# Graph Theory

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# Contents

# Chapter 1

# Spanning Tree

- A spanning tree is a subset of Graph $G$, which has all the vertices covered with minimum possible number of edges.

## 1.1 Theorems:

- $G$ is connected if and only if $G$ has a spanning tree.
  - Corllary: if $G$ has connected with $n$ vertices and $n-1$ edges, then $G$ is a tree.
    Proof:
    Since $G$ is connected, so $G$ has a spanning tree $T$.
    Then $T$ has $n-1$ edges. But $G$ has $n-1$ edges as well.
    So $G$ is a tree as well.
  - Suppose $G$ has $n$ vertices. If any 2 of 3 following conditions hold for $G$, then $G$ is a tree.
    * $G$ is connected.
    * $G$ has no cycle.
    * $G$ has $n-1$ edges.
- If $T$ is a spanning tree of $G$ and $e$ is an edge of $G$ not in $T$, then $T+e$ contains exactly one cycle $C$. If $e'$ is an edge of $C$, then $T+e-e'$ is a spanning tree.
  Proof:
  Let $e = uv$. Any cycle in $T+e$ must use $e$.
  Since $T$ has no cycle, such a cycle consist of $e$ and a $u-v$ path in $T$.
  Since there is a unique $uv$ path in $T$, there is exactly one cycle in $T+e$.
  Let $e'$ be an edge on $C$, then $e'$ is not a bridge.
  So $T+e-e'$ is connected. Removing $e'$ destroy the only cycle in $T+e$.
  Therefore $T+e-e'$ has no cycly and it is a tree.
  (Alternatively, $T+e-e'$ has the same number of edges as $T$, so it is a tree)

- No graph contains exactly two spanning tree.
- If $T$ is a spanning tree of $G$ and $e$ is an edge of $T$, then $T_e$ has two components, say $H$ is one of them. If $e'$ is in the cut induced by $V(H)$ in $G$, then $T - e + e'$ is a spanning tree of $G$.

# Chapter 2

# Bipartite characterization

## 2.1 Theorem: G is bipartite if and only if G does not contains an odd cycle.

Proof:
This theorem has an if and only if relationship, therefore we need to prove in both directions.
$\Rightarrow$
Suppose $G$ is bipartite with bipartition, say $A$ and $B$.
Let $C = v_1, v_2, \cdots, v_k, v_1$ be a cycle of odd length in $G$.
WLOG, $v_1 \in A$, then $v_2 \in B, v_3 \in A, v_4 \in B, \cdots$.
Since $k$ is odd, so $v_k \in A$, but $v_k - v_1$ is an edge and both $v_k, v_1 \in A$ that is contradiction.
Therefore $G$ does not contain an odd cycle.

$\Leftarrow$
Suppose $G$ is not bipartite, then there exist a non-bipartite component $H$.
Since $H$ is connected, it has a spanning tree $T$.
Since $T$ is partitle, thus it has bipartiton (say $A$, $B$)
Since $H$ is not bipartite, it contains an edge that ions two vertices in $B$ (WLOG) say that $e = uv$.
Let $v_1, v_2, \cdots, v_k$ e the unique $u - v$ path in $T (v_1 = u, v_k = v)$ Then $v_1 \in B$, $v_2 \in A$, $v_3 \in B$, $\cdots$
Since $v_k \in B$, $k$ is odd. Then $v_1, v_2, \cdots, v_k, v_1$ is a cycle of length $k$, which is odd.

# Chapter 3

# Minimum Spanning Tree(MST)

- Given connected graph $G$, weight function $w : E(G) \Rightarrow (R)$
- Goal: find a spanning tree whose total edge weight minimized.
- Notation: $w(e)$ is the weigth of $e$. If $H$ is a subgraph of $G$.

$$w(H) = \sum_{e \in E(G)} w(e)$$

# Chapter 4

# Prim's Algorithm

- Let $v$ be any vertex. Let $T$ be the tree with just $v$.

- While $T$ is not a spanning tree.
  - look at all the edges in the cut induced by $V(T)$
  - Pick $e = uv$ to be an edge of smallest weight in the cut (say $u \in V(T)$, $v \notin V(T)$)
  - Add $v$ to $V(T)$, add $e$ to $E(T)$

## 4.1 Theorem: Prim's algorithm produces a MST

Proof:
Let $T_1, T_2, T_3, \cdots, T_n$ be the trees produced by the algorithm at each step, where the order of the edges added is $e_1, e_2, \cdots, e_{n-1}$, i.e. $T_{k+1} = T_k + e_k$.
Prove by induction on $k$ that each $T_k$ is a subgraph of a MST. If so, then $T_n$ is a spinning tree contained in a MST, so $T_n$ is a MST.

Base case: For $k = 1$, $T_1$ is just one vertex, which is any MST.
Induction Hypotheses: Assume $T_k$ is a subgraph of a MST, $T_k$ for some $k$.
Induction Step:

- Show $T_{k+1}$ is subgraph of $T_k$ and we are done!

- Assume $e_k \in E(T^*)$, then $T_{k+1} + e_k - e'$ is also a spanning tree.

- The algorithm chose $e_k$ when looking at the cut induced by $V(T_k)$, so $w(e_k) \leq w(e')$.

- If $w(e_k) < w(e')$, then $w(T') = w(T^*) + w(e_k) - w(e') < w(T^*)$ which is impossibl, since $w(T')$ is smaller than the weight of a MST.

- If $w(e_k) = w(e')$, then $w(T) = w(T^*)$, so $T'$ is a MST. And $T'$ contains $T_{k+1}$ as a subgraph (since it contains $e_k$).
- This completes the induction.

# Chapter 5

# Travelling Salesman Problem(TSP)

- Goal: Find a Hamilton cycle (cycle that uses all vertices) of minimum weight.
- No polynomial time algorithm is known so far. Restrict weights to Euclidean distance
- **Approximate Algorithm**
  1. find the minimum spanning tree $T$
  2. double all edges
  3. find an Euclidean circuit
  4. find a hamilton cycle by skipping repeated vertices (take the short cut)

## 5.1 Theorem: The approx. algorithm produces a cycle whose weight is at most twice the best possible Hamilton cycle.

Proof:
Let $C^*$ be a minimum weight Hamilton cycle, $C^* - e$ is a spanning tree, so $w(c^*) \geq w(T)$.
If $C$ is the cycle from the approximate algorithm, then the $w(c) \leq 2w(T)$ since we are taking short cut suing trangle-inequality.
Therefoe $w(c) \leq 2w(c^*)$

# Chapter 6

# Planar Graph

## 6.1 Key Definations:

- A **planar embedding** of $G$ is a drawing of $G$ on the plane such that vertices are different points and edges are lines that join vertices such that they do not intersect (except at common endpoints).

- A graph that has a planar embedding is a planar graph.

- A face of a planar embedding is a connected region on the plane (not separated by lines). Two faces are adjacent if they share at least two edges.

- For a connected planar graph, the boundary walk of a face is a closed walk on the edges around the boundry of the region once around. The degree of the face is the length of its boundry walk.

- Handshaking lemma for Faces:
  - Let $G$ be a planar graph embedding when $F$ is the set of all faces. Then

  $$\sum_{f \in F} deg(f) = 2 \mid E(G) \mid$$

  Proof: each edge contributes 2 to the sum of face degrees. One for each side of the edge.

- Observation: if $e$ is a bridge, then two sides of $e$ are in the same face. Otherwise, they are in different faces, since $e$ is in a cycle.

## 6.2 Jordan Curve Theorem:

- Every single close curve (cycle) on the plane separted the plane into two parts, one insdide and one outside.

## 6.3 Euler's Formula

- A graph can be different embedding, consider n represents the number of vertices, m represents the number of edges, s represents the number of faces: then we have,

$$\begin{aligned} n - m + s &= 2 \\ n - m + s &= 1 + \#\text{ of copmonents} \end{aligned}$$

### 6.3.1 Theorem: For a connected planar graph with an embedding with $n$ vertices, $m$ edges and $s$ faces: $n - m + s = 2$

Proof by induction: (fix $n$) induced on $m$
Base case: $m = n - 1$(spanning tree), thus $s = 1$ (no cycles, only one face) $\Rightarrow$ the statements holds.
Inductive Hypotheses: Assume Euler's formula holds for any connected planar graph with $n$ vertices, $m - 1$ edges.
Inductive Step:
Suppose $G$ is connected, planar, $n$ vertices, $m$ edges and $s$ faces.
Let $e$ be an edge in a cycle of $G$, then $G - e$ is connected an planar.
The two sides of $e$ and different faces in $G$, and they merge into one faces in $G - e \rightarrow s - 1$ faces.
By inductive Hypotheses, the Euler's formula holds for $G - e$.
So $n - (m - 1) + (s - 1) = 2 \Rightarrow n - m + 1 + s - 1 = n - m + s = 2$

## 6.4 Platonic Solids

- If $G$ can be embeded on a planar, then $G$ can be embedded on a sphere. We then cut across each face to obtain a polyhedron.
- **Definition: a connected planar graph is platonic if it has an embedding where every vertex has same degree ($\geq 3$) and every face has some degree.**

$$\begin{cases} n \cdot d_v & = 2m \quad \text{basic handshaking lemma} \\ s \cdot d_f & = 2m \quad \text{handshaking lemma for faces} \\ n - m + s & = 2 \quad\;\; \text{Euler's Formula} \end{cases}$$

  - When $d_v = 3$, $d_f = 3$, $n = 4$, $m = 6$, $s = 4$, it is tetrahedron.
  - When $d_v = 3$, $d_f = 4$, $n = 8$, $m = 12$, $s = 6$, it is cube.
  - When $d_v = 4$, $d_f = 3$, $n = 6$, $m = 12$, $s = 8$, it is octahedron.
  - When $d_v = 3$, $d_f = 5$, $n = 20$, $m = 30$, $s = 12$, it is dodecahedron.
  - When $d_v = 5$, $d_f = 3$, $n = 12$, $m = 30$, $s = 20$, it is icosahedron.

# Chapter 7

# Non-planar Graph

- One way to prove non-planar graph is to show that it has more than $3n - 6$ edges.
- When $n \geq 3$, a connected planar graph on $n$ vertices has at most $3n - 6$ edges.
  Proof:
  Consider a planar embedding of $G$ with $n \geq 3$ vertices, $m$ edges and $s$ faces.
  If $G$ is a tree, then we have $m = n - 1$ edges, where $n - 1 \leq 3n - 6$ is satisfied for $n \geq 3$.
  If $G$ is a tree, then $G$ has a cycle, there are at least 2 faces, each face bounary must contains a cycle to separtate from other faces(Jordan curve theorem). So each face has degree at least 3.
  By handshaking lemma for faces, $2m = \sum_{f \in F}^{deg(f)} \geq 3 \cdot s$.
  So $2m \geq 3m = 3(2 - n + m)$ by Euler's formula, therefore $\leq 3n - 6$
- Note: Disconnected planar graphs have "fewer" edges that a connected one, so that applies to disconnected graph as well.

## 7.1 Corollary: $k_5$ is not planar, same with $k_{3,3}$

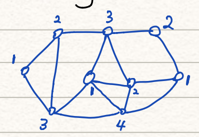
## 7.2 Theorem:

A connected bipartite planar graph has at least 3 vertices has at most 2n-4 edges.

## 7.3 Kuratowski's Theorem

- An edge subdivision of $G$ is obtained by replacing each edge of $G$ with a new path of length at least 1 (i.e. introducing new vertices of degree 2 to the edges). Observation: G is a planar iff every edge subdivision is a planar.
- $G$ is planar iff G does not contain an edge subdivision of $k_5$ and $k_{3,3}$. Note: do not repeat vertices or edges in the subdivision

# Colouring



4 - colouring

**Defn:** A k-colouring of $G$ is an assignment of a colour to each vertex using a pool of at most k colours.
A graph has a k-colouring is k-colourable.

**Note:** if $G$ is k-colourable, then $G$ is $(k+1)$-colourable.

General colouring question: How many colours are needed to colour a graph? we want to minimize this.
Applications: ① scheduling
② compilers assigning variables to registers

Suppose $G$ has $n$ vertices $\Rightarrow$ $G$ is $n$ colourable

$\boxed{\text{Thm: } K_n \text{ is } n\text{-colourable, but not } (n-1)\text{-colourable}}$
complete graph

$\boxed{\text{Thm: } G \text{ is 2 colourable iff } G \text{ is bipartite}}$

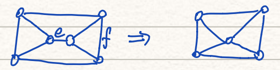
## Colouring planar graph

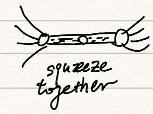
**Theorem:** Any planar has a vertex of degree at most 5.
**proof:** suppose $G$ is planar with $n$ vertices.
Suppose every vertex in $G$ has degree at least 6. Then $G$ has at least $\frac{6n}{2} = 3n$ edges (By handshaking)
Since it is a planar graph, it has at most $3n-6$ edges. $\Rightarrow$ contradictions.
So at least one vertex has degree at most 5.
■

**Theorem:** Any planar graph is 6-colourable.
**proof:** by induction on the # of vertices $n$.
B.C. $n=1$. then it is 6-colourable
I.H: Any planar graph with $n-1$ vertices is 6-colourable.
I.S: suppose $G$ is planar with $n$ vertices.
Let $v$ be a vertex of degree at most 5 in $G$. Obtain $G'$ from $G$ by removing $v$ and its incident edges.
Then $G'$ is planar with $n-1$ vertices.
By induction hypothesis, we know $G'$ is 6-colouring.
Keep the same colouring for $G$. For $v$, at most 5 colours are used by its neighbour.
Since we have 6 colours, at least one unused colour is available for $v$.
We get a 6-colouring for $G$.
■

Theorem: Any planar graph is 5-colourable

contraction $G/e$ if $G$ is planar, then $G/e$ is also planar.



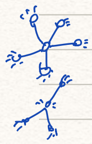




proof of 5-colour thm. strong induction on the number of vertices $n$.

Base case: Any planar graph with at most 5 vertices is 5-colourable after contraction, planarity needs to hold.

I.H. Assume any planar graph with fewer than $n$ vertices is 5-colourable.

I.S. Suppose $G$ is a planar graph with $n$ vertices.



Let $v$ be a vertex of degree at most 5.

if $\deg(v) \leq 4$, then apply the argument from 6-color theorem to prove that it is 5-colourable.

Assume that $\deg(v) = 5$. There exist two neighbours $x, y$ of $v$ that are not adjacent. for otherwise $G$ contains $K_5$ which is not planar.

Let $G'$ be obtained from $G$ by contracting $vx, vy$. Let $v^*$ be the contracted vertex.

Now $G'$ is planar with $n-2$ vertices. By induction $G'$ is 5-colourable.

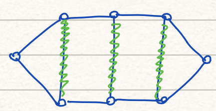
We keep the same colours for $G$ except $x, y$ receive the colour of $v^*$ (this is possible since $x, y$ are not adjacent) So the neighbour of $v$ use at most 4 colours.

Since there are 5 colours at least 1 is available for $v$.

$\square$

Theorem. Every planar graph is 4-colourable. ?!?!?! (compute prove :))

## Matching

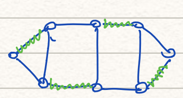


A matching in $G$ is set of edges where no two edges share a common $\boxed{\text{vertex}}$ (one edge also a matching)

(each vertex is incident with at most one edge in a matching)

~~ matching (empty set is a matching)

General Q: what is the maximum size of a matching in a graph?



~~ perfect matching. (matching every single vertex)

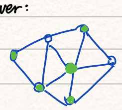
perfect match ⇒ maximum matching

Defn: A vertex incident within an edge in a matching is saturated. Otherwise It is unsaturated.

A matching that saturates every vertex is a perfect matching.

(odd graph does not have perfect matching)

Cover:



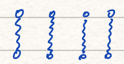
• → cover

Defn: A cover $C$ of a graph $G$ is a set of vertices where every edge in $G$ has at least one endpoint in $C$.

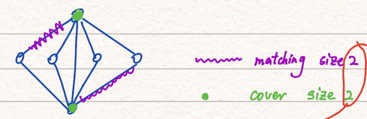
General Q: what is the minimum size of a cover in a graph?

## Matching vs covers

Suppose $M$ is a matching and $C$ is a cover

e.g.  $\Rightarrow$ minimum # cover = 4 (sharing no common vertex (matching))

$|C| \geq |M|$




**Thm:** If $M$ is a matching and $C$ is a cover of $G$, then $|M| \leq |C|$. (if $|M| = |C|$, then we find the max matching & min cover)

proof:

For each edge $uv$ in $M$, at least one of $u$ or $v$ in $C$. Since edges in $M$ do not share common vertices, $C$ must contain at least $|M|$ distinct vertices. □

**Corollary:** if $M$ is matching and $C$ is a cover of $G$ where $|M| = |C|$, then $M$ is a maximum matching and $C$ is a minimum cover.

proof: Let $M'$ be any matching. Then $|M'| \leq |C| = |M|$, so $M$ is maximum.
Let $C'$ be any cover. Then $|C'| \geq |M| = |C|$, so $C$ is minimum. □

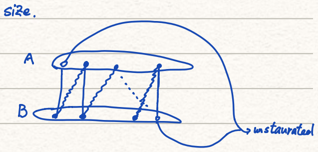
In general, size of max matching might not be equal to size of min. cover. e.g. 

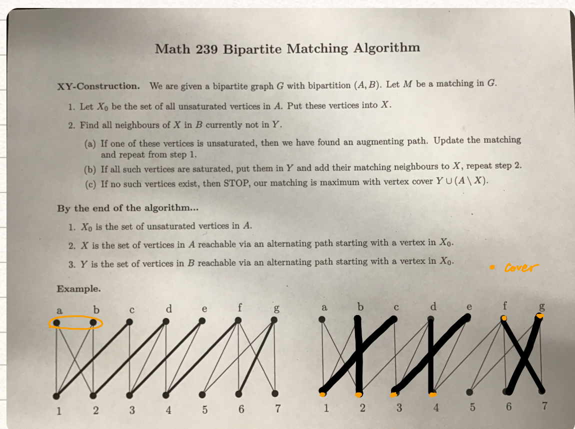
## König's Theorem:

In a bipartite graph, the size of a maximum matching is equal to the size of a minimum cover.

idea: attempt to find an argumenting path

if find one, update matching. if fail to find one, the matching is maximum and find a cover of same size.



### Math 239 Bipartite Matching Algorithm

**XY-Construction.** We are given a bipartite graph $G$ with bipartition $(A, B)$. Let $M$ be a matching in $G$.

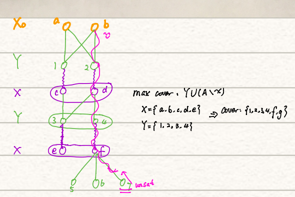
1. Let $X_0$ be the set of all unsaturated vertices in $A$. Put these vertices into $X$.
2. Find all neighbours of $X$ in $B$ currently not in $Y$.
   (a) If one of these vertices is unsaturated, then we have found an augmenting path. Update the matching and repeat from step 1.
   (b) If all such vertices are saturated, put them in $Y$ and add their matching neighbours to $X$, repeat step 2.
   (c) If no such vertices exist, then STOP, our matching is maximum with vertex cover $Y \cup (A \setminus X)$.

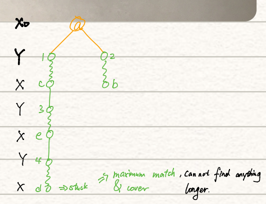
**By the end of the algorithm...**

1. $X_0$ is the set of unsaturated vertices in $A$.
2. $X$ is the set of vertices in $A$ reachable via an alternating path starting with a vertex in $X_0$.
3. $Y$ is the set of vertices in $B$ reachable via an alternating path starting with a vertex in $X_0$.

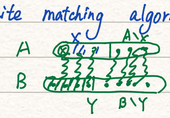
**Example.**






Proof: Suppose $M, X_0, X, Y$ are sets we get at the end of the bipartite matching algorithm




and $B \setminus Y$.

We see that there is no edge between $X$ and $B \setminus Y$. Since the algorithm would discover this edge and place the vertex in $B \setminus Y$ in $Y$.

Thus we claims that $Y \cup (A \setminus X)$ is a cover

Now every vertex in $Y$ is saturated, for otherwise the algorithm finds an augmenting path, and would have continued.

also every vertex in $A \setminus X$ is saturated, since all unsaturated vertices in $A$ are in $X_0$

No matching edges joins $Y$ to $A \setminus X$ (since matching neighbor of $Y$ are in $X$)

So each vertex in $Y \cup (A \setminus X)$ is saturated by a distinct matching edges.

Then $|M| = |Y \cup (A \setminus X)|$ □

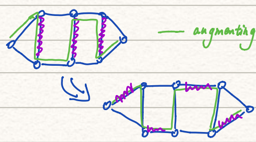
**Corollary:** A bipartite graph $G$ with $m$ edges and maximum degree $d$ has a matching of size at least $m/d$.

proof: we just need to prove that every cover has size at least $m/d$ (by Konig's Thm)

Let $C$ be a cover. Each vertex in $C$ covers at most $d$ edges. Then $C$ covers at most $d|C|$ edges.

But $d|C| \geq m$. So $|C| \geq \frac{m}{d}$ □

Augmenting paths:



— augmenting path (both end with unsaturated vertices) ⇒ matching edge & unmatching edge appear alternatively.

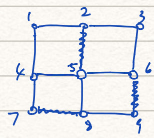
(switch edges in M)

if not finding augmenting path
⇒ we claim that it is the maximum matching

Defn: An alternating path $P$ with respect to a matching $M$ is a path where consecutive edges alternate between in $M$ and not in $M$

An alternating path is an alternative path that starts and ends with different unsaturated vertices (the first & last one are not in $M$)

e.g.




1, 4 is an alternating path

☆ If an augmenting path exist, then there is a larger matching.

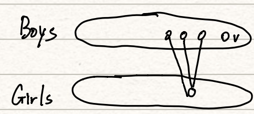
⇒ aug. path


⇒ 1 + even + 1 ⇒ even # of vertices ⇒ odd path. ⇒ switch get one larger.
↳ if $e \in M$, remove $e$ from; if $e \notin M$, add $e$ to $M$

Hall's theorem:

When does a max matching exist that saturates all of A?
What prevents us from saturating all of A?

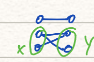
|Boys| > |girls|




$\deg(v) = 0$

$X \subseteq$ Boys. $|N(X)| < |X|$ ↳ neighbour of $X$.

Defn: Let $X$ be a set of vertices in $G$. The neighbour set $N(X)$ is the set of all vertices in $G$ adjacent to at least one vertex in $X$.




Hall's theorem. Let $G$ be bipartite with bipartition $(A, B)$. Then $G$ has a matching that saturates every vertex in $A$ iff for all $D \subseteq A$, $|N(D)| \geq |D|$

proof: ⇒ suppose $M$ is the matching that saturates $A$. Let $D \subseteq A$. Then each vertex in $D$ is matched to a distinct vertex in $N(D)$ using $M$. So $|N(D)| \geq |D|$

⇐ suppose a maximum matching $M$ does not saturate every vertex in $A$. Then $|M| < |A|$

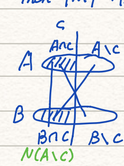
(Goal: Find a $D \subseteq A$. where $|N(D)| < |D|$).

By König's Theorem. there exists a cover $C$ where $|M| = |C|$

Since $C$ is a cover, there is no edge between $A \setminus C$ and $B \setminus C$.

So $N(A \setminus C) \subseteq B \setminus C$. $|C| = |A \cap C| + |B \cap C| = |M| < |A|$

Then $|N(A \setminus C)| \leq |B \cap C| < |A| - |A \cap C| = |A \setminus C|$. So $|N(A \setminus C)| < |A \setminus C|$ □




Corollary: A $k$-regular bipartite graph with $k \geq 1$ has a perfect matching.

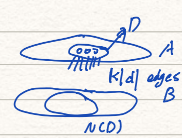
proof: Suppose $G$ has bipartition $(A, B)$. Let $D \subseteq A$

There are $k|D|$ edges incident with $D$, and the other endpoints are in $N(D)$.

There are $k \cdot |D|$ edges incident with $N(D)$

So $k|D| \leq k|N(D)|$, since $k \geq 1$. thus $|D| \leq |N(D)|$

By Hall's theorem, there exist a matching that saturates $A$. So $B$ is also saturated. Hence it is a perfect matching.




Corollary: Any $k$-regular bipartite graph can be partitioned into $k$ perfect matching.