Robust social planner with a single capital stock: notes

March 29, 2018

1 Adjustment cost model

Let K_t be capital and I_t be investment and C_t be capital. Suppose that

$$K_{t+1} = K_t \exp\left[\alpha_k + \left(\frac{I_t}{K_t}\right) - \phi\left(\frac{I_t}{K_t}\right)^2\right] \exp\left(\beta Z_t + \sigma_k \cdot W_{t+1} - \frac{1}{2}|\sigma_k|^2\right)$$

In addition:

$$C_t + I_t = AK_t$$

where A is a fixed parameter. The process Z_t is an AR1 normalized to have mean zero:

$$Z_{t+1} = \alpha_z + \exp(-\xi) Z_t + \sigma_z \cdot W_{t+1}$$

where $|\xi| < 1$.

Transform variables:

$$\log K_{t+1} = \log K_t + \alpha_k + \left(\frac{I_t}{K_t}\right) - \phi\left(\frac{I_t}{K_t}\right)^2 + \beta Z_t + -\frac{1}{2}|\sigma_k|^2 + \sigma_k \cdot W_{t+1}$$

and

$$\frac{C_t}{K_t} + \frac{I_t}{K_t} = A$$

Assume the period utility function is:

$$(1-\beta)\log C_t = (1-\beta)(\log C_t - \log K_t) + (1-\beta)\log K_t$$

where $0 < \beta < 1$. The $1-\beta$ scaling is done for convenience. Guess a date t continuation value:

$$V_t = \log K_t + f(Z_t)$$

where under expected utility:

$$V_t = (1-\beta)\log C_t + \beta E\left(V_{t+1}|\mathcal{F}_t\right)$$

and where $\frac{C_t}{K_t}$ and $\frac{I_t}{K_t}$ are chosen optimally subject to the constraint and the capital evolution equation.

Use the following parameters:

$$\begin{aligned} \alpha_y &= .373 & \beta &= 1 \\ \alpha_z &= 0 & \xi &= .017 \end{aligned}$$

$$\sigma = \begin{bmatrix} (\sigma_y)' \\ (\sigma_z)' \end{bmatrix} = \begin{bmatrix} .481 & 0 \\ .012 & .027 \end{bmatrix} \tag{1}$$

where $\sigma_y = \sigma_k$ and

$$\alpha_y = \alpha_k + \left(\frac{I_t}{K_t}\right) - \phi\left(\frac{I_t}{K_t}\right)^2 - \frac{1}{2}|\sigma_k|^2.$$

is the implied growth rate for consumption net of the contribution from Z. (The $\frac{I}{K}$ ratio turns out to be constant in this model.) Note: with this scaling one of the shocks has permanent consequences and the other is temporary.

Verify that both are state independent and that f is affine in the realized value z of Z_t.

Repeat the same calculation with a risk-sensitive or robustness adjustment for the the continuation value. What impact does risk sensitivity or robustness concern have on $\frac{C_t}{K_t}$ and $\frac{I_t}{K_t}$ for this economy? The continuation value under robustness is:

$$V_t = (1-\beta)\log C_t - \beta\theta\log E\left[\exp\left(-\frac{1}{\theta}V_{t+1}\right)|\mathcal{F}_t\right]$$

2 Permanent income model

We use a model of the type suggested by Hall and Flavin to capture a version of Friedman's permanent income model of consumption. Let $\{Y_t\}$ be the logarithm of an exogenous nonfinancial income process that is governed by an additive functional

$$Y_{t+1} - Y_t = \mathbb{D}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1} + \nu$$

where

$$X_{t+1} = \mathbb{A}_x X_t + \mathbb{B}_x W_{t+1}$$

and A is a stable matrix. Define $Y_t^0 = \bar{y}_0 + t\nu$ and

$$Y_{t+1}^1 - Y_t^1 = \mathbb{D}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1}.$$

Let $\widehat{K}_t$ be an asset stock that can be negative, meaning that we allow indebtedness. Combine an "asset return" $\exp(\rho)\widehat{K}_t$ and a time t exogenous nonfinancial income to deduce that the asset stock evolves according to

$$\widehat{K}_{t+1} + \exp\left(\widehat{C}_t\right) = \exp(\rho)\widehat{K}_t + \exp(Y_t), \tag{2}$$

where $\widehat{C}_t$ is the logarithm of consumption at date t and ρ parameterizes an exogenous constant rate of return on assets. It is convenient to scale variables by nonfinancial income, so we define $C_t = \widehat{C}_t - Y_t$ and $K_t = \frac{\widehat{K}_t}{\exp(Y_t)}$. Divide both sides of equation (2) by $\exp(Y_t)$ to obtain

$$K_{t+1} \exp\left(Y_{t+1} - Y_t\right) + \exp\left(C_t\right) = \exp(\rho) K_t + 1. \tag{3}$$

A representative consumer with time separable preferences, logarithmic one period utility of consumption, and subjective discount rate δ chooses a consumption process that respects the Euler equation

$$\exp(-\delta + \rho) E\left[\exp(\widehat{C}_t - \widehat{C}_{t+1}) | X_t\right] = 1,$$

or equivalently

$$\exp(-\delta + \rho) E\left[\exp(C_t - C_{t+1} + Y_t - Y_{t+1}) | X_t\right] = 1.$$

We assume that

$$\exp(-\delta + \rho - \nu) = 1,$$

a restriction that supports steady states in which the log consumption-log income ratio equals $\bar{c}$.

2.1 Steady state

Steady state means of asymptotically stationary components of (C_t, K_t) must satisfy

$$\bar{k}\exp(\nu) + \exp(\bar{c}) = \exp(\rho)\bar{k} + 1$$

or equivalently

$$\exp(\bar{c}) = [\exp(\rho) - \exp(\nu)]\,\bar{k} + 1,$$

where we assume that $\rho > \nu$. Notice that we are free to set $\bar{k}$. For convenience, in what follows we assume that $\bar{k} = 0$ and hence that $\exp(\bar{c}) = 1$.

2.2 First-order approximation

We take a first-order small-noise approximation that scales W_{t+1} by q and let q tend to zero. The processes with superscripts 1 are effectively first-order derivatives of the original processes with respect to q evaluated at $\mathsf{q} = 1$:

$$\begin{aligned} K_{t+1}^1 \exp(\nu) + \bar{k}\exp(\nu)\left(Y_{t+1}^1 - Y_t^1\right) + \exp(\bar{c})C_t^1 \\ = \exp(\rho)K_t^1. \end{aligned}$$

Then the restriction

$$K_{t+1}^1 = \exp(\rho - \nu)K_t^1 - \bar{k}\left(Y_{t+1}^1 - Y_t^1\right) - \exp(\bar{c} - \nu)\exp(\bar{c})C_t^1 \tag{4}$$

and $\bar{k} = 0$ make this equation become

$$K_{t+1}^1 = \exp(\rho - \nu)K_t^1 - \exp(-\nu)C_t^1 \tag{5}$$

Furthermore, a first-order approximation to the Euler equation is

$$E\left[C^1_{t+1} + Y^1_{t+1} | \mathcal{F}_t\right] = C^1_t + Y^1_t$$

We can solve equation (4) forward, with or without taking conditional expectations of all time indexed variables.[1] To derive a decision rule for consumption that solves a representative agent planning problem, we solve forward and take conditional expectations:

$$\begin{aligned}
\exp(\nu)K^1_t &= \sum_{j=0}^{\infty} \lambda^{j+1} E\left(C^1_{t+j} + Y^1_{t+j} \mid \mathcal{F}_t\right) - \sum_{j=0}^{\infty} \lambda^{j+1} E\left(Y^1_{t+j} \mid \mathcal{F}_t\right) \\
&= \left(\frac{\lambda}{1-\lambda}\right)(C^1_t + Y^1_t) - \left(\frac{\lambda}{1-\lambda}\right) \sum_{j=1}^{\infty} \lambda^j E\left(Y^1_{t+j} - Y^1_{t+j-1} \mid \mathcal{F}_t\right) - \left(\frac{\lambda}{1-\lambda}\right) Y^1_t \\
&= \left(\frac{\lambda}{1-\lambda}\right) C^1_t - \left(\frac{\lambda}{1-\lambda}\right) \sum_{j=1}^{\infty} \lambda^j E\left(Y^1_{t+j} - Y^1_{t+j-1} \mid \mathcal{F}_t\right) \qquad (6)
\end{aligned}$$

where $\lambda = \exp(\nu - \rho)$. Decision rule (6) implies that C^1_t and K^1_t are cointegrated with cointegrating vector $\begin{bmatrix} 1 & [\exp(\nu) - \exp(\rho)] \end{bmatrix}$.

From equation (6),

$$C^1_t = \frac{\exp(-\nu)(1-\lambda)}{\lambda} K^1_t + \exp(-\nu) \sum_{j=1}^{\infty} \lambda^j E\left(Y^1_{t+j} - Y^1_{t+j-1} \mid \mathcal{F}_t\right)$$

This links the log consumption-income ratio to the two income sources: financial income and non-financial income. The non-financial income contribution

$$\sum_{j=1}^{\infty} \lambda^j E\left(Y^1_{t+j} - Y^1_{t+j-1} \mid \mathcal{F}_t\right) = \lambda A \left(I - \lambda A\right)^{-1} X_t$$

Equation (6) corresponds to the solution of the planner's problem in the Hansen et al. (1999) economy without robustness (i.e., with their $\sigma = 0$).

2.3 Impulse responses

We evaluate the decision rules for consumption and assets at the following nonfinanical income processes adapted from Hansen et al. (1999), who assumed two components for

[1] Cite a reference to one of many places where we have done this in other publications.

nonfinancial income. Let

$$X^1_{1,t+1} = .704 X_{1,t} + \begin{bmatrix} .144 & 0 \end{bmatrix} W_{t+1}$$

where $Y^1_{1,t+1} = Y^1_t + X^1_{1,t+1}$. To construct the second component, let

$$X^1_{2,t+1} = X^1_{2,t} - .154 X^1_{2,t-1} + \begin{bmatrix} 0 & .206 \end{bmatrix} W_{t+1}.$$

and construct $Y^1_{2,t+1} = X^1_{2,t+1}$. Let $Y^1_{t+1} = (.01) Y^1_{1,t+1} + (.01) Y^2_{2,t+1}$.[2] We represent this $\{Y_t\}$ process as an additive functional. Set $\rho = .00663$ and $\nu = .00373$.

[2] We take income numbers from the first column of Table 2 of Hansen et al. (1999) with two exceptions. In Hansen et al. (1999), both income processes are stationary but one has an autoregressive root of .998. We set this to one here. This has a nontrivial impact on the consumption volatility, which Hansen et al. estimated in levels. We scale both innovation standard deviations by 1.33 to achieve a consumption growth rate volatility of .482 expressed as a percent (log differences multiplied by 100).

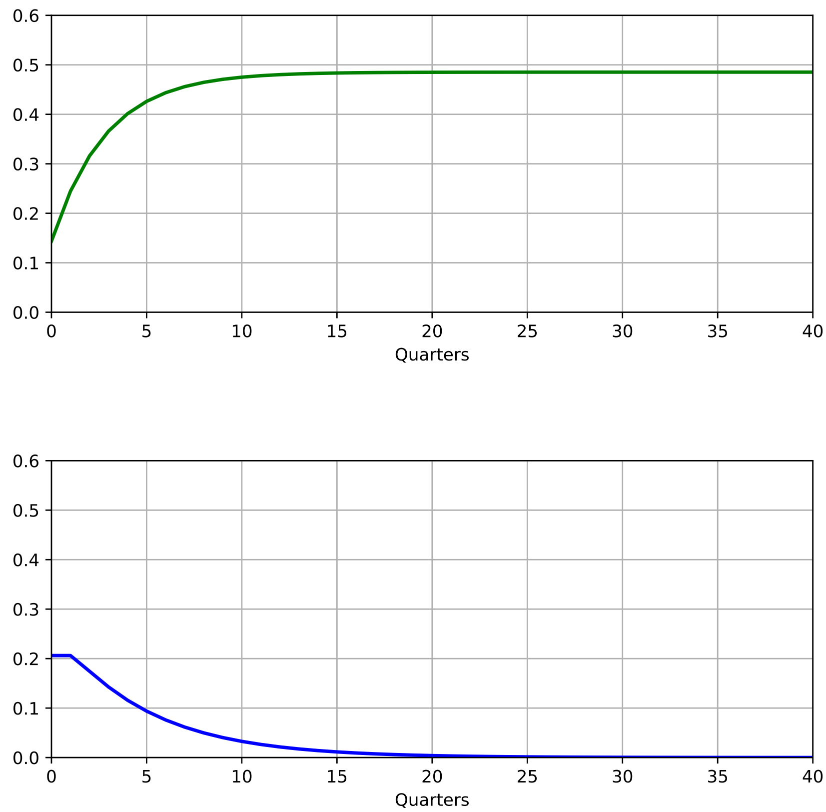

Figure 1: Impulse responses of log income to the two shock processes. Top panel: permanent shock. Bottom panel: transitory shock. Parameter settings from Hansen et al. (1999).

From Figure 1, the first shock has permanent consequences as is reflected by the limiting impulse response. The second has transitory implications as evident from the convergence of impulse responses to zero. The permanent shock cannot cannot be approximately diversified over time, while, via investment or savings adjustments, the transitory shock can be. Under time separability, the responses are constant across horizons. The two consumption responses are:

$$\text{permanent shock} = .482$$
$$\text{transitory shock} = .00383,$$

(when multiplied by 100) which shows the dramatic difference induced by the endogenous consumption savings responses.

There is a sense in which responses to both shocks are present-value neutral. Consumption is allowed to differ from income; but discounted-by-λ impulse responses for consumption and income should offset one another. As C_t^1 is the first-order approximation to the log consumption/income ratio, the infinite discounted sum of the responses should be zero. Figure 2 computes the discounted sum over the alternative horizons in order to check how quickly the sum converges to zero. From the time scale, we see that the convergence is indeed slow.

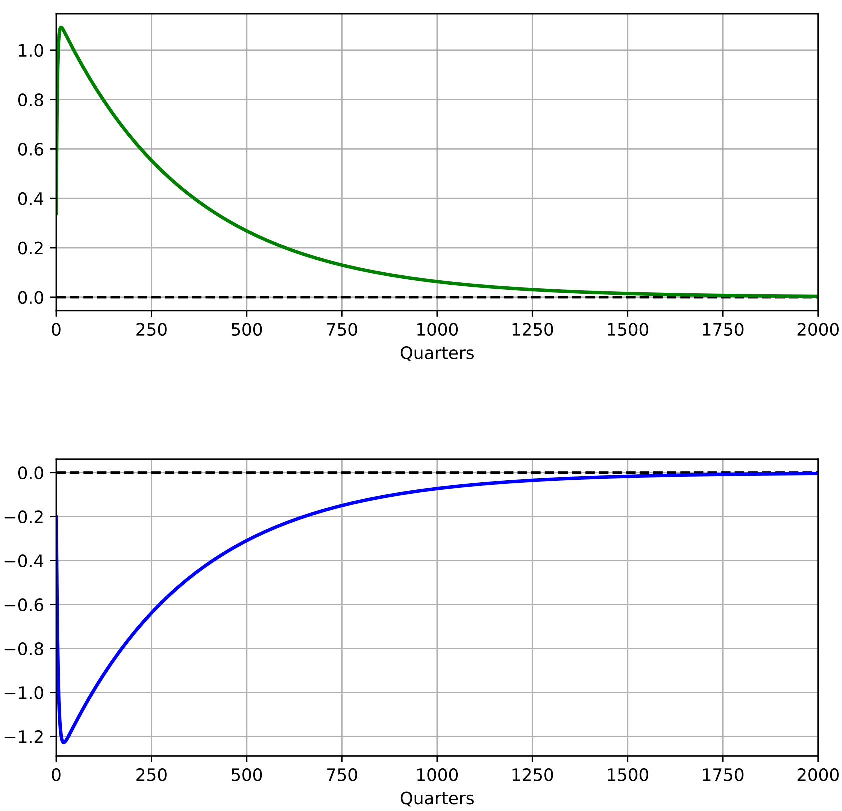

Figure 2: Present value responses for the two shock processes. Top panel: permanent shock. Bottom panel: transitory shock. Parameter settings from Hansen et al. (1999).

2.4 Robustness

Now activate robustness as in Hansen-Sargent-Tallarini. We use the utility recursion:

$$V_t = [1-\exp(-\delta)](C_t + Y_t) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right)|\mathcal{F}_t\right]$$

where V_t is the date t continuation value. Setting $\xi = \infty$ gives time-separable log utility. The parameter ξ is the inverse of the risk sensitivity parameter used in Hansen et al. (1999). The first-order approximation to a continuation value process in logarithms is of the form

$$V_t^1 = [1-\exp(-\delta)](C_t^1 + Y_t^1) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)|\mathcal{F}_t\right]$$

where $F_c \cdot W_{t+1}$ gives the response of C_{t+1}^1 to W_{t+1}. We maintain that the evolution for $C_{t+1}^1 + Y_{t+1}^1$ is:

$$C_{t+1}^1 + Y_{t+1}^1 = C_t^1 + Y_t^1 + (F_c + F_y) \cdot W_{t+1} \tag{7}$$

Then

$$V_t^1 = C_t^1 + Y_t^1 - \frac{|F_c + F_y|^2}{2\xi}$$

Notice that $(F_c + F_y) \cdot W_{t+1}$ denotes the exposure of V_{t+1}^1 to shocks. Under the implied change of measure associated with the positive random variable

$$M_{t+1}^0 = \frac{\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)}{E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)|\mathcal{F}_t\right]}$$

W_{t+1} is normally distributed with mean $-\frac{1}{\xi}(F_c + F_y)$ and covariance matrix I.

Thus, because a robust planner acts "as if" he is a nonrobust planner but one who takes conditional expectations with this distorted probability distribution instead of the benchmark distribution, the consumption Euler equation is now

$$\exp(-\delta + \rho - \nu)E\left[M_{t+1}^0\left(C_{t+1}^1 + Y_{t+1}^1\right)|\mathcal{F}_t\right] = C_t^1 + Y_t^1$$

Under the implied worst-case model, the expected growth rate in consumption is adjusted downward by:

$$-\frac{|F_c + F_y|^2}{\xi}$$

This would then impact the implied risk-free interest rate.

As an alternative, we follow Hansen et al. (1999) by fixing a target interest rate and adjusting δ accordingly. To be consistent with evolution equation (7), we therefore assume:

$$\delta = \rho - \nu - \frac{|F_c + F_y|^2}{\xi}$$

This gives an affine in $\frac{1}{\xi}$ counterpart to formula (28) in Hansen et al. (1999) with slope coefficient: $-\frac{|F_c+F_y|^2}{2}$ as is depicted in Figure 3.

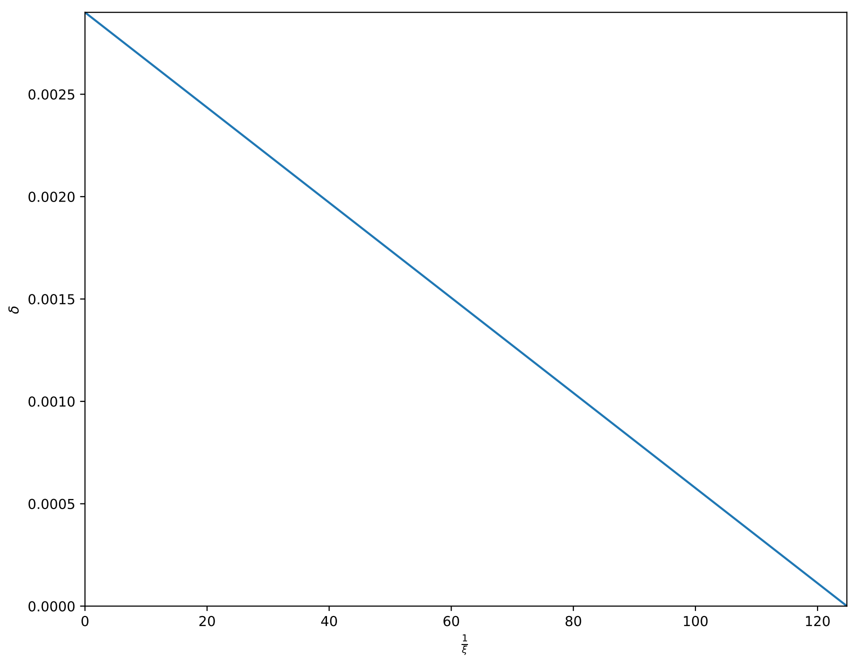

Figure 3: Subjective discount rates and robustness. This plot shows how to adjust the subjective discount rate δ for a given value of $\frac{1}{\xi}$ while leaving the implied riskless rate fixed.

The uncertainty price vector for the two shocks is:

$$-\frac{1}{\xi}(F_c + F_y) = -\frac{.01}{\xi}\begin{bmatrix} .482 \\ .00383 \end{bmatrix}$$

The exposure to the shock with permanent consequences requires much larger compensations as the robust planner fears the misspecification of that more.

3 Habit persistence

We aim now to construct a multiplicative-functional counterpart to HST's specification with habit persistence. (With different parameter values, the same specification could also capture durable consumption goods.)

Change preferences to be

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right)|\mathcal{F}_t\right]$$

where

$$\exp(H_{t+1} + Y_{t+1}) = \exp(-\psi)\left[\exp(H_t + Y_t)\right] + [1 - \exp(-\psi)]\left[\exp(C_t + Y_t)\right],$$

$0 \leqslant \psi < 1$,

$$\exp(U_t + Y_t) = \upsilon[\exp(C_t + Y_t), \exp(H_t + Y_t)],$$

and

$$\upsilon(c, h) = \left[(1-\alpha)c^{1-\eta} + \alpha h^{1-\eta}\right]^{\frac{1}{1-\eta}}.$$

The CES specification adopted here for υ is able to capture either durability of consumption goods or habit persistence in preferences or both.

3.1 H dynamics

Rewrite the H dynamics as

$$\exp(H_{t+1})\exp(Y_{t+1} - Y_t) = \exp(-\psi)\exp(H_t) + [1 - \exp(-\psi)]\exp(C_t)$$

where we divided through by $\exp(Y_t)$.

The steady state counterpart

$$\exp(\bar{h})\exp(\nu) = \exp(-\psi)\exp(\bar{h}) + [1 - \exp(-\psi)]\exp(\bar{c})$$

determines $\bar{h}$. The first-order approximation is

$$\exp(\nu)\exp(\bar{h})\left[H^1_{t+1} + (Y^1_{t+1} - Y^1_t)\right] = \exp(-\psi)\exp(\bar{h})H^1_t + [1 - \exp(-\psi)]\exp(\bar{c})C^1_t.$$

Equivalently,

$$H^1_{t+1} = \exp(-\nu - \psi)H^1_t + [\exp(-\nu) - \exp(-\psi - \nu)]\left[\frac{\exp(\bar{c})}{\exp(\bar{h})}\right]C^1_t - Y^1_{t+1} + Y^1_t,$$

which after simplification becomes

$$H^1_{t+1} = \exp(-\nu - \psi)H^1_t + [1 - \exp(-\nu - \psi)]C^1_t - Y_{t+1} + Y_t. \tag{8}$$

3.1.1 CES algebra

CES first derivatives:

$$\begin{aligned} mc &= (1-\alpha)u^\eta c^{-\eta} \\ mh &= \alpha u^\eta h^{-\eta} \end{aligned}$$

Define

$$\bar{u} = \frac{1}{1-\eta}\log\left((1-\alpha)\exp[(1-\eta)\bar{c}] + \alpha\exp[(1-\eta)\bar{h}]\right)$$

which is the steady-state version of U_t. The first-order approximation is:

$$U^1_t = (1-\alpha)\exp\left[(\eta-1)(\bar{u}-\bar{c})\right]C^1_t + \alpha\exp\left[(\eta-1)(\bar{u}-\bar{h})\right]H^1_t \tag{9}$$

3.1.2 Co-state evolution

There are two co-state equations and one set of first-order conditions for consumption:

$$\alpha\exp\left[(\eta-1)(U_t+Y_t) - \eta(H_t+Y_t)\right] - \exp\left(\widehat{MH}_t\right) + \exp(-\delta-\psi)E\left[\exp\left(\widehat{MH}_{t+1}\right) \mid \mathcal{F}_t\right] = 0$$

$$\begin{aligned} &(1-\alpha)\exp\left[(\eta-1)(U_t+Y_t) - \eta(C_t+Y_t)\right] \\ &+ \exp(-\delta)[1-\exp(-\psi)]E\left[\exp\left(\widehat{MH}_{t+1}\right) \mid \mathcal{F}_t\right] - \exp(-\delta)E\left[\exp\left(\widehat{MK}_{t+1}\right) \mid \mathcal{F}_t\right] = 0 \end{aligned}$$

$$\exp(-\delta+\rho)E\left[\exp\left(\widehat{MK}_{t+1}\right) \mid \mathcal{F}_t\right] - \exp\left(\widehat{MK}_t\right) = 0.$$

Multiply by $\exp(Y_t)$:

$$\exp(-\delta-\psi)E\left[\exp\left(MH_{t+1}+Y_t-Y_{t+1}\right)\mid\mathcal{F}_t\right]=\exp\left(MH_t\right)-\alpha\exp\left[(\eta-1)U_t-\eta H_t\right]$$

$$\begin{aligned}(1-\alpha)\exp\left[(\eta-1)U_t-\eta C_t\right]=&\exp(-\delta)E\left[\exp\left(MK_{t+1}+Y_t-Y_{t+1}\right)\mid\mathcal{F}_t\right]\\&-\exp(-\delta)[1-\exp(-\psi)]E\left[\exp\left(MH_{t+1}+Y_t-Y_{t+1}\right)\mid\mathcal{F}_t\right]\end{aligned}$$

$$\exp(-\delta+\rho)E\left[\exp\left(MK_{t+1}+Y_t-Y_{t+1}\right)\mid\mathcal{F}_t\right]=\exp\left(MK_t\right)$$

3.1.3 Additional steady state calculations

$$\exp(-\delta-\psi-\nu)\exp\left(\overline{mh}\right)=\exp\left(\overline{mh}\right)-\alpha\exp\left[(\eta-1)\bar{u}-\eta\bar{h}\right]$$

$$(1-\alpha)\exp\left[(\eta-1)\bar{u}-\eta\bar{c}\right]=\exp(-\delta-\nu)\exp\left(\overline{mk}\right)-\exp(-\delta-\nu)[1-\exp(-\psi)]\exp\left(\overline{mh}\right)$$

3.1.4 First-order approximation

$$\begin{aligned}&\exp\left(-\delta-\psi-\nu+\overline{mh}\right)E\left[MH^1_{t+1}+Y^1_t-Y^1_{t+1}\mid\mathcal{F}_t\right]\\&=\exp\left(\overline{mh}\right)MH^1_t-\alpha\exp\left[(\eta-1)\bar{u}-\eta\bar{h}\right]\left[(\eta-1)U^1_t-\eta H^1_t\right]\end{aligned}\tag{10}$$

$$\begin{aligned}&(1-\alpha)\exp\left[(\eta-1)\bar{u}-\eta\bar{c}\right]\left[(\eta-1)U^1_t-\eta C^1_t\right]\\&=\exp(-\delta-\nu)E\left[\exp(\overline{mk})MK^1_{t+1}-[1-\exp(-\psi)]\exp(\overline{mh})MH^1_{t+1}\mid\mathcal{F}_t\right)\\&+\exp(-\delta-\nu)\left[\exp(\overline{mk})-[1-\exp(-\psi)]\exp(\overline{mh})\right]E\left(Y^1_t-Y^1_{t+1}|\mathcal{F}_t\right).\end{aligned}\tag{11}$$

$$\exp(-\delta+\rho-\nu)E\left[MK^1_{t+1}+(Y^1_t-Y^1_{t+1})|\mathcal{F}_t\right]=MK^1_t.\tag{12}$$

3.1.5 Solution strategy

One approach is to use the deflating subspace calculations described in Hansen and Sargent (2008, ch. 4).

1. Construct

$$Z_t^1 = \begin{bmatrix} MK_t^1 \\ MH_t^1 \\ K_t^1 \\ H_t^1 \\ X_t \end{bmatrix}$$

2. Take equations (9) and (11) and solve for U_t^1 and C_t^1 in terms of Z_t^1 and Z_{t+1}^1.

3. Use equations (10), (11), (4), and (8) after substituting for U_t^1, C_t^1 and $E\left(Y_{t+1}^1 - Y_t^1 \mid \mathcal{F}_1\right) = D \cdot X_t$ and form the system:

$$\mathbb{L}Z_{t+1}^1 = \mathbb{J}Z_t^1$$

where we initially zero out the shocks and use $X_{t+1} = AX_t$.

4. Consider a solution of the co-states in terms of the states of the form:

$$\begin{bmatrix} MK_t^1 \\ MH_t^1 \end{bmatrix} = \mathbb{N}_{11} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} + \mathbb{N}_{12}X_t.$$

Substituting this into the system dynamics gives:

$$\mathbb{L} \begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} K_{t+1}^1 \\ H_{t+1}^1 \end{bmatrix} \\ X_{t+1} \end{bmatrix} = \mathbb{J} \begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix}$$

To compute $\mathbb{N}_{11}$ and $\mathbb{N}_{12}$ we will require that the dynamics for

$$\begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix}$$

be weakly stable. We accomplish this by first forming a generalized Schur decomposition. There will be a total of seven eigenvalues, three of which are associated with the exogenous dynamics. These three are all stable. There will be four eigenvalues associated with the endogenous dynamics, two of which are stable and two of which are unstable. One of the "endogenous" eigenvalues will be unity, and we will count this as (weakly) stable. This leads us to form:

$$\begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix}$$

by taking linear combinations of the five stable generalized eigenvalues. See Hansen and Sargent (2008, ch. 4) for elaboration.[3] To check the calculation verify that the eigenvalues of the resulting state dynamics are indeed weakly stable.

5. Perform the following check. I think but have not verified that $\mathbb{L}$ is nonsingular. Compute:

$$\mathbb{L}^{-1}\mathbb{J}$$

and thus

$$Z_{t+1}^1 = \mathbb{L}^{-1}\mathbb{J}Z_t^1$$

We know that

$$\begin{bmatrix} I & -\mathbb{N}_{11} & -\mathbb{N}_{12} \end{bmatrix} Z_{t+1} = 0.$$

Thus

$$\begin{bmatrix} I & -\mathbb{N}_{11} & -\mathbb{N}_{12} \end{bmatrix} \mathbb{L}^{-1}\mathbb{J} \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ I & 0 \\ 0 & I \end{bmatrix} = 0.$$

6. Compute the eigenvalues of the matrix:

$$\mathbb{A} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathbb{L}^{-1}\mathbb{J} \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ I & 0 \\ 0 & I \end{bmatrix}$$

and check that they coincide with the weakly stable eigenvalues.

7. Add the shocks back to the X evolution. Thus we have:

$$\begin{bmatrix} MK_t^1 \\ MH_t^1 \end{bmatrix} = \mathbb{N}_{11} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} + \mathbb{N}_{12}X_t,$$

[3]This will involve employing an ordered Schur decomposition. Evan Anderson wrote one in Matlab. We can translate it to Julia. Evan's program is described in Hansen and Sargent (2008, ch. 4) and available at the website for the Matlab programs for that book.

and

$$\begin{bmatrix} \begin{bmatrix} K_{t+1}^1 \\ H_{t+1}^1 \end{bmatrix} \\ X_{t+1} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix} + \mathbb{B} W_{t+1} \tag{13}$$

where

$$\mathbb{B} = \begin{bmatrix} 0 \\ \mathbb{B}_x \end{bmatrix}$$

The matrix $\mathbb{A}$ should be block triangular with $\mathbb{A}_x$ in the lower block.

The parameter η introduces a form of intertemporal complementarity into the analysis. This is achieved by letting ρ become large. The next two graphs plot the impulse responses for log consumption. Figure 4 investigates how the choice η alters the responses. For all these relatively large values of η, for the permanent shock the immediate response is muted relative to the long-term response as the response increases with the horizon. Larger values of η apparently induces a more sluggish consumption response. The qualitative nature of these responses looks very similar to those posed in long-run risk models with recursive utility. Here the consumption response is endogenous.

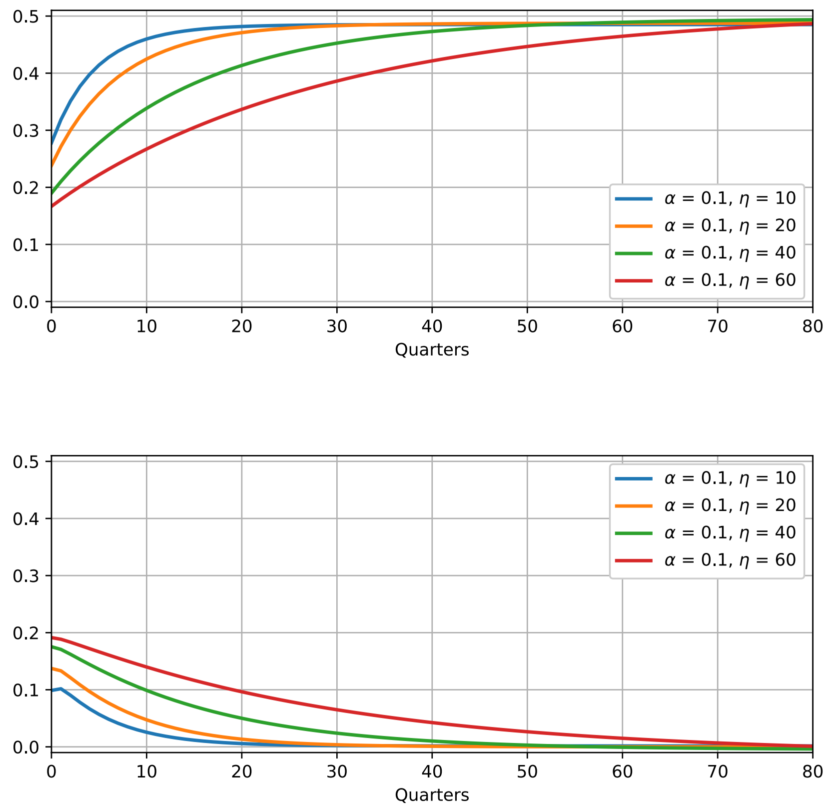

Figure 4: Consumption responses for the two shock processes for habit persistent preferences for $\alpha = .1$, $\psi = .3$ and alternative choice for η. Top panel: permanent shock. Bottom panel: transitory shock.

Figure **??** shows how changing α alters the impulse response for the logarithm of consumption for fixed values $\psi = .3$ and $\eta = 40$. While preserving the same qualitative response patterns for consumption, increasing α from .3 to .7 has little impact on consumption. At the more extreme $\alpha = .1$ and .9 there is substantial more curvature in the initial part of the responses and converges appears faster.

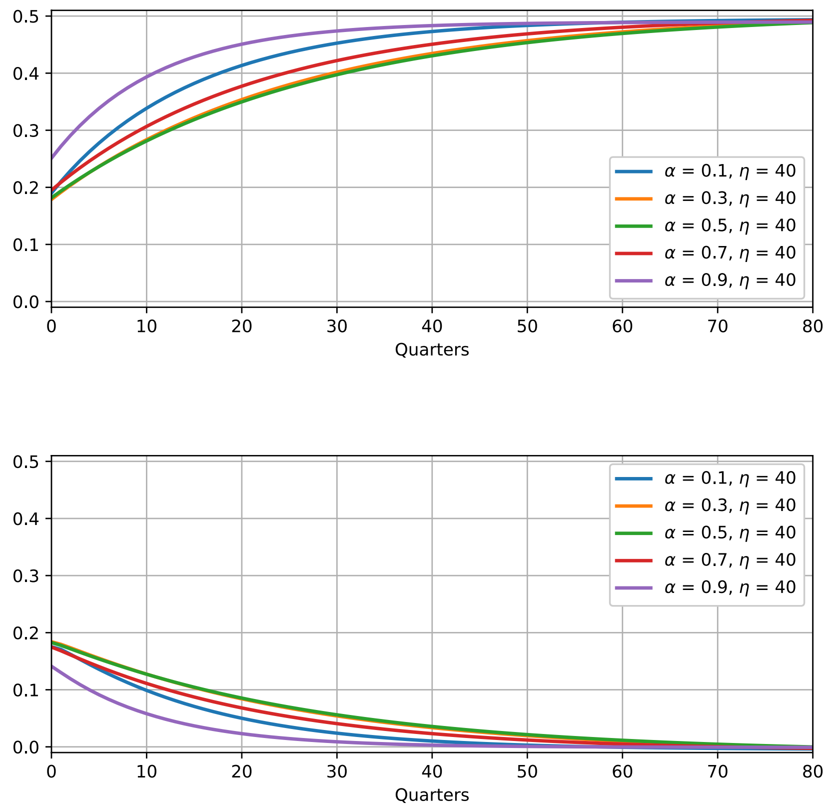

Figure 5: Consumption responses for the two shock processes for habit persistent preferences for $\eta = 40$, $\psi = .3$ and alternative choice for α. Top panel: permanent shock. Bottom panel: transitory shock.

For valuation purposes it the marginal utility that matters. For one-period asset pricing it the expose of the marginal utility to shocks that dictates risk-return tradeoffs. For $(\alpha, \rho) = (.1, 40), (.3, 40)$ we report the vector of exposures for $MK^1_{t+1} - Y^1_{t+1}$ to the two shocks.

Please compute the requested two dimensional vectors of shock loadings.

Next we add in a concern about robustness as in Hansen et al. (1999). This requires that we compute the first-order term for the continuation value process.

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right)|\mathcal{F}_t\right]$$

Thus

$$V_t^1 - Y_t^1 = [1 - \exp(-\delta)]U_t^1 - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1 - Y_t^1\right)|\mathcal{F}_t\right]$$

Represent:

$$\begin{aligned}
Y_{t+1}^1 - Y_t^1 &= \mathbb{S}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1} \\
U_t^1 &= \mathbb{S}_u \cdot \begin{bmatrix} K_t^1 \\ H_t^1 \\ X_t \end{bmatrix} \\
V_t^1 - Y_t^1 &= \mathbb{S}_v \cdot \begin{bmatrix} K_t^1 \\ H_t^1 \\ X_t \end{bmatrix} + \mathsf{s}_v
\end{aligned}$$

where $\mathbb{S}_u$ comes from the model solution using formula (9) and $\mathbb{S}_v$ and s_v are to be computed as in Hansen et al. (2008). In particular,

$$(\mathbb{S}_v)' = [1 - \exp(-\delta)](\mathbb{S}_u)' + \exp(-\delta)\left[(\mathbb{S}_v)'\mathbb{A} + \begin{bmatrix} 0 & 0 & (\mathbb{S}_y)' \end{bmatrix}\right],$$

and

$$\mathsf{s}_v = \exp(-\delta)\left[\mathsf{s}_v - \frac{\xi}{2}\left|(\mathbb{S}_v)'\mathbb{B} + (\mathbb{S}_y)'\mathbb{B}_x\right|^2\right]$$

The first equation is affine in $\mathbb{S}_v$ and may be solved prior to the second equation. Given $\mathbb{S}_v$, the second equation is affine in v and may be solved easily as well. We are particularly interested in computing

$$(\mathbb{S}_v)'\mathbb{B}$$

since the uncertainty prices are given by this two-dimensional vector scaled by $\frac{1}{\xi}$.

References

Hansen, Lars Peter and Thomas J. Sargent. 2008. *Robustness*. Princeton, New Jersey: Princeton University Press.

Hansen, Lars Peter, Thomas Sargent, and Thomas Tallarini. 1999. Robust Permanent Income and Pricing. *Review of Economic Studies* 66 (4):873–907.

Hansen, Lars Peter, John C. Heaton, and Nan Li. 2008. Consumption Strikes Back?: Measuring Long Run Risk. *Journal of Political Economy* .