The Master Method

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The "Master Theorem" provides a formula for the solution for many recurrence relations. Suppose that $a \geq 1$ and $b > 1$. Consider the recurrence

$$T(n) = aT(\frac{n}{b}) + \Theta(n^y)$$

in sloppy or exact form. Denote $x = log_b^a$. Then

$$T(n) \in \begin{cases} \Theta(n^x) & \text{if } y < x \\ \Theta(n^x logn) & \text{if } y = x \\ \Theta(n^y) & \text{if } y > x \end{cases}$$

The Master Theorem is really similar to Merge sort ($\Theta(nlogn)$).
Suppose that $a \geq 1$ and $b \geq 2$ are integers and

$$T(n) = aT(\frac{n}{b}) + cn^y, \quad T(1) = d$$

Table 1: Solution table:

Size of subproblem	**# nodes**	**cost/node**	**total cost**
$n = b^j$	1	cn^y	cn^y
$\frac{n}{b} = b^{j-1}$	a	$c(\frac{n}{b})^y$	$ca(\frac{n}{b})^y$
$\frac{n}{b^2} = b^{j-2}$	a^2	$c(\frac{n}{b^2})^y$	$ca^2(\frac{n}{b^2})^y$
$\cdots$	$\cdots$	$\cdots$	$cdots$
$\frac{n}{b^{j-1}} = b$	a^{j-1}	$c(\frac{n}{b^{j-1}})^y$	$ca^{j-1}(\frac{n}{b^{j-1}})^y$
$\frac{n}{b^j} = 1$	a^j	d	da^j

Proof:
Consider $a^j = (b^x)^j = (b^j)^x = n^x$
Then we have

$$\begin{aligned} & da^j + cn^y \sum_{i=0}^{j-1} (\frac{a}{b^y})^i \\ &= dn^x + cn^y \sum_{i=0}^{j-1} r^i \text{ where } r = \frac{a}{b^y} \end{aligned}$$

Consider $r = \frac{a}{b^y} = \frac{b^x}{b^y} = b^{x-y}$
Let

$$S = \sum_{i=0}^{j-1} r^i \in \Theta(r^j)$$

Since $1 + r + r^2 + \cdots + r^{j-1} = \frac{r^j}{j-1} \in \Theta(r^j)$
Thus $r^j = (b^{x-y})^j = (b^j)^{x-y} = n^{x-y}$
Therefore we have

$$T(n) = dn^x + cn^y n^{x-y}$$

$$\begin{cases} \text{If } r > 1 \ (i.e. x > y) & \rightarrow \Theta(dn^x + cn^y n^{x-y}) \in \Theta(n^x) \\ \text{If } r = 1 \ (i.e. x = y) & \rightarrow \Theta(dn^x + cn^y \sum_{i=0}^{j-1} r^i) \in \Theta(dn^x + cn^y j) \\ & \rightarrow \Theta(dcj(n^x)) \in \Theta(n^x(logn)) \\ \text{If } r < 1 \ (i.e. x < y) & \rightarrow \Theta(dn^x + cn^y) \in \Theta(n^y) \end{cases}$$