

# CS360 – Theory of Computing

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Important Notes:

- Computational problems and devices can be viewed as mathematical object
- Computation is very general
  - Computer running program
  - Networks of computers interacting
  - People performing computation by hand
  - Proofs of theorem
  - Biological process

Mathematical foundation:

- Set theory

$$\begin{cases} Naive\ theory \\ Axiomatic\ theory \end{cases}$$

Some notation:

1. TFAE = **T**he **F**ollowing **A**re **E**quivalent

## Lecture 1:

**Defn:** A set is countable if

- A is empty
- There exist an onto function  $f$  of this form:

$$f : \mathbb{N} \rightarrow A$$

TFAE:

1. A is countable.
2. There exist a one-to-one function  $g$  of the form:

$$g : A \rightarrow \mathbb{N}$$

3. Either A is finite or there exist a one-to-one onto function  $h$  of the form:

$$h : \mathbb{N} \rightarrow A$$

### Terminology:

If A is a set, then the power set  $P(A)$  is the **set** of all subsets of A.

### Cantor Theorem:

The power set of natural number is not countable.

*i.e.  $P(\mathbb{N})$  is not countable.*

Prove by contradiction:

Assume towards contradiction that  $P(N)$  is countable.

This implies that there exist an onto function  $f$ , such that

$$f : \mathbb{N} \rightarrow P(N)$$

Consider we define a set  $S = \{n \in \mathbb{N} : n \notin f(n)\}$

Since  $f$  is onto function, there must exist some number  $n \in N$ , such that  $S = f(n)$

We fix such a choice of  $n$ ! (i.e.  $S = f(n)$ )

We get

$$n \in S \Leftrightarrow n \notin f(n) \Leftrightarrow n \notin S$$

Thus it is a contradiction. Therefore  $P(\mathbb{N})$  is not countable.

## Lecture 2:

Terminologies:

Alphabet – symbols for using computation

*An alphabet is a finite and nonempty set.*

Typical names:  $\Sigma$ ,  $\Gamma$

Examples:

$\Sigma = \{0, 1\} \rightarrow$  *binary alphabet*

$\Sigma = \{0\} \rightarrow$  *unary alphabet*

$\Sigma = \{0, 1, 2, \dots, n-1\}$  *for some possible integer  $n$*

String – over some alphabet

*Let  $\Sigma$  be an alphabet, the string over  $\Sigma$  is a finite sequence of symbols from  $\Sigma$ .*

Typical variables names for symbols: a, b, c, d

Typical string names: u, v, w, x, y, z

Examples:

*under  $\Sigma = \{0, 1\}$  we have 0101, 0, 1,  $\epsilon$*

*under  $\Sigma = \{0, 1, 2\}$  we have 01, 1, 0*

Language

*Let  $\Sigma$  be an alphabet. A language over  $\Sigma$  is a set of strings over  $\Sigma$ .  $\Sigma^*$  language consisting of **EVERY** string over  $\Sigma$ .*

Typical names: A, B, C, D

Example(assume  $\Sigma = \{0, 1\}$ ):

$B = \{w \in \Sigma^* : w \text{ ends with } 0\}$

$C = \{w \in \Sigma^* : |w| \text{ is a prime number}\}$

### Question:

Consider  $\Sigma^*$ . Is it countable?

Yes, it is!

Consider we can find an onto function  $f$

$$f : \mathbb{N} \rightarrow \Sigma^*$$

We arrange the all the strings in  $\Sigma^*$  by its length.

i.e.  $\epsilon, 0, 1, 00, 01, 10, 11, \dots$

The sequence above has **lexicographic order**

Then we set the function to be:

$$f(0) = \epsilon$$

$$f(1) = 0$$

$$f(2) = 1$$

$$f(3) = 00$$

$\dots$

Thus we can form an onto function that is able to count all the strings.

Therefore  $\Sigma^*$  is countable.

Consider  $\Sigma = \{0, 1\}$ , how many languages over  $\Sigma$  are there?

TFAE:

1.  $A$  is language over  $\Sigma$
2.  $A \subset \Sigma^*$
3.  $A \in P(\Sigma^*)$

The answer is No. We can prove this by using Cantor Theorem.

From the previous questions, we know that there is an **one-to-one function**  $f$  that can map from  $\Sigma^*$  to  $\mathbb{N}$ .

$$i.e. f : \Sigma^* \rightarrow \mathbb{N}$$

Assume towards contradiction that  $P(\Sigma^*)$  is countable.

Therefore we can find an onto function, say  $g$ , that

$$g : \mathbb{N} \rightarrow P(\Sigma^*)$$

Since we can map  $\mathbb{N}$  to  $\Sigma^*$ , thus we can find an onto function, say  $h$ , such that

$$h : \mathbb{N} \rightarrow P(\mathbb{N})$$

That contradicts Cantor's Theorem.

Therefore we get the languages over  $\Sigma$  is uncountable.

## Deterministic Finite Automata (DFA)

### Definition:

A DFA is a 5 tuple  $m = (Q, \Sigma, \delta, q_0, F)$  where:

- $Q$  is a finite nonempty set of states.
- $\Sigma$  is an alphabet.
- $\delta$  is the transition function of the form

$$\delta : Q \times \Sigma \rightarrow Q$$

- $q_0$  is the starting state.
- $F$  is the set of accepting states.

### Important notes:

A DFA needs to accept all the strings in one language AND rejects all the strings that are not in that language.

### Formal Definition of DFA

Let  $M$  be a DFA and let  $w \in \Sigma^*$ . The DFA  $m$  accepts  $w$  if one of these following holds:

1.  $w = \epsilon$  and  $q_0 \in F$
2.  $w = a_1a_2\cdots a_n$  for  $n \geq 1$  and  $a_1, a_2, \dots, a_n \in \Sigma$  and there exist a sequence of states  $r_0, r_1, \dots, r_n \in Q$  such that  $r_0 = q_0$ ,  $r_n \in F$  and  $r_{k+1} = \delta(r_k, a_{k+1})$  for every  $k \in \{0, 1, 2, \dots, n-1\}$ . If  $m$  does not accept, then it rejects  $w$ .

### Convention Function:

Suppose  $m = (Q, \Sigma, \delta, q_0, F)$  is a DFA.

Define  $\delta^* : Q \times \Sigma^* \rightarrow Q$  as follows:

- $\delta^*(q, \epsilon) = q$
- $\delta^*(q, wa) = \delta(\delta^*(q, w), a)$

**Notation:**

$L(M) = \{w \in \Sigma^* : M \text{ accepts } w\} \rightarrow$  "language recognized by M"

**Regular Language Definition:**

Let  $\Sigma$  be an alphabet and let  $A$  be a language over  $\Sigma$ .  $A$  is regular if there exist a DFA such that  $L(M) = A$

Consider fixed  $\Sigma = \{0, 1\}$ , how many regular language over  $\Sigma$ ?

The answer is countable many.

Consider the number of states.

Similar to the approach above.

We can list all the DFA by counting their states.

Then we can claim that there exist an onto function  $f$  that maps from  $\mathbb{N}$  to all the DFA over  $\Sigma$ .

Thus we know that DFA is countable.

Therefore by the definition above, we know regular language is countable.

## Lecture 3:

**Non-Deterministic finite automatas(NFAs)**

— verification whether computer can get the result by trying multiple ways.

**Defn:**

NFA is a 5-tuple:

$$N = (Q, \Sigma, \delta, q, F)$$

where

- $Q$  is a finite and non-empty set of states
- $\Sigma$  is a transition function of the form

$$\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow P(Q)$$

- $q_0 \in Q$  is a start state
- $F \subset Q$  is a set of accept states

### Formal Definition

Let  $N = \{Q, \Sigma, \delta, q_0, F\}$  be an NFA and let  $w \in \Sigma^*$ .  $N$  accepts  $w$  if there exist  $m \in \mathbb{N}$ , a sequence of states,  $r_0 \cdots r_m \in Q$  and a sequence of alphabet (move),  $a_1, \cdots, a_m \in \Sigma \cup \{\epsilon\}$

- $r_0 = q_0, r_m \in F$
- $r_{k+1} \in \delta(r_k, a_{k+1}), \forall k \in \{0, 1, \cdots, m-1\}$
- $w = a_1 a_2 \cdots a_m$  If  $N$  does not accept  $w$ , then rejects  $w$

Now we need to check the base case when  $m = 0$ .

Consider  $m = 0$ , then we have

$$\begin{cases} r_0 \cdots r_m \rightarrow r_0 \in F \\ a_1 \cdots a_0 \rightarrow \text{empty sequence} \end{cases}$$

From above, we get the starting state is in the set of accepting states.

Therefore, we know the condition satisfy all situations.

### $\epsilon$ - closure

Let  $R \subseteq Q$  and let  $N = \{Q, \Sigma, \delta, q_0, F\}$  be an NFA. The  $\epsilon$  - closure of  $R$  denoted  $\epsilon(R)$  is the set of all states that can be reached from any states in  $R$  by following 0 or more  $\epsilon$  transition.

$$\epsilon : P(Q) \rightarrow P(Q)$$

Define  $\delta^*$  for an NFA

$$\begin{aligned} \delta^*(p, \epsilon) &= \epsilon(\{p\}) \\ \delta^*(p, wa) &= \epsilon(\cup_{r \in \epsilon^*(p, w)} \epsilon(r, a)) \end{aligned}$$

**Theorem:**

Let  $\Sigma$  be an alphabet,  $A \subseteq \Sigma^*$  be a language.

The language  $A$  is regular iff there exist an NFA,  $N = \{Q, \Sigma, \Delta, q_0, F\}$  such that  $A = L(N)$   
proof:

$\Rightarrow$  given that  $A$  is regular, then there exist a DFA  $M = (Q, \Sigma, \Delta, q_0, F)$  with  $A = L(N)$

Define  $N = (Q, \Sigma, \gamma, q_0, F)$  where

$$\begin{aligned}\gamma(q, a) &= \{\delta q, a\} \\ \gamma(q, \Sigma) &= \emptyset \quad \forall q \in Q, a \in \Sigma\end{aligned}$$

In this case,  $L(N)=L(M)=A$ .

$\Leftarrow$  given that NFA,  $N = (Q, \Sigma, \delta, q_0, F)$

idea: we will construct a DFA  $m$ , such that  $L(m) = L(N)$ . We will take the state set of  $m$  to be  $P(Q)$ .

define  $m$  as follows:

$$m = (P(Q), \Sigma, \gamma, \epsilon(\{q\}), G)$$

where  $\gamma$  and  $G$  are as follows.

$\gamma : P(Q) \times \Sigma \rightarrow P(Q)$  where  $\gamma(R, a) = \epsilon(\cup_{r \in R} \delta(r, a))$  for  $R \in P(Q), a \in \Sigma$

$G = \{R \in P(Q) : R \cap F \neq \emptyset\}$



## Lecture 4:

### Regular Operation:

Let  $A, B \subseteq \Sigma^*$  be language over a alphabet  $\Sigma$

The regular operation is:

1. Union:  $A \cup B = \{w : w \in A \text{ or } w \in B\}$
2. Concatenation:  $AB = \{wx : w \in A \text{ and } x \in B\}$
3. Kleene star:  $A^* = \{\epsilon\} \cup A \cup AA \cup AAA \dots$

### Theorem:

Let  $\Sigma$  be an alphabet and let  $A, B \subseteq \Sigma^*$  be regular language. The language  $A \cup B, AB, A^*$  are regular.

Proof:

Consider let  $M_a = (P, \Sigma, \delta, p_0, F)$  and let  $M_b = (Q, \Sigma, \gamma, q_0, G)$  be DFAs with  $A = L(M_a), B = L(M_b)$ . WLOG, let  $P \cap Q = \emptyset$

Consider  $A \cup B$ , we define a NFA as follows:

$$N = (P \cup Q \cup \{r_0\}, \Sigma, \mu, r_0, F \cup G)$$

where,

$$\begin{aligned}\mu(p, a) &= \{\delta(p, a)\} && \text{for } p \in P, a \in \Sigma \\ \mu(p, \epsilon) &= \emptyset && \text{for } p \in P, a \in \Sigma \\ \mu(q, a) &= \{\gamma(q, a)\} && \text{for } q \in Q, a \in \Sigma \\ \mu(q, \epsilon) &= \emptyset && \text{for } q \in Q, a \in \Sigma \\ \mu(r_0, \epsilon) &= \{q_0, p_0\} && \text{for } a \in \Sigma \\ \mu(r_0, a) &= \emptyset && \text{for } a \in \Sigma\end{aligned}$$

The proof for  $AB, A^*$  is similar.

### Questions:

$A \subseteq \Sigma^*$ , define  $\bar{A} = \Sigma^* \setminus A = \{x \in \Sigma^* : x \notin A\}$

If  $A$  is regular is  $\bar{A}$  regular?

Yes, by swapping all accepting state with non-accepting states.

Then we can form a new DFA/NFA, so we know  $\bar{A}$  is regular.

If  $A, B \subseteq \Sigma^*$  are regular is  $A \cap B$  regular?

$$m = (P \times Q, \Sigma, \mu, (p_0, q_0), F \times G)$$

$$\mu((p, q), a) = (\delta(p, a), \gamma(q, a))$$

Another way is using DeMorgan's Laws.

$$A \cap B = \overline{\bar{A} \cup \bar{B}}$$

## Regular Expression:

**Defn:** Let  $\Sigma$  be an alphabet.  $R$  is a regular expression over  $\Sigma$  if one of these following is true:

1.  $R = \emptyset$
2.  $R = \epsilon$
3.  $R = a$  for  $a \in \Sigma$
4.  $R = (R_1, R_2)$  for  $a \in \Sigma$
5.  $R = (R_1 R_2)$  for  $R_1, R_2$  regular expressions
6.  $R = (R_1^*)$  for  $R_1$  regular expressions

Let  $R$  be regular expression over  $\Sigma$ . The language recognized or method by  $R$  is defined as follows:

1. If  $R = \emptyset$ , then  $L(R) = \emptyset$
2. If  $R = \epsilon$ , then  $L(R) = \{\epsilon\}$
3. If  $R = a$  for  $a \in \Sigma$ , then  $L(R) = \{a\}$
4. If  $R = (R_1 \cup R_2)$ , then  $L(R) = L(R_1) \cup L(R_2)$
5. If  $R = (R_1 R_2)$ , then  $L(R) = L(R_1) L(R_2)$
6. If  $R = (R_1^*)$ , then  $L(R) = L(R_1)^*$

Order of precedence:

1. Kleene star
2. Concatenation
3. Union

**Theorem:** Let  $\Sigma$  be an alphabet and let  $A \subseteq \Sigma^*$  be a language over  $\Sigma$ .  $A$  is regular if and only if there exist a regular expression  $R$  with  $L(R) = A$ .

## Lecture 5

### Lemma: Pumping Lemma

Let  $\Sigma$  be an alphabet and let  $A \subseteq \Sigma^*$  be a regular language. There exist a positive integer (called a pumping length of  $A$ ) that possesses the following property. For every string  $w \in A$  with  $|w| \geq n$ . It is possible to write  $w = xyz$  for some choice of strings  $x, y, z \in \Sigma^*$  such that

1.  $y \neq \epsilon$
2.  $|xy| \leq n$
3.  $xy^iz \in A$ , for every  $i \in \mathbb{N}$

### Proof:

Let  $m = (Q, \Sigma, \delta, q_0, F)$  be a DFA with  $A = L(m)$ , and let  $n = |Q|$

If there is no strings  $w \in A$ , then the statement we need to prove is vacuously true.

Suppose  $w \in A$  and  $|w| \geq n$

Then we have  $w = a_1a_2 \cdots a_m$  where  $m \geq n$ . Therefore they must exist states  $r_0, r_1, \dots, r_m \in Q$  with  $r_0 = q_0, r_m \in F$ , and  $r_{k+1} = \delta(r_k, a_{k+1})$

Consider  $r_0, r_1, \dots, r_n$

By the pigeon hole principle, there must exist  $s, t \in \{0, 1, \dots, n\}$ , with  $s < t$  and  $r_s = r_t$

Let  $x = a_1a_2 \cdots a_s$ ,  $y = a_{s+1} \cdots a_t$  and  $z = a_{t+1} \cdots a_m$

We have  $y \neq \epsilon$ , since  $s < t$ , then item one is proved.

We have  $|xy| = t \leq n \Rightarrow$ , since the definition is claimed above.

We have  $xy^iz$  for any  $i \in \mathbb{N}$  must be accepted.

Consider the sequence,  $r_0, r_1, \dots, r_s, r_{s+1}, \dots, r_t, r_{t+1}, \dots, r_m$ , we can repeat  $i$  times, since  $r_s = r_t$ .

Proposition: Let  $\Sigma = \{0, 1\}$  and let  $A = \{0^m1^m : m \in \mathbb{N}\}$ , we want to show that  $A$  is non-regular.

Assume towards contradiction  $A$  is regular. By the pumping lemma, there exist a pumping length  $n \geq 1$  for  $A$ , such that the statements in that lemma are all true.

Fix such a choice of  $n$ .

Define  $w = 0^n1^n$

It is the case that  $w \in A$  and  $|w| = 2n \geq n$ . Therefore, by pumping lemma, there must exist  $x, y, z$  such that it satisfies 3 items.

We know that  $w = xyz$  and  $w = 0^n1^n$ , where  $|xy| \leq n$

This implies that  $y = 0^k$  for some  $k \in \mathbb{N}$

Since  $y \neq \epsilon$ , then  $k \geq 1$

Consider  $i = 2$ , in third item, then we have  $xy^2z = xyyz = 0^{n+k}1^n \in A$

However,  $0^{k+n}1^n \notin A$ , **contradiction!** Therefore, we have  $A$  is non-regular.

Normally, we can use Pumping lemma to prove a language is non-regular. However, there is also another way to prove as well.

Firstly, we need to introduce some concepts.

**Definition:**

$w^R$  is the reverse of  $w$ ,  $(aw)^R = w^R a$  for  $a \in \Sigma$  and  $w \in \Sigma^*$

We can simply prove the reverse of  $w$  is non-regular by using  $0^n 10^n$

But when comes to  $E = \{w \in \Sigma^* : w \neq w^R\}$ , we can easy to prove by using contradiction.

Then we assume towards contradiction  $E$  is regular, then we know that  $\bar{E}$  is regular (proved before), however we know that  $\bar{E} \equiv w = w^R$ , which is non-regular **contradiction!**

On the other hand, we can prove the above proposition in a easy way.

Consider  $F = \{w \in \Sigma^* : |w|_0 = |w|_1\}$

Assume  $F$  is regular,  $F \cap L(0^*1^*) = A$

We know that  $L(0^*1^*)$  is regular and  $A$  is non-regular. If  $F$  is regular, then we know that  $F \cap L(0^*1^*)$  is regular, which is not.

Therefore, we get  $F$  is non-regular.

## Lecture 6:

Proposition: Let  $\Sigma$  be an alphabet and let  $A \subseteq \Sigma^*$  be a regular language.

The language  $A^R = \{w^R : w \in A\}$  is regular.

**Proof:**

Consider there exist a DFA  $M = (Q, \Sigma, \delta, q_0, F)$ , such that  $L(M) = A$

We can construct an NFA  $N$ , such that  $L(N) = A^R$

Consider  $N = (Q \cup \{r_0\}, \Sigma, \eta, r_0, \{q_0\})$ , where

$$\begin{cases} \eta(r_0, \epsilon) &= F \\ \eta(r_0, q) &= \emptyset \\ \eta(q, \epsilon) &= \emptyset \\ \eta(q, a) &= \{p \in Q : \delta(p, a) = q\} \end{cases}$$

A is a regular language, there exist a regular expression  $S$ ,  $L(R) = A$   
Define the reverse of a regular expression in the natural way  $S$

$$L(S^R) = L(S)^R = A^R$$

The reverse of a regular expression  $S$  can be defined like this:

$$\begin{cases} 1. \emptyset^R = \emptyset \\ 2. \epsilon^R = \epsilon \\ 3. a^R = a \\ 4. S = (S_1 \cup S_2) \Rightarrow S^R = (S_1 \cup S_2)^R = (S_1^R \cup S_2^R) \\ 5. S = (S_1 S_2) \Rightarrow S^R = (S_1 S_2)^R = (S_2^R S_1^R) \\ 6. S = S_1^* \Rightarrow S^R = ((S_1^R)^*) \end{cases}$$

### Symmetric Difference

Consider  $A, B \subseteq \Sigma^*$  be language. Define  $A \triangle B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$

1. If  $A, B \subseteq \Sigma^*$  are regular, then  $A \triangle B$  is regular?

YES

Proof: we can prove this by using closure property and cartesian product.

TODO: NEED TO PROVE

2. If  $A, B \subseteq \Sigma^*$  are non-regular, then  $A \triangle B$  is non-regular?

NO

Proof: we can prove this by providing counter-example.

Consider  $A = B = \{0^n 1^n : n \in \mathbb{N}\}$

Then we have  $A \triangle B = \emptyset$  that is regular.

3. If  $A$  is regular,  $B$  is non-regular, then  $A \triangle B$  is non-regular.

YES

Proof: we can assume  $A \triangle B$  is regular

Then  $(A \triangle B) \triangle A = B$  is regular that is contradiction.

### Prefix, Suffix and Substring

Consider  $A \subseteq \Sigma^*$  be any language. We define:

$Prefix(A) = \{x \in \Sigma^*, \exists v \in \Sigma^* : xv \in A\}$

$Suffix(A) = \{x \in \Sigma^*, \exists v \in \Sigma^* : vx \in A\}$

$Substring(A) = \{x \in \Sigma^*, \exists u, v \in \Sigma^* : uxv \in A\}$

Let  $M = (Q, \Sigma, \delta, q_0, F)$  be a DFA with  $L(M) = A$

Prefix:

Define  $P \subseteq Q$  to be the set of all states from which we reach to accepting states.

$$P = \{q \in Q, \exists v \in \Sigma^* \text{ such that } \delta^*(q, v) \in F\}$$

We define  $K_1 = (Q, \Sigma, \delta, q_0, P)$  then  $L(K_1) = prefix(A)$

Suffix:

Define  $N = (Q \cup \{r_0, \}, \Sigma, \eta, r_0, F)$ , where

$$\begin{cases} \eta(q, a) &= \{\delta(q, a)\} \\ \eta(q, \epsilon) &= \emptyset \\ \eta(r_0, a) &= \emptyset \\ \eta(r_0, \epsilon) &= \{q \in Q : \exists a \in \Sigma^*, \delta^*(q_0, a) = q\} \end{cases}$$

Substring:

We can use closure property say that  $substring(A) = prefix(suffix(A))$  or  $substring(A) = suffix(prefix(A))$

**Prove regularity by constructing a regular language:**

Define  $M_{p,q} = (Q, \Sigma, \delta, q_0, F)$  be any DFA.

$A_{p,q} = L(M_{p,q}) \rightarrow$  any string starts from symbol  $p$  and ends with symbol  $q$

Consider  $B = \{uv : uv \in \{0, 1\}^*, uav \in A \text{ for some } a \in \{0, 1\}\}$

$$\bigcup_{r \in Q, a \in \{0, 1\}, q \in F} A_{q_0, r} A_{\delta(r, a), q} = B$$

Consider  $C = \{uav : uv \in \{0, 1\}^*, uav \in A \text{ for some } a \in \{0, 1\}\}$

$$\bigcup_{r \in Q, a \in \{0, 1\}, q \in F} A_{q_0, r} \{a\} A_{\delta(r, q)} = C$$