Cognitive Algorithms Assignment 1

Part 1 - Linear Algebra Recap

Due on Tuesday, May 2, 2017 10am via ISIS

Task 1 (8 points)

1. Compute the scalar product of the following vectors $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$.

   ☑ 3   ☐ 5   ☐ 7

2. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be two column vectors. Which of the following statements is always true?

   ☐ $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

   ☑ $\mathbf{v}^T \cdot \mathbf{w} = \mathbf{w}^T \cdot \mathbf{v}$

   ☐ $\mathbf{v} \cdot \mathbf{w}^T = \mathbf{w} \cdot \mathbf{v}^T$

3. The mapping $f : \mathbb{R}^2 \ni (x, y)^\top \mapsto (x + y, y - x)^\top \in \mathbb{R}^2$ is given by the following matrix:

   ☐ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

4. Which property does matrix multiplication *not* have?

   ☑ Associativity: $(AB)C = A(BC)$

   ☐ Commutativity: $AB = BA$

   ☑ Distributivity: $(A + B)C = AC + BC$

5. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ two column vectors with $A \cdot \mathbf{v} = \mathbf{w}$. Which of the following statements is always true?

   ☐ $A = \mathbf{w} \cdot \mathbf{v}^{-1}$ ☐ $\mathbf{v} = \mathbf{w} \cdot A^{-1}$ ☑ $\mathbf{v} = A^{-1} \cdot \mathbf{w}$

6. The rank of the matrix $\begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}$ is

   ☑ 1   ☐ 3   ☐ 4

7. For a square $n \times n$ matrix A holds

   ☐ $\operatorname{rank} A = n \Rightarrow A$ is invertible, but there are invertible A with $\operatorname{rank} A \neq n$

   ☐ A is invertible $\Rightarrow \operatorname{rank} A = n$, but there are A with $\operatorname{rank} A = n$, which are not invertible.

   ☑ $\operatorname{rank} A = n \Leftrightarrow A$ is invertible

8. Which of the following matrices is orthogonal:

☑ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Task 2 (3 points)

We consider two functions f and g which transform an input vector $\mathbf{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ into a scalar: $f(\mathbf{x}) = \mathbf{u}^\top \mathbf{x}$, $\mathbf{u} = (u_1, \ldots, u_d)^\top \in \mathbb{R}^d$ and $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x}$.

1. Compute the partial derivative of f and g with respect to one entry x_j ($j \in \{1, 2, \ldots, d\}$)
   (a) $\frac{\partial f(\mathbf{x})}{\partial x_j} =$ uj

   (b) $\frac{\partial g(\mathbf{x})}{\partial x_j} =$ 2xj

2. Compute the gradient $\nabla f(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_d}\right)^\top$ for f and g.
   (a) $\nabla f(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_d}\right)^\top =$ (u1,u2,...ud)

   (b) $\nabla g(\mathbf{x}) = \left(\frac{\partial g(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial g(\mathbf{x})}{\partial x_d}\right)^\top =$ (x1,x2,...,xd)

Task 3 (4 points)

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ Eigenvectors of A corresponding to Eigenvalues $\lambda, \mu \in \mathbb{R}$, with $\lambda \neq \mu$. (Recall the definition of Eigenvectors: $A\mathbf{v} = \lambda\mathbf{v}$ and $A\mathbf{w} = \mu\mathbf{w}$)
Show: $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, i.e. $\mathbf{v}^\top \mathbf{w} = 0$.
Hint: $\lambda \mathbf{v}^\top \mathbf{w} = \ldots$.

λv’w = (λv)’w = (Av)’w = (Av)’w = (v’A’)w

because of A is a symmetric matrix, A = A’

We can get λv’w = (v’A)w

μv’w = (Aw)v’

we can get λv’w = μv’w , so v’w = 0

shows the v and w are orthogonal

v' means the transpose of v