$$
\begin{aligned}
\mathrm{Im}[\chi_{\mathrm{v}}^{abc}(-2\omega;\omega,\omega)] &= \frac{\pi|e|^3}{2\hbar^2}\int\frac{d^3k}{8\pi^3}\Big[\sum_{vc}\frac{16}{(\omega_{cv}^S)^3}\Big(\sum_{c'}\frac{\mathrm{Im}[v_{vc}^{\Sigma,a}\{v_{cc'}^{\Sigma,b}v_{c'v}^{\Sigma,c}\}]}{\omega_{cv}^S-2\omega_{c'v}^S} \\
&- \sum_{v'}\frac{\mathrm{Im}[v_{vc}^{\Sigma,a}\{v_{cv'}^{\Sigma,b}v_{v'v}^{\Sigma,c}\}]}{\omega_{cv}^S-2\omega_{cv'}^S}\Big)\delta(\omega_{cv}^S-2\omega) \\
&+ \sum_{(vc)\neq\ell}\frac{1}{(\omega_{cv}^S)^3}\Big(\frac{\mathrm{Im}[v_{\ell c}^{\Sigma,a}\{v_{cv}^{\Sigma,b}v_{v\ell}^{\Sigma,c}\}]}{\omega_{c\ell}^S-2\omega_{cv}^S}-\frac{\mathrm{Im}[v_{v\ell}^{\Sigma,a}\{v_{\ell c}^{\Sigma,b}v_{cv}^{\Sigma,c}\}]}{\omega_{\ell v}^S-2\omega_{cv}^S}\Big)\delta(\omega_{cv}^S-\omega) \\
&- \sum_{vc}\frac{1}{(\omega_{cv}^S)^3}\Big(4\mathrm{Re}[\,v_{vc}^{\Sigma,a}\{\mathcal{F}_{cv}^{bc}\}]\delta(\omega_{cv}^S-2\omega)+\mathrm{Re}[\{\mathcal{F}_{vc}^{ab}v_{cv}^{\Sigma,c}\}]\delta(\omega_{cv}^S-\omega)\Big)\Big]. \qquad (1)
\end{aligned}
$$

Let's take $\omega_n \to \omega_n + \Delta_n$ that implies $\omega_n^S \to \omega_n^S + \Delta_n$, with Δ_n the energy difference of two calculations. Then, $\omega_{nm} \to \omega_{nm} + \Delta_{nm}$ and $\omega_{nm}^S \to \omega_{nm}^S + \Delta_{nm}$, with $\Delta_{nm} = \Delta_n - \Delta_m$. Then, $\omega_{nm}^S \to \omega_{nm}^S(1+\Delta_{nm}/\omega_{nm}^S) = \omega_{nm}^S(1+\delta_{nm}^S)$, with $\delta_{nm}^S = \Delta_{nm}/\omega_{nm}^S$. Likewise $\omega_{nm} \to \omega_{nm}(1+\delta_{nm})$, with $\delta_{nm} = \Delta_{nm}/\omega_{nm}$. Also, $\delta(\omega_{cv}-\omega) \to \delta(\omega_{cv}+\Delta_{cv}-\omega) \sim \delta(\omega_{cv}-\omega)$, since the difference Δ_{cv} is very small.

We expand to first order in δ_{nm}. The ω terms are

$$
\begin{aligned}
\frac{1}{(\omega_{cv}^S)^3}\frac{1}{\omega_{c\ell}^S-2\omega_{cv}^S} &\to \frac{1}{(\omega_{cv}^S(1+\delta_{cv}^S))^3}\frac{1}{\omega_{c\ell}^S(1+\delta_{c\ell}^S)-2\omega_{cv}^S(1+\delta_{cv}^S)} \\
&= \frac{1}{(\omega_{cv}^S)^3}(1+\delta_{cv}^S)^{-3}\frac{1}{\omega_{c\ell}^S-2\omega_{cv}^S+\Delta_{c\ell}-2\Delta_{cv}} \\
&= \frac{1}{(\omega_{cv}^S)^3}(1-3\delta_{cv}^S)\frac{(1+(\Delta_{c\ell}-2\Delta_{cv})/(\omega_{c\ell}^S-2\omega_{cv}^S))^{-1}}{\omega_{c\ell}^S-2\omega_{cv}^S} \\
&= \frac{1}{(\omega_{cv}^S)^3}\frac{1}{\omega_{c\ell}^S-2\omega_{cv}^S}(1-3\delta_{cv}^S)(1-\frac{\Delta_{c\ell}-2\Delta_{cv}}{\omega_{c\ell}^S-2\omega_{cv}^S}) \\
&= \frac{1}{(\omega_{cv}^S)^3}\frac{1}{\omega_{c\ell}^S-2\omega_{cv}^S}\Big(1-3\delta_{cv}^S-\frac{\Delta_{c\ell}-2\Delta_{cv}}{\omega_{c\ell}^S-2\omega_{cv}^S}\Big), \qquad (2)
\end{aligned}
$$

and

$$
\frac{1}{(\omega_{cv}^S)^3}\frac{1}{\omega_{\ell v}^S-2\omega_{cv}^S} \to \frac{1}{(\omega_{cv}^S)^3}\frac{1}{\omega_{\ell v}^S-2\omega_{cv}^S}\Big(1-3\frac{\Delta_{cv}}{\omega_{cv}^S}-\frac{\Delta_{\ell v}-2\Delta_{cv}}{\omega_{\ell v}^S-2\omega_{cv}^S}\Big). \qquad (3)
$$

$$
\begin{aligned}
\mathbf{v}_{nm}^{\Sigma} &= \frac{\omega_{nm}^S}{\omega_{nm}}\mathbf{v}_{nm} \\
&\to \frac{\omega_{nm}^S(1+\delta_{nm}^S)}{\omega_{nm}(1+\delta_{nm})}\mathbf{v}_{nm} \\
&= \frac{\omega_{nm}^S}{\omega_{nm}}\mathbf{v}_{nm}(1+\delta_{nm}^S)(1-\delta_{nm}) \\
&= \frac{\omega_{nm}^S}{\omega_{nm}}\mathbf{v}_{nm}(1+\delta_{nm}^S-\delta_{nm}) \\
\mathbf{v}_{nm}^{\Sigma} &\to \mathbf{v}_{nm}^{\Sigma}(1+\delta_{nm}^S-\delta_{nm}), \qquad (4)
\end{aligned}
$$

then

$$
\begin{aligned}
v^{\Sigma,\mathrm{a}}_{\ell c}\{v^{\Sigma,\mathrm{b}}_{cv}v^{\Sigma,c}_{v\ell}\} &\rightarrow v^{\Sigma,\mathrm{a}}_{\ell c}\{v^{\Sigma,\mathrm{b}}_{cv}v^{\Sigma,c}_{v\ell}\}(1+\delta^S_{\ell c}-\delta_{\ell c})(1+\delta^S_{cv}-\delta_{cv})(1+\delta^S_{v\ell}-\delta_{v\ell}) \\
&\rightarrow v^{\Sigma,\mathrm{a}}_{\ell c}\{v^{\Sigma,\mathrm{b}}_{cv}v^{\Sigma,c}_{v\ell}\}(1+\delta^S_{\ell c}+\delta^S_{cv}+\delta^S_{v\ell}-\delta_{\ell c}-\delta_{cv}-\delta_{v\ell}). \qquad (5)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{1}{(\omega^S_{cv})^3}\frac{\mathrm{Im}[v^{\Sigma,\mathrm{a}}_{\ell c}\{v^{\Sigma,\mathrm{b}}_{cv}v^{\Sigma,\mathrm{c}}_{v\ell}\}]}{\omega^S_{c\ell}-2\omega^S_{cv}} &\rightarrow \frac{1}{(\omega^S_{cv})^3}\frac{\mathrm{Im}[v^{\Sigma,\mathrm{a}}_{\ell c}\{v^{\Sigma,\mathrm{b}}_{cv}v^{\Sigma,\mathrm{c}}_{v\ell}\}]}{\omega^S_{c\ell}-2\omega^S_{cv}}\Big(1-3\delta^S_{cv}-\frac{\Delta_{c\ell}-2\Delta_{cv}}{\omega^S_{c\ell}-2\omega^S_{cv}}\Big) \\
&\times\ (1+\delta^S_{\ell c}+\delta^S_{cv}+\delta^S_{v\ell}-\delta_{\ell c}-\delta_{cv}-\delta_{v\ell}) \\
&\rightarrow \frac{1}{(\omega^S_{cv})^3}\frac{\mathrm{Im}[v^{\Sigma,\mathrm{a}}_{\ell c}\{v^{\Sigma,\mathrm{b}}_{cv}v^{\Sigma,\mathrm{c}}_{v\ell}\}]}{\omega^S_{c\ell}-2\omega^S_{cv}}\Big(1-3\delta^S_{cv}-\frac{\Delta_{c\ell}-2\Delta_{cv}}{\omega^S_{c\ell}-2\omega^S_{cv}} \\
&+\ \delta^S_{\ell c}+\delta^S_{cv}+\delta^S_{v\ell}-\delta_{\ell c}-\delta_{cv}-\delta_{v\ell}\Big), \qquad (6)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{(\omega^S_{cv})^3}\frac{\mathrm{Im}[v^{\Sigma,\mathrm{a}}_{v\ell}\{v^{\Sigma,\mathrm{b}}_{\ell c}v^{\Sigma,\mathrm{c}}_{cv}\}]}{\omega^S_{\ell v}-2\omega^S_{cv}} &\rightarrow \frac{1}{(\omega^S_{cv})^3}\frac{\mathrm{Im}[v^{\Sigma,\mathrm{a}}_{v\ell}\{v^{\Sigma,\mathrm{b}}_{\ell c}v^{\Sigma,\mathrm{c}}_{cv}\}]}{\omega^S_{\ell v}-2\omega^S_{cv}}\Big(1-3\delta^S_{cv}-\frac{\Delta_{\ell v}-2\Delta_{cv}}{\omega^S_{\ell v}-2\omega^S_{cv}} \\
&+\ \delta^S_{v\ell}+\delta^S_{\ell c}+\delta^S_{cv}-\delta_{v\ell}-\delta_{\ell c}-\delta_{cv}\Big), \qquad (7)
\end{aligned}
$$

For the matrix elements of the position operator we have that

$$
\begin{aligned}
\mathbf{r}_{nm} &= \frac{\mathbf{p}_{nm}}{im_e\omega_{nm}} \rightarrow \frac{\mathbf{p}_{nm}}{im_e\omega_{nm}(1+\delta_{nm})} = \mathbf{r}_{nm}(1-\delta_{nm}) \\
\mathbf{r}_{nm} &\rightarrow \mathbf{r}_{nm}(1-\delta_{nm}). \qquad (8)
\end{aligned}
$$

where we recall that $\mathbf{r}^S_{nm}=\mathbf{r}_{nm}$, and $\omega_{nm}\mathbf{r}_{nm}$ is invariant under the energy differences.

$$
\begin{aligned}
r^b_{nm;k^a} &= \frac{r^a_{nm}\mathcal{V}^b_{mn}+r^b_{nm}\mathcal{V}^a_{mn}}{\omega_{nm}}+\frac{i}{\omega_{nm}}\sum_\ell\left(\omega_{\ell m}r^a_{n\ell}r^b_{\ell m}-\omega_{n\ell}r^b_{n\ell}r^a_{\ell m}\right) \\
&\rightarrow \frac{r^a_{nm}\mathcal{V}^b_{mn}+r^b_{nm}\mathcal{V}^a_{mn}}{\omega_{nm}(1+\delta_{nm})}(1-\delta_{nm}) \\
&+\ \frac{i}{\omega_{nm}(1+\delta_{nm})}\sum_\ell\left(\omega_{\ell m}r^b_{\ell m}r^a_{n\ell}(1-\delta_{n\ell})-\omega_{n\ell}r^b_{n\ell}r^a_{\ell m}(1-\delta_{\ell m})\right) \\
r^b_{nm;k^a}\rightarrow\tilde{r}^b_{nm;k^a} &= \Big[\frac{r^a_{nm}\mathcal{V}^b_{mn}+r^b_{nm}\mathcal{V}^a_{mn}}{\omega_{nm}}(1-\delta_{nm}) \\
&+\ \frac{i}{\omega_{nm}}\sum_\ell\left(\omega_{\ell m}r^b_{\ell m}r^a_{n\ell}(1-\delta_{n\ell})-\omega_{n\ell}r^b_{n\ell}r^a_{\ell m}(1-\delta_{\ell m})\right)\Big](1-\delta_{nm}) \\
\tilde{r}^b_{nm;k^a} &= \tilde{\tilde{r}}^b_{nm;k^a}(1-\delta_{nm}). \qquad (9)
\end{aligned}
$$

Then, $\mathcal{F}_{vc}^{\mathrm{ab}}$ is given by

$$
\begin{aligned}
\mathcal{F}_{vc}^{\mathrm{ab}} &= -i\Delta \sum_{\ell\neq(vc)} \Big(r_{v\ell}^{\mathrm{a}} r_{\ell c}^{\mathrm{b}} - r_{v\ell}^{\mathrm{b}} r_{\ell c}^{\mathrm{a}}\Big)(1-\delta_{v\ell}-\delta_{\ell c}) + \Delta\tilde{\tilde{r}}_{vc;k^{\mathrm{a}}}^{\mathrm{b}}(1-\delta_{vc}) \\
&= \Big[-i\Delta \sum_{\ell\neq(vc)} \Big(r_{v\ell}^{\mathrm{a}} r_{\ell c}^{\mathrm{b}} - r_{v\ell}^{\mathrm{b}} r_{\ell c}^{\mathrm{a}}\Big)(1-\delta_{v\ell}-\delta_{\ell c}+\delta_{vc}) + \Delta\tilde{\tilde{r}}_{vc;k^{\mathrm{a}}}^{\mathrm{b}}\Big](1-\delta_{vc}) \\
&= \tilde{\mathcal{F}}_{vc}^{\mathrm{ab}}(1-\delta_{vc}),
\end{aligned}
\tag{10}
$$

and

$$
\frac{1}{(\omega_{cv}^{S})^3}\mathcal{F}_{vc}^{\mathrm{ab}} v_{cv}^{\Sigma,\mathrm{c}} \to \frac{1}{(\omega_{cv}^{S})^3}(1-3\delta_{cv}^{S})\tilde{\mathcal{F}}_{vc}^{\mathrm{ab}} v_{cv}^{\Sigma,\mathrm{c}}(1+\delta_{cv}^{S}) = \frac{1}{(\omega_{cv}^{S})^3}\tilde{\mathcal{F}}_{vc}^{\mathrm{ab}} v_{cv}^{\Sigma,\mathrm{c}}(1-2\delta_{cv}^{S}), \tag{11}
$$

since $\delta_{cv} - \delta_{vc} = \Delta_{cv}/\omega_{cv} - \Delta_{vc}/\omega_{vc} = \Delta_{cv}/\omega_{cv} - (-\Delta_{cv})/(-\omega_{cv}) = 0$.