12
CHAPTER

DIMENSIONAL AND MODEL ANALYSIS

▶ 12.1 INTRODUCTION

Dimensional analysis is a method of dimensions. It is a mathematical technique used in research work for design and for conducting model tests. It deals with the dimensions of the physical quantities involved in the phenomenon. All physical quantities are measured by comparison, which is made with respect to an arbitrarily fixed value. Length L, mass M and time T are three fixed dimensions which are of importance in Fluid Mechanics. If in any problem of fluid mechanics, heat is involved then temperature is also taken as fixed dimension. These fixed dimensions are called fundamental dimensions or fundamental quantity.

▶ 12.2 SECONDARY OR DERIVED QUANTITIES

Secondary or derived quantities are those quantities which possess more than one fundamental dimension. For example, velocity is denoted by distance per unit time (L/T), density by mass per unit volume $\left(\frac{M}{L^3}\right)$ and acceleration by distance per second square (L/T^2). Then velocity, density and acceleration become as secondary or derived quantities. The expressions (L/T), $\left(\frac{M}{L^3}\right)$ and $\left(\frac{L}{T^2}\right)$ are called the dimensions of velocity, density and acceleration respectively. The dimensions of mostly used physical quantities in Fluid Mechanics are given in Table 12.1.

Table 12.1

S. No.	*Physical Quantity*	*Symbol*	*Dimensions*
	(*a*) **Fundamental**		
1.	Length	L	L
2.	Mass	M	M
3.	Time	T	T

Contd...

S.No.	Physical Quantity	Symbol	Dimensions
	(*b*) **Geometric**		
4.	Area	A	L^2
5.	Volume	$\forall$	L^3
	(*c*) **Kinematic Quantities**		
6.	Velocity	v	LT^{-1}
7.	Angular Velocity	ω	T^{-1}
8.	Acceleration	a	LT^{-2}
9.	Angular Acceleration	α	T^{-2}
10.	Discharge	Q	L^3T^{-1}
11.	Acceleration due to Gravity	g	LT^{-2}
12.	Kinematic Viscosity	ν	L^2T^{-1}
	(*d*) **Dynamic Quantities**		
13.	Force	F	MLT^{-2}
14.	Weight	W	MLT^{-2}
15.	Density	ρ	ML^{-3}
16.	Specific Weight	w	$ML^{-2}T^{-2}$
17.	Dynamic Viscosity	μ	$ML^{-1}T^{-1}$
18.	Pressure Intensity	p	$ML^{-1}T^{-2}$
19.	Modulus of Elasticity	$\begin{cases} K \\ E \end{cases}$	$ML^{-1}T^{-2}$
20.	Surface Tension	σ	MT^{-2}
21.	Shear Stress	τ	$ML^{-1}T^{-2}$
22.	Work, Energy	W or E	ML^2T^{-2}
23.	Power	P	ML^2T^{-3}
24.	Torque	T	ML^2T^{-2}
25.	Momentum	M	MLT^{-1}

Problem 12.1 *Determine the dimensions of the quantities given below : (i) Angular velocity, (ii) Angular acceleration, (iii) Discharge, (iv) Kinematic viscosity, (v) Force, (vi) Specific weight, and (vii) Dynamic viscosity.*

Solution. (*i*) Angular velocity $= \dfrac{\text{Angle covered in radians}}{\text{Time}} = \dfrac{1}{T} = T^{-1}$.

(*ii*) Angular acceleration $= \text{rad/sec}^2 = \dfrac{\text{rad}}{T^2} = \dfrac{1}{T^2} = T^{-2}$.

(*iii*) Discharge $= \text{Area} \times \text{Velocity} = L^2 \times \dfrac{L}{T} = \dfrac{L^3}{T} = L^3T^{-1}$.

(*iv*) Kinematic viscosity (ν) $= \dfrac{\mu}{\rho}$, where μ is given by $\tau = \mu \dfrac{\partial u}{\partial y}$

$$\therefore \qquad \mu = \frac{\tau}{\dfrac{\partial u}{\partial y}} = \frac{\text{Shear Stress}}{\dfrac{L}{T} \times \dfrac{1}{L}} = \frac{\dfrac{\text{Force}}{\text{Area}}}{\dfrac{1}{T}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Area} \times \text{Time}} = \frac{M \times \frac{L}{T^2}}{L^2 \times \frac{1}{T}} = \frac{ML}{L^2T^2 \times \frac{1}{T}} = \frac{M}{LT} = ML^{-1}T^{-1}$$

and $$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{L^3} = ML^{-3}$$

$\therefore$ Kinematic viscosity $$(\nu) = \frac{\mu}{\rho} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = L^2T^{-1}.$$

(*v*) Force $$= \text{Mass} \times \text{Acceleration} = M \times \frac{\text{Length}}{(\text{Time})^2} = \frac{ML}{T^2} = MLT^{-2}.$$

(*vi*) Specific weight $$= \frac{\text{Weight}}{\text{Volume}} = \frac{\text{Force}}{\text{Volume}} = \frac{MLT^{-2}}{L^3} = ML^{-2}T^{-2}.$$

(*vii*) Dynamic viscosity, μ is derived in (*iv*) as $\mu = ML^{-1}T^{-1}$.

▶ 12.3 DIMENSIONAL HOMOGENEITY

Dimensional homogeneity means the dimensions of each terms in an equation on both sides are equal. Thus if the dimensions of each term on both sides of an equation are the same the equation is known as dimensionally homogeneous equation. The powers of fundamental dimensions (*i.e.*, L, M, T) on both sides of the equation will be identical for a dimensionally homogeneous equation. Such equations are independent of the system of units.

Let us consider the equation, $V = \sqrt{2gH}$

Dimension of L.H.S. $= V = \frac{L}{T} = LT^{-1}$

Dimension of R.H.S. $= \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} = LT^{-1}$

Dimension of L.H.S. $=$ Dimension of R.H.S. $= LT^{-1}$

$\therefore$ Equation $V = \sqrt{2gH}$ is dimensionally homogeneous. So it can be used in any system of units.

▶ 12.4 METHODS OF DIMENSIONAL ANALYSIS

If the number of variable involved in a physical phenomenon are known, then the relation among the variables can be determined by the following two methods :

1. Rayleigh's method, and
2. Buckingham's π-theorem.

12.4.1 Rayleigh's Method. This method is used for determining the expression for a variable which depends upon maximum three or four variables only. If the number of independent variables becomes more than four, then it is very difficult to find the expression for the dependent variable.

Let X is a variable, which depends on X_1, X_2 and X_3 variables. Then according to Rayleigh's method, X is function of X_1, X_2 and X_3 and mathematically it is written as $X = f[X_1, X_2, X_3]$.

This can also be written as $X = KX_1^a \cdot X_2^b \cdot X_3^c$
where K is constant and a, b and c are arbitrary powers.

The values of a, b and c are obtained by comparing the powers of the fundamental dimension on both sides. Thus the expression is obtained for dependent variable.

Problem 12.2 *The time period (t) of a pendulum depends upon the length (L) of the pendulum and acceleration due to gravity (g). Derive an expression for the time period.*

Solution. Time period t is a function of (*i*) L and (*ii*) g

$\therefore$ $$t = KL^a \cdot g^b, \text{ where } K \text{ is a constant} \quad ...(i)$$

Substituting the dimensions on both sides $T^1 = KL^a \cdot (LT^{-2})^b$

Equating the powers of M, L and T on both sides, we have

Power of T, $1 = -2b \quad \therefore \quad b = -\frac{1}{2}$

Power of L, $0 = a + b \quad \therefore \quad a = -b = -\left(-\frac{1}{2}\right) = \frac{1}{2}$

Substituting the values of a and b in equation (*i*),

$$t = KL^{1/2} \cdot g^{-1/2} = K\sqrt{\frac{L}{g}}$$

The value of K is determined from experiments which is given as

$$K = 2\pi$$

$\therefore$ $$t = \mathbf{2\pi}\sqrt{\frac{\mathbf{L}}{\mathbf{g}}}. \textbf{ Ans.}$$

Problem 12.3 *Find an expression for the drag force on smooth sphere of diameter D, moving with a uniform velocity V in a fluid of density ρ and dynamic viscosity μ.*

Solution. Drag force F is a function of

(*i*) Diameter, D (*ii*) Velocity, V (*iii*) Density, ρ
(*iv*) Viscocity, μ

$\therefore$ $$F = KD^a \cdot V^b \cdot \rho^c \cdot \mu^d \quad ...(i)$$

where K is non-dimensional factor.

Substituting the dimensions on both sides,

$$MLT^{-2} = KL^a \cdot (LT^{-1})^b \cdot (ML^{-3})^c \cdot (ML^{-1}T^{-1})^d$$

Equating the powers of M, L and T on both sides,

Power of M, $1 = c + d$
Power of L, $1 = a + b - 3c - d$
Power of T, $-2 = -b - d$.

There are four unknowns (a, b, c, d) but equations are three. Hence it is not possible to find the values of a, b, c and d. But three of them can be expressed in terms of fourth variable which is most important. Here viscosity is having a vital role and hence a, b, c are expressed in terms of d which is the power to viscosity.

$\therefore$
$$c = 1 - d$$
$$b = 2 - d$$
$$a = 1 - b + 3c + d = 1 - 2 + d + 3(1 - d) + d$$

$$= 1 - 2 + d + 3 - 3d + d = 2 - d$$

Substituting these values of a, b and c in (i), we get

$$F = KD^{2-d} \cdot V^{2-d} \cdot \rho^{1-d} \cdot \mu^{d}$$

$$= KD^2V^2\rho\,(D^{-d} \cdot V^{-d} \cdot \rho^{-d} \cdot \mu^{d}) = K\rho D^2V^2 \left(\frac{\mu}{\rho VD}\right)^{d}$$

$$= \mathbf{K\rho D^2V^2\phi\left(\frac{\mu}{\rho VD}\right)} \textbf{. Ans.}$$

Problem 12.4 *Find the expression for the power P, developed by a pump when P depends upon the head H, the discharge Q and specific weight w of the fluid.*

Solution. Power P is a function of

(i) Head, H (ii) Discharge, Q

(iii) Specific weight, w

$$\therefore \quad P = KH^a \cdot Q^b \cdot w^c \quad ...(i)$$

where K = Non-dimensional constant.

Substituting the dimensions on both sides of equation (i)

$$ML^2T^{-3} = KL^a \cdot (L^3T^{-1})^b \cdot (ML^{-2}T^{-2})^c$$

Equating the powers of M, L and T on both sides,

Power of M, $1 = c$, $\therefore$ $c = 1$

Power of L, $2 = a + 3b - 2c$, $a = 2 - 3b + 2c = 2 - 3 + 2 = 1$

Power of T, $-3 = -b - 2c$ $\therefore$ $b = 3 - 2c = 3 - 2 = 1$

Substituting the values of a, b and c in (i)

$$P = KH^1 \cdot Q^1 \cdot w^1 = \mathbf{KHQw}\textbf{. Ans.}$$

Problem 12.5 *The efficiency η of a fan depends on the density ρ, the dynamic viscosity μ of the fluid, the angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters.*

Solution. The efficiency η depends on

(i) density, ρ (ii) viscosity, μ

(iii) angular velocity, ω (iv) diameter, D

(v) discharge, Q

$$\therefore \quad \eta = K\rho^a \cdot \mu^b \cdot \omega^c \cdot D^d \cdot Q^e \quad ...(i)$$

where K = Non-dimensional constant.

Substituting the dimensions on both sides of equation (i)

$$M^0L^0T^0 = K\,(ML^{-3})^a \cdot (ML^{-1}T^{-1})^b \cdot (T^{-1})^c \cdot (L)^d \cdot (L^3T^{-1})^e$$

Equating powers of M, L, T on both sides,

Power of M, $0 = a + b$

Power of L, $0 = -3a - b + d + 3e$

Power of T, $0 = -b - c - e$.

There are five unknowns but equations are three. Express the three unknowns in terms of the other two unknowns which are more important. Viscosity and discharge are more important in this problem. Hence expressing a, c and d in terms of b and e, we get

$$a = -b$$

$$c = -(b + e)$$

$$d = +3a + b - 3e = -3b + b - 3e = -2b - 3e.$$

Substituting these values in equation (*i*), we get

$$\eta = K\rho^{-b}.\ \mu^{b}.\ \omega^{-(b+e)}\ .\ D^{-2b-3e}\ .\ Q^{e}$$
$$= K\rho^{-b}\ .\ \mu^{b}\ .\ \omega^{-b}\ .\ \omega^{-e}\ .\ D^{-2b}\ .\ D^{-3e}\ .\ Q^{e}$$
$$= \mathbf{K}\left(\frac{\mu}{\rho\omega \mathbf{D}^2}\right)^{\mathbf{b}}\cdot\left(\frac{\mathbf{Q}}{\omega\mathbf{D}^3}\right)^{\mathbf{e}} = \phi\left[\left(\frac{\mu}{\rho\omega\mathbf{D}^2}\right)\cdot\left(\frac{\mathbf{Q}}{\omega\mathbf{D}^3}\right)\right].\ \textbf{Ans.}$$

Problem 12.6 *The resisting force R of a supersonic plane during flight can be considered as dependent upon the length of the aircraft l, velocity V, air viscosity μ, air density ρ and bulk modulus of air K. Express the functional relationship between these variables and the resisting force.*

Solution. The resisting force R depends upon

(*i*) density, l, (*ii*) velocity, V,
(*iii*) viscosity, μ, (*iv*) density, ρ,
(*v*) Bulk modulus, K.

$$\therefore \quad R = Al^{a}\ .\ V^{b}\ .\ \mu^{c}\ .\ \rho^{d}\ .\ K^{e} \quad ...(i)$$

where A is the non-dimensional constant.

Substituting the dimensions on both sides of the equation (*i*),

$$MLT^{-2} = AL^{a}\ .\ (LT^{-1})^{b}\ .\ (ML^{-1}T^{-1})^{c}\ .\ (ML^{-3})^{d}\ .\ (ML^{-1}T^{-2})^{e}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 1 = c + d + e$

Power of L, $\quad 1 = a + b - c - 3d - e$

Power of T, $\quad -2 = -b - c - 2e.$

There are five unknowns but equations are only three. Expressing the three unknowns in terms of two unknowns (μ and K).

$\therefore$ Express the values of a, b and d in terms of c and e.

Solving,

$$d = 1 - c - e$$
$$b = 2 - c - 2e$$
$$a = 1 - b + c + 3d + e = 1 - (2 - c - 2e) + c + 3\ (1 - c - e) + e$$
$$= 1 - 2 + c + 2e + c + 3 - 3c - 3e + e = 2 - c.$$

Substituting these values in (*i*), we get

$$R = A\ l^{2-c}\ .\ V^{2-c-2e}\ .\ \mu^{c}\ .\ \rho^{1-c-e}\ .\ K^{e}$$
$$= A\ l^{2}\ .\ V^{2}\ .\ \rho(l^{-c}\ V^{-c}\ \mu^{c}\ \rho^{-c})\ .\ (V^{-2e}\ .\ \rho^{-e}\ .\ K^{e})$$
$$= A\ l^{2}V^{2}\rho\left(\frac{\mu}{\rho VL}\right)^{c}\cdot\left(\frac{K}{\rho V^{2}}\right)^{e}$$
$$= \mathbf{A\rho l^{2}V^{2}}\ \phi\left[\left(\frac{\mu}{\rho\mathbf{VL}}\right)\cdot\left(\frac{\mathbf{K}}{\rho\mathbf{V}^{2}}\right)\right].\ \textbf{Ans.}$$

Problem 12.7 *A partially sub-merged body is towed in water. The resistance R to its motion depends on the density ρ, the viscosity μ of water, length l of the body, velocity v of the body and the acceleration due to gravity g. Show that the resistance to the motion can be expressed in the form*

$$R = \rho\, L^{2}V^{2}\ \phi\left[\left(\frac{\mu}{\rho VL}\right)\cdot\left(\frac{lg}{V^{2}}\right)\right].$$

Solution. The resistance R depends on

(*i*) density, ρ, (*ii*) viscosity, μ,
(*iii*) length, l, (*iv*) velocity, V,
(*v*) acceleration, g.

$$\therefore \quad R = K\rho^a \cdot \mu^b \cdot l^c \cdot V^d \cdot g^e \quad \ldots(i)$$

where K = Non-dimensional constant.

Substituting the dimensions on both sides of the equation (*i*),

$$MLT^{-2} = K(ML^{-3})^a \cdot (ML^{-1}T^{-1})^b \cdot L^c \cdot (LT^{-1})^d \cdot (LT^{-2})^e$$

Equating the powers of M, L, T on both sides

Power of M, $1 = a + b$

Power of L, $1 = -3a - b + c + d + e$

Power of T, $-2 = -b - d - 2e$.

There are five unknowns and equations are only three. Expressing the three unknowns in terms of two unknowns (μ and g). Hence express a, c and d in terms of b and e. Solving, we get

$$a = 1 - b$$
$$d = 2 - b - 2e$$
$$c = 1 + 3a + b - d - e = 1 + 3(1 - b) + b - (2 - b - 2e) - e$$
$$= 1 + 3 - 3b + b - 2 + b + 2e - e = 2 - b + e.$$

Substituting these values in equation (*i*), we get

$$R = K\rho^{1-b} \cdot \mu^b \cdot l^{2-b+e} \cdot V^{2-b-2e} \cdot g^e$$
$$= K\rho l^2 \cdot V^2 \cdot (\rho^{-b}\mu^b l^{-b} V^{-b}) \cdot (l^e \cdot V^{-2e} \cdot g^e)$$
$$= K\rho l^2 V^2 \cdot \left(\frac{\mu}{\rho V l}\right)^b \cdot \left(\frac{lg}{V^2}\right)^e = \mathbf{K\rho l^2 V^2 \phi\left[\left(\frac{\mu}{\rho V l}\right) \cdot \left(\frac{lg}{V^2}\right)\right]}. \quad \textbf{Ans.}$$

12.4.2 Buckingham's π-Theorem. The Rayleigh's method of dimensional analysis becomes more laborious if the variables are more than the number of fundamental dimensions (M, L, T). This difficulty is overcame by using Buckingham's π-theorem, which states, "If there are n variables (independent and dependent variables) in a physical phenomenon and if these variables contain m fundamental dimensions (M, L, T), then the variables are arranged into $(n - m)$ dimensionless terms. Each term is called π-term".

Let $X_1, X_2, X_3, \ldots, X_n$ are the variables involved in a physical problem. Let X_1 be the dependent variable and $X_2, X_3, \ldots, X_n$ are the independent variables on which X_1 depends. Then X_1 is a function of $X_2, X_3, \ldots, X_n$ and mathematically it is expressed as

$$X_1 = f(X_2, X_3, \ldots, X_n) \quad \ldots(12.1)$$

Equation (12.1) can also be written as

$$f_1(X_1, X_2, X_3, \ldots, X_n) = 0. \quad \ldots(12.2)$$

Equation (12.2) is a dimensionally homogeneous equation. It contains n variables. If there are m fundamental dimensions then according to Buckingham's π-theorem, equation (12.2) can be written in terms of number of dimensionless groups or π-terms in which number of π-terms is equal to $(n - m)$. Hence equation (12.2) becomes as

$$f(\pi_1, \pi_2, \ldots, \pi_{n-m}) = 0. \quad \ldots(12.3)$$

Each of π-terms is dimensionless and is independent of the system. Division or multiplication by a constant does not change the character of the π-term. Each π-term contains $m + 1$ variables, where m is the number of fundamental dimensions and is also called repeating variables. Let in the above case X_2, X_3 and X_4 are repeating variables if the fundamental dimension m (M, L, T) = 3. Then each π-term is written as

$$\left.\begin{aligned} \pi_1 &= X_2^{a_1} . X_3^{b_1} . X_4^{c_1} . X_1 \\ \pi_2 &= X_2^{a_2} . X_3^{b_2} . X_4^{c_2} . X_5 \\ &\vdots \\ \pi_{n-m} &= X_2^{a_{n-m}} . X_3^{b_{n-m}} . X_4^{c_{n-m}} . X_n \end{aligned}\right\} \quad ...(12.4)$$

Each equation is solved by the principle of dimensional homogeneity and values of a_1, b_1, c_1 etc., are obtained. These values are substituted in equation (12.4) and values of $\pi_1, \pi_2, ..., \pi_{n-m}$ are obtained. These values of π's are substituted in equation (12.3). The final equation for the phenomenon is obtained by expressing any one of the π-terms as a function of others as

$$\pi_1 = \phi\, [\pi_2, \pi_3, ..., \pi_{n-m}]$$

or
$$\pi_2 = \phi_1\, [\pi_1, \pi_3, ..., \pi_{n-m}] \quad ...(12.5)$$

12.4.3 Method of Selecting Repeating Variables. The number of repeating variables are equal to the number of fundamental dimensions of the problem. The choice of repeating variables is governed by the following considerations :

1. As far as possible, the dependent variable should not be selected as repeating variable.
2. The repeating variables should be choosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Variables with Geometric Property are

(*i*) Length, l (*ii*) d (*iii*) Height, H etc.

Variables with flow property are

(*i*) Velocity, V (*ii*) Acceleration etc.

Variables with fluid property : (*i*) μ, (*ii*) ρ, (*iii*) ω etc.

3. The repeating variables selected should not form a dimensionless group.
4. The repeating variables together must have the same number of fundamental dimensions.
5. No two repeating variables should have the same dimensions.

In most of fluid mechanics problems, the choice of repeating variables may be (*i*) d, v, ρ (*ii*) l, v, ρ or (*iii*) l, v, μ or (*iv*) d, v, μ.

12.4.4 Procedure for Solving Problems by Buckingham's π-theorem. The procedure for solving problems by Buckingham's π-theorem is explained by considering the problem 12.6 which is also solved by the Rayleigh's method. The problem is :

The resisting force R of a supersonic plane during flight can be considered as dependent upon the length of the aircraft l, velocity V, air viscosity μ, air density ρ and bulk modulus of air K. Express the functional relationship between these variables and the resisting force.

Solution. Step 1. The resisting force R depends upon (*i*) l, (*ii*) V, (*iii*) μ, (*iv*) ρ and (*v*) K. Hence R is a function of l, V, μ, ρ and K. Mathematically,

$$R = f(l, V, \mu, \rho, K) \quad ...(i)$$

or it can be written as $f_1\,(R, l, V, \mu, \rho, K) = 0$ ...(*ii*)

$\therefore$ Total number of variables, $n = 6$.

Number of fundamental dimensions, $m = 3$.

[m is obtained by writing dimensions of each variables as $R = MLT^{-2}$, $V = LT^{-1}$, $\mu = ML^{-1}T^{-1}$, $\rho = ML^{-3}$, $K = ML^{-1}T^{-2}$. Thus as fundamental dimensions in the problem are M, L, T and hence $m = 3$.]

Number of dimensionless π-terms $= n - m = 6 - 3 = 3$.

Thus three π-terms say π_1, π_2 and π_3 are formed. Hence equation (*ii*) is written as

$$f_1\,(\pi_1, \pi_2, \pi_3) = 0. \quad ...(iii)$$

Step 2. Each π term $= m + 1$ variables, where m is equal to 3 and also called repeating variables. Out of six variables R, l, V, μ, ρ and K, three variables are to be selected as repeating variable. R is a dependent variable and should not be selected as a repeating variable. Out of the five remaining variables, one variable should have geometric property, the second variable should have flow property and third one fluid property. These requirements are fulfilled by selecting l, V and ρ as repeating variables. The repeating variables themselves should not form a dimensionless term and should have themselves fundamental dimensions equal to m, *i.e.*, 3 here. Dimensions of l, V and ρ are L, LT^{-1}, ML^{-3} and hence the three fundamental dimensions exist in l, V and ρ and they themselves do not form dimensionless group.

Step 3. Each π-term is written as according to equation (12.4)

$$\left.\begin{aligned} \pi_1 &= l^{a_1} . V^{b_1} . \rho^{c_1} . R \\ \pi_2 &= l^{a_2} . V^{b_2} . \rho^{c_2} . \mu \\ \pi_3 &= l^{a_3} . V^{b_3} . \rho^{c_3} . K \end{aligned}\right\} \quad \ldots(iv)$$

Step 4. Each π-term is solved by the principle of dimensional homogeneity. For the first π-term, we have

$$\pi_1 = M^0L^0T^0 = L^{a_1} . (LT^{-1})^{b_1} . (ML^{-3})^{c_1} . MLT^{-2}.$$

Equating the powers of M, L, T on both sides, we get

Power of M, $\quad 0 = c_1 + 1 \quad \therefore \quad c_1 = -1$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 + 1,$

$\therefore \quad a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$

Power of T, $\quad 0 = -b_1 - 2 \quad \therefore \quad b_1 = -2$

Substituting the values of a_1, b_1 and c_1 in equation (iv),

$$\pi_1 = l^{-2} . V^{-2} . \rho^{-1} . R$$

or

$$\pi_1 = \frac{R}{l^2V^2\rho} = \frac{R}{\rho l^2 V^2} \quad \ldots(v)$$

Similarly for the 2nd π-term, we get $\pi_2 = M^0L^0T^0 = L^{a_2} . (LT^{-1})^{b_2} . (ML^{-3})^{c_2} . ML^{-1}T^{-1}$.

Equating the powers of M, L, T on both sides

Power of M, $\quad 0 = c_2 + 1, \quad \therefore \quad c_2 = -1$

Power of L, $\quad 0 = a_2 + b_2 - 3c_2 - 1,$

$a_2 = -b_2 + 3c_2 + 1 = 1 - 3 + 1 = -1$

Power of T, $\quad 0 = -b_2 - 1, \quad \therefore \quad b_2 = -1$

Substituting the values of a_2, b_2 and c_2 in π_2 of (iv)

$$\pi_2 = l^{-1} . V^{-1} . \rho^{-1} . \mu = \frac{\mu}{lV\rho}.$$

3rd π-term

$$\pi_3 = l^{a_3} . V^{b_3} . \rho^{c_3} . K$$

or

$$M^0L^0T^0 = L^{a_3} . (LT^{-3})^{b_3} . (ML^{-3})^{a_3} . ML^{-1}T^{-2}$$

Equating the powers of M, L, T on both sides, we have

Power of M, $\quad 0 = c_3 + 1, \quad \therefore \quad c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore \quad a_3 = -b_3 + 3c_3 + 1 = 2 - 3 + 1 = 0$

Power of T, $\quad 0 = -b_3 - 2, \quad \therefore \quad b_3 = -2$

Substituting the values of a_3, b_3 and c_3 in π_3 term

$$\pi_3 = l^0 \cdot V^{-2} . \rho^{-1} . K = \frac{K}{V^2\rho}.$$

Step 5. Substituting the values of π_1, π_2 and π_3 in equation (*iii*), we get

$$f_1\left(\frac{R}{\rho l^2V^2}, \frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right) = 0 \quad \text{or} \quad \frac{R}{\rho l^2V^2} = \phi\left[\frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right]$$

or
$$R = \rho \mathbf{l^2V^2}\phi\left[\frac{\mu}{\mathbf{lV}\rho}, \frac{\mathbf{K}}{\mathbf{V^2}\rho}\right]. \textbf{ Ans.}$$

Problem 12.8 *(a) State Buckingham's π-theorem.*
(b) The efficiency η of a fan depends on density ρ, dynamic viscosity μ of the fluid, angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters.

Solution. (*a*) Statement of Buckingham's π-theorem is given in Article 12.4.2.

(*b*) Given : η is a function of ρ, μ, ω, D and Q

$\therefore$ $$\eta = f(\rho, \mu, \omega, D, Q) \quad \text{or} \quad f_1(\eta, \rho, \mu, \omega, D, Q) = 0 \qquad ...(i)$$

Hence total number of variables, $n = 6$.

The value of m, *i.e.*, number of fundamental dimensions for the problem is obtained by writing dimensions of each variable. Dimensions of each variable are

$$\eta = \text{Dimensionless}, \rho = ML^{-3}, \mu = ML^{-1}T^{-1}, \omega = T^{-1}, D = L \text{ and } Q = L^3T^{-1}$$

$\therefore$ $m = 3$

Number of π-terms $= n - m = 6 - 3 = 3$

Equation (*i*) is written as $f_1(\pi_1, \pi_2, \pi_3) = 0$ ...(*ii*)

Each π-term contains $m + 1$ variables, where m is equal to three and is also repeating variable.

Choosing D, ω and ρ as repeating variables, we have

$$\pi_1 = D^{a_1} . \omega^{b_1} . \rho^{c_1} . \eta$$
$$\pi_2 = D^{a_2} . \omega^{b_2} . \rho^{c_2} . \mu$$
$$\pi_3 = D^{a_3} . \omega^{b_3} . \rho^{c_3} . Q$$

First π-term $\quad \pi_1 = D^{a_1} . \omega^{b_1} . \rho^{c_1} . \eta$

Substituting dimensions on both sides of π_1,

$$M^0L^0T^0 = L^{a_1} . (T^{-1})^{b_1} . (ML^{-3})^{c_1} . M^0L^0T^0$$

Equating the powers of M, L, T on both sides

Power of M, $\quad 0 = c_1 + 0, \quad \therefore \quad c_1 = 0$

Power of L, $\quad 0 = a_1 + 0, \quad \therefore \quad a_1 = 0$

Power of T, $\quad 0 = -b_1 + 0, \quad \therefore \quad b_1 = 0$

Substituting the values of a_1, b_1 and c_1 in π_1, we get

$$\pi_1 = D^0\omega^0\rho^0 . \eta = \eta$$

[If a variable is dimensionless, it itself is a π-term. Here the variable η is a dimensionless and hence η is a π-term. As it exists in first π-term and hence $\pi_1 = \eta$. Then there is no need of equating the powers. Directly the value can be obtained.]

Second π-term $\quad \pi_2 = D^{a_2} . \omega^{b_2} . \rho^{c_2} . \mu$

Substituting the dimensions on both sides

$$M^0L^0T^0 = L^{a_2} . (T^{-1})^{b_2} . (ML^{-3})^{c_2} . ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_2 + 1$, $\therefore$ $c_2 = -1$
Power of L, $0 = a_2 - 3c_2 - 1$, $\therefore$ $a_2 = 3c_2 + 1 = -3 + 1 = -2$
Power of T, $0 = -b_2 - 1$, $\therefore$ $b_2 = -1$
Substituting the values of a_2, b_2 and c_2 in π_2,

$$\pi_2 = D^{-2} \cdot \omega^{-1} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{D^2 \omega \rho}$$

Third π-term $\pi_3 = D^{a_3} \cdot \omega^{b_3} \cdot \rho^{c_3} \cdot Q$
Substituting the dimensions on both sides

$$M^0 L^0 T^0 = L^{a_3} \cdot (T^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot L^3 T^{-1}$$

Equating the powers of M, L and T on both sides
Power of M, $0 = c_3$, $\therefore$ $c_3 = 0$
Power of L, $0 = a_3 - 3c_3 + 3$, $\therefore$ $a_3 = 3c_3 - 3 = -3$
Power of T, $0 = -b_{3\,-1}$, $\therefore$ $b_3 = -1$
Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = D^{-3} \cdot \omega^{-1} \cdot \rho^0 \cdot Q = \frac{Q}{D^2 \omega}$$

Substituting the values of π_1, π_2 and π_3 in equation (*ii*)

$$f_1\left(\eta, \frac{\mu}{D^2\omega\rho}, \frac{Q}{D^2\omega}\right) = 0 \text{ or } \eta = \phi\left[\frac{\mu}{\mathbf{D^2\omega\rho}}, \frac{\mathbf{Q}}{\mathbf{D^2\omega}}\right]. \textbf{ Ans.}$$

Problem 12.9 *Using Buckingham's π-theorem, show that the velocity through a circular orifice is given by* $V = \sqrt{2gH}\, \phi\left[\frac{D}{H}, \frac{\mu}{\rho V H}\right]$, *where H is the head causing flow, D is the diameter of the orifice, μ is co-efficient of viscosity, ρ is the mass density and g is the acceleration due to gravity.*

Solution. Given :
V is a function of H, D, μ, ρ and g

$\therefore$ $V = f(H, D, \mu, \rho, g)$ or $f_1(V, H, D, \mu, \rho, g) = 0$

$\therefore$ Total number of variable, $n = 6$ ...(*i*)

Writing dimension of each variable, we have

$$V = LT^{-1},\ H = L,\ D = L,\ \mu = ML^{-1}T^{-1},\ \rho = ML^{-3},\ g = LT^{-2}.$$

Thus number of fundamental dimensions, $m = 3$

$\therefore$ Number of π-terms $= n - m = 6 - 3 = 3.$

Equation (*i*) can be written as $f_1(\pi_1, \pi_2, \pi_3) = 0$ ...(*ii*)

Each π-term contains $m + 1$ variables, where $m = 3$ and is also equal to repeating variables. Here V is a dependent variable and hence should not be selected as repeating variable. Choosing H, g, ρ as repeating variable, we get three π-terms as

$$\pi_1 = H^{a_1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V$$
$$\pi_2 = H^{a_2} \cdot g^{b_2} \cdot \rho^{c_2} \cdot D$$
$$\pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu$$

First π-term $\pi_1 = H^{a_1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V$

Substituting dimensions on both sides

$$M^0L^0T^0 = L^{a_1} \, . \, (LT^{-2})^{b_1} \, . \, (MT^{-3})^{c_1} \, . \, (LT^{-1})$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c, \quad \therefore \quad c_1 = 0$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 + 1, \quad \therefore \quad a_1 = -b_1 + 3c_1 - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$

Power of T, $\quad 0 = -2b_1 - 1, \quad \therefore \quad b_1 = -\frac{1}{2}$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = H^{-\frac{1}{2}} . g^{-\frac{1}{2}} \, . \, \rho^0 \, . \, V = \frac{V}{\sqrt{gH}}.$$

Second π-term $\quad \pi_2 = H^{a_2} \, . \, g^{b_2} \, . \, \rho^{c_2} \, . \, D$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} \, . \, (LT^{-2})^{b_2} . \, (ML^{-3})^{c_2} \, . \, L$$

Equating the powers of M, L, T,

Power of M, $\quad 0 = c_2 \quad \therefore \quad c_2 = 0$

Power of L, $\quad 0 = a_2 + b_2 - 3c_2 + 1, \; a_2 = -b_2 + 3c_2 - 1 = -1$

Power of T, $\quad 0 = -2b_2, \quad \therefore \quad b_2 = 0$

Substituting the values of a_2, b_2, c_2 in π_2,

$$\pi_2 = H^{-1} \, . \, g^0\rho^0 \, . \, D = \frac{D}{H}.$$

Third π-term $\quad \pi_3 = H^{a_3} \, . \, g^{b_3} \, . \, \rho^{c_3} \, . \, \mu$

Substituting the dimensions on both sides

$$M^0L^0T^0 = L^{a_3} \, . \, (LT^{-2})^{b_3} \, . \, (ML^{-3})^{c_3} \, . \, ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides

Power of M, $\quad 0 = c_3 + 1, \quad \therefore \quad c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore \quad a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1 = -\frac{3}{2}$

Power of T, $\quad 0 = -2b_3 - 1, \quad \therefore \quad b_3 = -\frac{1}{2}$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = H^{-3/2} \, . \, g^{-1/2} \, . \, \rho^{-1} \, . \, \mu = \frac{\mu}{H^{3/2}\rho\sqrt{g}}$$

$$= \frac{\mu}{H\rho\sqrt{gH}} = \frac{\mu V}{H\rho V\sqrt{gH}} \qquad \text{[Multiply and Divide by } V\text{]}$$

$$= \frac{\mu}{H\rho V} \, . \, \pi_1 \qquad \left\{\because \; \frac{V}{\sqrt{gH}} = \pi_1\right\}$$

Substituting the values of π_1, π_2 and π_3 in equation (*ii*),

$$f_1\left(\frac{V}{\sqrt{gH}}, \frac{D}{H}, \pi_1 \frac{\mu}{H\rho V}\right) = 0 \text{ or } \frac{V}{\sqrt{gH}} = \phi\left[\frac{D}{H}, \pi_1 \frac{\mu}{H\rho V}\right]$$

or $$V = \sqrt{2gH}\ \phi\left[\frac{D}{H}, \frac{\mu}{\rho VH}\right].\ \textbf{Ans.}$$

Multiplying by a constant does not change the character of π-terms.

Problem 12.10 *The pressure difference Δp in a pipe of diameter D and length l due to turbulent flow depends on the velocity V, viscosity μ, density ρ and roughness k. Using Buckingham's π-theorem, obtain an expression for Δp.*

Solution. Given :

Δp is a function of D, l, V, μ, ρ, k

$\therefore$ $$\Delta p = f(D, l, V, \mu, \rho, k) \text{ or } f_1(\Delta p, D, l, V, \mu, \rho, k) = 0 \quad ...(i)$$

$\therefore$ Total number of variables, $n = 7$.

Writing dimensions of each variable,

Dimension of Δp = Dimension of pressure = $ML^{-1}T^{-2}$

$$D = L, l = L, V = LT^{-1}, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, k = L$$

$\therefore$ Number of fundamental dimensions, $m = 3$

Number of π-terms $= n - m = 7 - 3 = 4.$

Now equation (i) can be grouped in 4 π-terms as

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \quad ...(ii)$$

Each π-term contains $m + 1$ or $3 + 1 = 4$ variables. Out of four variables, three are repeating variables. Choosing D, V, ρ as the repeating variables, we have the four π-terms as

$$\pi_1 = D^{a_1} . V^{b_1} . \rho^{c_1} . \Delta p$$
$$\pi_2 = D^{a_2} . V^{b_2} . \rho^{c_2} . l$$
$$\pi_3 = D^{a_3} . V^{b_3} . \rho^{c_3} . \mu$$
$$\pi_4 = D^{a_4} . V^{b_4} . \rho^{c_4} . k$$

First π-term $\quad \pi_1 = D^{a_1} . V^{b_1} . \rho^{c_1} . \Delta p$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_1}. (LT^{-1})^{b_1} . (ML^{-3})^{c_1} . ML^{-1}T^{-2}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_1 + 1 \quad \therefore\ c_1 = -1$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 - 1, \quad \therefore\ a_1 = -b_1 + 3c_1 + 1 = 2 - 3 + 1 = 0$

Power of T, $\quad 0 = -b_1 - 2, \quad \therefore\ b_1 = -2$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = D^0 . V^{-2} . \rho^{-1} . \Delta p = \frac{\Delta p}{\rho V^2}$$

Second π-term $\quad \pi_2 = D^{a_2} . V^{b_2} . \rho^{c_2} . l$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} . (LT^{-1})^{b_2} . (ML^{-3})^{c_2} . L$$

Equating powers of M, L, T on both sides,

Power of M, $\quad 0 = c_2, \quad \therefore\ c_2 = 0$

Power of L, $\quad 0 = a_2 - b_2 - 3c_2 + 1, \quad \therefore\ a_2 = b_2 + 3c_2 - 1 = -1$

Power of T, $\quad 0 = -b_2, \quad \therefore\ b_2 = 0$

Substituting the values of a_2, b_2, c_2 in π_2,

$$\pi_2 = D^{-1} \,.\, V^0 \,.\, \rho^0 \,.\, l = \frac{l}{D}$$

Third π-term $\quad \pi_3 = D^{a_3} . V^{b_3} \,.\, \rho^{c_3} \,.\, \mu$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_3} \,.\, (LT^{-1})^{b_3} \,.\, (ML^{-3})^{c_3} \,.\, ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_3 + 1, \quad \therefore \; c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore \; a_3 = -b_3 + 3c_3 + 1 = 1 - 3 + 1 = -1$

Power of T, $\quad 0 = -b_3 - 1, \quad \therefore \; b_3 = -1$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = D^{-1} \,.\, V^{-1} \,.\, \rho^{-1} \,.\, \mu = \frac{\mu}{DV\rho}.$$

Fourth π-term $\quad \pi_4 = D^{a_4} \,.\, V^{b_4} \,.\, \rho^{c_4} \,.\, k$

or $\quad M^0L^0T^0 = L^{a_4} \,.\, (LT^{-1})^{b_4} \,.\, (ML^{-3})^{c_4} \,.\, L \qquad$ {Dimension of $k = L$}

Equating the power of M, L, T on both sides,

Power of M, $\quad 0 = c_4, \quad \therefore \; c_4 = 0$

Power of L, $\quad 0 = a_4 - b_4 - 3c_4 + 1, \quad \therefore \; a_4 = b_4 + 3c_4 - 1 = -1$

Power of T, $\quad 0 = -b_4, \quad \therefore \; b_4 = 0$

Substituting the values of a_4, b_4, c_4 in π_4,

$$\pi_4 = D^{-1} \,.\, V^0 \,.\, \rho^0 \,.\, k = \frac{k}{D}$$

Substituting the values of π_1, π_2, π_3 and π_4 in (*ii*),we get

$$f_1\left(\frac{\Delta p}{\rho V^2}, \frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right) = 0 \quad \text{or} \quad \frac{\Delta p}{\rho V^2} = \phi\left[\frac{\mathbf{l}}{\mathbf{D}}, \frac{\mu}{\mathbf{DV\rho}}, \frac{\mathbf{k}}{\mathbf{D}}\right].$$ **Ans.**

Expression for h_f (Difference of pressure-head). From experiments, it was observed that pressure difference, Δp is a linear function of $\frac{l}{D}$ and hence it is taken out of function

$$\therefore \qquad \frac{\Delta p}{\rho V^2} = \frac{l}{D}\phi\left[\frac{\mu}{DV\rho}, \frac{k}{D}\right]$$

$$\therefore \qquad \frac{\Delta p}{\rho} = V^2 \,.\, \frac{l}{D}\phi\left[\frac{\mu}{DV\rho}, \frac{k}{D}\right]$$

Dividing by g to both sides, we have $\dfrac{\Delta p}{\rho g} = \dfrac{V^2 \,.\, l}{g \,.\, D}\phi\left[\dfrac{\mu}{DV\rho}, \dfrac{k}{D}\right].$

Now $\phi\left[\dfrac{\mu}{DV\rho}, \dfrac{k}{D}\right]$ contains two terms. First one is $\dfrac{\mu}{DV\rho}$ which is $\dfrac{1}{\text{Reynolds number}}$ or $\dfrac{1}{R_e}$ and second one is $\dfrac{k}{D}$ which is called roughness factor. Now $\phi\left[\dfrac{1}{R_e}, \dfrac{k}{D}\right]$ is put equal to f, where f is the co-efficient of friction which is a function of Reynolds number and roughness factor.

$$\therefore \qquad \frac{\Delta p}{\rho g} = \frac{4f}{2} \cdot \frac{V^2 l}{gD} \qquad \left\{\because \; f = \phi\left(\frac{\mu}{DV\rho}, \frac{K}{D}\right)\right\}$$

Multiplying or dividing by any constant does not change the character of π-terms.

$$\therefore \qquad \frac{\Delta p}{\rho g} = h_f = \frac{\mathbf{4f \cdot LV^2}}{\mathbf{D \times 2g}}. \textbf{ Ans.}$$

Problem 12.11 *The pressure difference Δp in a pipe of diameter D and length l due to viscous flow depends on the velocity V, viscosity μ and density ρ. Using Buckingham's π-theorem, obtain an expression for Δp.*

Solution. This problem is similar to problem 12.10. The only difference is that Δp is to be calculated for viscous flow. Then in the repeating variable instead of ρ, the fluid property μ is to be chosen.

Now Δp is a function of D, l, V, μ, ρ or $\Delta p = f(D, l, V, \mu, \rho)$

or $\qquad f_1(\Delta p, D, l, V, \mu, \rho) = 0 \qquad$ ...(*i*)

Total number of variables, $n = 6$

Number of fundamental dimensions, $m = 3$

Number of π-terms $= n - 3 = 6 - 3 = 3$

Hence equation (*i*) is written as $f_1(\pi_1, \pi_2, \pi_3) = 0 \qquad$ ...(*ii*)

Each π-term contains $m + 1$ variables, *i.e.*, $3 + 1 = 4$ variables. Out of four variables, three are repeating variables.

Choosing D, V, μ as repeating variables, we have π-terms as

$$\pi_1 = D^{a_1} \cdot V^{b_1} \cdot \mu^{c_1} \cdot \Delta p$$

$$\pi_2 = D^{a_2} \cdot V^{b_2} \cdot \mu^{c_2} \cdot l$$

$$\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \mu^{c_3} \cdot \rho$$

First π-term $\qquad \pi_1 = D^{a_1} \cdot V^{b_1} \cdot \mu^{c_1} \cdot \Delta p$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_1} \cdot (LT^{-1})^{b_1} \cdot (ML^{-1}T^{-1})^{c_1} \cdot ML^{-1}T^{-2}.$$

Equating the powers of M, L, T on both sides,

Power of M, $\qquad 0 = c_1 + 1, \qquad \therefore \; c_1 = -1$

Power of L, $\qquad 0 = a_1 + b_1 - c_1 - 1, \qquad \therefore \; a_1 = -b_1 + c_1 + 1 = 1 - 1 + 1 = 1$

Power of T, $\qquad 0 = -b_1 - c_1 - 2, \qquad \therefore \; b_1 = -c_1 - 2 = 1 - 2 = -1$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = D^1 \cdot V^{-1} \cdot \mu^{-1} \cdot \Delta p = \frac{D\Delta p}{\mu V}.$$

Second π-term $\qquad \pi_2 = D^{a_2} \cdot V^{b_2} \cdot \mu^{c_2} \cdot l$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-1}T^{-1})^{c_2} \cdot L.$$

Equating the powers of M, L, T on both sides

Power of M, $\qquad 0 = c_2, \qquad \therefore \; c_2 = 0$

Power of L, $\qquad 0 = a_2 + b_2 - c_2 + 1, \qquad \therefore \; a_2 = -b_2 + c_2 - 1 = -1$

Power of T, $\qquad 0 = -b_2 - c_2, \qquad \therefore \; b_2 = -c_2 = 0$

Substituting the values of a_2, b_2 and c_2 in π_2,

$$\pi_2 = D^{-1} \cdot V^0 \cdot \mu^0 \cdot l = \frac{l}{D}.$$

Third π-term $\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \mu^{c_3} \cdot \rho$

Substituting the dimension on both sides,

$$M^0L^0T^0 = L^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-1}T^{-1})^{c_3} \cdot ML^{-3}.$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_3 + 1$, $\therefore$ $c_3 = -1$

Power of L, $0 = a_3 + b_3 - c_3 - 3$, $\therefore$ $a_3 = -b_3 + c_3 + 3 = -1 - 1 + 3 = 1$

Power of T, $0 = -b_3 - c_3$, $\therefore$ $b_3 = -c_3 = -(-1) = 1$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = D^1 \cdot V^1 \cdot \mu^{-1} \cdot \rho = \frac{\rho DV}{\mu}.$$

Substituting the values of π_1, π_2 and π_3 in equation (*ii*),

$$f_1\left(\frac{D\Delta p}{\mu V}, \frac{l}{D}, \frac{\rho DV}{\mu}\right) = 0 \quad \text{or} \quad \frac{D\Delta p}{\mu V} = \phi\left[\frac{l}{D}, \frac{\rho DV}{\mu}\right] \quad \text{or} \quad \Delta p = \frac{\mu V}{D}\phi\left[\frac{l}{D}, \frac{\rho DV}{\mu}\right]$$

Experiments show that the pressure difference Δp is a linear function $\frac{l}{D}$. Hence $\frac{l}{D}$ can be taken out of the functional as

$$\Delta p = \frac{\mu \mathbf{V}}{\mathbf{D}} \times \frac{l}{\mathbf{D}}\phi\left[\frac{\rho \mathbf{DV}}{\mu}\right]. \textbf{ Ans.}$$

Expression for difference of pressure head for viscous flow

$$h_f = \frac{\Delta p}{\rho g} = \frac{\mu V}{D} \times \frac{l}{D} \times \frac{1}{\rho g}\,\phi\,[R_e] \qquad \left\{\because \frac{\rho DV}{\mu} = R_e\right\}$$

$$= \frac{\mu \mathbf{V}\mathbf{l}}{\mathbf{w}\mathbf{D}^2}\phi\,[\mathbf{R_e}]. \textbf{ Ans.}$$

Problem 12.12 *Derive on the basis of dimensional analysis suitable parameters to present the thrust developed by a propeller. Assume that the thrust P depends upon the angular velocity ω, speed of advance V, diameter D, dynamic viscosity μ, mass density ρ, elasticity of the fluid medium which can be denoted by the speed of sound in the medium C.*

Solution. Thrust P is a function of ω, V, D, μ, ρ, C

or $$P = f(\omega, V, D, \mu, \rho, C)$$

or $$f_1 = (P, \omega, V, D, \mu, \rho, C) = 0 \qquad \text{...}(i)$$

$\therefore$ Total number of variables, $n = 7$

Writing dimensions of each variable, we have

$$P = MLT^{-2}, \omega = T^{-1}, V = LT^{-1}, D = L,$$
$$\mu = ML^{-1}T^{-1}, \rho = ML^{-3}, C = LT^{-1}$$

$\therefore$ Number of fundamental dimensions, $m = 3$

$\therefore$ Number of π-terms $= n - m = 7 - 3 = 4$

Hence, equation (*i*) can be written as $f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0$ ...(*ii*)

Each π-term contains $m + 1$, *i.e.*, $3 + 1 = 4$ variables. Out of four variables, three are repeating variables.

Choosing D, V, ρ as repeating variables, we get π-terms as

$$\pi_1 = D^{a_1} . V^{b_1} . \rho^{c_1} . P$$
$$\pi_2 = D^{a_2} . V^{b_2} . \rho^{c_2} . \omega$$
$$\pi_3 = D^{a_3} . V^{b_3} . \rho^{c_3} . \mu$$
$$\pi_4 = D^{a_4} . V^{b_4} . \rho^{c_4} . C$$

First π-term $\quad \pi_1 = D^{a_1} . V^{b_1} . \rho^{c_1} . P$

Writing dimensions on both sides,

$$M^0L^0T^0 = L^{a_1} . (LT^{-1})^{b_1} . (ML^{-3})^{c_1} . MLT^{-2}.$$

Equating powers of M, L, T on both sides,

Power of M, $\quad 0 = c_1 + 1, \quad \because \quad c_1 = -1$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 + 1,$

$\quad a_1 = -b_1 + 3c_1 - 1 = 2 - 3 - 1 = -2$

Power of T, $\quad 0 = -b_1 - 2, \quad \therefore \quad b_1 = -2$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = D^{-2} . V^{-2} . \rho^{-1} P = \frac{P}{D^2V^2\rho}.$$

Second π-term $\quad \pi_2 = D^{a_2} . V^{b_2} . \Delta^{c_2} . \omega$

Writing dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} . (LT^{-1})^{b_2} . (ML^{-3})^{c_2} . T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_2, \quad \therefore \quad c_2 = 0$

Power of L, $\quad 0 = a_2 + b_2 - 3c_2, \quad \therefore \quad a_2 = -b_2 + 3c_2 = 1 + 0 = 1$

Power of T, $\quad 0 = -b_2 - 1, \quad \therefore \quad b_2 = -1$

Substituting the values of a_2, b_2, c_2 in π_2,

$$\pi_2 = D^1 . V^{-1} . \rho^0 . \omega = \frac{D\omega}{V}.$$

Third π-term $\quad \pi_3 = D^{a_3} . V^{b_3} . \rho^{c_3} . \mu.$

Writing dimensions on both sides,

$$M^0L^0T^0 = D^{a_3} . (LT^{-1})^{b_3} . (ML^{-3})^{c_3} . ML^{-1}T^{-1}.$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_3 + 1, \quad \therefore \quad c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore \quad a_3 = -b_3 + 3c_3 + 1 = 1 - 3 + 1 = -1$

Power of T, $\quad 0 = -b_3 - 1 \quad \therefore \quad b_3 = -1$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = D^{-1} . V^{-1} . \rho^{-1} . \mu = \frac{\mu}{DV\rho}.$$

Fourth π-term $\quad \pi_4 = D^{a_4} . V^{b_4} . \rho^{c_4} C.$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_4} . (LT^{-1})^{b_4} . (ML^{-3})^{c_4} . LT^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_4$, $\therefore$ $c_4 = 0$
Power of L, $0 = a_4 + b_4 - 3c_4 + 1$, $\therefore$ $a_4 = -b_4 + 3c_4 - 1 = 1 + 0 - 1 = 0$
Power of T, $0 = -a_4 - 1$, $\therefore$ $b_4 = -1$

Substituting the values of a_4, b_4 and c_4 in π_4,

$$\pi_4 = D^0 . V^{-1} . \rho^0 . C = \frac{C}{V}.$$

Substituting the values of π_1, π_2, π_3 and π_4 in equation (*ii*),

$$f_1\left(\frac{P}{D^2V^2\rho}, \frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V}\right) = 0 \quad \text{or} \quad \frac{P}{D^2V^2\rho} = \phi\left(\frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V}\right)$$

or
$$\mathbf{P = D^2V^2\rho\,\phi\left(\frac{D\omega}{V}, \frac{\mu}{DV\rho}, \frac{C}{V}\right).}\ \textbf{Ans.}$$

Problem 12.13 *The frictional torque T of a disc of diameter D rotating at a speed N in a fluid of viscosity μ and density ρ in a turbulent flow is given by* $T = D^5N^2\rho\phi\left[\frac{\mu}{D^2N\rho}\right]$.

Prove this by the method of dimensions.

Solution. Given : $T = f(D, N, \mu, \rho)$ or $f_1(T, D, N, \mu, \rho) = 0$ ...(*i*)

$\therefore$ Total number of variables, $n = 5$

Dimensions of each variable are expressed as

$$T = ML^2T^{-2},\ D = L,\ N = T^{-1},\ \mu = ML^{-1}T^{-1},\ \rho = ML^{-3}$$

$\therefore$ Number of fundamental dimensions, $m = 3$

Number of π-terms $= n - m = 5 - 3 = 2$

Hence equation (*i*) can be written as $f_1(\pi_1, \pi_2) = 0$ ...(*ii*)

Each π-term contains $m + 1$ variable, *i.e.*, $3 + 1 = 4$ variables. Three variables are repeating variables.

Choosing D, N, ρ as repeating variables, the π-terms are

$$\pi_1 = D^{a_1} . N^{b_1} . \rho^{c_1} . T$$
$$\pi_2 = D^{a_2} . N^{b_2} . \rho^{c_2} . \mu$$

Dimensional Analysis of π_1

$$\pi_1 = D^{a_1} . N^{b_1} . \rho^{c_1} . T$$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_1} . (T^{-1})^{b_1} . (ML^{-3})^{c_1} . ML^2T^{-2}.$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_1 + 1$, $\therefore$ $c_1 = -1$
Power of L, $0 = a_1 - 3c_1 + 2$, $\therefore$ $a_1 = 3c_1 - 2 = -3 - 2 = -5$
Power of T, $0 = -b_1 - 2$, $\therefore$ $b_1 = -2$

Substituting the values of a_1, b_1, c_1 in π,

$$\pi_1 = D^{-5} . N^{-2} . \rho^{-1} . T = \frac{T}{D^5N^2\rho}.$$

Dimensional Analysis of π_2

$$\pi_2 = D^{a_2} . N^{b_2} . \rho^{c_2} . \mu$$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} \cdot (T^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot ML^{-1}T^{-1}.$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_2 + 1, \quad \therefore \quad c_2 = -1$

Power of L, $\quad 0 = a_2 - 3c_2 - 1, \quad \therefore \quad a_2 = 3c_2 + 1 = -3 + 1 = -2$

Power of T, $\quad 0 = -b_2 - 1, \quad \therefore \quad b_2 = -1$

Substituting the values of a_2, b_2 and c_2 in π_2,

$$\pi_2 = D^{-2}N^{-1}\rho^{-1} \cdot \mu = \frac{\mu}{D^2N\rho}.$$

Substituting the values of π_1 and π_2 in equation (*ii*),

$$f_1\left(\frac{T}{D^5N^2\rho}, \frac{\mu}{D^2N\rho}\right) = 0 \quad \text{or} \quad \frac{T}{D^5N^2\rho} = \phi\left(\frac{\mu}{D^2N\rho}\right)$$

or

$$T = \mathbf{D^5N^2}\ \rho\ \phi\left[\frac{\mu}{\mathbf{D^2N}\rho}\right]. \textbf{ Ans.}$$

Problem 12.14 *Using Buckingham's π-theorem, show that the discharge Q consumed by an oil ring is given by*

$$Q = Nd^3\phi\left[\frac{\mu}{\rho Nd^2}, \frac{\sigma}{\rho N^2d^3}, \frac{w}{\rho N^2d}\right]$$

where d is the internal diameter of the ring, N is rotational speed, ρ is density, μ is viscosity, σ is surface tension and w is the specific weight of oil.

Solution. Given : $\quad Q = f(d, N, \rho, \mu, \sigma, w)$ or $f_1(Q, d, N, \rho, \mu, \sigma, w) = 0 \quad$ ...(*i*)

$\therefore$ Total number of variables, $n = 7$

Dimensions of each variables are

$$Q = L^3T^{-1},\ d = L,\ N = T^{-1},\ \rho = ML^{-3},\ \mu = ML^{-1}T^{-1},\ \sigma = MT^{-2}$$

and $\quad w = ML^{-2}T^{-2}$

$\therefore$ Total number of fundamental dimensions, $m = 3$

$\therefore$ Total number of π-terms $= n - m = 7 - 3 = 4$

$\therefore$ Equation (*i*) becomes as $f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \quad$ ...(*ii*)

Choosing d, N, ρ as repeating variables, the π-terms are

$$\pi_1 = d^{a_1} \cdot N^{b_1} \cdot \rho^{c_1} \cdot Q$$
$$\pi_2 = d^{a_2} \cdot N^{b_2} \cdot \rho^{c_2} \cdot \mu$$
$$\pi_3 = d^{a_3} \cdot N^{b_3} \cdot \rho^{c_3} \cdot \sigma$$
$$\pi_4 = d^{a_4} \cdot N^{b_4} \cdot \rho^{c_4} \cdot w.$$

First π-term $\quad \pi_1 = d^{a_1} \cdot N^{b_1} \cdot \rho^{c_1} \cdot Q.$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_1} \cdot (T^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot L^3T^{-1}.$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c_1, \quad \therefore \quad c_1 = 0$

Power of L, $\quad 0 = a_1 - 3c_1 + 3, \quad \therefore \quad a_1 = 3c_1 - 3 = 0 - 3 = -3$

Power of T, $\quad 0 = -b_1 - 1, \quad \therefore \quad b_1 = -1$

Substituting a_1, b_1, c_1 in π_1, $\pi_1 = d^{-3} \cdot N^{-1} \cdot \rho^0 \cdot Q = \dfrac{Q}{d^3N}$

Second π-term $\pi_2 = d^{a_2} . N^{b_2} . \rho^{c_2} . \mu.$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} . (T^{-1})^{b_2} . (ML^{-3})^{c_2} . ML^{-1}\, T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_2 + 1,$ $\therefore$ $c_2 = -1$

Power of L, $0 = a_2 - 3c_2 - 1,$

$\therefore$ $a_2 = 3c_2 + 1 = -3 + 1 = -2$

Power of T, $0 = -b_2 - 1,$ $\therefore$ $b_2 = -1$

Substituting the values of a_2, b_2, c_2 in π_2,

$$\pi_2 = d^{-2} . N^{-1} . \rho^{-1} . \mu = \frac{\mu}{d^2 N\rho} \quad \text{or} \quad \frac{\mu}{\rho N d^2}.$$

Third π-term $\pi_3 = d^{a_3} . N^{b_3} . \rho^{c_3} . \sigma.$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_3} . (T^{-1})^{b_3} . (ML^{-3})^{c_3} . MT^{-2}.$$

Equating the powers of M, L, T on the sides,

Power of M, $0 = c_3 + 1,$ $\therefore$ $c_3 = -1$

Power of L, $0 = a_3 - 3c_3,$ $\therefore$ $a_3 = 3c_3 = -3$

Power of T, $0 = -b_3 - 2,$ $\therefore$ $b_3 = -2$

Substituting the values of a_3, b_3, c_3 in π_3,

$$\pi_3 = d^{-3} . N^{-2} . \rho^{-1} . \sigma = \frac{\sigma}{d^3 N^2 \rho}.$$

Fourth π-term $\pi_4 = d^{a_4} . N^{b_4} . \rho^{c_4} . w$

Substituting dimensions on both sides,

$$M^0L^0T^0 = L^{a_4} . (T^{-1})^{b_4} . (ML^{-3})^{c_4} . ML^{-2}T^{-2}.$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_4 + 1,$ $\therefore$ $c_4 = -1$

Power of L, $0 = a_4 - 3c_4 - 2,$ $\therefore$ $a_4 = 3c_4 + 2 = -3 + 2 = -1$

Power of T, $0 = -b_4 - 2,$ $\therefore$ $b_4 = -2$

Substituting the values of a_4, b_4 and c_4 in π_4,

$$\pi_4 = d^{-1} . N^{-2} . \rho^{-1} . w = \frac{w}{dN^2\rho}.$$

Now substituting the values of π_1, π_2, π_3, π_4 in (*ii*),

$$f\left(\frac{Q}{d^3N}, \frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3N^2\rho}, \frac{w}{dN^2\rho}\right) = 0 \quad \text{or} \quad \frac{Q}{d^3N} = f_1\left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3N^2\rho}, \frac{w}{dN^2\rho}\right]$$

or

$$Q = \mathbf{d^3N\phi}\left[\frac{\mu}{\mathbf{\rho Nd^2}}, \frac{\sigma}{\mathbf{d^3N^2d}}, \frac{\mathbf{w}}{\mathbf{dN^2\rho}}\right]. \textbf{ Ans.}$$

▶ 12.5 MODEL ANALYSIS

For predicting the performance of the hydraulic structures (such as dams, spillways etc.) or hydraualic machines (such as turbines, pumps etc.), before actually constructing or manufacturing, models of the structures or machines are made and tests are performed on them to obtain the desired information.

The **model** is the small scale replica of the actual structure or machine. The actual structure or machine is called **Prototype**. It is not necessary that the models should be smaller than the prototypes (though in most of cases it is), they may be larger than the prototype. The study of models of actual machines is called **Model analysis**. Model analysis is actually an experimental method of finding solutions of complex flow problems. Exact analytical solutions are possible only for a limited number of flow problems. The followings are the advantages of the dimensional and model analysis :

1. The performance of the hydraulic structure or hydraulic machine can be easily predicted, in advance, from its model.

2. With the help of dimensional analysis, a relationship between the variables influencing a flow problem in terms of dimensionless parameters is obtained. This relationship helps in conducting tests on the model.

3. The merits of alternative designs can be predicted with the help of model testing. The most economical and safe design may be, finally, adopted.

4. The tests performed on the models can be utilized for obtaining, in advance, useful information about the performance of the prototypes only if a complete similarity exists between the model and the prototype.

▶ 12.6 SIMILITUDE-TYPES OF SIMILARITIES

Similitude is defined as the similarity between the model and its prototype in every respect, which means that the model and prototype have similar properties or model and prototype are completely similar. Three types of similarities must exist between the model and prototype. They are

1. Geometric Similarity, 2. Kinematic Similarity, and 3. Dynamic Similarity.

1. Geometric Similarity. The geometric similarity is said to exist between the model and the prototype. The ratio of all corresponding linear dimension in the model and prototype are equal.

Let L_m = Length of model, b_m = Breadth of model,
D_m = Diameter of model, A_m = Area of model,
$\forall_m$ = Volume of model,

and $L_P, b_P, D_P, A_P, \forall_P$ = Corresponding values of the prototype.

For geometric similarity between model and prototype, we must have the relation,

$$\frac{L_P}{L_m} = \frac{b_P}{b_m} = \frac{D_P}{D_m} = L_r \qquad ...(12.6)$$

where L_r is called the scale ratio.

For area's ratio and volume's ratio the relation should be as given below :

$$\frac{A_P}{A_m} = \frac{L_P \times b_P}{L_m \times b_m} = L_r \times L_r = L_r^2 \qquad ...(12.7)$$

and
$$\frac{\forall_P}{\forall_m} = \left(\frac{L_P}{L_m}\right)^3 = \left(\frac{b_P}{b_m}\right)^3 = \left(\frac{D_P}{D_m}\right)^3 \qquad ...(12.8)$$

2. Kinematic Similarity. Kinematic similarity means the similarity of motion between model and prototype. Thus kinematic similarity is said to exist between the model and the prototype if the ratios of the velocity and acceleration at the corresponding points in the model and at the corresponding points in the prototype are the same. Since velocity and acceleration are vector quantities, hence not only the ratio of magnitude of velocity and acceleration at the corresponding points in model and prototype should be same ; but the directions of velocity and accelerations at the corresponding points in the model and prototype also should be parallel.

Let V_{P_1} = Velocity of fluid at point 1 in prototype,

V_{P_2} = Velocity of fluid at point 2 in prototype,

a_{P_1} = Acceleration of fluid at point 1 in prototype,

a_{P_2} = Acceleration of fluid at point 2 in prototype, and

$V_{m_1}, V_{m_2}, a_{m_1}, a_{m_2}$ = Corresponding values at the corresponding points of fluid velocity and acceleration in the model.

For kinematic similarity, we must have

$$\frac{V_{P_1}}{V_{m_1}} = \frac{V_{P_2}}{V_{m_2}} = V_r \qquad \text{...(12.9)}$$

where V_r is the velocity ratio.

For acceleration, we must have $\dfrac{a_{P_1}}{a_{m_1}} = \dfrac{a_{P_2}}{a_{m_2}} = a_r$ ...(12.10)

where a_r is the acceleration ratio.

Also the directions of the velocities in the model and prototype should be same.

3. Dynamic Similarity. Dynamic similarity means the similarity of forces between the model and prototype. Thus dynamic similarity is said to exist between the model and the prototype if the ratios of the corresponding forces acting at the corresponding points are equal. Also the directions of the corresponding forces at the corresponding points should be same.

Let $(F_i)_P$ = Inertia force at a point in prototype,

$(F_v)_P$ = Viscous force at the point in prototype,

$(F_g)_P$ = Gravity force at the point in prototype,

and $(F_i)_m$ $(F_v)_m$, $(F_g)_m$ = Corresponding values of forces at the corresponding point in model.

Then for dynamic similarity, we have

$$\frac{(F_i)_P}{(F_i)_m} = \frac{(F_v)_P}{(F_v)_m} = \frac{(F_g)_P}{(F_g)_m} \ldots = F_r, \text{ where } F_r \text{ is the force ratio.}$$

Also the directions of the corresponding forces at the corresponding points in the model and prototype should be same.

▶ 12.7 TYPES OF FORCES ACTING IN MOVING FLUID

For the fluid flow problems, the forces acting on a fluid mass may be any one, or a combination of the several of the following forces :

1. Inertia force, F_i.
2. Viscous force, F_v.
3. Gravity force, F_g.
4. Pressure force, F_p.
5. Surface tension force, F_s.
6. Elastic force, F_e.

1. **Inertia Force** (F_i). It is equal to the product of mass and acceleration of the flowing fluid and acts in the direction opposite to the direction of acceleration. It is always existing in the fluid flow problems.

2. **Viscous Force** (F_v). It is equal to the product of shear stress (τ) due to viscosity and surface area of the flow. It is present in fluid flow problems where viscosity is having an important role to play.

3. **Gravity Force** (F_g). It is equal to the product of mass and acceleration due to gravity of the flowing fluid. It is present in case of open surface flow.

4. **Pressure Force** (F_p). It is equal to the product of pressure intensity and cross-sectional area of the flowing fluid. It is present in case of pipe-flow.

5. **Surface Tension Force** (F_s). It is equal to the product of surface tension and length of surface of the flowing fluid.

6. **Elastic Force** (F_e). It is equal to the product of elastic stress and area of the flowing fluid.

For a flowing fluid, the above-mentioned forces may not always be present. And also the forces, which are present in a fluid flow problem, are not of equal magnitude. There are always one or two forces which dominate the other forces. These dominating forces govern the flow of fluid.

▶ 12.8 DIMENSIONLESS NUMBERS

Dimensionless numbers are those numbers which are obtained by dividing the inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force. As this is a ratio of one force to the other force, it will be a dimensionless number. These dimensionless numbers are also called non-dimensional parameters. The followings are the important dimensionless numbers :

1. Reynold's number,
2. Froude's number,
3. Euler's number,
4. Weber's number,
5. Mach's number.

12.8.1 Reynold's Number (R_e).

It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. The expression for Reynold's number is obtained as

Inertia force (F_i) = Mass × Acceleration of flowing fluid

$$= \rho \times \text{Volume} \times \frac{\text{Velocity}}{\text{Time}} = \rho \times \frac{\text{Volume}}{\text{Time}} \times \text{Velocity}$$

$$= \rho \times AV \times V \quad \{\because \text{ Volume per sec} = \text{Area} \times \text{Velocity} = A \times V\}$$

$$= \rho AV^2 \qquad \text{...(12.11)}$$

Viscous force (F_v) = Shear stress × Area $\left\{\because \ \tau = \mu \dfrac{du}{dy} \ \therefore \ \text{Force} = \tau \times \text{Area}\right\}$

$$= \tau \times A$$

$$= \left(\mu \frac{du}{dy}\right) \times A = \mu \,.\, \frac{V}{L} \times A \qquad \left\{\because \ \frac{du}{dy} = \frac{V}{L}\right\}$$

By definition, Reynold's number,

$$R_e = \frac{F_i}{F_v} = \frac{\rho AV^2}{\mu \,.\, \dfrac{V}{L} \times A} = \frac{\rho VL}{\mu}$$

$$= \frac{V \times L}{(\mu / \rho)} = \frac{V \times L}{\nu} \qquad \left\{\because \ \frac{\mu}{\rho} = \nu = \text{Kinematic viscosity}\right\}$$

In case of pipe flow, the linear dimension L is taken as diameter, d. Hence Reynold's number for pipe flow,

$$R_e = \frac{V \times d}{\nu} \quad \text{or} \quad \frac{\rho V d}{\mu}. \qquad \text{...(12.12)}$$

12.8.2 Froude's Number (F_e). The Froude's number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravity force. Mathematically, it is expressed as

$$F_e = \sqrt{\frac{F_i}{F_g}}$$

where F_i from equation (12.11) $= \rho A V^2$

and F_g = Force due to gravity

= Mass × Acceleration due to gravity

$= \rho \times \text{Volume} \times g = \rho \times L^3 \times g$ $\{\because \text{Volume} = L^3\}$

$= \rho \times L^2 \times L \times g = \rho \times A \times L \times g$ $\{\because L^2 = A = \text{Area}\}$

$$\therefore \quad F_e = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho A V^2}{\rho A L g}} = \sqrt{\frac{V^2}{Lg}} = \frac{V}{\sqrt{Lg}} \qquad \text{...(12.13)}$$

12.8.3 Euler's Number (E_u). It is defined as the square root of the ratio of the inertia force of a flowing fluid to the pressure force. Mathematically, it is expressed as

$$E_u = \sqrt{\frac{F_i}{F_P}}$$

where F_P = Intensity of pressure × Area $= p \times A$

and $F_i = \rho A V^2$

$$\therefore \quad E_u = \sqrt{\frac{\rho A V^2}{p \times A}} = \sqrt{\frac{V^2}{p/\rho}} = \frac{V}{\sqrt{p/\rho}} \qquad \text{...(12.14)}$$

12.8.4 Weber's Number (W_e). It is defined as the square root of the ratio of the inertia force of a flowing fluid to the surface tension force. Mathematically, it is expressed as

$$\text{Weber's Number,} \quad W_e = \sqrt{\frac{F_i}{F_s}}$$

where F_i = Inertia force $= \rho A V^2$

and F_s = Surface tension force

= Surface tension per unit length × Length $= \sigma \times L$

$$\therefore \quad W_e = \sqrt{\frac{\rho A V^2}{\sigma \times L}} = \sqrt{\frac{\rho \times L^2 \times V^2}{\sigma \times L}} \qquad \{\because A = L^2\}$$

$$= \sqrt{\frac{\rho L \times V^2}{\sigma}} = \sqrt{\frac{V^2}{\sigma / \rho L}} = \frac{V}{\sqrt{\sigma / \rho L}}. \qquad \text{...(12.15)}$$

12.8.5 Mach's Number (M). Mach's number is defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force. Mathematically, it is defined as

$$M = \sqrt{\frac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\frac{F_i}{F_e}}$$

where $F_i = \rho A V^2$

and F_e = Elastic force = Elastic stress × Area

$= K \times A = K \times L^2$ $\{\because \quad K = \text{Elastic stress}\}$

$$\therefore \qquad M = \sqrt{\frac{\rho A V^2}{K \times L^2}} = \sqrt{\frac{\rho \times L^2 \times V^2}{K \times L^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}}$$

But $\sqrt{\frac{K}{\rho}} = C$ = Velocity of sound in the fluid

$$\therefore \qquad M = \frac{V}{C}. \qquad \text{...(12.16)}$$

▶ 12.9 MODEL LAWS OR SIMILARITY LAWS

For the dynamic similarity between the model and the prototype, the ratio of the corresponding forces acting at the corresponding points in the model and prototype should be equal. The ratio of the forces are dimensionless numbers. It means for dynamic similarity between the model and prototype, the dimensionless numbers should be same for model and the prototype. But it is quite difficult to satisfy the condition that all the dimensionless numbers (*i.e.*, R_e, F_e, W_e, E_u and M) are the same for the model and prototype. Hence models are designed on the basis of ratio of the force, which is dominating in the phenomenon. The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity. The followings are the model laws :

1. Reynold's model law,
2. Froude model law,
3. Euler model law,
4. Weber model law,
5. Mach model law.

12.9.1 Reynold's Model Law. Reynold's model law is the law in which models are based on Reynold's number. Models based on Reynold's number includes :

(*i*) Pipe flow

(*ii*) Resistance experienced by sub-marines, airplanes, fully immersed bodies etc.

As defined earlier that Reynold number is the ratio of inertia force and viscous force, and hence fluid flow problems where viscous forces alone are predominent, the models are designed for dynamic similarity on Reynolds law, which states that the Reynold number for the model must be equal to the Reynold number for the prototype.

Let V_m = Velocity of fluid in model,

ρ_m = Density of fluid in model,

L_m = Length or linear dimension of the model,

μ_m = Viscosity or fluid in model,

and V_P, ρ_P, L_P and μ_P are the corresponding values of velocity, density, linear dimension and viscosity of fluid in prototype. Then according to Reynold's model law,

$$[R_e]_m = [R_e]_P \text{ or } \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_P V_P L_P}{\mu_P} \qquad \text{...(12.17)}$$

or $$\frac{\rho_P \cdot V_P \cdot L_P}{\rho_m \cdot V_m \cdot L_m} \times \frac{1}{\frac{\mu_P}{\mu_m}} = 1 \quad \text{or} \quad \frac{\rho_r \cdot V_r \cdot L_r}{\mu_r} = 1$$

$$\left\{\text{where} \quad \rho_r = \frac{\rho_P}{\rho_m}, V_r = \frac{V_P}{V_m} \text{ and } L_r = \frac{L_P}{L_m}, \frac{\mu_P}{\mu_m} = \mu_r\right\}$$

And also ρ_r, V_r, L_r and μ_r are called the scale ratios for density, velocity, linear dimension and viscosity.

The scale ratios for time, acceleration, force and discharge for Reynold's model law are obtained as

$$t_r = \text{Time scale ratio} = \frac{L_r}{V_r} \qquad \left\{\because \quad V = \frac{L}{t} \quad \therefore \quad t = \frac{L}{V}\right\}$$

$$a_r = \text{Acceleration scale ratio} = \frac{V_r}{t_r}$$

$$F_r = \text{Force scale ratio} = (\text{Mass} \times \text{Acceleration})_r$$

$$= m_r \times a_r = \rho_r A_r\, V_r \times a_r \qquad \{A_r = \text{Area ratio}\}$$

$$= \rho_r L_r^2 V_r \times a_r$$

$$Q_r = \text{Discharge scale ratio} = (\rho A V)_r$$

$$= \rho_r A_r\, V_r = \rho_r \,.\, L_r^2 \,.\, V_r.$$

Problem 12.15 *A pipe of diameter 1.5 m is required to transport an oil of sp. gr. 0.90 and viscosity 3×10^{-2} poise at the rate of 3000 litre/s. Tests were conducted on a 15 cm diameter pipe using water at 20°C. Find the velocity and rate of flow in the model. Viscosity of water at 20°C = 0.01 poise.*

Solution. Given :

Dia. of prototype, $D_P = 1.5$ m

Viscosity of fluid, $\mu_P = 3 \times 10^{-2}$ poise

Q for prototype, $Q_P = 3000$ lit/s $= 3.0$ m^3/s

Sp. gr. of oil, $S_P = 0.9$

$\therefore$ Density of oil, $\rho_P = S_P \times 1000 = 0.9 \times 1000 = 900$ kg/m^3

Dia. of the model, $D_m = 15$ cm $= 0.15$ m

Viscosity of water at 20°C $= .01$ poise $= 1 \times 10^{-2}$ poise or $\mu_m = 1 \times 10^{-2}$ poise

Density of water or $\rho_m = 1000$ kg/m^3.

For pipe flow, the dynamic similarity will be obtained if the Reynold's number in the model and prototype are equal

Hence using equation (12.17), $$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_P V_P D_P}{\mu_P} \qquad \{\text{For pipe, linear dimension is } D\}$$

$$\therefore \qquad \frac{V_m}{V_P} = \frac{\rho_P}{\rho_m} \cdot \frac{D_P}{D_m} \cdot \frac{\mu_m}{\mu_P}$$

$$= \frac{900}{1000} \times \frac{1.5}{0.15} \times \frac{1 \times 10^{-2}}{3 \times 10^{-2}} = \frac{900}{1000} \times 10 \times \frac{1}{3} = 3.0$$

But $$V_P = \frac{\text{Rate of flow in prototype}}{\text{Area of prototype}} = \frac{3.0}{\frac{\pi}{4}(D_P)^2} = \frac{3.0}{\frac{\pi}{4}(1.5)^2}$$

$$= \frac{3.0 \times 4}{\pi \times 2.25} = 1.697 \text{ m/s}$$

$\therefore$ $V_m = 3.0 \times V_P = 3.0 \times 1.697 =$ **5.091 m/s. Ans.**

Rate of flow through model, $Q_m = A_m \times V_m = \frac{\pi}{4}(D_m)^2 \times V_m = \frac{\pi}{4}(0.15)^2 \times 5.091 \text{ m}^3/\text{s}$

$= 0.0899 \text{ m}^3/\text{s} = 0.0899 \times 1000 \text{ lit/s} =$ **89.9 lit/s. Ans.**

Problem 12.16 *Water is flowing through a pipe of diameter 30 cm at a velocity of 4 m/s. Find the velocity of oil flowing in another pipe of diameter 10 cm, if the condition of dynamic similarity is satisfied between the two pipes. The viscosity of water and oil is given as 0.01 poise and .025 poise. The sp. gr. of oil = 0.8.*

Solution. Given :

Two pipes having different liquids.

Let for pipe 1, Liquid = Water

Dia. of pipe, $d_1 = 30 \text{ cm} = 0.30 \text{ m}$

Velocity of flow, $V_1 = 4 \text{ m/s}$

Viscosity, $\mu_1 = 0.01 \text{ poise} = \frac{0.01}{10}$ (S.I. Units)

Density, $\rho_1 = 1000 \text{ kg/m}^3$

For pipe 2, Liquid = Oil

Dia. of pipe, $d_2 = 10 \text{ cm} = 0.1 \text{ m}$

Velocity of flow, $V_2 = ?$

Viscosity, $\mu_2 = 0.025 \text{ poise} = \frac{0.025}{10}$ (S.I. Units)

Sp. gr. of oil $= 0.8$

$\therefore$ Density, $\rho_2 = 0.8 \times 1000 = 800 \text{ kg/m}^3$

If the pipes are dynamically similar, the Reynold's number for both the pipes should be same.

$$\therefore \quad \frac{\rho_1 V_1 d_1}{\mu_1} = \frac{\rho_2 V_2 d_2}{\mu_2} \quad \text{or} \quad V_2 = \frac{\rho_1}{\rho_2} \cdot \frac{d_1}{d_2} \cdot \frac{\mu_2}{\mu_1} V_1$$

$$= \frac{1000}{800} \times \frac{0.30}{0.10} \times \frac{\frac{.025}{10}}{\frac{.01}{10}} \times 4.0 = \frac{1000}{800} \times 3 \times \frac{.025}{.01} \times 4.0$$

= **37.5 m/s. Ans.**

Problem 12.17 *The ratio of lengths of a sub-marine and its model is 30 : 1. The speed of sub-marine (prototype) is 10 m/s. The model is to be tested in a wind tunnel. Find the speed of air in wind tunnel. Also determine the ratio of the drag (resistance) between the model and its prototype. Take the value of kinematic viscosities for sea water and air as .012 stokes and .016 stokes respectively. The density for sea-water and air is given as 1030 kg/m³ and 1.24 kg/m³ respectively.*

Solution. Given :

Prototype (sub-marine) and its model.

For prototype, Speed $V_P = 10$ m/s

Fluid = Sea - water

Kinematic viscosity, $\nu_P = 0.012$ stokes $= .012$ cm^2/s

$= .012 \times 10^{-4}$ m^2/s $\{\because$ Stoke = cm^2/s$\}$

Density, $\rho_P = 1030$ kg/m^3

For model Fluid = Air

Kinematic viscosity, $\nu_m = 0.016$ stokes $= 0.016$ cm^2/s $= .016 \times 10^{-4}$ m^2/s

Density, $\rho_m = 1.24$ kg/m^3

Also $\dfrac{\text{Length of prototype}}{\text{Length of model}} = \dfrac{L_P}{L_m} = 30.0$

Let the velocity of air in model $= V_m$.

For dynamic similarity between model and sub-marine, the viscous resistance is to be overcame and hence for fully submerged sub-marine, the Reynold's number for model and prototype should be same.

$$\therefore \quad \frac{\rho_P V_P D_P}{\mu_P} = \frac{\rho_m V_m D_m}{\mu_m} \quad \text{or} \quad \frac{V_P D_P}{(\mu / \rho)_P} = \frac{V_m D_m}{(\mu / \rho)_m} \; ; \; \frac{V_P D_P}{\nu_P} = \frac{V_m D_m}{\nu_m}$$

$$\therefore \quad V_m = \frac{\nu_m}{\nu_P} \times \frac{D_P}{D_m} \times V_P$$

$$= \frac{0.016 \times 10^{-4}}{.012 \times 10^{-4}} \times 30 \times 10 \text{ m/s} \qquad \left\{\because \frac{D_P}{D_m} = \frac{L_P}{L_m} = 30\right\}$$

$$= \frac{0.016}{.012} \times 30 \times 10 = \textbf{400 m/s. Ans.}$$

Ratio of drag force (resistance) :

Drag force $=$ Mass $\times$ Acceleration

$$= \rho L^3 \times \frac{V}{t} = \rho \,.\, L^2 \,.\, \frac{L}{t} \times V = \rho\, L^2 V^2 \qquad \left\{\because \frac{L}{t} = V\right\}$$

Let F_P and F_m denote the drag force for the prototype and for the model respectively then,

$$\frac{F_P}{F_m} = \frac{\rho_P L_P^2 V_P^2}{\rho_m L_m^2 V_m^2} = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2$$

$$= \frac{1030}{1.24} \times 30^2 \times \left(\frac{10}{400}\right)^2 = \textbf{467.22. Ans.}$$

Problem 12.18 *A ship 300 m long moves in sea-water, whose density is 1030 kg/m^3, A 1 : 100 model of this ship is to be tested in a wind tunnel. The velocity of air in the wind tunnel around the model is 30 m/s and the resistance of the model is 60 N. Determine the velocity of ship in sea-water and also the resistance of the ship in sea-water. The density of air is given as 1.24 kg/m^3. Take the kinematic viscosity of sea-water and air as 0.012 stokes and 0.018 stokes respectively.*

Solution. Given :

For Prototype,

Length, $L_P = 300$ m

Fluid = Sea-water

Density of water $= 1030 \text{ kg/m}^3$

Kinematic viscosity, $\nu_P = 0.012$ stokes $= 0.012 \times 10^{-4} \text{ m}^2/\text{s}$

Let velocity of ship $= V_P$

Resistance $= F_P$

For model

Length, $L_m = \dfrac{1}{100} \times 300 = 3$ m

Velocity, $V_m = 30$ m/s

Resistance, $F_m = 60$ N

Density of air, $\rho_m = 1.24 \text{ kg/m}^3$

Kinematic viscosity of air, $\nu_m = 0.018$ stokes $= .018 \times 10^{-4} \text{ m}^2/\text{s}$.

For dynamic similarity between the prototype and its model, the Reynolds number for both of them should be equal.

$$\therefore \quad \frac{V_P \times L_P}{\nu_P} = \frac{V_m \times L_m}{\nu_m} \quad \text{or} \quad V_P = \frac{\nu_P}{\nu_m} \times \frac{L_m}{L_P} \times V_m$$

$$= \frac{.012 \times 10^{-4}}{.018 \times 10^{-4}} \times \frac{3}{300} \times 30 = \frac{1}{1.5} \times \frac{1}{100} \times 30 = \mathbf{0.2 \text{ m/s. Ans.}}$$

Resistance $=$ Mass $\times$ Acceleration

$$= \rho L^3 \times \frac{V}{t} = \rho L^2 \times \frac{V}{1} \times \frac{L}{t} = \rho L^2 V^2$$

Then
$$\frac{F_P}{F_m} = \frac{\left(\rho L^2 V^2\right)_P}{\left(\rho L^2 V^2\right)_m} = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2$$

But
$$\frac{\rho_P}{\rho_m} = \frac{1030}{1.24}$$

$$\therefore \quad \frac{F_P}{F_m} = \frac{1030}{1.24} \times \left(\frac{300}{3}\right)^2 \times \left(\frac{0.2}{30}\right)^2 = 369.17$$

$$\therefore \quad F_P = 369.17 \times F_m = 369.17 \times 60 = \mathbf{22150.2 \text{ N. Ans.}}$$

12.9.2 Froude Model Law. Froude model law is the law in which the models are based on Froude number which means for dynamic similarity between the model and prototype, the Froude number for both of them should be equal. Froude model law is applicable when the gravity force is only predominant force which controls the flow in addition to the force of inertia. Froude model law is applied in the following fluid flow problems :

1. Free surface flows such as flow over spillways, weirs, sluices, channels etc.,
2. Flow of jet from an orifice or nozzle,
3. Where waves are likely to be formed on surface,
4. Where fluids of different densities flow over one another.

Let V_m = Velocity of fluid in model,
L_m = Linear dimension or length of model,
g_m = Acceleration due to gravity at a place where model is tested.

and V_P, L_P and g_P are the corresponding values of the velocity, length and acceleration due to gravity for the prototype. Then according to Froude model law,

$$(F_e)_{model} = (F_e)_{prototype} \text{ or } \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_P}{\sqrt{g_P L_P}} \quad ...(12.18)$$

If the tests on the model are performed on the same place where prototype is to operate, then $g_m = g_P$ and equation (12.18) becomes as

$$\frac{V_m}{\sqrt{L_m}} = \frac{V_P}{\sqrt{L_P}} \quad ...(12.19)$$

or
$$\frac{V_m}{V_P} \times \frac{1}{\sqrt{\frac{L_m}{L_P}}} = 1$$

$$\frac{V_P}{V_m} = \sqrt{\frac{L_P}{L_m}} = \sqrt{L_r} \quad \left\{\because \frac{L_P}{L_m} = L_r\right\}$$

where L_r = Scale ratio for length

$\frac{V_P}{V_m} = V_r$ = Scale ratio for velocity.

$$\therefore \quad \frac{V_P}{V_m} = V_r = \sqrt{L_r}\,. \quad ...(12.20)$$

Scale ratios for various physical quantities based on Froude model law are :

(*a*) **Scale ratio for time**

As $\text{time} = \frac{\text{Length}}{\text{Velocity}}$,

then ratio of time for prototype and model is

$$T_r = \frac{T_P}{T_m} = \frac{\left(\frac{L}{V}\right)_P}{\left(\frac{L}{V}\right)_m} = \frac{\frac{L_P}{V_P}}{\frac{L_m}{V_m}} = \frac{L_P}{L_m} \times \frac{V_m}{V_P} = L_r \times \frac{1}{\sqrt{L_r}} \quad \left\{\because \frac{V_P}{V_m} = \sqrt{L_r}\right\}$$

$$= \sqrt{L_r}\,. \quad ...(12.21)$$

(*b*) **Scale ratio for acceleration**

$$\text{Acceleration} = \frac{V}{T}$$

$$\therefore \quad a_r = \frac{a_P}{a_m} = \frac{\left(\frac{V}{T}\right)_P}{\left(\frac{V}{T}\right)_m} = \frac{V_P}{T_P} \times \frac{T_m}{V_m} = \frac{V_P}{V_m} \times \frac{T_m}{T_P}$$

$$= \sqrt{L_r} \times \frac{1}{\sqrt{L_r}} \qquad \left\{\because \frac{V_P}{V_m} = \sqrt{L_r}, \frac{T_P}{T_m} = \sqrt{L_r}\right\}$$

$$= 1. \qquad \text{...(12.22)}$$

(*c*) **Scale ratio for discharge**

$$Q = A \times V = L^2 \times \frac{L}{T} = \frac{L^3}{T}$$

$$\therefore \qquad Q_r = \frac{Q_P}{Q_m} = \frac{\left(\frac{L^3}{T}\right)_P}{\left(\frac{L^3}{T}\right)_m} = \left(\frac{L_P}{L_m}\right)^3 \times \left(\frac{T_m}{T_P}\right) = L_r^{\,3} \times \frac{1}{\sqrt{L_r}} = L_r^{\,2.5} \qquad \text{...(12.23)}$$

(*d*) **Scale ratio for force**

As $$\text{Force} = \text{Mass} \times \text{Acceleration} = \rho L^3 \times \frac{V}{T} = \rho L^2 \,.\, \frac{L}{T} \,.\, V = \rho L^2 V^2$$

$\therefore$ Ratio for force, $$F_r = \frac{F_P}{F_m} = \frac{\rho_P L_P^2 V_P^2}{\rho_m L_m^2 V_m^2} = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2.$$

If the fluid used in model and prototype is same, then

$$\frac{\rho_P}{\rho_m} = 1 \quad \text{or} \quad \rho_P = \rho_m$$

and hence $$F_r = \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2 = L_r^{\,2} \times \left(\sqrt{L_r}\right)^2 = L_r^{\,2} \,.\, L_r = L_r^{\,3}. \qquad \text{...(12.24)}$$

(*e*) **Scale ratio for pressure intensity**

As $$p = \frac{\text{Force}}{\text{Area}} = \frac{\rho L^2 V^2}{L^2} = \rho V^2$$

$\therefore$ Pressure ratio, $$p_r = \frac{p_P}{p_m} = \frac{\rho_P V_P^2}{\rho_m V_m^2}$$

If fluid is same, then $\rho_P = \rho_m$

$$\therefore \qquad p_r = \frac{V_P^2}{V_m^2} = \left(\frac{V_P}{V_m}\right)^2 = L_r. \qquad \text{...(12.25)}$$

(*f*) **Scale ratio for work, energy, torque, moment etc.**

$$\text{Torque} = \text{Force} \times \text{Distance} = F \times L$$

$\therefore$ Torque ratio, $$T_r^* = \frac{T_P^*}{T_m^*} = \frac{(F \times L)_P}{(F \times L)_m} = F_r \times L_r = L_r^3 \times L_r = L_r^{\,4}. \qquad \text{...(12.26)}$$

(*g*) **Scale ratio for power**

As $$\text{Power} = \text{Work per unit time}$$

$$= \frac{F \times L}{T}$$

$\therefore$ Power ratio,
$$P_r = \frac{P_P}{P_m} = \frac{\frac{F_P \times L_P}{T_P}}{\frac{F_m \times L_m}{T_m}} = \frac{F_P}{F_m} \times \frac{L_P}{L_m} \times \frac{1}{\frac{T_P}{T_m}}$$

$$= F_r \,.\, L_r \,.\, \frac{1}{T_r} = L_r^3 \,.\, L_r \,.\, \frac{1}{\sqrt{L_r}} = L^{3.5}. \qquad \text{...(12.27)}$$

Problem 12.19 *In 1 in 40 model of a spillway, the velocity and discharge are 2 m/s and 2.5 m^3/s. Find the corresponding velocity and discharge in the prototype.*

Solution. Given :

Scale ratio of length, $L_r = 40$

Velocity in model, $V_m = 2$ m/s

Discharge in model, $Q_m = 2.4$ m^3/s

Let V_P and Q_P are the velocity and discharge in prototype.

Using equation (12.20) for velocity ratio, $\frac{V_P}{V_m} = \sqrt{L_r} = \sqrt{40}$

$\therefore$ $V_P = V_m \times \sqrt{40} = 2 \times \sqrt{40} =$ **12.65 m/s. Ans.**

Using equation (12.23) for discharge ratio,

$$\frac{Q_P}{Q_m} = L_r^{2.5} = (40)^{2.5}$$

$\therefore$ $Q_P = Q_m \times 40^{2.5} = 2.5 \times 40^{2.5} =$ **25298.2 m^3/s. Ans.**

Problem 12.20 *A ship model of scale $\frac{1}{50}$ is towed through sea water at a speed of 1 m/s. A force of 2 N is required to tow the model. Determine the speed of ship and the propulsive force on the ship, if prototype is subjected to wave resistance only.*

Solution. Given :

Scale ratio of length, $L_r = 50$

Speed of model, $V_m = 1$ m/s

Force required for model, $F_m = 2$ N

Let the speed of ship $= V_P$

and the propulsive force for ship $= F_P$.

As prototype is subjected to wave resistance only for dynamic similarity, the Froude number should be same for model and prototype. Hence for velocity ratio, for Froude model law using equation (12.20), we have

$\therefore$ $$\frac{V_P}{V_m} = \sqrt{L_r} = \sqrt{50}$$

$\therefore$ $V_P = \sqrt{50} \times V_m = \sqrt{50} \times 1 =$ **7.071 m/s. Ans.**

Force scale ratio is given by equation (12.24),

$\therefore$ $$F_r = \frac{F_P}{F_m} = L_r^3$$

$\therefore$ $F_P = F_m \times L_r^3 = 2 \times (50)^3 =$ **250000 N. Ans.**

Problem 12.21 *In the model test of a spillway the discharge and velocity of flow over the model were 2 m³/s and 1.5 m/s respectively. Calculate the velocity and discharge over the prototype which is 36 times the model size.*

Solution. Given :

Discharge over model, $Q_m = 2 \text{ m}^3/\text{s}$

Velocity over model, $V_m = 1.5 \text{ m/s}$

Linear scale ratio, $L_r = 36.$

For dynamic similarity, Froude model law is used. Using equation (12.20), we have

$$\frac{V_P}{V_m} = \sqrt{L_r} = \sqrt{36} = 6.0$$

$\therefore$ V_P = Velocity over prototype = $V_m \times 6.0 = 1.5 \times 6.0 =$ **9 m/s. Ans.**

For discharge, using equation (12.23), we get

$$\frac{Q_P}{Q_m} = L_r^{2.5} = (36)^{2.5}.$$

$$\therefore \quad Q_P = Q_m \times (36)^{2.5} = 2 \times 36^{2.5} = \mathbf{15552\ m^3/s.\ Ans.}$$

Problem 12.22 *In a geometrically similar model of spillway the discharge per metre length is $\frac{1}{6}$ m³/s. If the scale of the model is $\frac{1}{36}$, find the discharge per metre length of the prototype.*

Solution. Given :

Discharge per metre length for model, $q_m = \frac{1}{6} \text{ m}^3/\text{s}$

Linear scale ratio, $L_r = 36$

Discharge per metre length for prototype, $q_P = ?$

The discharge ratio for spillway is given by equation (12.23), $\frac{Q_P}{Q_m} = L_r^{2.5}.$

But discharge ratio per metre length is given as

$$\frac{q_P}{q_m} = \frac{Q_P / L_P}{Q_m / L_m} = \frac{Q_P}{Q_m} \times \frac{L_m}{L_P} = L_r^{2.5} \times \frac{1}{L_r} = L_r^{1.5}$$

$$\therefore \quad q_P = q_m \times L_r^{1.5} = \frac{1}{6} \times (36)^{1.5} = \frac{1}{6} \times 6^{2 \times 1.5}$$

$$= 6^{3-1} = 6^2 = \mathbf{36\ m^3/s\ per\ metre\ length.\ Ans.}$$

Problem 12.23 *A spillway model is to be built to a geometrically similar scale of $\frac{1}{50}$ across a flume of 600 mm width. The prototype is 15 m high and maximum head on it is expected to be 1.5 m. (i) What height of model and what head on the model should be used ? (ii) If the flow over the model at a particular head is 12 litres per second, what flow per metre length of the prototype is expected ? (iii) If the negative pressure in the model is 200 mm, what is the negative pressure in prototype ? Is it practicable ?*

Solution. Given :

Scale ratio for length, $L_r = 50$

Width of model, $B_m = 600 \text{ mm} = 0.6 \text{ m}$

Flow over model, $Q_m = 12$ litres/s

Pressure in model, $h_m = -\ 200$ mm of water $= -\ 0.2$ m

Height of prototype, $H_P = 15$ m

Head on prototype, $H_P{}^* = 1.5$ m

(*i*) Let the height of model $= H_m$

and head on model $= H_m{}^*$

Linear scale ratio, $$L_r = \frac{H_P}{H_m} = \frac{H_{P^*}}{H_{m^*}} = 50$$

$\therefore$ Height of model, $$H_m = \frac{H_P}{50} = \frac{15}{50} = \mathbf{0.3\ m.\ Ans.}$$

And head on model, $$H_m{}^* = \frac{H_{P^*}}{50} = \frac{1.50}{50} = \mathbf{0.03\ m.\ Ans.}$$

Width of prototype, $B_P = L_r \times B_m = 50 \times 0.6 = 30$ m.

(*ii*) Discharge ratio is given by equation (12.23) as

$$\frac{Q_P}{Q_m} = L_r^{2.5} = (50)^{2.5} = 17677.67$$

$$\therefore \quad Q_P = Q_m \times 17677.67 = 12 \times 17677.67 = 212132.04 \text{ lit/s}$$

Discharge per metre length of prototype $= \dfrac{Q_P}{\text{Length of prototype}} = \dfrac{212132.04}{\text{Width of prototype}}$

$$= \frac{212132.04}{30} = \mathbf{7071.078\ litres/s.\ Ans.}$$

(*iii*) Negative pressure head in prototype,

$$h_P = L_r \times h_m = 50 \times (-\ 0.2) = \mathbf{-\ 10.0\ m.\ Ans.}$$

This negative pressure is not practicable. Maximum practicable negative pressure head is – 7.50 m of water.

Problem 12.24 *In a 1 in 20 model of stilling basin, the height of the hydraulic jump in the model is observed to be 0.20 metre. What is the height of the hydraulic jump in the prototype ? If the energy dissipated in the model is $\frac{1}{10}$ kW, what is the corresponding value in prototype ?*

Solution. Given :

Linear scale ratio, $L_r = 20$

Height of hydraulic jump in model, $h_m = 0.20$ m

Energy dissipated in model, $P_m = \dfrac{1}{10}$ kW

(*i*) Let the height of hydraulic jump in the prototype $= h_P$

Then $$\frac{h_P}{h_m} = L_r = 20$$

$$\therefore \quad h_P = h_m \times 20 = 0.20 \times 20 = \mathbf{4\ m.\ Ans.}$$

(*ii*) Let the energy dissipated in prototype $= F_P$

Using equation (12.27) for power ratio, $\dfrac{P_P}{P_m} = L_r^{3.5} = 20^{3.5} = 35777.088$

$$\therefore \quad P_P = P_m \times 35777.088 = \frac{1}{10} \times 35777.088 = \mathbf{3577.708\ kW.\ Ans.}$$

Problem 12.25 *The characteristics of the spillway are to be studied by means of a geometrically similar model constructed to the scale ratio of 1 : 10.*

(i) If the maximum rate of flow in the prototype is 28.3 cumecs, what will be the corresponding flow in model ?

(ii) If the measured velocity in the model at a point on the spillway is 2.4 m/s, what will be the corresponding velocity in prototype ?

(iii) If the hydraulic jump at the foot of the model is 50 mm high, what will be the height of jump in prototype ?

(iv) If the energy dissipated per second in the model is 3.5 Nm, what energy is dissipated per second in the prototype ?

Solution. Given :

$$\frac{\text{Linear dimension of model}}{\text{Linear dimension of prototype}} = \frac{1}{10}$$

$\therefore$ Scale ratio, $L_r = 10.$

(*i*) Discharge in prototype, $Q_P = 28.3 \text{ m}^3/\text{s}$

Let Q_m = Discharge in model

For discharge using equation (12.23), we get

$$\frac{Q_P}{Q_m} = L_r^{2.5}$$

$$\therefore \quad Q_m = \frac{Q_P}{L_r^{2.5}} = \frac{28.3}{10^{2.5}} = \mathbf{0.0895\ m^3/s.\ Ans.}$$

(*ii*) Velocity in the model, $V_m = 2.4$ m/s

Let V_P = Velocity in the prototype

For velocity using equation (12.20), we get

$$\frac{V_P}{V_m} = \sqrt{L_r}$$

$$\therefore \quad V_P = V_m \times \sqrt{L_r} = 2.4 \times \sqrt{10} = \mathbf{7.589\ m/s.\ Ans.}$$

(*iii*) Hydraulic jump in model, $H_m = 50$ mm

Let H_P = Hydraulic jump in prototype

Now scale ratio $= \dfrac{H_P}{H_m}$

$$\therefore \quad H_P = H_m \times \text{Scale ratio} = 50 \times 10 = \mathbf{500\ mm.\ Ans.}$$

(*iv*) Energy dissipated/s in model, $E_m = 3.5$ N m/s

Let E_P = Energy dissipated/s in prototype

Now using equation (12.27), we get $\dfrac{E_P}{E_m} = L_r^{3.5}$

$$\therefore \quad E_P = E_m \times L_r^{3.5} = 3.5 \times 10^{3.5} = \mathbf{11067.9\ N\ m/s.\ Ans.}$$

Problem 12.26 *A 1 : 64 model is constructed of an open channel in concrete which has Manning's N = 0.014. Find the value of N for the model.*

Solution. Given :

Linear scale ratio, $L_r = 64$

Value of N for prototype, $N_P = 0.014$

Let N_m = Value of N for model.

The Manning's formula* is given by, $V = \dfrac{1}{N} m^{3/2} \,.\, i^{1/2}$

in which m = Hydraulic mean depth in m

i = Slope of the bed of the channel

Now for the model, the Manning's formula becomes as

$$V_m = \frac{1}{N_m} \,.\, (m_m)^{2/3} \,.\, (i_m)^{1/2} \qquad \text{...}(i)$$

and for the prototype, the Manning's formula is written as

$$V_P = \frac{1}{N_P} \,.\, (m_p)^{2/3} \,.\, (i_p)^{1/2} \qquad \text{...}(ii)$$

Dividing equation (*ii*) by equation (*i*), we get

$$\frac{V_P}{V_m} = \frac{\dfrac{1}{N_P} \cdot (m_P)^{2/3} \cdot (i_P)^{1/2}}{\dfrac{1}{N_m} \cdot (m_m)^{2/3} \cdot (i_m)^{1/2}} = \frac{N_m}{N_P} \cdot \left(\frac{m_P}{m_m}\right)^{2/3} \cdot \left(\frac{i_P}{i_m}\right)^{1/2} \qquad \text{...}(iii)$$

For dynamic similarity, Froude model law is used. Using equation (12.20), we have

$$\frac{V_P}{V_m} = \sqrt{L_r} = \sqrt{64} = 8$$

But $\dfrac{m_p}{m_m} = L_r$ and $\dfrac{i_p}{i_m} = 1$ as i_p and i_m are dimensionless.

Substituting these values in equation (*iii*), we get

$$8 = \frac{N_m}{N_P} \times (L_r)^{2/3} \times 1 = \frac{N_m}{0.014} \times (64)^{2/3} \qquad (\because N_P = 0.014)$$

$$\therefore \qquad N_m = \frac{8 \times 0.014}{64^{2/3}} = \frac{8 \times 0.014}{16} = \mathbf{0.007.\ Ans.}$$

Problem 12.27 *A 7.2 m height and 15 m long spillway discharges 94 m^3/s discharge under a head of 2.0 m. If a 1 : 9 scale model of this spillway is to be constructed, determine model dimensions, head over spillway model and the model discharge. If model experiences a force of 7500 N (764.53 kgf), determine force on the prototype.*

Solution. Given :

For prototype : Height $h_P = 7.2$ m

Length, $L_P = 15$ m

Discharge, $Q_P = 94\ m^3/s$

*See chapter 16 where Manning's formula for velocity through an open channel flow is given.

Head, $H_P = 2.0$ m

Size of model $= \frac{1}{9}$ of the size of prototype.

$\therefore$ Linear scale ratio, $L_r = 9$

Force experienced by model, $F_P = 7500$ N

Find : (*i*) Model dimensions *i.e.*, height and length of model (h_m and L_m)

(*ii*) Head over model *i.e.*, H_m

(*iii*) Discharge through model *i.e.*, Q_m

(*iv*) Force on prototype (*i.e.*, F_p)

(*i*) *Model dimensions (h_m and L_m)*

$$\frac{h_P}{h_m} = \frac{L_P}{L_m} = L_r = 9$$

$\therefore$
$$h_m = \frac{h_P}{9} = \frac{7.2}{9} = \mathbf{0.8\ m.\ Ans.}$$

And
$$L_m = \frac{L_P}{9} = \frac{15}{9} = \mathbf{1.67\ m.\ Ans.}$$

(*ii*) *Head over model (H_m)*

$$\frac{H_P}{H_m} = L_r = 9$$

$\therefore$
$$H_m = \frac{H_P}{9} = \frac{2}{9} = \mathbf{0.222\ m.\ Ans.}$$

(*iii*) *Discharge through model (Q_m)*

Using equation (12.23), we get $\frac{Q_P}{Q_m} = L_r^{2.5}$

$\therefore$
$$Q_m = \frac{Q_P}{L_r^{2.5}} = \frac{94}{9^{2.5}} = \frac{94}{243} = \mathbf{0.387\ m^3/s.\ Ans.}$$

(*iv*) *Force on the Prototype (F_P)*

Using equation (12.24), we get $F_r = \frac{F_P}{F_m} = L_r^3$

$\therefore$
$$F_P = F_m \times L_r^3 = 7500 \times 9^3 = \mathbf{5467500\ N.\ Ans.}$$

12.9.3 Euler's Model Law. Euler's model law is the law in which the models are designed on Euler's number which means for dynamic similarity between the model and prototype, the Euler number for model and prototype should be equal. Euler's model law is applicable when the pressure forces are alone predominant in addition to the inertia force. According to this law :

$$(E_u)_{\text{model}} = (E_u)_{\text{prototype}} \qquad \text{...(12.28)}$$

If V_m = Velocity of fluid in model,

p_m = Pressure of fluid in model,

ρ_m = Density of fluid in model,

and V_P, p_P, ρ_P = Corresponding values in prototype, then

Substituting these values in equation (12.28), we get

$$\frac{V_m}{\sqrt{p_m/\rho_m}} = \frac{V_P}{\sqrt{p_P/\rho_P}} \qquad ...(12.29)$$

If fluid is same in model and prototype, then equation (12.29) becomes as

$$\frac{V_m}{\sqrt{p_m}} = \frac{V_P}{\sqrt{p_P}} \qquad ...(12.30)$$

Euler's model law is applied for fluid flow problems where flow is taking place in a closed pipe in which case turbulence is fully developed so that viscous forces are negligible and gravity force and surface tension force is absent. This law is also used where the phenomenon of cavitation takes place.

12.9.4 Weber Model Law. Weber model law is the law in which models are based on Weber's number, which is the ratio of the square root of inertia force to surface tension force. Hence where surface tension effects predominate in addition to inertia force, the dynamic similarity between the model and prototype is obtained by equating the Weber number of the model and its prototype. Hence according to this law :

$$(W_e)_{\text{model}} = (W_e)_{\text{prototype,}} \qquad \text{where } W_e \text{ is Weber number and } = \frac{V}{\sqrt{\sigma/\rho L}}$$

If V_m = Velocity of fluid in model,
σ_m = Surface tensile force in model,
ρ_m = Density of fluid in model,
L_m = Length of surface in model,
and $V_P, \sigma_P, \rho_P, L_P$ = Corresponding values of fluid in prototype.

Then according to Weber law, we have

$$\frac{V_m}{\sqrt{\sigma_m/\rho_m L_m}} = \frac{V}{\sqrt{\sigma_P/\rho_P L_P}} \qquad ...(12.31)$$

Weber model law is applied in following cases :

1. Capillary rise in narrow passages,
2. Capillary movement of water in soil,
3. Capillary waves in channels,
4. Flow over weirs for small heads.

12.9.5 Mach Model Law. Mach model law is the law in which models are designed on Mach number, which is the ratio of the square root of inertia force to elastic force of a fluid. Hence where the forces due to elastic compression predominate in addition to inertia force, the dynamic similarity between the model and its prototype is obtained by equating the Mach number of the model and its prototype. Hence according to this law :

$$(M)_{\text{model}} = (M)_{\text{prototype}}$$

where M = Mach number = $\dfrac{V}{\sqrt{K/\rho}}$

If V_m = Velocity of fluid in model,
K_m = Elastic stress for model,
ρ_m = Density of fluid in model,
and V_P, K_P and ρ_P = Corresponding values for prototype. Then according to Mach law,

$$= \frac{V_m}{\sqrt{K_m/\rho_m}} = \frac{V}{\sqrt{K_P/\rho_P}} \qquad ...(12.32)$$

Mach model law is applied in the following cases :

1. Flow of aeroplane and projectile through air at supersonic speed, *i.e.*, at a velocity more than the velocity of sound,
2. Aerodynamic testing,
3. Under water testing of torpedoes,
4. Water-hammer problems.

Problem 12.28 *The pressure drop in an aeroplane model of size $\frac{1}{10}$ of its prototype is 80 N/cm^2. The model is tested in water. Find the corresponding pressure drop in the prototype. Take density of air = 1.24 kg/m^3. The viscosity of water is 0.01 poise while the viscosity of air is 0.00018 poise.*

Solution. Given :

Pressure drop in model, $p_m = 80 \text{ N/cm}^2 = 80 \times 10^4 \text{ N/m}^2$

Linear scale ratio, $L_r = 40$

Fluid in model = Water, while in prototype = Air

Viscosity of water, $\mu_m = 0.01$ poise

Density of water, $\rho_m = 1000 \text{ kg/m}^3$

Viscosity of air, $\mu_P = .00018$ poise

Density of air, $\rho_P = 1.24 \text{ kg/m}^3$

Let the corresponding pressure drop in prototype = p_P.

As the problem involves pressure force and viscous force and hence for dynamic similarity between the model and prototype, Euler's number and Reynold's number should be considered. Making first of all, Reynold's number equal, we get from equation (12.17)

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_P V_P L_P}{\mu_P} \quad \text{or} \quad \frac{V_m}{V_P} = \frac{\rho_P}{\rho_m} \times \frac{L_P}{L_m} \times \frac{\mu_m}{\mu_P}$$

But $$\frac{\rho_P}{\rho_m} = \frac{1.24}{1000}$$

$$\frac{L_P}{L_m} = L_r = 40, \quad \frac{\mu_m}{\mu_P} = \frac{0.01}{.00018}$$

$\therefore$ $$\frac{V_m}{V_P} = \frac{1.24}{1000} \times 40 \times \frac{.01}{.00018} = 2.755.$$

Now making Euler's number equal, we get from equation (12.29) as

$$\frac{V_m}{\sqrt{\frac{p_m}{\rho_m}}} = \frac{V_P}{\sqrt{\frac{p_P}{\rho_P}}} \quad \text{or} \quad \frac{V_m}{V_P} = \frac{\sqrt{p_m/\rho_m}}{\sqrt{p_P/\rho_P}} = \sqrt{\frac{p_m}{p_P}} \times \sqrt{\frac{\rho_P}{\rho_m}}$$

But $$\frac{V_m}{V_P} = 2.755 \text{ and } \frac{\rho_P}{\rho_m} = \frac{1.24}{1000}$$

$\therefore$ $$2.755 = \sqrt{\frac{p_m}{p_P}} \times \sqrt{\frac{1.24}{1000}} = \sqrt{\frac{p_m}{p_P}} \times .0352$$

$\therefore$ $$\sqrt{\frac{p_m}{p_P}} = \frac{2.755}{.0352} = 78.267$$

$\therefore$ $$\frac{p_m}{p_P} = (78.267)^2 \text{ or } p_P = \frac{p_m}{(78.267)^2} = \frac{80}{(78.267)^2}$$

$$= \mathbf{0.01306 \text{ N/cm}^2. \text{ Ans.}}$$

▶ 12.10 MODEL TESTING OF PARTIALLY SUB-MERGED BODIES

Let us consider the testing of a ship model (ship is a partially sub-merged body) in a water-tunnel in order to find the drag force F or resistance experienced by a ship. The drag experienced by a ship consists of :

1. The wave resistance, which is the resistance offered by the waves on the free sea-surface, and
2. The frictional or viscous resistance, which is offered by the water on the surface of contact of the ship with water.

Thus in this case three forces namely inertia, gravity and viscous forces are present. Then for dynamic similarity between the model and its prototype, the Reynold's number (which is ratio of inertia force to viscous force) and the Froude number (which is the ratio of inertia force to gravity force) should be taken into account. This means that in this case, the Reynold model law and Froude model law should be applied.

But for Reynold model law, the condition is

Reynold number of model= Reynold number of prototype

or $$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_P V_P L_P}{\mu_P}$$

If fluid is same for the model and prototype, then $\rho_m = \rho_P$ and $\mu_m = \mu_P$

$$\therefore \quad V_m L_m = V_P L_P$$

$$V_m = \frac{V_P L_P}{L_m} = L_r V_P \qquad \left\{\because \frac{L_P}{L_m} = L_r\right\} \qquad ...(12.33)$$

For Froude model law, have from equation (12.18) as $\dfrac{V_m}{\sqrt{g_m L_m}} = \dfrac{V_P}{\sqrt{g_P L_P}}$

If fluid is same for model and prototype and test is conducted at the same place where prototype is to operate, then $g_m = g_P$

$$\therefore \quad \frac{V_m}{\sqrt{L_m}} = \frac{V_P}{\sqrt{L_P}}$$

$$\therefore \quad V_m = \sqrt{\frac{L_m}{L_P}} \times V_P = V_P \times \frac{1}{\sqrt{\dfrac{L_P}{L_m}}} = V_P \times \frac{1}{\sqrt{L_r}} \qquad \left\{\because \frac{L_P}{L_m} = L_r\right\} \qquad ...(12.34)$$

From equations (12.33) and (12.34), we observe that the velocity of fluid in model for Reynold model law and Froude model law is different. Thus it is quite impossible to satisfy both the laws together, which means the dynamic similarity between the model and its prototype will not exist. To overcome this difficulty, the method suggested by William Froude is adopted for testing the ship model (or partially sub-merged bodies) as :

Step 1. The total resistance experienced by a ship is equal to the wave resistance plus frictional or viscous resistance.

Let $(R)_P$ = Total resistance experienced by prototype,

$(R_w)_P$ = Wave resistance experienced by prototype,

$(R_f)_P$ = Frictional resistance experienced by prototype, and

$(R)_m$, $(R_w)_m$, $(R_f)_m$ = Corresponding values for model.

Then, we have for prototype, $(R)_P = (R_w)_P + (R_f)_P$ ...(12.35)

and for model, $(R)_m = (R_w)_m + (R_f)_m$ ...(12.36)

Step 2. The frictional resistances for the model and the ship [*i.e.*, $(R_f)_m$ and $(R_f)_P$] are calculated from the expressions given below :

$$(R_f)_P = f_P A_P V_P^{\,n} \qquad \text{...(12.37)}$$

and

$$(R_f)_m = f_m A_m V_m^{\,n} \qquad \text{...(12.38)}$$

where f_P = Frictional resistance per unit area per unit velocity of prototype,

A_P = Wetted surface area of the prototype,

V_P = Velocity of prototype,

n = Constant, and

f_m, A_m, V_m = Corresponding values of frictional resistance, wetted area and velocity of model.

The values of f_P and f_m are determined from experiments.

Step 3. The model is tested by towing it in water contained in a towing tank such that the dynamic similarity for Froude number is satisfied *i.e.*, $(F_e)_m = (F_e)_P$. The total resistance of the model (R_m) is measured for this condition.

Step 4. The total resistance (R_m) for the model is known from step 3 and frictional resistance of the model $(R_f)_m$ is calculated from equation (12.37). Then the wave resistance for the model is known from equation (12.36) as

$$(R_w)_m = R_m - (R_f)_m \qquad \text{...(12.39)}$$

Step 5. The resistance experienced by a ship of length L, flowing with velocity V in fluid of viscosity μ, density ρ depends upon g, the acceleration due to gravity. By dimensional analysis, the expression for resistance is given by

$$\frac{R}{\rho L^2 V^2} = \phi\left[\frac{\rho V L}{\mu}, \frac{V^2}{gL}\right] = \phi\,[R_e, F_e^{\,2}]$$

Thus resistance is a function of Reynold number (R_e) and Froude number (F_e).

For dynamic similarity for model and prototype for wave resistance only, we have

$$\frac{(R_w)_P}{\rho_P L_P^2 V_P^2} = \frac{(R_w)_m}{\rho_m L_m^2 V_m^2}$$

or wave resistance for prototype is given as

$$(R_w)_P = \frac{\rho_P}{\rho_m} \times \frac{L_P^2}{L_m^2} \times \frac{V_P^2}{V_m^2} \times (R_w)_m \qquad \text{...(12.40)}$$

But from Step 3, $(F_e)_m = (F_e)_P$ or $\dfrac{V_m}{\sqrt{L_m g_m}} = \dfrac{V_P}{\sqrt{L_P g_P}}$

If the model and ship are at the same place, $g_m = g_P$

$$\therefore \quad \frac{V_m}{\sqrt{L_m}} = \frac{V_P}{\sqrt{L_P}} \quad \text{or} \quad V_m = \sqrt{\frac{L_m}{L_P}}\,.\,V_P$$

Substituting the value of V_m in equation (12.40), we have

$$(R_w)_P = \frac{\rho_P}{\rho_m} \times \frac{L_P^2}{L_m^2} \times \frac{V_P^2}{V_P^2 \times \dfrac{L_m}{L_P}} \times (R_w)_m$$

$$= \frac{\rho_P}{\rho_m} \times \frac{L_P^3}{L_m^3} \times (R_w)_m. \qquad ...(12.41)$$

Step 6. The total resistance of the ship is given by adding $(R_w)_P$ from equation (12.41) to $(R_f)_P$ given by equation (12.37) as

$$R_P = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^3 \times (R_w)_m + f_P A_P V_P^2. \qquad ...(12.42)$$

Problem 12.29 *A 1 in 20 model of a naval ship having a sub-merged surface area of 5 m² and length 8 m has a total drag of 20 N when towed through water at a velocity of 1.5 m/s. Calculate the total drag on the prototype when moving at the corresponding speed. Use the relation* $F_f = \frac{1}{2} C_f \rho A V^2$ *for calculating the skin (frictional) resistance. The value of* C_f *is given by* $C_f = \frac{0.0735}{(R_e)^{1.5}}$.

Take kinematic viscosity of water (or sea-water) as 0.01 stoke and density of water (or sea-water) as 1000 kg/m³.

Solution. Given :

Linear scale ratio, $L_r = 20$

Sub-merged area of model, $A_m = 5.0 \text{ m}^2$

Length of model, $L_m = 8.0$ m

Total drag of model, $R_m = 20$ N

Velocity of model, $V_m = 1.5$ m/s

Let A_P, L_P, R_P, V_P = Corresponding values for prototype.

Fluid in model is the same as in prototype and is sea-water.

Kinematic viscosity of sea-water, $\nu_m = \nu_P = 0.01$ stokes $= .01 \text{ cm}^2/\text{s} = .01 \times 10^{-4} \text{ m}^2/\text{s}$

Density of water, $\rho_m = 1000 \text{ kg/m}^3$

The skin (frictional) resistance of model is given by

$$(F_f)_m = \frac{1}{2} C_{f_m} \rho_m A_m V_m^2 \qquad ...(i)$$

where $$C_{f_m} = \frac{0.0735}{\left[(R_e)_m\right]^{1/5}} \qquad ...(ii)$$

where $(R_e)_m$ = Reynold's number for model

$$= \frac{\rho_m V_m L_m}{\mu_m} \text{ or } \frac{V_m L_m}{\nu_m} \qquad \left\{\because \nu = \frac{\mu}{\rho}\right\}$$

$$= \frac{1.5 \times 8.0}{.01 \times 10^{-4}} = 1.2 \times 10^7.$$

Substituting this value in equation (*ii*), we get

$$C_{f_m} = \frac{0.0735}{(1.2 \times 10^7)^{1/5}} = \frac{.0735}{26.0517} = 2.82 \times 10^{-3} \qquad ...(iii)$$

Substituting the value of C_{f_m} in equation (i), we get

$$(F_f)_m = \frac{1}{2} \times 2.82 \times 10^{-3} \times 1000 \times 5.0 \times (1.5)^2 = 15.8617 = 15.862 \text{ N}$$

Using equation (12.36), we get $\quad R_m = (R_w)_m + (R_f)_m$

where $(R_f)_m = (F_f)_m = 15.862$ or $20 = (R_w)_m + 15.862$

$\therefore$ Wave resistance for model, $(R_w)_m = 20 - 15.862 = 4.138$ N ...(iv)

The wave resistance experienced by the ship is given by equation (12.41) as

$$(R_w)_P = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^3 \times (R_w)_m$$

$$= 1 \times L_r^3 \times 4.138 \text{ N} \qquad \left\{\because \frac{\rho_P}{\rho_m} = 1 \text{ for same fluid}\right\}$$

$$= 1 \times 20^3 \times 4.138 = 33104 \text{ N}$$

and skin (frictional) resistance of prototype is given by

$$(R_f)_P = (F_f)_P = \frac{1}{2} C_{f_P} \times \rho_P \times A_P \times V_P^2 \qquad \text{...}(v)$$

where V_P is the velocity of prototype and is given by Froude model law,

i.e., $\quad (F_e)_m = (F_e)_P \quad$ or $\quad \dfrac{V_m}{\sqrt{L_m g}} = \dfrac{V}{\sqrt{L_P g}} \quad$ or $\quad \dfrac{V_m}{\sqrt{L_m}} = \dfrac{V_P}{\sqrt{L_P}}$

$$\therefore \quad V_P = \sqrt{\frac{L_P}{L_m}} \times V_m = \sqrt{L_r} \times V_m \qquad \left\{\because \frac{L_P}{L_m} = L_r\right\}$$

$$= \sqrt{20} \times 1.5 = 6.708 \text{ m/s}$$

Now $\quad \dfrac{A_P}{A_m} = L_r^2 = 20^2$

$\therefore \quad A_P = A_m \times 20^2 = 5 \times 400 = 2000 \text{ m}^2$

and $\quad L_P = L_m \times L_r = 8 \times 20 = 160$ m

In equation (v), the value of C_{f_P} is given by $C_{f_P} = \dfrac{0.0735}{\left[(R_e)_P\right]^{1/5}}$

where $(R_e)_P$ = Reynolds number for prototype

$$= \frac{V_P \times L_P}{\nu_P} = \frac{6.708 \times 160}{.01 \times 10^{-4}} = 1.073 \times 10^9$$

$$\therefore \quad C_{f_P} = \frac{0.0735}{(1.073 \times 10^9)^{1/5}} = \frac{0.0735}{63.99} = 1.1486 \times 10^{-3}$$

Substituting this value of C_{f_P} in equation (v), we get

$$(R_f)_P = (F_f)_P = \frac{1}{2} \times 1.1486 \times 10^{-3} \times 1000 \times 2000 \times (6.708)^2 = 51683.8 \text{ N}$$

$\therefore$ Total drag on prototype is obtained by using equation (12.35).

$$\therefore \quad R_P = (R_w)_P + (R_f)_P = 33104 + 51683.8 = \mathbf{84787.8\ N.\ Ans.}$$

Problem 12.30 *A 1 : 15 model of a flying boat is towed through water. The prototype is moving in sea-water of density 1024 kg/m³ at a velocity of 20 m/s. Find the corresponding speed of the model. Also determine the resistance due to waves on model if the resistance due to waves of prototype is 600 N.*

Solution. Given :

Linear scale ratio, $L_r = 15$

Velocity of prototype, $V_p = 20$ m/s

Fluid in prototype is sea-water while in model it is water

Density of sea-water, $\rho_P = 1024$ kg/m^3

Density of water, $\rho_m = 1000$ kg/m^3

Resistance due to waves for prototype is, $(R_w)_P = 600$ N.

Find V_m and $(R_w)_m$.

(i) The velocity, V_m from model is given by Froude model law,

$$\therefore \quad \frac{V_m}{\sqrt{L_m g}} = \frac{V_P}{\sqrt{L_P \times g}}$$

$$\therefore \quad V_m = \sqrt{\frac{L_m}{L_P}} \times V_P = \frac{V_P}{\sqrt{L_P / L_m}} = \frac{20}{\sqrt{15}} \qquad \left\{\because \frac{L_P}{L_m} = L_r = 15\right\}$$

$$= \frac{20}{3.872} = \mathbf{5.165\ m/s.\ Ans.}$$

(ii) For dynamic similarity between model and its prototype for wave resistance only, we have equation (12.41) as

$$(R_w)_P = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^3 \times (R_w)_m$$

Substituting the known values, $600 = \frac{1024}{1000} \times L_r^3 \times (R_w)_m = \frac{1024}{1000} \times 15^3 \times (R_w)_m$

$$\therefore \quad (R_w)_m = \frac{600 \times 1000}{1024 \times 15^3} = \mathbf{0.1736\ N.\ Ans.}$$

Problem 12.31 *A 1 : 40 model of an ocean tanker is dragged through fresh water at 2 m/s with a total measured drag of 12 N. The skin (frictional) drag co-efficient 'f' for model and prototype are 0.03 and 0.002 respectively in the equation $R_f = f \cdot AV^2$. The wetted surface area of the model is 25 m². Determine the total drag on the prototype and the power required to drive the prototype.*

Take $\rho_P = 1030$ kg/m³ and $\rho_m = 1000$ kg/m³.

Solution. Given :

Linear scale ratio, $L_r = 40$

Velocity of model, $V_m = 2$ m/s
Total drag of model, $R_m = 12$ N
Wetted area of model, $A_m = 25$ m^2
Co-efficient of friction for model, $f_m = .03$
for prototype, $f_P = .002$.
Let the total drag on prototype $= R_P$
And power required to drive the prototype $= P$
Frictional drag on model, $(R_f)_m = f_m A_m V_m^2 = .03 \times 25 \times 2^2 = 3$ N
$\therefore$ Wave drag on model, $(R_w)_m = R_m - (R_f)_m = 12 - 3 = 9$ N.

The waves drag on prototype is obtained from equation (12.41) as

$$(R_w)_P = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^3 \times (R_w)_m = \frac{1030}{1000} \times L_r^3 \times 9 \qquad \left\{\because \frac{L_P}{L_m} = L_r = 40\right\}$$

$$= \frac{1030}{1000} \times 40^3 \times 9 = 593291.8 \text{ N} \qquad ...(i)$$

The frictional drag on prototype is given by

$$(R_f)_P = f_P \times A_P \times V_P^2 \qquad ...(ii)$$

where the velocity of prototype V_P is obtained from Froude model law as

$$\frac{V_m}{\sqrt{L_m \times g}} = \frac{V_P}{\sqrt{L_P \times g}} \quad \text{or} \quad \frac{V_m}{\sqrt{L_m}} = \frac{V_P}{\sqrt{L_P}}$$

$$\therefore \qquad V_p = \sqrt{\frac{L_P}{L_m}} \times V_m = \sqrt{L_r} \times V_m = \sqrt{40} \times 2 = 12.65 \text{ m/s}$$

and

$$\frac{A_P}{A_m} = L_r^2 = 40 \times 40 \text{ or } A_P = 40 \times 40 \times A_m$$

$$= 40 \times 40 \times 25 = 40000 \text{ m}^2.$$

Substituting these values in (*ii*), we get

$$(R_f)_P = .002 \times 40000 \times (12.65)^2 = 12801.8 \text{ N} \qquad ...(iii)$$

Total drag on the prototype is obtained by adding equations (*i*) and (*ii*) as

$$R_P = (R_w)_P + (R_f)_P$$

$$= 593291.8 + 12801.8 = \mathbf{606093.6 \text{ N. Ans.}}$$

Power required to drive the prototype,

$$P = \frac{(\text{Total drag on prototype}) \times \text{Velocity of prototype}}{1000}$$

$$= \frac{606093.6 \times 12.65}{1000} = \mathbf{7667 \text{ kW. Ans.}}$$

Problem 12.32 *Resistance R, to the motion of a completely sub-merged body is given by*

$$R = \rho V^2 l^2 \, \phi\left(\frac{Vl}{\nu}\right),$$

where ρ and ν are density and kinematic viscosity of the fluid while l is the length of the body and V is the velocity of flow. If the resistance of a one-eight scale air-ship model when tested in water at

12 m/s is 22 N, what will be the resistance in air of the air-ship at the corresponding speed ? Kinematic viscosity of air is 13 times that of water and density of water is 810 times of air.

Solution. Given :

Linear scale ratio, $L_r = 8$

Velocity of model, $V_m = 12$ m/s

Resistance to model, $R_m = 22$ N

The fluid for model is water and for prototype the fluid is air.

Kinematic viscosity of air = 13 × Kinematic viscosity of water

∴ $\nu_P = 13 \times \nu_m$

Density of water = 810 × Density of air

∴ $\rho_m = 810 \times \rho_P$

Let V_P = Velocity of the air-ship (Prototype)

R_P = Resistance of the air-ship

The resistance, R, is given by $R = \rho V^2 l^2 \phi\left(\frac{Vl}{\nu}\right)$

∴ The non-dimensional terms $\frac{R}{\rho V^2 l^2}$ and $\frac{Vl}{\nu}$ should be same for the prototype and its model.

$$\therefore \quad \left(\frac{Vl}{\nu}\right)_{\text{prototype}} = \left(\frac{Vl}{\nu}\right)_{\text{model}} \quad \text{or} \quad \frac{V_P l_P}{\nu_P} = \frac{V_m l_m}{\nu_m}$$

$$\therefore \quad V_P = V_m \frac{l_m}{l_P} \times \frac{\nu_P}{\nu_m} = 12 \times \frac{1}{L_r} \times 13 \qquad \left\{\because \frac{l_P}{l_m} = L_r\right\}$$

$$= 12 \times \frac{1}{8} \times 13 = 19.5 \text{ m/s}$$

Also

$$\left(\frac{R}{\rho V^2 l^2}\right)_{\text{prototype}} = \left(\frac{R}{\rho V^2 l^2}\right)_{\text{model}} \quad \text{or} \quad \frac{R_P}{\rho_P V_P{}^2 l_P{}^2} = \frac{R_m}{\rho_m V_m{}^2 l_m{}^2}$$

$$\therefore \quad R_p = R_m \times \frac{\rho_P}{\rho_m} \times \frac{V_P^2}{V_m^2} \times \frac{l_P^2}{l_m^2}$$

$$= 22 \times \frac{1}{810} \times \frac{(19.5)^2}{12^2} \times 8^2 \qquad \left(\because \frac{l_P}{l_m} = l_r = 8.0\right)$$

$$= \mathbf{4.59 \text{ N. Ans.}}$$

▶ 12.11 CLASSIFICATION OF MODELS

The hydraulic models are classified as :

1. Undistorted models, and 2. Distorted models.

12.11.1 Undistorted Models. Undistorted models are those models which are geometrically similar to their prototypes or in other words if the scale ratio for the linear dimensions of the model and its prototype is same, the model is called undistorted model. The behaviour of the prototype can be easily predicted from the results of undistorted model.

12.11.2 Distorted Models. A model is said to be distorted if it is not geometrically similar to its prototype. For a distorted model different scale ratios for the linear dimensions are adopted. For example, in case of rivers, harbours, reservoirs etc., two different scale ratios, one for horizontal dimensions and other for vertical dimensions are taken. Thus the models of rivers, harbours and reservoirs will become as distorted models. If for the river, the horizontal and vertical scale ratios are taken to be same so that the model is undistorted, then the depth of water in the model of the river will be very-very small which may not be measured accurately. The following are the advantage of distorted models :

1. The vertical dimensions of the model can be measured accurately.
2. The cost of the model can be reduced.
3. Turbulent flow in the model can be maintained.

Though there are some advantages of the distorted model, yet the results of the distorted model cannot be directly transferred to its prototype. But sometimes from the distorted models very useful information can be obtained.

12.11.3 Scale Ratios for Distorted Models. As mentioned above, two different scale ratios, one for horizontal dimensions and other for vertical dimensions, are taken for distorted models.

Let $(L_r)_H$ = Scale ratio for horizontal dimension

$$= \frac{L_P}{L_m} = \frac{B_P}{B_m} = \frac{\text{Linear horizontal dimension of prototype}}{\text{Linear horizontal dimension of model}}$$

$(L_r)_V$ = Scale ratio for vertical dimension

$$= \frac{\text{Linear vertical dimension of prototype}}{\text{Linear vertical dimension of model}} = \frac{h_P}{h_m}$$

Then the scale ratios of velocity, area of flow, discharge etc., in terms of $(L_r)_H$ and $(L_r)_V$ can be obtained for distorted models as given below :

1. Scale ratio for velocity

Let V_P = Velocity in prototype

V_m = Velocity in model.

Then
$$\frac{V_P}{V_m} = \frac{\sqrt{2gh_P}}{\sqrt{2gh_m}} = \sqrt{\frac{h_P}{h_m}} = \sqrt{(L_r)_V} \qquad \left(\because \frac{h_P}{h_m} = (L_r)_V\right)$$

2. Scale ratio for area of flow

Let A_P = Area of flow in prototype = $B_P \times h_P$

A_m = Area of flow in model = $B_m \times h_m$

$$\therefore \quad \frac{A_P}{A_m} = \frac{B_P \times h_P}{B_m \times h_m} = \frac{B_P}{B_m} \times \frac{h_P}{h_m} = (L_r)_H \times (L_r)_V$$

3. Scale ratio for discharge

Let Q_P = Discharge through prototype = $A_P \times V_P$

Q_m = Discharge through model = $A_m \times V_m$

$$\therefore \quad \frac{Q_P}{Q_m} = \frac{A_P \times V_P}{A_m \times V_m} = (L_r)_H \times (L_r)_V \times \sqrt{(L_r)_V} = (L_r)_H \times [(L_r)_V]^{3/2} \qquad \text{...(12.43)}$$

Problem 12.33 *The discharge through a weir is 1.5 m³/s. Find the discharge through the model of the weir if the horizontal dimension of the model =* $\frac{1}{50}$ *the horizontal dimension of the prototype and vertical dimension of the model =* $\frac{1}{10}$ *the vertical dimension of the prototype.*

Solution. Given :

Discharge through weir (prototype), $Q_P = 1.5\ \text{m}^3/\text{s}$

Horizontal dimension of model $= \frac{1}{50} \times$ Horizontal dimension of prototype

$$\therefore \quad \frac{\text{Horizontal dimension of prototype}}{\text{Horizontal dimension of model}} = 50 \text{ or } (L_r)_H = 50$$

Vertical dimension of model $= \frac{1}{10} \times$ Vertical dimension of prototype

$$\therefore \quad \frac{\text{Vertical dimension of prototype}}{\text{Vertical dimension of model}} = 10$$

$$\therefore \quad (L_r)_V = 10.$$

Using equation (12.43), we get
$$\frac{Q_P}{Q_m} = (L_r)_H \times [(L_r)_V]^{3/2} = 50 \times 10^{3/2} = 1581.14$$

$$Q_m = \frac{Q_P}{1581.14} = \frac{1.50}{1581.14} = .000948\ \text{m}^3/\text{s}$$

$$= \textbf{0.948 litres/s. Ans.}$$

HIGHLIGHTS

1. Dimensional analysis is the method of dimensions, in which fundamental dimensions are M, L and T.
2. Dimensional analysis is performed by two methods namely Rayleigh's Method and Buckingham's π-theorem.
3. Rayleigh's method is used for finding an expression for a variable which depends on maximum three or four variables while there is no restriction on the number of variables for Buckingham's π-theorem.
4. Model analysis is an experimental method of finding solutions of complex flow problems. A model is a small scale replica of the actual machine or structure. The actual machine or structure is called prototype.
5. Three types of similarities must exist between the model and prototype. They are : (*i*) Geometric Similarity, (*ii*) Kinematic Similarity, and (*iii*) Dynamic Similarity.
6. For geometric similarity, the ratio of all linear dimensions of the model and of the prototype should be equal.
7. Kinematic similarity means the similarity of motion between model and prototype.
8. Dynamic similarity means the similarity of forces between the model and prototype.
9. Reynold's number is defined as the ratio of inertia force and viscous force of a flowing fluid. It is given by,

$$R_e = \frac{\rho VL}{\mu} = \frac{VL}{\nu} = \frac{V \times d}{\nu} \text{ for pipe flow}$$

where V = Velocity of flow, d = Diameter of pipe and
ν = Kinematic viscosity of fluid.

10. Froude's Number is the ratio of the square root of inertia force and gravity force and is given by

$$F_e = \sqrt{\frac{F_i}{F_g}} = \frac{V}{\sqrt{Lg}}.$$

11. Euler's number is the ratio of the square root of inertia force and pressure force and is given by,

$$E_u = \sqrt{\frac{F_i}{F_P}} = \frac{V}{\sqrt{p/\rho}}.$$

12. Mach number is the ratio of the square root of inertia force and elastic force and is given by

$$M = \sqrt{\frac{F_i}{F_e}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C}.$$

13. The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity. The model laws are (*i*) Reynold's Model Law, (*ii*) Froude Model Law, (*iii*) Euler Model Law, (*iv*) Weber Model Law, (*v*) Mach Model Law.

14. The drag experienced by a ship model (or partially sub-merged body) is obtained by Froude's method.

15. Hydraulic models are classified as (*i*) undistorted models and (*ii*) distorted models.

16. If the models are geometrically similar to its prototype, the models are known as undistorted model. And if the models are having different scale ratio for horizontal and vertical dimensions, the models are known as distorted model.

EXERCISE

(A) THEORETICAL PROBLEMS

1. Define the terms dimensional analysis and model analysis.

2. What do you mean by fundamental units and derived units ? Give examples.

3. Explain the term, 'dimensionally homogeneous equation'.

4. What are the methods of dimensional analysis ? Describe the Rayleigh's method for dimensional analysis.

5. State Buckingham's π-theorem. Why this theorem is considered superior over the Rayleigh's method for dimensional analysis ?

6. What do you mean by repeating variables ? How are the repeating variables selected for dimensional analysis ?

7. Define the terms : model, prototype, model analysis, hydraulic similitude.

8. Explain the different types of hydraulic similarities that must exist between a prototype and its model.

9. What do you mean by dimensionless numbers ? Name any four dimensionless numbers.
Define and explain Reynold's number, Froude's number's and Mach number. Derive expressions for any above two numbers.

10. What is meant by geometric, kinematic and dynamic similarities ? Are these similarities truly attainable ? If not why ?

11. Define the following non-dimensional numbers : Reynold's number, Froude's number and Mach's number. What are their significances for fluid flow problems ?

12. What are the different laws on which models are designed for dynamic similarity ? Where are they used ?

13. How will you determine the total drag of a ship or partially sub-merged bodies ?

14. Explain the terms : distorted models and undistorted models. What is the use of distorted models ?

15. Prove that the scale ratio for discharge for a distorted model is given as

$$\frac{Q_P}{Q_m} = (L_r)_H \times (L_r)_V^{3/2}$$

where Q_P = Discharge through prototype
Q_m = Discharge through model
$(L_r)_H$ = Horizontal scale ratio
$(L_r)_V$ = Vertical scale ratio.

16. Show that ratio of inertia force to viscous force gives the Reynolds number.

17. State Buckingham's π-theorem. What do you mean by repeating variables ? How are the repeating variables selected in dimensional analysis ?

18. What is the significance of the non-dimensional numbers : Reynolds number, Froude number and Mach number in the theory of similarity ? What is the dimensional analysis ? How is this analysis related to the theory of similarity ? (*Delhi University, December 2001*)

19. Define and explain : (*i*) Froude's number, (*ii*) Mach number, (*iii*) Hydraulic similarities (*iv*) Distorted and undistorted models. (*Delhi University, December 2002*)

(B) NUMERICAL PROBLEMS

1. Give the dimensions of : (*i*) Force (*ii*) Viscosity (*iii*) Power and (*iv*) Kinematic viscosity.
[**Ans.** MLT^{-2}, $ML^{-1}T^{-1}$, $ML^{+2}T^{-3}$, L^2T^{-1}]

2. The variables controlling the motion of a floating vessel through water are the drag force F, the speed V, the length L, the density ρ and dynamic viscosity μ of water and acceleration due to gravity g. Derive an expression for F by dimensional analysis. $\left[\textbf{Ans. } F = \rho L^2 V^2 \phi\left[\frac{\mu}{\rho VL}, \frac{Lg}{V^2}\right]\right]$

3. The resistance R, to the motion of a completely sub-merged body depends upon the length of the body L, velocity of flow V, mass density of fluid ρ and kinematic viscosity of fluid ν. By dimensional analysis prove that

$$R = \rho V^2 L^2 \phi\left(\frac{VL}{\nu}\right).$$

4. A pipe of diameter 1.8 m is required to transport an oil of sp. gr. 0.8 and viscosity .04 poise at the rate of 4 m^3/s. Tests were conducted on a 20 cm diameter pipe using water at 20°C. Find the velocity and rate of flow in the model. Viscosity of water at 20°C = .01 poise. [**Ans.** 2.829 m/s, 88.8 litres/s]

5. A model of a sub-marine of scale $\frac{1}{40}$ is tested in a wind tunnel. Find the speed of air in wind tunnel if the speed of sub-marine in sea-water is 15 m/s. Also find the ratio of the resistance between the model and its prototype. Take the values of kinematic viscosities for sea-water and air as .012 stokes and 0.016 stokes respectively. The density of sea-water and of air are given as 1030 kg/m^3 and 1.24 kg/m^3 respectively.

$\left[\textbf{Ans}.\ 800 \text{ m/s}, \frac{F_m}{F_P} = 0.00214\right]$

6. A ship 250 m long moves in sea-water, whose density is 1030 kg/m^3. A 1 : 125 model of this ship is to be tested in wind tunnel. The velocity of air in the wind tunnel around the model is 20 m/s and the resistance of the model is 50 N. Determine the velocity of ship in sea-water and also the resistance of the ship in sea-water. The density of air is given as 1.24 kg/m^3. Take the kinematic viscosity of sea-water and air as 0.012 stokes and 0.018 stokes respectively. [**Ans.** 0.106 m/s, 18228.7 N]