13

CHAPTER

BOUNDARY LAYER FLOW

▶ 13.1 INTRODUCTION

When a real fluid flows past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs. This means that the velocity of fluid close to the boundary will be same as that of the boundary. If the boundary is stationary, the velocity of fluid at the boundary will be zero. Farther away from the boundary, the velocity will be higher and as a result of this variation of velocity, the velocity gradient $\frac{du}{dy}$ will exist. The velocity of fluid increases from zero velocity on the stationary boundary to free-stream velocity (U) of the fluid in the direction normal to the boundary. This variation of velocity from zero to free-stream velocity in the direction normal to the boundary takes place in a narrow region in the vicinity of solid boundary. This narrow region of the fluid is called boundary layer. The theory dealing with boundary layer flows is called boundary layer theory.

According to boundary layer theory, the flow of fluid in the neighbourhood of the solid boundary may be divided into two regions as shown in Fig. 13.1.

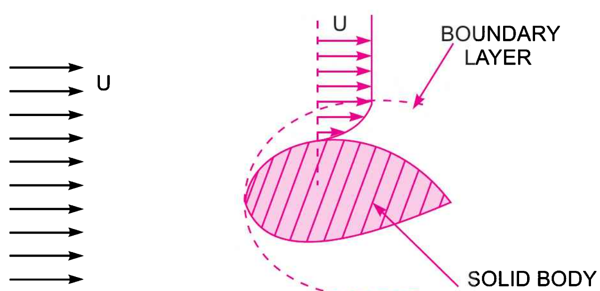

Fig. 13.1 *Flow over solid body.*

1. A very thin layer of the fluid, called the boundary layer, in the immediate neighbourhood of the solid boundary, where the variation of velocity from zero at the solid boundary to free-stream velocity in the direction normal to the boundary takes place. In this region, the velocity gradient $\frac{du}{dy}$ exists and hence the fluid exerts a shear stress on the wall in the direction of motion. The value of shear stress is given by

$$\tau = \mu \frac{du}{dy}.$$

2. The remaining fluid, which is outside the boundary layer. The velocity outside the boundary layer is constant and equal to free-stream velocity. As there is no variation of velocity in this region, the velocity gradient $\frac{du}{dy}$ becomes zero. As a result of this the shear stress is zero.

▶ 13.2 DEFINITIONS

13.2.1 Laminar Boundary Layer. For defining the boundary layer (*i.e.*, laminar boundary layer or turbulent boundary layer) consider the flow of a fluid, having free-stream velocity (U), over a smooth thin plate which is flat and placed parallel to the direction for free stream of fluid as shown in Fig. 13.2. Let us consider the flow with zero pressure gradient on one side of the plate, which is stationary.

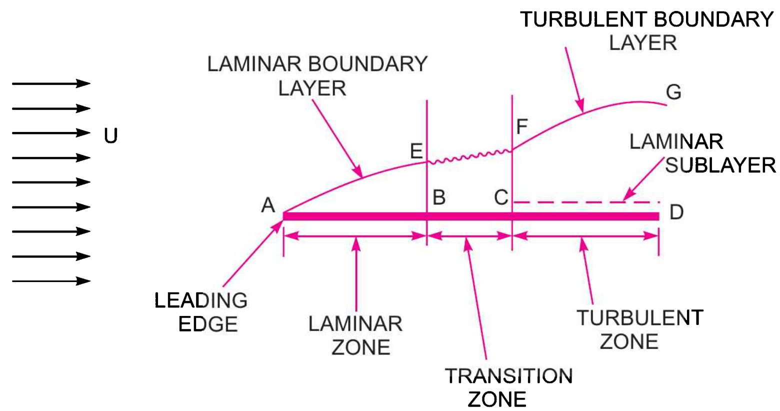

Fig. 13.2 *Flow over a plate.*

The velocity of fluid on the surface of the plate should be equal to the velocity of the plate. But plate is stationary and hence velocity of fluid on the surface of the plate is zero. But at a distance away from the plate, the fluid is having certain velocity. Thus a velocity gradient is set up in the fluid near the surface of the plate. This velocity gradient develops shear resistance, which retards the fluid. Thus the fluid with a uniform free stream velocity (U) is retarded in the vicinity of the solid surface of the plate and the boundary layer region begins at the sharp leading edge. At subsequent points downstream the leading edge, the boundary layer region increases because the retarded fluid is further retarded. This is also referred as the growth of boundary layer. Near the leading edge of the surface of the plate, where the thickness is small, the flow in the boundary layer is laminar though the main flow is turbulent. This layer of the fluid is said to be laminar boundary layer. This is shown by AE in Fig. 13.2. The length of the plate from the leading edge, upto which laminar boundary layer exists, is called laminar zone. This is shown by distance AB. The distance of B from leading edge is obtained from Reynold number equal to 5×10^5 for a plate. Because upto this Reynold number the boundary layer is laminar. The Reynold number is given by $(R_e)_x = \frac{U \times x}{\nu}$

where x = Distance from leading edge,
U = Free-stream velocity of fluid,
ν = Kinematic viscosity of fluid,

Hence for laminar boundary layer, we have $5 \times 10^5 = \dfrac{U \times x}{\nu}$ ...(13.1)

If the values of U and ν are known, x or the distance from the leading edge upto which laminar boundary layer exists can be calculated.

13.2.2 Turbulent Boundary Layer. If the length of the plate is more than the distance x, calculated from equation (13.1), the thickness of boundary layer will go on increasing in the down-stream direction. Then the laminar boundary layer becomes unstable and motion of fluid within it, is disturbed and irregular which leads to a transition from laminar to turbulent boundary layer. This short length over which the boundary layer flow changes from laminar to turbulent is called transition zone. This is shown by distance BC in Fig. 13.2. Further downstream the transition zone, the boundary layer is turbulent and continues to grow in thickness. This layer of boundary is called turbulent boundary layer, which is shown by the portion FG in Fig. 13.2.

13.2.3 Laminar Sub-layer. This is the region in the turbulent boundary layer zone, adjacent to the solid surface of the plate as shown in Fig. 13.2. In this zone, the velocity variation is influenced only by viscous effects. Though the velocity distribution would be a parabolic curve in the laminar sub-layer zone, but in view of the very small thickness we can reasonably assume that velocity variation is linear and so the velocity gradient can be considered constant. Therefore, the shear stress in the laminar sub-layer would be constant and equal to the boundary shear stress τ_0. Thus the shear stress in the sub-layer is

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{u}{y} \qquad \left\{\because \text{ For linear variation, } \frac{\partial u}{\partial y} = \frac{u}{y}\right\}$$

13.2.4 Boundary Layer Thickness (δ). It is defined as the distance from the boundary of the solid body measured in the y-direction to the point, where the velocity of the fluid is approximately equal to 0.99 times the free stream velocity (U) of the fluid. It is denoted by the symbol δ. For laminar and turbulent zone it is denoted as :

1. δ_{lam} = Thickness of laminar boundary layer,
2. δ_{tur} = Thickness of turbulent boundary layer, and
3. δ' = Thickness of laminar sub-layer.

13.2.5 Displacement Thickness (δ^*). It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in flow rate on account of boundary layer formation. It is denoted by δ^*. It is also defined as :

"The distance perpendicular to the boundary, by which the free-stream is displaced due to the formation of boundary layer".

Expression for δ^*.

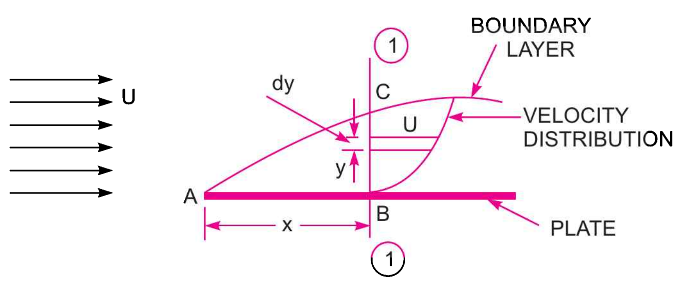

Fig. 13.3 *Displacement thickness.*

Consider the flow of a fluid having free-stream velocity equal to U over a thin smooth plate as shown in Fig. 13.3. At a distance x from the leading edge consider a section 1-1. The velocity of fluid at B is zero and at C, which lies on the boundary layer, is U. Thus velocity varies from zero at B to U at C, where BC is equal to the thickness of boundary layer *i.e.*,

Distance $BC = \delta$

At the section 1-1, consider an elemental strip.

Let y = distance of elemental strip from the plate,

dy = thickness of the elemental strip,

u = velocity of fluid at the elemental strip,

b = width of plate.

Then area of elemental strip, $dA = b \times dy$

Mass of fluid per second flowing through elemental strip

$$= \rho \times \text{Velocity} \times \text{Area of elemental strip}$$

$$= \rho u \times dA = \rho u \times b \times dy \quad \text{...(}i\text{)}$$

If there had been no plate, then the fluid would have been flowing with a constant velocity equal to free-stream velocity (U) at the section 1-1. Then mass of fluid per second flowing through elemental strip would have been

$$= \rho \times \text{Velocity} \times \text{Area} = \rho \times U \times b \times dy \quad \text{...(}ii\text{)}$$

As U is more than u, hence due to the presence of the plate and consequently due to the formation of the boundary layer, there will be a reduction in mass flowing per second through the elemental strip.

This reduction in mass/sec flowing through elemental strip

$$= \text{mass/sec given by equation (}ii\text{)} - \text{mass/sec given by equation (}i\text{)}$$

$$= \rho U b dy - \rho u b dy = \rho b(U - u)dy$$

$\therefore$ Total reduction in mass of fluid/s flowing through BC due to plate

$$= \int_0^\delta \rho b(U - u)dy = \rho b \int_0^\delta (U - u)dy \quad \text{...(}iii\text{)}$$

{if fluid is incompressible}

Let the plate is displaced by a distance δ^* and velocity of flow for the distance δ^* is equal to the free-stream velocity (*i.e.*, U). Loss of the mass of the fluid/sec flowing through the distance δ^*

$$= \rho \times \text{Velocity} \times \text{Area}$$

$$= \rho \times U \times \delta^* \times b \quad \{\because \text{Area} = \delta^* \times b\} \text{ ...(}iv\text{)}$$

Equating equation (iii) and (iv), we get

$$\rho b \int_0^\delta (U - u)dy = \rho \times U \times \delta^* b$$

Cancelling ρb from both sides, we have

$$\int_0^\delta (U - u)dy = U \times \delta^*$$

or

$$\delta^* = \frac{1}{U}\int_0^\delta (U - u)dy = \int_0^\delta \frac{(U - u)dy}{U} \quad \left\{\begin{array}{l}\because U \text{ is constant and can} \\ \text{be taken inside the integral}\end{array}\right\}$$

$$\therefore \quad \delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy. \quad \text{...(13.2)}$$

13.2.6 Momentum Thickness (θ). Momentum thickness is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in **momentum** of the flowing fluid on account of boundary layer formation. It is denoted by θ.

Consider the flow over a plate as shown in Fig. 13.3. Consider the section 1-1 at a distance x from leading edge. Take an elemental strip at a distance y from the plate having thickness (dy). The mass of fluid flowing per second through this elemental strip is given by equation (i) and is equal to $\rho ubdy$.

Momentum of this fluid = Mass × Velocity = $(\rho ubdy)u$

Momentum of this fluid in the absence of boundary layer = $(\rho ubdy)U$

$\therefore$ Loss of momentum through elemental strip = $(\rho ubdy)U - (\rho ubdy) \times u = \rho bu(U - u)dy$

$\therefore$ Total loss of momentum/sec through $BC = \int_0^\delta \rho bu(U - u)dy$ ...(13.3)

Let θ = distance by which plate is displaced when the fluid is flowing with a constant velocity U

$\therefore$ Loss of momentum/sec of fluid flowing through distance θ with a velocity U

$$= \text{Mass of fluid through } \theta \times \text{velocity}$$
$$= (\rho \times \text{area} \times \text{velocity}) \times \text{velocity}$$
$$= [\rho \times \theta \times b \times U] \times U \qquad \{\because \text{Area} = \theta \times b\}$$
$$= \rho\theta bU^2 \qquad ...(13.4)$$

Equating equations (13.4) and (13.3), we have

$$\rho\theta bU^2 = \int_0^\delta \rho bu(U - u)dy = \rho b \int_0^\delta u(U - u)dy \qquad \{\text{If fluid is assumed incompressible}\}$$

or
$$\theta U^2 = \int_0^\delta u(U - u)dy \qquad \{\text{cancelling } \rho b \text{ from both sides}\}$$

or
$$\theta = \frac{1}{U^2}\int_0^\delta u(U - u)dy = \int_0^\delta \frac{u(U - u)dy}{U^2}$$

$\therefore$
$$\theta = \int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right] dy. \qquad ...(13.5)$$

13.2.7 Energy Thickness (δ^{**}). It is defined as the distance measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in kinetic energy of the flowing fluid on account of boundary layer formation. It is denoted by δ^{**}.

Consider the flow over the plate as shown in Fig. 13.3 having section 1-1 at a distance x from leading edge. The mass of fluid flowing per second through the elemental strip of thickness 'dy' at a distance y from the plate as given by equation (i) = $\rho ubdy$

Kinetic energy of this fluid $= \frac{1}{2} m \times \text{velocity}^2 = \frac{1}{2} (\rho ubdy)\, u^2$

Kinetic energy of this fluid in the absence of boundary layer

$$= \frac{1}{2} (\rho ubdy)U^2$$

$\therefore$ Loss of K.E. through elemental strip

$$= \frac{1}{2} (\rho ubdy)U^2 - \frac{1}{2} (\rho ubdy)\, u^2 = \frac{1}{2} \rho ub\, [U^2 - u^2]\, dy$$

$\therefore$ Total loss of K.E. of fluid passing through BC

$$= \int_0^\delta \frac{1}{2} \rho u b\, [U^2 - u^2]\, dy = \frac{1}{2}\rho b \int_0^\delta u\, (U^2 - u^2)\, dy$$

{If fluid is considered incompressible}

Let δ^{**} = distance by which the plate is displaced to compensate for the reduction in K.E.

$\therefore$ Loss of K.E. through δ^{**} of fluid flowing with velocity U

$$= \frac{1}{2} \text{ (mass)} \times \text{velocity}^2 = \frac{1}{2} (\rho \times \text{area} \times \text{velocity}) \times \text{velocity}^2$$

$$= \frac{1}{2} (\rho \times b \times \delta^{**} \times U) U^2 \qquad \{\because \text{ Area} = b \times \delta^{**}\}$$

$$= \frac{1}{2} \rho b \delta^{**} U^3$$

Equating the two losses of K.E., we get

$$\frac{1}{2} \rho b \delta^{**} U^3 = \frac{1}{2}\rho b \int_0^\delta u\, (U^2 - u^2)\, dy$$

or

$$\delta^{**} = \frac{1}{U^3} \int_0^\delta u\, (U^2 - u^2)\, dy$$

$\therefore$

$$\delta^{**} = \int_0^\delta \frac{u}{U} \left[1 - \frac{u^2}{U^2}\right] dy. \qquad \text{...(13.6)}$$

Problem 13.1 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = \frac{y}{\delta}$, *where u is the velocity at a distance y from the plate and u = U at y =* δ, *where* δ *= boundary layer thickness. Also calculate the value of* δ^*/θ.

Solution. Given :

Velocity distribution $\quad \dfrac{u}{U} = \dfrac{y}{\delta}$

(*i*) Displacement thickness δ^* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy \qquad \left\{\because \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \left[y - \frac{y^2}{2\delta} \right]_0^\delta \qquad \{\delta \text{ is constant across a section}\}$$

$$= \delta - \frac{\delta^2}{2\delta} = \delta - \frac{\delta}{2} = \mathbf{\frac{\delta}{2}}. \quad \textbf{Ans.}$$

(*ii*) Momentum thickness, θ is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = \frac{y}{\delta}$,

$$\theta = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy$$

$$= \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{3\delta - 2\delta}{6} = \frac{\delta}{\mathbf{6}}. \textbf{ Ans.}$$

(*iii*) Energy thickness δ^{**} is given by equation (13.6), as

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy = \int_0^\delta \frac{y}{\delta}\left[1 - \frac{y^2}{\delta^2}\right] dy \qquad \left\{\because \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \int_0^\delta \left[\frac{y}{\delta} - \frac{y^3}{\delta^3}\right] dy = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2} - \frac{\delta}{4} = \frac{2\delta - \delta}{4} = \frac{\delta}{\mathbf{4}}. \textbf{ Ans.}$$

(*iv*)
$$\frac{\delta^*}{\theta} = \frac{\left(\frac{\delta}{2}\right)}{\left(\frac{\delta}{6}\right)} = \frac{\delta}{2} \times \frac{6}{\delta} = \mathbf{3}. \textbf{ Ans.}$$

Problem 13.2 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$.

Solution. Given :

Velocity distribution
$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

(*i*) Displacement thickness δ^* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$, we have

$$\delta^* = \int_0^\delta \left\{1 - \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right\} dy$$

$$= \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right\} dy = \left[y - \frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta$$

$$= \delta - \frac{\delta^2}{\delta} + \frac{\delta^3}{3\delta^2} = \delta - \delta + \frac{\delta}{3} = \frac{\delta}{\mathbf{3}}. \quad \textbf{Ans.}$$

(*ii*) Momentum thickness θ, is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U}\left\{1-\frac{u}{U}\right\}dy = \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left[1-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\right]dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right]dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy = \left[\frac{2y^2}{2\delta}-\frac{5y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{\delta^2}{\delta}-\frac{5\delta^3}{3\delta^2}+\frac{\delta^4}{\delta^3}-\frac{\delta^5}{5\delta^4}\right] = \delta-\frac{5\delta}{3}+\delta-\frac{\delta}{5}$$

$$= \frac{15\delta-25\delta+15\delta-3\delta}{15} = \frac{30\delta-28\delta}{15} = \mathbf{\frac{2\delta}{15}}. \quad \textbf{Ans.}$$

(*iii*) Energy thickness δ** is given by equation (13.6),

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1-\frac{u^2}{U^2}\right]dy = \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]^2\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\left[\frac{4y^2}{\delta^2}+\frac{y^4}{\delta^4}-\frac{4y^3}{\delta^3}\right]\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{4y^2}{\delta^2}-\frac{y^4}{\delta^4}+\frac{4y^3}{\delta^3}\right)dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta}-\frac{8y^3}{\delta^3}-\frac{2y^5}{\delta^5}+\frac{8y^4}{\delta^4}-\frac{y^2}{\delta^2}+\frac{4y^4}{\delta^4}+\frac{y^6}{\delta^6}-\frac{4y^5}{\delta^5}\right)dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}-\frac{8y^3}{\delta^3}+\frac{12y^4}{\delta^4}-\frac{6y^5}{\delta^5}+\frac{y^6}{\delta^6}\right]dy$$

$$= \left[\frac{2y^2}{2\delta}-\frac{y^3}{3\delta^2}-\frac{8y^4}{4\delta^3}+\frac{12y^5}{5\delta^4}-\frac{6y^6}{6\delta^5}+\frac{y^7}{7\delta^6}\right]_0^\delta$$

$$= \frac{\delta^2}{\delta}-\frac{\delta^3}{3\delta^2}-\frac{2\delta^4}{\delta^3}+\frac{12\delta^5}{5\delta^4}-\frac{\delta^6}{\delta^5}+\frac{\delta^7}{7\delta^6} = \delta-\frac{\delta}{3}-2\delta+\frac{12}{5}\delta-\delta+\frac{\delta}{7}$$

$$= -2\delta-\frac{\delta}{3}+\frac{12}{5}\delta+\frac{\delta}{7} = \frac{-210\delta-35\delta+252\delta+15\delta}{105}$$

$$= \frac{-245\delta+267\delta}{105} = \mathbf{\frac{22\delta}{105}}. \quad \textbf{Ans.}$$

▶ 13.3 DRAG FORCE ON A FLAT PLATE DUE TO BOUNDARY LAYER

Consider the flow of a fluid having free-stream velocity equal to U, over a thin plate as shown in Fig. 13.4. The drag force on the plate can be determined if the velocity profile near the plate is known. Consider a small length Δx of the plate at a distance of x from the leading edge as shown in Fig. 13.4 (a). The enlarged view of the small length of the plate is shown in Fig. 13.4 (b).

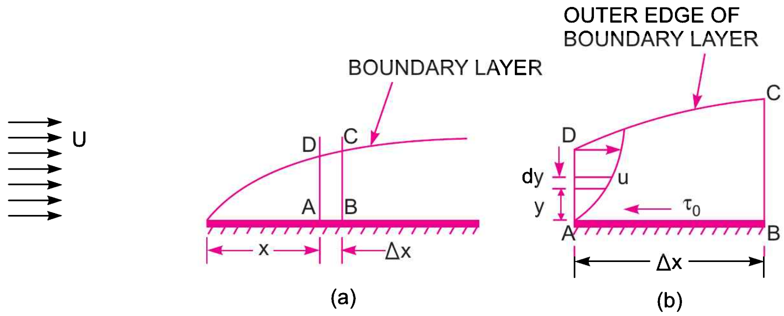

Fig. 13.4 *Drag force on a plate due to boundary layer.*

The shear stress τ_0 is given by $\tau_0 = \mu \left(\frac{du}{dy}\right)_{y=0}$, where $\left(\frac{du}{dy}\right)_{y=0}$ is the velocity distribution near the plate at $y = 0$.

Then drag force or shear force on a small distance Δx is given by

$$\Delta F_D = \text{shear stress} \times \text{area}$$
$$= \tau_0 \times \Delta x \times b \qquad \text{...(13.7)} \quad \{\text{Taking width of plate} = b\}$$

where ΔF_D = drag force on distance Δx

The drag force ΔF_D must also be equal to the rate of change of momentum over the distance Δx.

Consider the flow over the small distance Δx. Let $ABCD$ is the control volume of the fluid over the distance Δx as shown in Fig. 13.4 (b). The edge DC represents the outer edge of the boundary layer.

Let u = velocity at any point within the boundary layer

b = width of plate

Then mass rate of flow entering through the side AD

$$= \int_0^\delta \rho \times \text{velocity} \times \text{area of strip of thickness } dy$$

$$= \int_0^\delta \rho \times u \times b \times dy \qquad \{\because \text{ Area of strip} = b \times dy\}$$

$$= \int_0^\delta \rho u b dy$$

Mass rate of flow leaving the side BC

$$= \text{mass through } AD + \frac{\partial}{\partial x}(\text{mass through } AD) \times \Delta x$$

$$= \int_0^\delta \rho u b dy \ \frac{\partial}{\partial x}\left[\int_0^\delta (\rho u b dy)\right] \times \Delta x$$

From continuity equation for a steady incompressible fluid flow, we have

Mass rate of flow entering AD + mass rate of flow entering DC

= mass rate of flow leaving BC

$\because$ Mass rate of flow entering DC = mass rate of flow through BC – mass rate of flow through AD

$$= \int_0^\delta \rho ubdy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho ubdy\right] \times \Delta x - \int_0^\delta \rho ubdy$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho ubdy\right] \times \Delta x$$

The fluid is entering through side DC with a uniform velocity U.

Now let us calculate momentum flux through control volume.

Momentum flux entering through AD

$$= \int_0^\delta \text{momentum flux through strip of thickness } dy$$

$$= \int_0^\delta \text{mass through strip} \times \text{velocity} = \int_0^\delta (\rho ubdy) \times u = \int_0^\delta \rho u^2 bdy$$

Momentum flux leaving the side $BC = \int_0^\delta \rho u^2 bdy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 bdy\right] \times \Delta x$

Momentum flux entering the side DC = mass rate through DC $\times$ velocity

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho ubdy\right] \times \Delta x \times U \qquad (\because \ \text{Velocity} = U\}$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho uUbdy\right] \times \Delta x$$

As U is constant and so it can be taken inside the differential and integral.

$\therefore$ Rate of change of momentum of the control volume

= Momentum flux through BC – Momentum flux through AD

– momentum flux through DC

$$= \int_0^\delta \rho u^2 bdy + \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 bdy\right] \times \Delta x - \int_0^\delta \rho u^2 bdy - \frac{\partial}{\partial x}\left[\int_0^\delta \rho uUbdy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 bdy\right] \times \Delta x - \frac{\partial}{\partial x}\left[\int_0^\delta \rho uUbdy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho u^2 bdy - \int_0^\delta \rho uUbdy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\rho u^2 b - \rho uUb\right)dy\right] \times \Delta x$$

$$= \frac{\partial}{\partial x}\left[\rho b \int_0^\delta \left(u^2 - uU\right)dy\right] \times \Delta x$$

{For incompressible fluid ρ is constant}

$$= \rho b \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right] \times \Delta x \qquad ...(13.8)$$

Now the rate of change of momentum on the control volume *ABCD* must be equal to the total force on the control volume in the same direction according to the momentum principle. But for a flat plate $\frac{\partial p}{\partial x} = 0$, which means there is no external pressure force on the control volume. Also the force on the side *DC* is negligible as the velocity is constant and velocity gradient is zero approximately. The only external force acting on the control volume is the shear force acting on the side *AB* in the direction from *B* to *A* as shown in Fig. 13.4 (*b*). The value of this force is given by equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

∴ Total external force in the direction of rate of change of momentum

$$= - \tau_0 \times \Delta x \times b \qquad ...(13.9)$$

According to momentum principle, the two values given by equations (13.9) and (13.8) should be the same.

$$\therefore \qquad - \tau_0 \times \Delta x \times b = \rho b \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right] \times \Delta x$$

Cancelling $\Delta x \times b$, to both sides, we have

$$- \tau_0 = \rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right]$$

or

$$\tau_0 = - \rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(u^2 - uU\right)dy\right] = \rho \frac{\partial}{\partial x}\left[\int_0^\delta \left(uU - u^2\right)dy\right]$$

$$= \rho \frac{\partial}{\partial x}\left[\int_0^\delta U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right] = \rho U^2 \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

or

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right] \qquad ...(13.10)$$

In equation (13.10), the expression $\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy$ is equal to momentum thickness θ. Hence equation (13.10) is also written as

$$\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x} \qquad ...(13.11)$$

Equation (13.11) is known as **Von Karman momentum integral equation** for boundary layer flows.

This is applied to :

1. Laminar boundary layers,
2. Transition boundary layers, and
3. Turbulent boundary layer flows.

For a given velocity profile in laminar zone, transition zone or turbulent zone of a boundary layer, the shear stress τ_0 is obtained from equation (13.10) or (13.11). Then drag force on a small distance Δx of the plate is obtained from equation (13.7) as

$$\Delta F_D = \tau_0 \times \Delta x \times b$$

Then total drag on the plate of length L on one side is

$$F_D = \int \Delta F_D = \int_0^L \tau_0 \times b \times dx \quad \{\text{change } \Delta x = dx\}. \qquad ...(13.12)$$

13.3.1 Local Co-efficient of Drag [C_D*]. It is defined as the ratio of the shear stress τ_0 to the quantity $\frac{1}{2}\rho U^2$. It is denoted by C_D^*

Hence
$$C_D^* = \frac{\tau_0}{\frac{1}{2}\rho U^2}. \qquad ...(13.13)$$

13.3.2 Average Co-efficient of Drag [C_D]. It is defined as the ratio of the total drag force to the quantity $\frac{1}{2}\rho A U^2$. It is also called co-efficient of drag and is denoted by C_D.

Hence
$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2} \qquad ...(13.14)$$

where A = Area of the surface (or plate)
U = Free-stream velocity
ρ = Mass density of fluid.

13.3.3 Boundary Conditions for the Velocity Profiles. The followings are the boundary conditions which must be satisfied by any velocity profile, whether it is in laminar boundary layer zone, or in turbulent boundary layer zone :

1. At $y = 0$, $u = 0$ and $\frac{du}{dy}$ has some finite value
2. At $y = \delta$, $u = U$
3. At $y = \delta$, $\frac{du}{dy} = 0$.

Problem 13.3 *For the velocity profile for laminar boundary layer flows given as*

$$\frac{u}{U} = 2(y/\delta) - (y/\delta)^2$$

find an expression for boundary layer thickness (δ), shear stress (τ_0) and co-efficient of drag (C_D) in terms of Reynold number.

Solution. Given :

(*i*) The velocity distribution $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ ...(*i*)

Substituting this value of $\frac{u}{U}$ in equation (13.10), we get

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] = \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right]dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy\right]$$

$$= \frac{\partial}{\partial x}\int_0^\delta \left[\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy = \frac{\partial}{\partial x}\left[\frac{2y^2}{2\delta}-\frac{5\times y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{\delta^2}{\delta}-\frac{5}{3}\frac{\delta^3}{\delta^2}+\frac{\delta^4}{\delta^3}-\frac{\delta^5}{5\delta^4}\right] = \frac{\partial}{\partial x}\left[\delta-\frac{5}{3}\delta+\delta-\frac{\delta}{5}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{15\delta-25\delta+15\delta-3\delta}{15}\right] = \frac{\partial}{\partial x}\left[\frac{30\delta-28\delta}{15}\right] = \frac{\partial}{\delta x}\left[\frac{2\delta}{15}\right] = \frac{2}{15}\frac{\partial}{\partial x}[\delta]$$

$\therefore$
$$\tau_0 = \rho U^2 \times \frac{2}{15}\frac{\partial}{\partial x}[\delta] = \frac{2}{15}\rho U^2 \frac{\partial[\delta]}{\partial x} \qquad ...(13.15)$$

The shear stress at the boundary in laminar flow is also given by Newton's law of viscosity as

$$\tau_0 = \mu\left(\frac{du}{dy}\right)_{y=0} \qquad ...(ii)$$

But from equation (*i*), $u = U\left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]$

$\therefore$ $\frac{du}{dy} = U\left[\frac{2}{\delta}-\frac{2y}{\delta^2}\right]$ $\{\because$ U is constant$\}$

$\therefore$ $\left(\frac{du}{dy}\right)_{y=0} = U\left[\frac{2}{\delta}-\frac{2\times(0)}{\delta^2}\right] = \frac{2U}{\delta}$

Substituting this value in (*ii*), we get

$$\tau_0 = \mu \times \frac{2U}{\delta} = \frac{2\mu U}{\delta} \qquad ...(iii)$$

Equating the two values of τ_0 given by equation (13.15) and (*iii*)

$$\frac{2}{15}\rho U^2 \frac{\partial}{\partial x}[\delta] = \frac{2\mu U}{\delta}$$

or
$$\frac{\delta\partial}{\partial x}[\delta] = \frac{15\mu U}{\rho U^2} = \frac{15\mu}{\rho U} \quad \text{or} \quad \delta\partial[\delta] = \frac{15\mu}{\rho U}\partial x$$

As the boundary layer thickness (δ) is a function of x only.
Hence partial derivative can be changed to total derivative

$$\therefore \qquad \delta d[\delta] = \frac{15\mu}{\rho U}\, dx$$

On integration, we get $$\frac{\delta^2}{2} = \frac{15\mu}{\rho U}\, x + C \qquad \left\{\frac{\mu}{\rho U} \text{ is constant}\right\}$$

$x = 0$, $\delta = 0$ and hence $C = 0$

$$\therefore \qquad \frac{\delta^2}{2} = \frac{15\mu x}{\rho U}$$

$$\therefore \qquad \delta = \sqrt{\frac{2 \times 15\mu x}{\rho U}} = \sqrt{\frac{30\mu x}{\rho U}} = 5.48\sqrt{\frac{\mu x}{\rho U}} \qquad \text{...(13.16)}$$

$$= 5.48\sqrt{\frac{\mu x \times x}{\rho U \times x}} = 5.48\sqrt{\frac{x^2}{R_{e_x}}} \qquad \left\{\because \; R_{e_x} = \frac{\rho U x}{\mu}\right\}$$

$$= 5.48\frac{x}{\sqrt{R_{e_x}}} \qquad \text{...(13.17)}$$

In equation (13.16), μ, ρ and U are constant and hence it is clear from this equation that thickness of laminar boundary layer is proportional to the square root of the distance from the leading edge. Equation (13.17) gives the thickness of laminar boundary layer in terms of Reynolds number.

(*ii*) **Shear stress (τ_0) in terms of Reynolds number**

From equation (*iii*), we have $\tau_0 = \dfrac{2\mu U}{\delta}$

Substituting the value of δ from equation (13.17), in the above equation, we get

$$\tau_0 = \frac{2\mu U}{5.48\dfrac{x}{\sqrt{R_{e_x}}}} = \frac{2\mu U\sqrt{R_{e_x}}}{5.48x} = 0.365\,\frac{\mu U}{x}\sqrt{R_{e_x}}$$

(*iii*) **Co-efficient of Drag (C_D)**

From equation (13.14), we have $C_D = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$

where F_D is given by equation (13.12) as

$$F_D = \int_0^L \tau_0 \times b \times dx = \int_0^L 0.365\frac{\mu U}{x}\sqrt{R_{e_x}} \times b \times dx$$

$$= 0.365\int_0^L \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times b \times dx \qquad \left\{\because \; R_{e_x} = \frac{\rho U x}{\mu}\right\}$$

$$= 0.365\int_0^L \mu U\sqrt{\frac{\rho U}{\mu}} \times \frac{1}{\sqrt{x}} \times b \times dx$$

$$= 0.365\,\mu U\sqrt{\frac{\rho U}{\mu}} \times b \int_0^L x^{-1/2}\,dx = 0.365\,\mu U\sqrt{\frac{\rho U}{\mu}} \times b \times \left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^L$$

$$= 0.365 \times 2\mu U\sqrt{\frac{\rho U}{\mu}} \times b \times \sqrt{L}$$

$$= 0.73\,b\mu U\sqrt{\frac{\rho UL}{\mu}} \qquad \text{...(13.18)}$$

$$\therefore \qquad C_D = \frac{0.73\,b\mu U\sqrt{\frac{\rho UL}{\mu}}}{\frac{1}{2}\rho A U^2}$$

where A = Area of plate = Length of plate × width = $L \times b$

$$\therefore \qquad C_D = \frac{0.73\,b\mu U}{\frac{1}{2}\rho \times L \times b \times U^2}\sqrt{\frac{\rho UL}{\mu}} = \frac{1.46\,\mu}{\rho LU}\sqrt{\frac{\rho UL}{\mu}}$$

$$= \frac{1.46\sqrt{\mu}}{\sqrt{\rho UL}} = 1.46\sqrt{\frac{\mu}{\rho UL}} = \frac{1.46}{\sqrt{R_{e_L}}} \quad \text{...(13.19)} \qquad \left\{\because \sqrt{\frac{\mu}{\rho UL}} = \frac{1}{\sqrt{R_{e_L}}}\right\}$$

Problem 13.4 *For the velocity profile given in problem 13.3, find the thickness of boundary layer at the end of the plate and the drag force on one side of a plate 1 m long and 0.8 m wide when placed in water flowing with a velocity of 150 mm per second. Calculate the value of co-efficient of drag also. Take μ for water = 0.01 poise.*

Solution. Given :

Length of plate, $L = 1$ m

Width of plate, $b = 0.8$ m

Velocity of fluid (water), $U = 150$ mm/s $= 0.15$ m/s

μ for water $= 0.01 \text{ poise} = \dfrac{0.01}{10}\dfrac{\text{Ns}}{\text{m}^2} = 0.001\dfrac{\text{Ns}}{\text{m}^2}$

Reynold number at the end of the plate *i.e.*, at a distance of 1 m from leading edge is given by

$$R_{e_L} = \frac{\rho UL}{\mu} = 1000 \times \frac{0.15 \times 1.0}{.001} \qquad (\because \quad \rho = 1000)$$

$$= \frac{1000 \times .15 \times 1.0}{0.001} = 150000$$

(*i*) As laminar boundary layer exists upto Reynold number = 2×10^5. Hence this is the case of laminar boundary layer. Thickness of boundary layer at $x = 1.0$ m is given by equation (13.17) as

$$\delta = 5.48\frac{x}{\sqrt{R_{e_x}}} = \frac{5.48 \times 1.0}{\sqrt{150000}} = 0.01415 \text{ m} = \mathbf{14.15\ mm.\ Ans.}$$

(*ii*) Drag force on one side of the plate is given by equation (13.18)

$$F_D = 0.73\, b\mu U \sqrt{\frac{\rho UL}{\mu}}$$

$$= 0.73 \times 0.8 \times 0.001 \times 0.15 \times \sqrt{150000} \qquad \left\{\because \frac{\rho UL}{\mu} = R_{e_L}\right\}$$

$$= \mathbf{0.0338\ N.\ Ans.}$$

(*iii*) Co-efficient of drag, C_D is given by equation (13.19) as

$$C_D = \frac{1.46}{\sqrt{R_{e_L}}} = \frac{1.46}{\sqrt{150000}} = \mathbf{.00376.\ Ans.}$$

Problem 13.5 *For the velocity profile for laminar boundary layer* $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$. *Determine the boundary layer thickness, shear stress, drag force and co-efficient of drag in terms of Reynold number.*

Solution. Given :

Velocity distribution, $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$

Using equation (13.10), we have $\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right]$

Substituting the value of $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ in the above equation

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \left\{\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}\right] dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{3y}{2\delta} - \frac{y^3}{2\delta^3}\right)\left(1 - \frac{3y}{2\delta} + \frac{y^3}{2\delta^3}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \left(\frac{3y}{2\delta} - \frac{9y^2}{4\delta^2} + \frac{3y^4}{4\delta^4} - \frac{y^3}{2\delta^3} + \frac{3y^4}{4\delta^4} - \frac{y^6}{4\delta^6}\right) dy\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{3y^2}{2\times 2\delta} - \frac{9y^3}{3\times 4\delta^2} + \frac{3y^5}{5\times 4\delta^4} - \frac{y^4}{4\times 2\delta^3} + \frac{3y^5}{5\times 4\delta^4} - \frac{y^7}{7\times 4\delta^6}\right]_0^\delta$$

$$= \frac{\partial}{\partial x}\left[\frac{3\delta^2}{4\delta} - \frac{3\delta^3}{4\delta^2} + \frac{3}{20}\frac{\delta^5}{\delta^4} - \frac{1}{8}\frac{\delta^4}{\delta^3} + \frac{3}{20}\frac{\delta^5}{\delta^4} - \frac{1}{28}\frac{\delta^7}{\delta^6}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{3}{4}\delta - \frac{3}{4}\delta + \frac{3}{20}\delta - \frac{1}{8}\delta + \frac{3}{20}\delta - \frac{1}{28}\delta\right]$$