Algorithmic Game Theory
COMP6207

Lecture 4: Basics of Mechanism Design

Bahar Rastegari
b.rastegari@soton.ac.uk
Electronics and Computer Science
University of Southampton

Learning Outcomes

By the end of this lecture, you should be able to

- **Define** what is a *mechanism*, what is a *direct mechanism*, and what is a *quasilinear mechanism*
- **Describe** the relationship between a *Bayesian game setting*, a *Bayesian game*, and a *mechanism*.
- **Define** what does it mean for a mechanism to be *dominant-strategy truthful* and **identify** whether a mechanism is *dominant-strategy truthful*.

Recap: Bayesian Game

A tuple (N, A, Θ, p, u) where

- $N = \{1, \ldots, n\}$ is a finite set of agents
- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$ where Θ_i is the type space of player i
- $p : \Theta \mapsto [0, 1]$ is a common-prior probability distribution on Θ
- $u = (u_1, \ldots, u_n)$, where $u_i : A \times \Theta \mapsto \mathbb{R}$ is the utility function for player i.

Recap: Bayesian Game with **Strict Incomplete Information**

is a tuple (N, A, Θ, u) where

- $N = \{1, \ldots, n\}$ is a finite set of agents
- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$ where Θ_i is the type space of player i
- $p : \Theta \mapsto [0, 1]$ is a common-prior probability distribution on Θ
- $u = (u_1, \ldots, u_n)$, where $u_i : A \times \Theta \mapsto \mathbb{R}$ is the utility function for player i.

Recap: Bayesian Game Setting

A tuple (N, O, Θ, p, u)

- $N = \{1, \ldots, n\}$ is a finite set of agents
- O is a set of outcomes
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$ is a set of possible joint type vector
- p is a common-prior probability distribution on Θ
- $u = (u_1, \ldots, u_n)$, where $u_i : O \times \Theta \mapsto \mathbb{R}$ is the utility function for player i.

The key difference with Bayesian Game is that the Bayesian Game Setting does **not include actions** for the agents, and instead defines the utility function over the **set of possible outcomes**.

Mechanism

Definition (Mechanism)

A (deterministic) mechanism (for a Bayesian game setting (N, O, Θ, p, u)) is a pair (A, M), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i, and
- $M : A \mapsto O$ maps each action profile to an outcome.

Mechanism

Definition (Mechanism)

A (deterministic) mechanism (for a Bayesian game setting (N, O, Θ, p, u)) is a pair (A, M), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i, and
- $M : A \mapsto O$ maps each action profile to an outcome.

Footnote: Mechanisms need not to be deterministic (they can be randomised) in which case $M : A \mapsto \Pi(O)$. For now, however, we only focus on deterministic mechanisms.

Direct Mechanism

A direct mechanism is a mechanism in which the only action available to each agent is to announce his private information.

Direct Mechanism

A direct mechanism is a mechanism in which the only action available to each agent is to announce his private information.

So in a direct mechanism $A_i = \Theta_i$ for each i.

Direct Mechanism

A direct mechanism is a mechanism in which the only action available to each agent is to announce his private information.

So in a direct mechanism $A_i = \Theta_i$ for each i.

A (deterministic) direct mechanism (for a Bayesian game setting (N, O, Θ, p, u)) is a pair (A, M), where

- $A = A_1 \times \ldots \times A_n$, where $A_i = \Theta_i$ is the set of actions available to agent i, and
- $M : A \mapsto O$ maps each action profile to an outcome.

Quasilinear Utility settings

In a quasilinear utility setting:

- the set of outcomes is $O = X \times \mathbb{R}^n$ for a finite set of choices X, and
- when outcome $o = (x, (p_1, \ldots, p_n))$ is chosen, the utility of an agent i given joint type θ is $u_i(o, \theta) = u_i(x, \theta) - p_i$.

Quasilinear Utility settings

In a quasilinear utility setting:

- the set of outcomes is $O = X \times \mathbb{R}^n$ for a finite set of choices X, and
- when outcome $o = (x, (p_1, \ldots, p_n))$ is chosen, the utility of an agent i given joint type θ is $u_i(o, \theta) = u_i(x, \theta) - p_i$.

When we assume independent private values (IPVs), we can write agent i's utility function as $u_i(x, \theta_i)$.

Quasilinear Utility settings

In a quasilinear utility setting:

- the set of outcomes is $O = X \times \mathbb{R}^n$ for a finite set of choices X, and
- when outcome $o = (x, (p_1, \dots, p_n))$ is chosen, the utility of an agent i given joint type θ is $u_i(o, \theta) = u_i(x, \theta) - p_i$.

When we assume independent private values (IPVs), we can write agent i's utility function as $u_i(x, \theta_i)$.

We can then also refer to an agent's valuation for choice x, written $v_i(x) = u_i(x, \theta_i)$.

- v_i is the maximum amount of money i is willing to pay for the mechanism to choose x.
- The notation of v_i does not explicitly refer to θ_i, but an agent's valuation does depend on her type (θ_i is dropped for simplicity of notation and because it can be inferred from the context).

Mechanisms in the quasilinear settings

In a quasilinear utility setting:

- the set of outcomes is $O = X \times \mathbb{R}^n$ for a finite set of choices X, and
- when outcome $o = (x, (p_1, \ldots, p_n))$ is chosen, the utility of an agent i given joint type θ is $u_i(o, \theta) = u_i(x, \theta) - p_i$.

Mechanisms in the quasilinear settings

In a quasilinear utility setting:

- the set of outcomes is $O = X \times \mathbb{R}^n$ for a finite set of choices X, and
- when outcome $o = (x, (p_1, \ldots, p_n))$ is chosen, the utility of an agent i given joint type θ is $u_i(o, \theta) = u_i(x, \theta) - p_i$.

Definition (Quasilinear mechanism)

A quasilinear mechanism is a triple (A, χ, p), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i,
- Choice function $\chi : A \mapsto X$ maps each action profile to a choice in X, and
- Payment function $p : A \mapsto \mathbb{R}^n$ maps each action profile to a payment for each agent.

Question: Single-item auctions as quasilinear mechanisms

A quasilinear mechanism is a triple (A, χ, p), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i,
- Choice function $\chi : A \mapsto X$ maps each action profile to a choice in X, and
- Payment function $p : A \mapsto \mathbb{R}^n$ maps each action profile to a payment for each agent.

Question: Single-item auctions as quasilinear mechanisms

A quasilinear mechanism is a triple (A, χ, p), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i,
- Choice function $\chi : A \mapsto X$ maps each action profile to a choice in X, and
- Payment function $p : A \mapsto \mathbb{R}^n$ maps each action profile to a payment for each agent.

In a sealed-bid single-item auction:

- What is the set of actions A_i available to agent i?

Question: Single-item auctions as quasilinear mechanisms

A quasilinear mechanism is a triple (A, χ, p), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i,
- Choice function $\chi : A \mapsto X$ maps each action profile to a choice in X, and
- Payment function $p : A \mapsto \mathbb{R}^n$ maps each action profile to a payment for each agent.

In a sealed-bid single-item auction:

- What is the set of actions A_i available to agent i?
- What is the finite set X?

Question: Single-item auctions as quasilinear mechanisms

A quasilinear mechanism is a triple (A, χ, p), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i,
- Choice function $\chi : A \mapsto X$ maps each action profile to a choice in X, and
- Payment function $p : A \mapsto \mathbb{R}^n$ maps each action profile to a payment for each agent.

In a sealed-bid single-item auction:

- What is the set of actions A_i available to agent i?
- What is the finite set X?
- What is the size of X (or, how many elements does X have)?

Question: Single-item auctions as quasilinear mechanisms

A quasilinear mechanism is a triple (A, χ, p), where

- $A = A_1 \times \ldots \times A_n$, where A_i is the set of actions available to agent i,
- Choice function $\chi : A \mapsto X$ maps each action profile to a choice in X, and
- Payment function $p : A \mapsto \mathbb{R}^n$ maps each action profile to a payment for each agent.

In a sealed-bid single-item auction:

- What is the set of actions A_i available to agent i?
- What is the finite set X?
- What is the size of X (or, how many elements does X have)?
- How is the payment vector $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ calculated?

A Bayesian game setting $+$ a mechanism $=$

A Bayesian game setting + a mechanism =

- A Bayesian game setting gives us (N, O, Θ, p, u) where $u_i : O \times \Theta \mapsto \mathbb{R}$.
- A mechanism gives us the set A of actions available to agents, and a mapping $M : A \mapsto O$ from action profiles to outcomes.
- So we get (N, A, Θ, p, u) where $u_i : A \times \Theta \mapsto \mathbb{R}$,

A Bayesian game setting + a mechanism =

- A Bayesian game setting gives us (N, O, Θ, p, u) where $u_i : O \times \Theta \mapsto \mathbb{R}$.
- A mechanism gives us the set A of actions available to agents, and a mapping $M : A \mapsto O$ from action profiles to outcomes.
- So we get (N, A, Θ, p, u) where $u_i : A \times \Theta \mapsto \mathbb{R}$,

a Bayesian game.

Bayes-Nash Equilibrium

- In normal-form games we have Nash equilibrium.
- In Bayesian games we have Bayes-Nash equilibrium.

Bayes-Nash Equilibrium

- In normal-form games we have Nash equilibrium.
- In Bayesian games we have Bayes-Nash equilibrium.
- Conceptually they are equivalent: A Bayes-Nash equilibrium is a mixed-strategy profile s such that each s_i is a best response to s_{-i}.

Bayes-Nash Equilibrium

- In normal-form games we have Nash equilibrium.
- In Bayesian games we have Bayes-Nash equilibrium.
- Conceptually they are equivalent: A Bayes-Nash equilibrium is a mixed-strategy profile s such that each s_i is a best response to s_{-i}.
- It's only that the calculation is more involved as to compute her best response, each agent has to take into account the probability distribution p over the set of possible joint type vectors Θ.

Mechanism design

Perform an optimisation problem, given that the values of (some of) the inputs are unknown (withheld by the agents)

Mechanism design

Perform an optimisation problem, given that the values of (some of) the inputs are unknown (withheld by the agents)

, or equivalently

design a game that implements a particular **social choice function** in equilibrium, given that the designer does not know agents' preferences and the agents might lie.

Social Choice setting

- O: a set of outcomes, e.g.
  - possible locations for a bridge or library
  - possible meeting times
  - candidates for local MPs
- N: a set of agents who have preferences over the outcomes
- Goal: designing/picking a social choice function f that maps from agents' preferences to a particular outcome, which is enforced.
- Question: how to pick function f? what properties do we desire from the function, i.e. the outcome it picks?

Bayesian Game setting versus Social Choice setting

Bayesian game setting extends the social choice setting to a new setting where agents are strategic and hence cannot be relied upon to disclose their preferences honestly.

Definition (Bayesian Game setting)

A tuple (N, O, Θ, p, u)

- $N = \{1, \ldots, n\}$ is a finite set of agents
- O is a set of outcomes
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$ is a set of possible joint type vector
- p is a common-prior probability distribution on Θ
- $u = (u_1, \ldots, u_n)$, where $u_i : O \times \Theta \mapsto \mathbb{R}$ is the utility function for player i.

Recall: Dominated and dominant strategies

Let s_i and s'_i be two strategies of agent i, and S_{-i} the set of all strategy profiles of the remaining agents.

1. s_i strictly dominates s'_i if for all $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.
2. s_i weakly dominates s'_i if for all $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$, and for at least one $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.
3. s_i very weakly dominates s'_i if for all $s_{-i} \in S_{-i}$ it is the case that $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

In all cases above we say that s'_i is a strictly (resp. weakly; very weakly) dominated strategy.

A strategy is strictly (resp. weakly; very weakly) dominant for an agent if it strictly (resp. weakly; very weakly) dominates any other strategy for that agent.

Implementation in Dominant Strategies

Roughly speaking,

a mechanism M implements a social choice function f in dominant strategies if for some dominant strategy equilibrium of the induced game, we have that the mechanism M chooses the same outcome (given action profile of the agents) as does f (given true utilities of the agents).

Dominant-Strategy Truthful mechanism

Dominant-Strategy Truthful mechanism

A strategy profile $(s_1, \ldots, s_n)$ in which every s_i is a dominant strategy for agent i is called an equilibrium in dominant strategies.

Dominant-Strategy Truthful mechanism

A strategy profile $(s_1, \ldots, s_n)$ in which every s_i is a dominant strategy for agent i is called an equilibrium in dominant strategies.

Definition (Dominant-strategy truthfulness)

A direct mechanism is dominant-strategy truthful if, for every agent i, telling the truth (i.e. revealing the true valuation) maximises i's utility, no matter what strategy the other players pick.

Also known as

- **strategy-proof** mechanism
- **dominant-strategy incentive-compatible** mechanism
- **truthful** mechanism

Dominant-Strategy Truthful mechanism

A strategy profile $(s_1, \dots, s_n)$ in which every s_i is a dominant strategy for agent i is called an equilibrium in dominant strategies.

Definition (Dominant-strategy truthfulness)

A direct mechanism is dominant-strategy truthful if, for every agent i, telling the truth (i.e. revealing the true valuation) maximises i's utility, no matter what strategy the other players pick.

Also known as

- **strategy-proof** mechanism
- **dominant-strategy incentive-compatible** mechanism
- **truthful** mechanism

Question: If Mechanism M is dominant-strategy truthful then truth-telling is a .... strategy for each agent.

Dominant-Strategy Truthful mechanism

A strategy profile $(s_1, \ldots, s_n)$ in which every s_i is a dominant strategy for agent i is called an equilibrium in dominant strategies.

Definition (Dominant-strategy truthfulness)

A direct mechanism is dominant-strategy truthful if, for every agent i, telling the truth (i.e. revealing the true valuation) maximises i's utility, no matter what strategy the other players pick.

Also known as

- **strategy-proof** mechanism
- **dominant-strategy incentive-compatible** mechanism
- **truthful** mechanism

Question: If Mechanism M is dominant-strategy truthful then truth-telling is a (very weakly) dominant strategy for each agent.

(Dominant strategy) Truthfulness in single-item auctions

Recall that we assume quasilinear utilities and IPVs.

(Dominant strategy) Truthfulness in single-item auctions

Recall that we assume quasilinear utilities and IPVs.

A single-item auction (Θ, χ, p) is dominant-strategy truthful if for each bidder i, bidding $b_i = \theta_i$ maximises i's utility, no matter what bids the other bidders place:

$$v_i(\chi(\theta_i, b_{-i})) - p(\theta_i, b_{-i}) \geq v_i(\chi(b_i, b_{-i})) - p(b_i, b_{-i}),\ \forall b_i\ \forall b_{-i}\ \forall \theta_i$$

(Dominant strategy) Truthfulness in single-item auctions

Recall that we assume quasilinear utilities and IPVs.

A single-item auction (Θ, χ, p) is dominant-strategy truthful if for each bidder i, bidding $b_i = \theta_i$ maximises i's utility, no matter what bids the other bidders place:

$$v_i(\chi(\theta_i, b_{-i})) - p(\theta_i, b_{-i}) \geq v_i(\chi(b_i, b_{-i})) - p(b_i, b_{-i}),\ \forall b_i\ \forall b_{-i}\ \forall \theta_i$$

We can actually write the above inequality as

$$v_i(\chi(\theta_i, \theta_{-i})) - p(\theta_i, \theta_{-i}) \geq v_i(\chi(b_i, \theta_{-i})) - p(b_i, \theta_{-i}),\ \forall b_i\ \forall \theta_{-i} \forall \theta_i$$

Do you see why doing so is OK?

(Dominant strategy) Truthfulness in single-item auctions

Recall that we assume quasilinear utilities and IPVs.

A single-item auction (Θ, χ, p) is dominant-strategy truthful if for each bidder i, bidding $b_i = \theta_i$ maximises i's utility, no matter what bids the other bidders place:

$$v_i(\chi(\theta_i, b_{-i})) - p(\theta_i, b_{-i}) \geq v_i(\chi(b_i, b_{-i})) - p(b_i, b_{-i}), \; \forall b_i \; \forall b_{-i} \; \forall \theta_i$$

We can actually write the above inequality as

$$v_i(\chi(\theta_i, \theta_{-i})) - p(\theta_i, \theta_{-i}) \geq v_i(\chi(b_i, \theta_{-i})) - p(b_i, \theta_{-i}), \; \forall b_i \; \forall \theta_{-i} \forall \theta_i$$

Do you see why doing so is OK?

To simplify notation, sometimes we write v_i as a function of action profiles instead of outcomes, e.g. write $v_i(b_i, b_{-i})$ instead of $v_i(\chi(b_i, b_{-i}))$.

Quiz: Which of these single-item auctions is dominant-strategy truthful?

1. English auction
2. Dutch auction
3. First-price sealed-bid auction
4. Vickrey auction

Vickrey auction is dominant-strategy truthful

Proof.

Assume that the other bidders bid in some arbitrary way. We show that i maximises her utility by bidding truthfully. We break the proof into two cases:

1. By bidding honestly, i wins the auction.
2. By bidding honestly, i loses the auction.

Case 1: By bidding honestly, i wins

Vickrey auction is dominant-strategy truthful

Proof.

Assume that the other bidders bid in some arbitrary way. We show that i maximises her utility by bidding truthfully. We break the proof into two cases:

1. By bidding honestly, i wins the auction.
2. By bidding honestly, i loses the auction.

Case 1: By bidding honestly, i wins

- If i submitted a higher bid, she would still win and pay the same amount

Vickrey auction is dominant-strategy truthful

Proof.

Assume that the other bidders bid in some arbitrary way. We show that i maximises her utility by bidding truthfully. We break the proof into two cases:

1. By bidding honestly, i wins the auction.
2. By bidding honestly, i loses the auction.

Case 1: By bidding honestly, i wins

- If i submitted a higher bid, she would still win and pay the same amount
- If i submitted a lower bid, she would either still win and pay the same amount... or lose and get utility zero.

Case 2: By bidding honestly, i loses

Vickrey auction is dominant-strategy truthful

Proof.

Assume that the other bidders bid in some arbitrary way. We show that i maximises her utility by bidding truthfully. We break the proof into two cases:

1. By bidding honestly, i wins the auction.
2. By bidding honestly, i loses the auction.

Case 1: By bidding honestly, i wins

- If i submitted a higher bid, she would still win and pay the same amount
- If i submitted a lower bid, she would either still win and pay the same amount... or lose and get utility zero.

Case 2: By bidding honestly, i loses

- If i submitted a lower bid, she would still lose and pay nothing

Vickrey auction is dominant-strategy truthful

Proof.

Assume that the other bidders bid in some arbitrary way. We show that i maximises her utility by bidding truthfully. We break the proof into two cases:

1. By bidding honestly, i wins the auction.
2. By bidding honestly, i loses the auction.

Case 1: By bidding honestly, i wins

- If i submitted a higher bid, she would still win and pay the same amount
- If i submitted a lower bid, she would either still win and pay the same amount... or lose and get utility zero.

Case 2: By bidding honestly, i loses

- If i submitted a lower bid, she would still lose and pay nothing
- If i submitted a higher bid, she would either lose and still pay nothing... or win and pay more than her valuation.

Vickrey auction is dominant-strategy truthful

Proof.

Assume that the other bidders bid in some arbitrary way. We show that i maximises her utility by bidding truthfully. We break the proof into two cases:

1. By bidding honestly, i wins the auction.
2. By bidding honestly, i loses the auction.

Case 1: By bidding honestly, i wins

- If i submitted a higher bid, she would still win and pay the same amount
- If i submitted a lower bid, she would either still win and pay the same amount... or lose and get utility zero.

Case 2: By bidding honestly, i loses

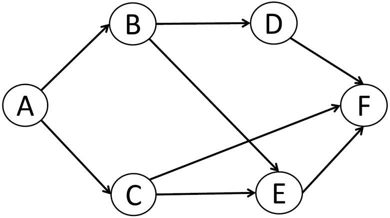
- If i submitted a lower bid, she would still lose and pay nothing
- If i submitted a higher bid, she would either lose and still pay nothing... or win and pay more than her valuation.

In either case, bidding truthfully maximises i's utility. □

Why do we so much care about
dominant-strategy truthfulness?

Fun Game

Fun game: Selfish routing



- A network with 6 vertices and 8 edges.
- Each edge has a cost and there is an agent associated with each edge.
- 8 students play as agents; others act as mediators.
- Agents' utility functions: $u_i =$ payment - cost if your edge is chosen; 0 otherwise.
- Mediators: find a path from A to F at the lowest cost you can.
- Agents: agree to be paid whatever you like; claim whatever you like; don't show your paper to anyone.