agent can say which of them they prefer.
- We give a formal definition of the notion of preference.

- We use slightly different interpretations of preferences, depending on whether the decision-making setting is one of certainty or uncertainty.
- In decision-making under certainty, we know exactly what the consequences of our choices will be.
- In decision-making under uncertainty, we don't know exactly what the consequences of our choices will be: for every possible choice, there are multiple possible consequences, each with an attached probability (more on this later!).

A **preference** is a binary relation $\succeq$ on $\Omega \times \Omega$. We assume this relation is:

- Reflexive:

$$\omega \succeq \omega \text{ for all } \omega \in \Omega$$

- Total:

$$\text{for all } \omega, \omega' \in \Omega, \text{ either } \omega \succeq \omega' \text{ or } \omega' \succeq \omega$$

- Transitive:

$$\text{for all } \omega, \omega', \omega'' \in \Omega, \text{ if } \omega \succeq \omega' \text{ and } \omega' \succeq \omega'', \text{ then } \omega \succeq \omega''.$$

Given a pair of outcomes ω, ω',

$$\omega \succeq \omega'$$

means that an agent **prefers** ω at least as much as ω'.

If both

$$\omega \succeq \omega' \text{ and } \omega' \succeq \omega$$

then we are **indifferent** between the two. We denote it by

$$\omega \sim \omega'$$

If both

$$\omega \succeq \omega' \text{ but not } \omega' \succeq \omega$$

then we **strictly prefer** ω over ω', and denote it by

$$\omega \succ \omega'$$

- To build a mathematical model we define functions to model preferences.
- A **utility function** is a mapping

$$u : \Omega \to \mathbb{R}$$

- A utility function u over Ω is said to **represent** a preference relation $\succeq$ iff, for all $\omega, \omega' \in \Omega$:

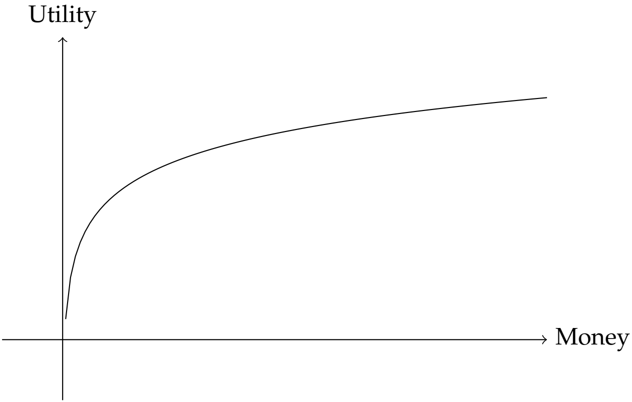
$$\omega \succeq \omega' \qquad \text{iff} \qquad u(\omega) \geq u(\omega')$$

- Given a finite set Ω, for every preference relation $\succeq$ on $\Omega \times \Omega$ there exists a utility function $u : \Omega \to \mathbb{R}$ that represents it.

UTILITY

- Utility functions are a numerical representation of preferences.
- An agent chooses an outcome over another not because the numerical value of the utility, but because of their preference.
- An agent choosing the best outcome is maximising their utility, as long as the representation is the correct one.
- The numerical values are useful for adopting numerical techniques for the mathematical models, but the values themselves are not important. Their relative order is. It is how agents rank their preferences.
- Utilities of different agents cannot in general be compared.

- Utility values are not money.
- They can sometimes be interpreted as such, but they measure different things.
- Numerical representation of individual preferences vs objective numerical scale.



- We can formally represent decision-making as a tuple

$$\langle S, \Omega, g, \succeq, u \rangle$$

where

  - S and Ω are the set of choices and consequences
  - $g : S \to \Omega$ is an outcome function that specifies the consequence of each choice.
  - $\succeq \subseteq \Omega \times \Omega$ is the agent's preference relation.
  - $u : \Omega \to \mathbb{R}$ is the utilty function representing $\succeq$

- A rational decision maker is one that makes the best possible choice
- This means that the agent makes the choice that maximises their utility, i.e.

$$s \in \operatorname*{argmax}_{s \in S} u(g(s)).$$

- It is worth noticing that using numerical utilities allows us to express decision problems as optimisation problems.

- In many settings, we don't know exactly what outcome will result by making a choice.
- For every option $s \in S$, there will typically be a range of possible outcomes, with differing probabilities of occurring.
- Such settings require more complex machinery for preferences and utilities.
- In particular, preference relations $\succeq$ over outcomes are not enough: we need preferences over **lotteries**.

- To represent the uncertainty of an outcome we use probability distributions.
- A **probability distribution** over a non-empty set of outcomes Ω is a function

  $$\mu : \Omega \to [0, 1]$$

  such that

  $$\sum_{\omega \in \Omega} \mu(\omega) = 1.$$

- A probability distribution represents the likelihood of the occurrence of a certain outcome.

- A **lottery** over a set of outcomes Ω is a probability distribution over Ω.
- We denote a lottery by L.
- Example: suppose

$$\Omega = \{\text{chocolate}, \text{vanilla}, \text{strawberry}\}$$

and μ is a probability distribution over Ω, such that

$$\mu(\text{chocolate}) = 0.2 \quad \mu(\text{vanilla}) = 0.5 \quad \mu(\text{strawberry}) = 0.3$$

we have a lottery

$$L = [0.2(\text{chocolate}), 0.5(\text{vanilla}), 0.3(\text{strawberry})]$$

- Notice that the concept of a lottery generalises the concept of an outcome.
- If ω is the consequence of a choice s, this corresponds to the lottery where $\mu(\omega) = 1$ and, for all $\omega' \in \Omega$ such that $\omega \neq \omega'$, $\mu(\omega') = 0$.
- This is what is called a **degenerate lottery**.
- Similar to the previous case, we can define a preference relation $\succeq$ over a set of lotteries $\mathcal{L}$
- Again, we can define a utility function as a function that represents the preferences of an agent over their set of lotteries.

- We also define the notion of a **compound lottery**, i.e. a lottery of lotteries.
- Given a set $\mathcal{L} = \{L_1, \ldots, L_n\}$ of lotteries, a compound lottery is a probability distribution μ over $\mathcal{L}$ and is given by

$$L^\star = [q_1(L_1), \ldots, q_n(L_n)]$$

where $q_i = \mu(L_i)$.

- Imagine the path you choose to go to work changes depending on whether it rains or not.
- When you choose a path it takes different times to reach your destination depending on the circumstances.
- Let Ω be the possible minutes it takes to go to work.
- Define two lotteries L_{rain} and L_{sun} over Ω that represent the probability of reaching work in a certain amount of time.
- Define the lottery
$$L^\star = [pL_{\text{rain}}, (1-p)L_{\text{sun}}\}$$
where p is the probability it will rain.
- $L^\star$ is a compound lottery that describes the uncertainty of reaching the destination in a certain time taking into account the weather.

- Given a set of outcomes Ω and a utility function $u : \Omega \to \mathbb{R}$, the expected utility of a lottery L, with probability distribution μ, is given by

$$EU(L) = \sum_{\omega \in \Omega} u(\omega)\mu(\omega).$$

- EU is the expected value of the function u given the probability distribution μ.
- EU is the average utility we could expect from the lottery.

- We have introduced preferences over lotteries, utility functions over outcomes, and expected utility.
- How are these connected?
- What is the relation between preferences over lotteries and expected utility?
- von Neumann and Morgenstern discovered a link between preferences over lotteries and expected utility based on utility functions over outcomes.
- Their result connects the ordinal representation of preferences with the numerical concept of an expected utility.
- Preferences are required to satisfy the following axioms...

Axiom (Continuity)

For every lotteries $L_1 \succ L_2 \succ L_3$, there exist $\alpha, \beta \in]0,1[$ such that

$$[\alpha L_1, (1-\alpha)L_3] \succ L_2 \succ [\beta L_1, (1-\beta)L_3]$$

Axiom (Independence)

For all lotteries L_1, L_2, L_3 and $\alpha \in]0,1]$

$$L_1 \succeq L_2 \quad \textit{IFF} \quad [\alpha L_1, (1-\alpha)L_3] \succeq [\alpha L_2, (1-\alpha)L_3]$$

Theorem

Let $\hat{\mathcal{L}}$ be the set of compound lotteries over a finite set Ω and let $\succeq$ be a preference relation over $\hat{\mathcal{L}}$. Then, the following are equivalent:

1. *$\succeq$ satisfies continuity and independence.*
2. *There exists a utility function over Ω such that for all $L_1, L_2 \in \hat{\mathcal{L}}$*

$$L_1 \succeq L_2 \quad IFF \quad EU(L_1) \geq EU(L_2)$$

References

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