# Expansion Notes

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## 1 Introduction

We will discuss an explicit solution for the second-order approximation. This second-order approximation features prominently in our asset pricing applications. We first show how to expand all quantities that are of our interest and then we incorporate these expansions into the approximation of the set of equilibrium conditions that describe our dynamic model.

## 2 Small noise expansion

Here we follow Lombardo and Uhlig (2018). Consider a class of models indexed by the perturbation parameter $\mathsf{h}$:

$$X_{t+1}\left(\mathsf{h}\right) = \psi\left[X_t\left(\mathsf{h}\right), \mathsf{h}W_{t+1}, \mathsf{h}\right]. \tag{1}$$

where $X$ is an $n$-dimensional stochastic process. Deonte the $\mathsf{h} = 0$ limit as:

$$X^0_{t+1} = \psi\left(X^0_t, 0, 0\right),$$

and assume that there exists a second-order expansion of $X_t$ around $\mathsf{h} = 0$:

$$X_t \approx X_t^0 + \mathsf{h} X_t^1 + \frac{\mathsf{h}^2}{2} X_t^2.$$

The processes $X^j$, $j = 0, 1, 2$ are essentially stochastic process derivatives of $X$ with respect to the perturbation parameter. of $\psi$, $\psi^i_{x'}$ denote the row vector of first derivatives with respect to the vector $x$, and similarly for $\psi^i_w$ and $\psi_{\mathsf{h}}$. The implied recursion for the first-derivative process is

$$X^1_{t+1} = \begin{bmatrix} \psi^1_{x'} \\ \psi^2_{x'} \\ .... \\ \psi^n_{x'} \end{bmatrix} X^1_t + \begin{bmatrix} \psi^1_{w'} \\ \psi^2_{w'} \\ ... \\ \psi^n_{w'} \end{bmatrix} W_{t+1} + \begin{bmatrix} \psi^1_{\mathsf{h}} \\ \psi^2_{\mathsf{h}} \\ ... \\ \psi^n_{\mathsf{h}} \end{bmatrix}$$

Write this compactly as:

$$X^1_{t+1} = \psi_x X^1_t + \psi_w W_{t+1} + \psi_{\mathsf{h}}$$

which is a first-order vector autoregression. We presume that that matrix $\psi_x$ has stable eigenvalues.

Analogously, we denote second derivatives with double subscripts. Differentiating the first-derivative recursion gives:

$$\begin{aligned} X^2_{t+1} = & \psi_x X^2_t + \begin{bmatrix} X^{1\prime}_t \psi^1_{xx'} X^1_t \\ X^{1\prime}_t \psi^2_{xx'} X^1_t \\ .... \\ X^{1\prime}_t \psi^n_{xx'} X^1_t \end{bmatrix} + 2 \begin{bmatrix} X^{1\prime}_t \psi^1_{xw'} W_{t+1} \\ X^{1\prime}_t \psi^2_{xw'} W_{t+1} \\ .... \\ X^{1\prime}_t \psi^n_{xw'} W_{t+1} \end{bmatrix} + \begin{bmatrix} W_{t+1}{}' \psi^1_{ww'} W_{t+1} \\ W_{t+1}{}' \psi^2_{ww'} W_{t+1} \\ .... \\ W_{t+1}{}' \psi^n_{ww'} W_{t+1} \end{bmatrix} \\ & + 2 \begin{bmatrix} \psi^1_{\mathsf{h}x'} X^1_t \\ \psi^2_{\mathsf{h}x'} X^1_t \\ ... \\ \psi^n_{\mathsf{h}x'} X^2_t \end{bmatrix} + 2 \begin{bmatrix} \psi^1_{\mathsf{h}w'} W_{t+1} \\ \psi^2_{\mathsf{h}w'} W_{t+1} \\ ... \\ \psi^n_{\mathsf{h}x'} W_{t+1} \end{bmatrix} + \begin{bmatrix} \psi^1_{\mathsf{hh}} \\ \psi^2_{\mathsf{hh}} \\ ... \\ \psi^n_{\mathsf{hh}} \end{bmatrix} \end{aligned}$$

Observe that the expansion has a recursively linear structure, an inherent feature of the series expansion method. The law of motion for $X^0$ is deterministic and in stationary models, $X^0$ is constant (invariant over time). The dynamics for $X^2$ is nonlinear only in $X^1$ and $W_{t+1}$. Therefore, stable dynamics for $X^1$ also implies stable dynamics for $X^2$ (and this is also true for higher-order terms).

It is well known that in standard rational-expectations perturbation solutions (see, for instance, Schmitt-Grohé and Uribe (2004)), the partial derivatives of $\psi$ with respect to $\mathsf{q}$ are zero. This is not the case for the robust or recursive utility approximation we describe in what follows.

# 3 Recursive utility

As a precursor to a complete solution to a model, suppose that consumption satisfies an approximation of the type that we just described. We discuss value function and stochastic discount factor approximation.

Write the logarithm of the continuation value as:

$$v_t = \frac{1}{1-\rho} \log \left[(1-\delta) \exp[(1-\rho)c_t] + \delta \exp\left[(1-\rho)r_t\right]\right]$$

where $c_t$ is the logarithm of consumption, and

$$r_t = \frac{1}{1-\gamma} \log \left(E \exp\left[(1-\gamma)v_{t+1}\right] \mid \mathfrak{F}_t\right)$$

To facilitate limits, suppose that

$$1-\gamma = \frac{1-\gamma_o}{\mathsf{h}}.$$

We will eventually formally link $\mathsf{h}$ with $\mathsf{q}$ by making them proportional, or in the simplest case equal.

Suppose that consumption is stationary in growth rates. That is the first-difference in the logarithm of consumption is stationary.

# 4 Continuation Value Expansion

## 4.1 Order zero

For order zero we have a constant growth rate:

$$c_{t+1}^0 - c_t^0 = \eta_c^0$$

The zero order equation is:

$$v_t^0 - c_t^0 = \frac{1}{1-\rho} \log \left( (1-\delta) + \delta \exp \left[ (1-\rho) \left( v_{t+1}^0 - c_{t+1}^0 + \eta_c \right) \right] \right)$$

We now look for a solution for which $v_t^0 - c_t^0 = \eta_{v-c}^0$. Thus

$$\exp \left[ (1-\rho) \left( \eta_{v-c}^0 \right) \right] = \frac{1-\delta}{1 - \delta \exp \left[ (1-\rho) \eta_c \right]} \tag{2}$$

## 4.2 Order one

Now suppose that $\{c_t\}$ has first order (in $\mathsf{h}$) contribution $\{c_t^1\}$. We now confront the stochastic evolution component to the evolution. To facilitate the limits that interest us, construct:

$$\tilde{v}_t = \frac{v_t - v_t^0}{\mathsf{h}}$$

$$\tilde{r}_t = \frac{r_t - v_{t+1}^0}{\mathsf{h}}$$

which we assume to be well defined as $\mathsf{h}$ declines to zero, with the limit denoted by $\tilde{v}_t^0$. Multiplying through by $\mathsf{h}$ and differentiating we see that

$$\begin{aligned}
\mathsf{h}\frac{d}{d\mathsf{h}}\tilde{v}_t + \tilde{v}_t &= \frac{d}{d\mathsf{h}}v_t \\
\mathsf{h}\frac{d^2}{d\mathsf{h}^2}\tilde{v}_t + 2\frac{d}{d\mathsf{h}}\tilde{v}_t &= \frac{d^2}{d^2\mathsf{h}}v_t \\
\mathsf{h}\frac{d^3}{d\mathsf{h}^3}\tilde{v}_t + 3\frac{d}{d\mathsf{h}}\tilde{v}_t &= \frac{d^3}{d^3\mathsf{h}}v_t
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde{v}_t^0 &= v_t^1 \\
2\tilde{v}_t^1 &= v_t^2 \\
3\tilde{v}_t^2 &= v_t^3
\end{aligned}$$

The analogous formulas relate $\tilde{r}_t$ and its derivatives to the derivatives of $r_t$.

Write:

$$\tilde{r}_t = \left(\frac{1}{1-\gamma_o}\right)\log\left(E\exp\left[(1-\gamma_o)\tilde{v}_{t+1}\right] \mid \mathfrak{F}_t\right),$$

and note that

$$\tilde{r}_t^0 = \left(\frac{1}{1-\gamma_o}\right)\log\left(E\exp\left[(1-\gamma_o)\tilde{v}_{t+1}^0\right] \mid \mathfrak{F}_t\right) \tag{3}$$

Equivalently,

$$r_t^1 = \left(\frac{1}{1-\gamma_o}\right)\log\left(E\exp\left[(1-\gamma_o)v_{t+1}^1\right] \mid \mathfrak{F}_t\right)$$

We use this as input into a forward recursions for $v_t^1$:

$$v_t^1 = \left( \frac{(1-\delta)\exp\left[(1-\rho)c_t^0\right]}{(1-\delta)\exp[(1-\rho)c_t^0] + \delta\exp\left[(1-\rho)v_{t+1}^0\right]} \right) c_t^1 + \left( \frac{\delta\exp\left[(1-\rho)v_{t+1}^0\right]}{(1-\delta)\exp[(1-\rho)c_t^0] + \delta\exp\left[(1-\rho)v_{t+1}^0\right]} \right) r_t^1$$

It is most convenient to rewrite this as:

$$\begin{aligned}
v_t^1 = &\left( \frac{(1-\delta)}{(1-\delta) + \delta\exp\left[(1-\rho)\left(\eta_{v-c}^0 + \eta_c^0\right)\right]} \right) c_t^1 \\
&+ \left( \frac{\delta\exp\left[(1-\rho)\left(\eta_{v-c}^0 + \eta_c^0\right)\right]}{(1-\delta) + \delta\exp\left[(1-\rho)\left(\eta_{v-c}^0 + \eta_c^0\right)\right]} \right) r_t^1 \\
= &\frac{(1-\delta)\left(1-\delta\exp\left[(1-\rho)\eta_c^0\right]\right)}{(1-\delta)\left(1-\delta\exp\left[(1-\rho)\eta_c^0\right]\right) + (1-\delta)\delta\exp[(1-\rho)\eta_c^0]} c_t^1 \\
&+ \frac{(1-\delta)\delta\exp[(1-\rho)\eta_c]}{(1-\delta)\left(1-\delta\exp\left[(1-\rho)\eta_c^0\right]\right) + (1-\delta)\delta\exp[(1-\rho)\eta_c]} r_t^1 \\
= &\frac{\left(1-\delta\exp\left[(1-\rho)\eta_c^0\right]\right)}{\left(1-\delta\exp\left[(1-\rho)\eta_c^0\right]\right) + \delta\exp[(1-\rho)\eta_c^0]} c_t^1 \\
&+ \frac{\delta\exp[(1-\rho)\eta_c^0]}{\left(1-\delta\exp\left[(1-\rho)\eta_c^0\right]\right) + \delta\exp[(1-\rho)\eta_c^0]} r_t^1.
\end{aligned}$$

where the second equality follows from (2). We rewrite the third equality as:

$$v_t^1 = (1-\lambda)c_t^1 + \lambda r_t^1$$

where

$$\lambda \doteq \delta\exp[(1-\rho)\eta_c^0]$$

acts as a discount factor adjusted for growth. Thus $v_t^1$ is a weighted average of $c_t^1$ and $r_t^1$. Notice that the parameter $\rho$ impacts the weight $\lambda$ provided that there is growth in the consumption process, that is provided that $\eta_c > 0$.

Combining this with (3),

$$v_t^1 = (1-\lambda)c_t^1 + \lambda\left(\frac{1}{1-\gamma_o}\right)\log E\left(\exp\left[(1-\gamma_o)v_{t+1}^1\right] \mid \mathfrak{F}_t\right). \tag{4}$$

which is the familiar risk-sensitive recursion obtained as a first-order approximation. It was used, for instance, in Tallarini (2000)'s paper on risk-sensitive business cycles and in Hansen et al. (2008)'s paper on the measurement challenges related to long-term risk. Both of these feature log utility. Here the $\rho \neq 1$ impact shows up in the implied discount factor $\lambda$.

When the underlying shocks are normally distributed, the first-order approximation for the logarithm of consumption is Gaussian. In this case, the parameter $\gamma_o$ only contributes to a constant term added to the first-order approximation for the logarithm of the continuation value process relative to the case in which $\gamma_o = 1$. This case is particularly easy to solve. Suppose that

$$c_{t+1}^1 - c_t^1 = \mathbb{D}_c^1 \cdot X_t^1 + \mathbb{F}_c^1 \cdot W_{t+1} + \eta_c^1.$$

Now guess that

$$v_t^1 - c_t^1 = \mathbb{S}_{v-c}^1 \cdot X_t^1 + \eta_{s-c}^1.$$

Then from (4)

$$\begin{aligned}
\mathbb{S}_{v-c}^1 &= \lambda\left(\psi_{x'}\right)' \mathbb{S}_{s-v}^1 + \lambda\mathbb{D}_c^1 \\
\eta_{v-c}^1 &= \lambda\eta_{v-c}^1 + \lambda(1-\gamma_o)\frac{|\sigma_v^1|^2}{2} + \lambda\eta_c^1
\end{aligned}$$

where the second equation uses the familiar log normal formula. Solving these equations in sequence, we see that

$$\begin{aligned}
\mathbb{S}_{v-c}^1 &= \lambda\left[I - \lambda\left(\Psi_{x'}\right)'\right]^{-1}\mathbb{D}_c^1 \\
\eta_{v-c}^1 &= \left(\frac{\lambda}{1-\lambda}\right)\left[(1-\gamma_o)\frac{|\sigma_v^1|^2}{2} + \eta_c^1\right]
\end{aligned}$$

Finally, note write the date $t+1$ dependence on the shock as $\sigma_v^1 \cdot W_{t+1}$ and note taht

$$\sigma_v^1 = \mathbb{F}_c^1 + (\psi_{w'})' \mathbb{S}_{v-c}^1$$

For computations, it is advantageous to construct the positive random variable:

$$\frac{M_{t+1}}{M_t} = \frac{\exp\left[(1-\gamma_o)\tilde{v}_{t+1}\right]}{E\left(\exp\left[(1-\gamma_o)\tilde{v}_{t+1}\right] \mid \mathfrak{F}_t\right)}$$

and it is first-order expansion. The conditional expectation of this random variable (indexed by $\mathsf{h}$) is unity as is the order zero:

$$\begin{aligned}\frac{M_{t+1}^0}{M_t^0} =& \frac{\exp\left[(1-\gamma_o)\tilde{v}_{t+1}^0\right]}{E\left(\exp\left[(1-\gamma_o)\tilde{v}_{t+1}^0\right] \mid \mathfrak{F}_t\right)} \\ =& \frac{\exp\left[(1-\gamma_o)v_{t+1}^1\right]}{E\left(\exp\left[(1-\gamma_o)v_{t+1}^1\right] \mid \mathfrak{F}_t\right)}\end{aligned}$$

This random variable will be featured prominently in the valuation expansions that interest us. Since its conditional expectation is one, it functions as a change in probability measure. We construct this random variable from the first-order expansion of the value function. When the underlying shocks are distributed as a multivariate standard normal, this change of probability measure appends a nonzero mean to shocks. Writing the dependence of $v_{t+1}^1$ onto the shock vector $w_{t+1}$ as $\sigma_v^1 \cdot w_{t+1}$, the implied conditional distribution for the shocks under the change of measure is

$$w_{t+1} \sim \text{Normal}\left((1-\gamma_o)\sigma_v^1, I\right) \tag{5}$$

This change of measure has the alternative interpretation from robust control whereby $\frac{1}{1-\gamma_0}$ is a penalization parameter and the change of distribution implements caution induced by a concern form model misspecification.

## 4.3 Order two

Recall

$$\tilde{r}_t = \left(\frac{1}{1-\gamma_o}\right) \log E\left(\exp\left[(1-\gamma_o)\tilde{v}_{t+1}\right] \mid \mathfrak{F}_t\right)$$

Note that

$$\frac{d\tilde{r}_t}{d\mathsf{h}} = \frac{E\left(\exp\left[(1-\gamma_o)\tilde{v}_{t+1}\right] \frac{d\tilde{v}_{t+1}}{d\mathsf{h}} \mid \mathfrak{F}_t\right)}{E\left(\exp\left[(1-\gamma_o)\tilde{v}_{t+1}\right] \mid \mathfrak{F}_t\right)},$$

and thus

$$\begin{aligned} r_t^2 = 2\tilde{r}_t^1 &= 2E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)\tilde{v}_{t+1}^1 \mid \mathfrak{F}_t\right] \\ &= E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right) v_{t+1}^2 \mid \mathfrak{F}_t\right] \end{aligned}$$

which gives $r_t^2$ as a distorted expectation of $v_{t+1}^2$.

Differentiating a second time the CES recursion for value function updating gives

$$\begin{aligned} v_t^2 =& (1-\lambda)c_t^2 + \lambda r_t^2 \\ &+ (1-\rho)\left[(1-\lambda)\left(c_t^1\right)^2 + \lambda\left(r_t^1\right)^2\right] \\ &- (1-\rho)\left[(1-\lambda)c_t^1 + \lambda r_t^1\right]^2. \end{aligned}$$

Rearranging the quadratic terms gives;

$$\begin{aligned} v_t^2 =& (1-\lambda)c_t^2 + \lambda r_t^2 + (1-\rho)\lambda(1-\lambda)\left(c_t^1 - r_t^1\right)^2 \\ =& (1-\lambda)c_t^2 + \lambda E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right) v_{t+1}^2 \mid \mathfrak{F}_t\right] + (1-\rho)\lambda(1-\lambda)\left(c_t^1 - r_t^1\right)^2 \end{aligned}$$

Finally, we use $v^2$ to compute

$$\frac{d}{d\mathsf{h}}\frac{M_{t+1}}{M_t} = \left(\frac{M_{t+1}}{M_t}\right)\left[\frac{d}{d\mathsf{h}}\left(\log M_{t+1} - \log M_t\right)\right]$$

Thus

$$\begin{aligned} \frac{M^1_{t+1}}{M^1_t} =& (1-\gamma)\left(\frac{M^0_{t+1}}{M^0_t}\right)\left(\tilde{v}^1_{t+1} - E\left[\left(\frac{M^0_{t+1}}{M^0_t}\right)\tilde{v}^1_{t+1} \mid \mathfrak{F}_t\right]\right) \\ =& \frac{(1-\gamma)}{2}\left(\frac{M^0_{t+1}}{M^0_t}\right)\left(v^2_{t+1} - E\left[\left(\frac{M^0_{t+1}}{M^0_t}\right)v^2_{t+1} \mid \mathfrak{F}_t\right]\right) \end{aligned} \qquad (6)$$

Observe that this random variable has expectation zero.

# 5 Stochastic Discount Factor Expansion

Next we consider the expansion of the stochastic discount factor:

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\exp\left[-\rho(c_{t+1} - c_t)\right]\left[\frac{\exp[(1-\gamma)v_{t+1}]}{E\left(\exp[(1-\gamma)v_{t+1}] \mid \mathfrak{F}_t\right)}\right]^{\frac{\rho-\gamma}{1-\gamma}}$$

Take logarithms:

$$\begin{aligned} s_{t+1} - s_t =& -\delta - \rho(c_{t+1} - c_t) + (\rho-\gamma)\, v_{t+1} \\ & - \left(\frac{\rho-\gamma}{1-\gamma}\right)\log E\left(\exp[(1-\gamma)v_{t+1}]\right). \end{aligned}$$

To support our representation, write:

$$\begin{aligned} \rho - \gamma &= \rho - 1 + \frac{1-\gamma_o}{\mathsf{h}} \\ \frac{\rho-\gamma}{1-\gamma} &= \mathsf{h}\left(\frac{\rho-1}{1-\gamma_o}\right) + 1 \end{aligned}$$

## 5.1 Order zero

Then the order zero approximation is:

$$
\begin{aligned}
s_{t+1}^{0} - s_{t}^{0} &= -\delta - \rho\eta_c^0 + (1-\gamma_o)\tilde{v}_{t+1}^{0} - \log E\left(\exp[(1-\gamma_o)\tilde{v}_{t+1}^{0}] \mid \mathfrak{F}_t\right), \\
&= -\delta - \rho\eta_c^0 + \log\left(\frac{M_{t+1}^{0}}{M_t^0}\right)
\end{aligned}
$$

## 5.2 Order one

The first-order approximation is:

$$
\begin{aligned}
s_{t+1}^{1} - s_{t}^{1} =& -\rho(c_{t+1}^{1} - c_t^1) + (\rho-1)\tilde{v}_{t+1}^{0} + (1-\gamma_o)\tilde{v}_{t+1}^{1} \\
& + \left(\frac{1-\rho}{1-\gamma_o}\right)\log E\left(\exp[(1-\gamma_o)\tilde{v}_{t+1}^{0}] \mid \mathfrak{F}_t\right) \\
& - (1-\gamma_o)E\left[\left(\frac{M_{t+1}^{0}}{M_t^0}\right)\tilde{v}_{t+1}^{1} \mid \mathcal{F}_t\right] \\
=& -\rho(c_{t+1}^{1} - c_t^1) \\
& + (\rho-1)\left[v_{t+1}^{1} - \frac{1}{1-\gamma_o}\log E\left(\exp[(1-\gamma_o)v_{t+1}^{1}] \mid \mathfrak{F}_t\right)\right] \\
& + \frac{(1-\gamma_o)}{2}\left(v_{t+1}^{2} - E\left[\left(\frac{M_{t+1}^{0}}{M_t^0}\right)v_{t+1}^{2} \mid \mathcal{F}_t\right]\right)
\end{aligned}
$$

### 5.3 Order two

The second-order expansion is

$$
\begin{aligned}
s_{t+1}^2 - s_t^2 = & - \rho(c_{t+1}^2 - c_t^2) \\
& + (\rho - 1)\left(\tilde{v}_{t+1}^1 - E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)\tilde{v}_{t+1}^1 \mid \mathfrak{F}_t\right]\right) \\
& + (1-\gamma_o)\left(\tilde{v}_{t+1}^2 - E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)\tilde{v}_{t+1}^2 \mid \mathcal{F}_t\right]\right) \\
& - (1-\gamma_o)E\left[\left(\frac{M_{t+1}^1}{M_t^1}\right)\tilde{v}_{t+1}^1 \mid \mathfrak{F}_t\right] \\
= & - \rho(c_{t+1}^2 - c_t^2) \\
& + (\rho - 1)\left(v_{t+1}^2 - E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)v_{t+1}^2 \mid \mathfrak{F}_t\right]\right) \\
& + \frac{(1-\gamma_o)}{3}\left(v_{t+1}^3 - E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)v_{t+1}^3 \mid \mathcal{F}_t\right]\right) \\
& - \frac{(1-\gamma_o)}{2}E\left[\left(\frac{M_{t+1}^1}{M_t^1}\right)v_{t+1}^2 \mid \mathfrak{F}_t\right]
\end{aligned} \tag{7}
$$

where $\frac{M_{t+1}^1}{M_t^1}$ is given by (6). After substituting from (6) we find that

$$
\begin{aligned}
& \frac{(1-\gamma_o)}{2}E\left[\left(\frac{M_{t+1}^1}{M_t^1}\right)v_{t+1}^2 \mid \mathfrak{F}_t\right] \\
& \quad = \frac{(1-\gamma_o)^2}{4}\left[E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)\left(v_{t+1}^2\right)^2 \mid \mathfrak{F}_t\right] - E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)v_{t+1}^2 \mid \mathfrak{F}_t\right]^2\right]
\end{aligned}
$$

The right-hand side of this equality is the variance of $v_{t+1}^2$ under the change of measure induced by $\frac{M_{t+1}^0}{M_t^0}$ scaled by $\frac{(1-\gamma_o)^2}{4}$.

# 6 Approximate Valuation

This subsection builds from and extends Borovička and Hansen (2014). Consider a growth process $G$ that is stationary in log differences and has a second-order expansion.

$$\log G_{t+1} - \log G_t \approx \eta_g^0 + \mathsf{h}\left(g_{t+1}^1 - g_t^1\right) + \frac{\mathsf{h}^2}{2}\left(g_{t+1}^2 - g_t^2\right)$$

Write

$$\log S_{t+1} - \log S_t \approx \eta_s^0 + \mathsf{h}\left(s_{t+1}^1 - s_t^1\right) + \frac{\mathsf{h}^2}{2}\left(s_{t+1}^2 - s_t^2\right) + \log M_{t+1}^0 - \log M_t^0$$

where $\eta_s^0 = -\delta - \rho\eta_c^0$ Finally,

$$\log \Phi_{t+1} \approx \phi^0 + \mathsf{h}\phi_{t+1}^1 + \frac{\mathsf{h}^2}{2}\phi_{t+1}^2.$$

We assume that $\phi^0$ is a constant process. (Note there is a notational conflict with section two that will have to be fixed.)

Let $Q = SG$. Combining the two approximations, write

$$\log Q_{t+1} - \log Q_t \approx \eta_q^0 + \mathsf{h}\left(q_{t+1}^1 - q_t^1\right) + \frac{\mathsf{h}^2}{2}\left(q_{t+1}^2 - q_t^2\right) + \log M_{t+1}^0 - \log M_t^0$$

We are interested in representing: $\upsilon$ versus $\psi$

$$\log E\left[\frac{Q_{t+1}}{Q_t}\Phi_{t+1}|\mathfrak{F}_t\right] \approx \psi^0 + \mathsf{h}\psi_t^1 + \frac{\mathsf{h}^2}{2}\psi_t^2.$$

View this a mapping from $(\phi^0, \phi^1, \phi^2)$ into $(\psi^0, \psi^1, \psi^2)$.

## 6.1 Order zero

$$\psi^0 = \phi^0 + \eta_q^0$$

## 6.2 Order one

Observe

$$\begin{aligned}
\frac{d}{d\mathsf{h}} \log E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right] &= \frac{\frac{d}{d\mathsf{h}} E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]}{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]} \\
&= \frac{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \frac{d}{d\mathsf{h}} \log\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]}{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \Phi_{t+1} \mid \mathfrak{F}_t\right]}
\end{aligned}$$

where we used the familiar relation between the derivative of a positive function and the derivative of the logarithm of that function. Notice that

$$\left.\frac{\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right)}{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]}\right|_{\mathsf{h}=0} = \frac{M^0_{t+1}}{M^0_t}$$

Thus

$$\psi^1_t = E\left[\left(\frac{M^0_{t+1}}{M^0_t}\right)\left(q^1_{t+1} - q^1_t + \phi^1_{t+1}\right) | \mathfrak{F}_t\right]$$

## 6.3 Order two

We base this calculation on

$$\begin{aligned}
\frac{d^2}{d\mathsf{h}} \log E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right] =& \frac{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \frac{d^2}{d\mathsf{h}} \log\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]}{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]} \\
&+ \frac{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \left[\frac{d}{d\mathsf{h}} \log\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right)\right]^2 \mid \mathfrak{F}_t\right]}{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]} \\
&- \left(\frac{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \frac{d}{d\mathsf{h}} \log\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]}{E\left[\left(\frac{Q_{t+1}\Phi_{t+1}}{Q_t}\right) \mid \mathfrak{F}_t\right]}\right)^2
\end{aligned}$$

Evaluating the right-hand side at the $\mathsf{h} = 0$ limits gives:

$$
\begin{aligned}
\psi_t^2 =& E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)\left(q_{t+1}^2 - q_t^2 + \phi_{t+1}^2\right)|\mathfrak{F}_t\right] \\
&+ E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)\left(q_{t+1}^1 - q_t^1 + \phi_{t+1}^1\right)^2|\mathfrak{F}_t\right] \\
&- \left(E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)\left(q_{t+1}^1 - q_t^1 + \phi_{t+1}^1\right)|\mathfrak{F}_t\right]\right)^2
\end{aligned}
\tag{8}
$$

### 6.3.1 Observations

- As noted previously, the $\left(\frac{M_{t+1}^0}{M_t^0}\right)$ is treated most conveniently as a change of the shock distribution. See (5).

- There is no need to compute $v_{t+1}^3$ in the second-order term for the stochastic discount factor because this term enters as:

$$
\left(v_{t+1}^3 - E\left[\left(\frac{M_{t+1}^0}{M_t^0}\right)v_{t+1}^3 \mid \mathfrak{F}_t\right]\right)
$$

  and thus expectation under the change of probability measure is zero. See (7).

- The second two terms on the right-hand side of (8) combined are the conditional variance for $q_{t+1}^1 - q_t^1 + \phi_{t+1}^1$ under the change in probability distribution.

## 6.4 Computing a present value

Let $\pi_t = \pi(X_t, \mathsf{h})$ denote date $t$ the log price-dividend ratio associated with the process $G$. Then

$$
\pi_t = \log \mathbb{E}\left(\frac{Q_{t+1}}{Q_t}\left[1 + \exp\left(\pi_{t+1}\right)\right] \mid \mathfrak{F}_t\right)
$$

Thus let

$$\phi_{t+1} = \log \Phi_{t+1} = \log \left[1 + \exp\left(\pi_{t+1}\right)\right]$$

Write:

$$\begin{aligned}
\phi^0 &= \log \left[1 + \exp\left(\pi^0\right)\right] \\
\phi^1_{t+1} &= \left[\frac{\exp\left(\pi^0\right)}{1 + \exp\left(\pi^0\right)}\right] \pi^1_{t+1} \\
\phi^2_{t+1} &= \left[\frac{\exp\left(\pi^0\right)}{1 + \exp\left(\pi^0\right)}\right] \pi^2_{t+1} + \left[\frac{\exp\left(\pi^0\right)}{\left[1 + \exp\left(\pi^0\right)\right]^2}\right] \left(\pi^1_{t+1}\right)^2
\end{aligned}$$

We now look for a fixed point as we have a mapping from $\pi_{t+1}$ to $\pi_t$ and we know that $\pi_t = \pi(x_t, \mathsf{h})$, a time invariant function of $x_t$. Thus we use this to compute order zero, order one and order two approximations for $\pi$. Specifically,

$$\begin{aligned}
\pi^0 =& \phi(X^0, 0) \\
\pi^1_t =& \frac{\partial \phi}{\partial x}(X^0, 0) \cdot X^1_t + \frac{\partial \phi}{\partial \mathsf{h}}(X^0, 0) \\
\pi^2_t =& \frac{\partial \phi}{\partial x}(X^0, 0) \cdot X^2_t + (X^1_t)' \frac{\partial^2 \phi}{\partial x \partial x'}(X^0, 0) X^1_t \\
&+ \frac{\partial \phi}{\partial x \partial \mathsf{h}}(X^0, 0) \cdot X^1_t + \frac{\partial \phi}{\partial \mathsf{h}^2}(X^0, 0)
\end{aligned}$$

The $\pi$ function, its derivative vectors and matrices evaluated at $(X^0, 0)$ will not be computed directly but will deduced from the fixed point solution. The distortions in mean of $W_{t+1}$ will contribute the derivatives with respect to $\mathsf{h}$.

- Specify the second order approximations for $\log G_{t+1} - \log G_t$ and $\log C_{t+1} - \log C_t$ as constant, linear and quadratic functions of $(X_t, W_{t+1})$.
- Compute the log continuation value relative to consumption for $\gamma_o = 1$. Then compute the more general value function expansion to deduce second-order approximation for $\log S_{t+1} - \log S_t$ and implied mean distortion for the shock vector.

- Use the valuation recursion to produce a matrix equation to solve for the first-order approximation.
- Use the first-order approximation and the valuation recursion to produce a matrix equation to solve for the coefficients for the second-order approximation.
- Be sure the use the change of mean for the shock vector for this computation.

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