Stochastic growth models*

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1 Adjustment cost model

Let K_t be capital, I_t investment, C_t capital, and Z_t a shock process governed by

$$Z_{t+1} = \exp\left(-\kappa\right) Z_t + \sigma_z \cdot W_{t+1}$$

where $\kappa > 0$, $\{W_{t+1}\}$ is a two dimensional i.i.d. process with mean vector zero and covariance matrix I. The unconditional mean of Z_t in a stochastic steady state is zero. The feasibility constraint is

$$C_t + I_t = \mathbb{A} K_t \tag{1}$$

where $\mathbb{A}$ is a fixed parameter. Dividing the above equation by K_t gives

$$\frac{C_t}{K_t} + \frac{I_t}{K_t} = \mathbb{A}. \tag{2}$$

Where Φ is a concave function and $\Phi'(0) = 1$, the capital accumulation technology is

$$K_{t+1} = K_t \left[1 + \Phi\left(\frac{I_t}{K_t}\right)\right] \exp\left(-\alpha_k + Z_t + \sigma_k \cdot W_{t+1} - \frac{1}{2}|\sigma_k|^2\right), \tag{3}$$

where $\alpha_k > 0$ captures a rate of depreciation and σ_k is a 2×1 vector contributing to a random increment in the capital accumulation. Taking logarithms of both sides of the

*We thank Dongchen Zou for helping with computations and Balint Szoke for comments.

preceding equation gives

$$\log K_{t+1} = \log K_t + \log \left[1 + \Phi \left(\mathbb{A} - \frac{C_t}{K_t} \right) \right] - \alpha_k + Z_t - \frac{1}{2} |\sigma_k|^2 + \sigma_k \cdot W_{t+1}. \tag{4}$$

1.1 Unitary elasticity of substitution

We impute to a planner the one-period utility function $[1 - \exp(-\delta)] \log C_t$, where $\delta > 0$ is a discount rate. A sequence $\{V_t\}$ of discounted sums of current and future expected utilities satisfies

$$\log V_t = [1 - \exp(-\delta)] \log C_t - \exp(-\delta)\theta \log \mathcal{E} \left[\exp \left(-\frac{\log V_{t+1}}{\theta} \right) | \mathfrak{F}_t \right].$$

where $\theta \in [\underline{\theta}, +\infty]$ is the multiplier parameter in the recursive multiplier preferences of **?**. We can also interpret $\frac{1}{\theta}$ as the risk-sensitivity parameter in the risk-sensitive recursion of **?**. This recursion for values imposes a unitary intertemporal elasticity of substitution between current period utility and a one-period-ahead continuation value, a specification we will later generalize to allow for a non-unitary elasticity of substitution. A planner maximizes V_0 by choosing $\frac{C_t}{K_t}$ and $\frac{I_t}{K_t}$ for $t \geqslant 0$ subject to constraints (2) and (4) and initial conditions for K_0 and Z_0.

We can find an optimal value sequence $\{V_t\}$ and decision rules that attain it by applying the following guess-and-verify method. It is convenient to represent the one-period utility function as

$$[1 - \exp(-\delta)] \log C_t = [1 - \exp(-\delta)] (\log C_t - \log K_t) + [1 - \exp(-\delta)] \log K_t.$$

Let f be an affine function

$$f(Z) = f_0 + f_z Z$$

and guess that

$$\log V_t = \log K_t + f(Z_t)$$

where

$$f(Z_t) = \max_c \{ [1 - \exp(-\delta)] \log c$$
$$+ \exp(-\delta) \left[f_0 + \exp(-\kappa) Z_t f_z + \frac{1}{2\theta} \mid f_z \sigma_z + \sigma_k \mid^2 \right]$$

$$
\begin{aligned}
&+ \exp(-\delta)\left(\log\left[1+\Phi\left(\mathbb{A}-c\right)\right]\right) \\
&+ \exp(-\delta)\left(-\alpha_k + Z_t - \frac{1}{2}|\sigma_k|^2\right)\Bigg\}.
\end{aligned}
$$

The optimal choice of c is independent of the state, which implies that consumption-capital and investment-capital ratios are both constant. Furthermore

$$
f_z = \frac{\exp(-\delta)}{1-\exp(-\delta)\exp(-\kappa)}.
$$

Net of the contribution from Z, the implied growth rate for consumption is

$$
\alpha_c = -\alpha_k + \log\left[1+\Phi\left(\mathbb{A}-c^*\right)\right] - \frac{1}{2}|\sigma_k|^2
$$

where c^* is the optimal consumption capital ratio and the shock exposure vector for the logarithm of consumption is $\sigma_c = \sigma_k$. Associated with the planner's optimal value function $\log V_t = \log K_t + f(Z_t)$ is the log stochastic discount factor ratio

$$
\begin{aligned}
\log S_{t+1} - \log S_t = &-\delta + \left(\log K_t - \log K_{t+1}\right) \\
&- \frac{1}{\theta}\left(f_z\sigma_z\cdot W_{t+1} + \sigma_k\cdot W_{t+1}\right) - \frac{1}{2\theta^2}\mid f_z\sigma_z + \sigma_k \mid^2 .
\end{aligned}
$$

1.2 Relaxing the unitary elasticity assumption

Let ρ be the intertemporal elasticity of substitution. Up to now we have assumed that $\rho = 1$. We now extend preferences to allow ρ not to equal one. We solve the model by using a small noise approximation in which we scale the shock vector by q. To do so, we begin with the specification

$$
\begin{aligned}
V_t &= \left(\left[1-\exp(-\delta)\right]\left(C_t\right)^{1-\rho} + \exp(-\delta)\left[\mathcal{R}\left(V_{t+1}\mid\mathfrak{F}_t\right)\right]^{1-\rho}\right)^{\frac{1}{1-\rho}} \\
\mathcal{R}\left(V_{t+1}\mid\mathfrak{F}_t\right) &= \left(E\left[\left(V_{t+1}\right)^{1-\gamma}\mid\mathfrak{F}_t\right]\right)^{\frac{1}{1-\gamma}}
\end{aligned}
$$

Take logarithms and write:

$$
\begin{aligned}
\log V_t = &\frac{1}{1-\rho}\log\left(\left[1-\exp(-\delta)\right]\exp[(1-\rho)\log C_t)\right] \\
&+ \exp(-\delta)\exp\left[(1-\rho)\log\mathcal{R}\left(V_{t+1}\mid\mathfrak{F}_t\right)\right])
\end{aligned}
$$

where

$$\log \mathcal{R}\left(V_{t+1} \mid \mathfrak{F}_t\right) = -\mathsf{q}\theta \log \mathcal{E}\left(\exp\left[-\frac{1}{\mathsf{q}\theta} \log V_{t+1}\right] \mid \mathfrak{F}_t\right)$$

where $\mathsf{q}\theta = \frac{1}{\gamma-1}$.

In deriving the first-order conditions, it is helpful to construct the following: Form the corresponding marginal utility processes

$$\begin{aligned} MC_t &= [(1-\exp(-\delta)](C_t)^{-\rho}(V_t)^{\rho} \\ MR_t &= \exp(-\delta)(R_t)^{-\rho}(V_t)^{\rho} \end{aligned}$$

where $R_t \equiv \mathcal{R}\left(V_{t+1} \mid \mathfrak{F}_t\right)$. Let $\mathcal{L}_t$ be the multiplier on the capital evolution (3) (the co-state) and $\mathcal{M}_t$ be the multiplier on the feasibility restriction (1).

We have the following three first-order conditions.

- First-order conditions for consumption:

$$MC_t = \mathcal{M}_t \tag{5}$$

- First-order conditions for investment:

$$\mathcal{M}_t = (MR_t)E\left[\left(\frac{V_{t+1}}{R_t}\right)^{-\gamma} \mathcal{L}_{t+1}G_{t+1} \mid \mathfrak{F}_t\right] \Phi'\left(\frac{I_t}{K_t}\right)$$

- Co-state evolution:

$$\mathcal{L}_t = (MR_t)E\left[\left(\frac{V_{t+1}}{R_t}\right)^{-\gamma} \mathcal{L}_{t+1}G_{t+1} \mid \mathfrak{F}_t\right]\left[1 + \Phi\left(\frac{I_t}{K_t}\right) - \left(\frac{I_t}{K_t}\right)\Phi'\left(\frac{I_t}{K_t}\right)\right] + \mathbb{A}\mathcal{M}_t$$

Using the first-order conditions for consumption and resource constraint, we find

$$\mathcal{L}_t = (MR_t)E\left[\left(\frac{V_{t+1}}{R_t}\right)^{-\gamma} \mathcal{L}_{t+1}Z_{t+1} \mid \mathfrak{F}_t\right]\left[1 + \Phi\left(\frac{I_t}{K_t}\right)\right] + \left(\frac{C_t}{K_t}\right)\mathcal{M}_t$$

Substituting for Z_{t+1} from the capital evolution:

$$\mathcal{L}_t = (MR_t)E\left[\left(\frac{V_{t+1}}{R_t}\right)^{-\gamma}\mathcal{L}_{t+1}\left(\frac{K_{t+1}}{K_t}\right) \mid \mathfrak{F}_t\right] + \left(\frac{C_t}{K_t}\right)\mathcal{M}_t.$$

Finally, multiplying by K_t gives:

$$\mathcal{L}_t K_t = (MR_t)E\left[\left(\frac{V_{t+1}}{R_t}\right)^{-\gamma}\mathcal{L}_{t+1}K_{t+1} \mid \mathfrak{F}_t\right] + C_t\mathcal{M}_t$$

This equation is satisfied by setting $\mathcal{L}_t K_t = V_t$ and applying Euler's Theorem applied to the utility recursion. Thus the co-state evolution and the forward looking continuation-value evolution are redundant. For purposes of computation, we need only to include the continuation-value evolution.

Next rewrite the first-order conditions for investment by substituting from the capital evolution equation:

$$\begin{aligned}
\mathcal{M}_t &= (MR_t)E\left[(V_{t+1})^{-\gamma}(R_t)^{\gamma}\left(\frac{V_{t+1}}{K_{t+1}}\right)Z_{t+1}|\mathfrak{F}_t\right]\Phi'\left(\frac{I_t}{K_t}\right) \\
&= \left(\frac{MR_t}{K_t}\right)(R_t)^{\gamma}E\left[(V_{t+1})^{1-\gamma}|\mathfrak{F}_t\right]\left[\frac{\Phi'\left(\frac{I_t}{K_t}\right)}{1+\Phi\left(\frac{I_t}{K_t}\right)}\right] \\
&= \exp(-\delta)\left(\frac{R_t}{K_t}\right)^{1-\rho}\left(\frac{V_t}{K_t}\right)^{\rho}\left[\frac{\Phi'\left(\frac{I_t}{K_t}\right)}{1+\Phi\left(\frac{I_t}{K_t}\right)}\right]
\end{aligned}$$

Thus from the first-order conditions (5) for consumption:

$$\left[1+\Phi\left(\frac{I_t}{K_t}\right)\right][\exp(\delta)-1]\left(\frac{C_t}{K_t}\right)^{-\rho}\left(\frac{R_t}{K_t}\right)^{\rho-1} = \Phi'\left(\frac{I_t}{K_t}\right) \tag{6}$$

where

$$\left(\frac{R_t}{K_t}\right)^{\rho-1} = \left(E\left[\left(\frac{V_{t+1}}{K_{t+1}}\right)^{1-\gamma}\left(\frac{K_{t+1}}{K_t}\right)^{1-\gamma} \mid \mathfrak{F}_t\right]\right)^{\frac{\rho-1}{1-\gamma}}$$

To construct approximations, we transform variables as follows. There is one control, $\log C_t - \log K_t$, one exogenous state variable, Z_t, and one forward-looking variable, namely, $\log V_t - \log K_t$. Note that $\log C_t - \log K_t$ determines the evolution of the key component to the increment $\log K_{t+1} - \log K_t$ needed to update the continuation value.

1.3 Order zero approximation

Let $g^0 = \log K^0_{t+1} - \log K^0_t$, $c^0 = \log C^0_t - \log K^0_t$, $v^0 = V^0_t - \log K^0_t$ and $Z_t = z = 0$. From the capital evolution:

$$\exp(g^0) = \left(1 + \Phi\left[\mathbb{A} - \exp(c^0)\right]\right)\exp(-\alpha_k)$$

From relation (6),

$$\exp(-\rho c^0)\exp[(\rho-1)v^0]\exp[(\rho-1)g^0)][\exp(\delta)-1][1+\Phi(\mathbb{A}-\exp(c^0)] = \Phi'[\mathbb{A}-\exp(c^0)],$$

From the utility recursion we have the equation

$$\exp\left[(1-\rho)v^0\right] = (1-\exp(-\delta))\exp\left[(1-\rho)c^0\right] + \exp(-\delta)\exp\left[(1-\rho)v^0\right]\exp\left[(1-\rho)g^0\right].$$

We solve these three equations for $\exp(c^0)$, $\exp(k^0)$, and $\exp[(1-\rho)v^0]$.

1.4 First order adjustment for the continuation values

The first-order approximation to the continuation value is special in how it induces constant terms beyond those that occur in order zero approximations. In effect, a precautionary term emerges here.

First represent the order one approximation for the continuation values recursively as:

$$\begin{aligned} V^1_t - \log K^1_t = &\frac{[1-\exp(-\delta)]\exp\left[(1-\rho)c^o\right]\left(\log C^1_t - \log K^1_t\right)}{[1-\exp(-\delta)]\exp\left[(1-\rho)c^0\right] + \exp(-\delta)\exp\left[(1-\rho)(v^0+g^0)\right]} \\ &+ \frac{\exp(-\delta)\exp\left[(1-\rho)(v^0+g^0)\right]\mathcal{E}\left(\log V^1_{t+1} - \log K^1_{t+1} + \log K^1_{t+1} - \log K^1_t \mid \mathfrak{F}_t\right)}{[1-\exp(-\delta)]\exp\left[(1-\rho)c^0\right] + \exp(-\delta)\exp\left[(1-\rho)(v^0+g^0)\right]} \end{aligned} \tag{7}$$

which gives the approximation net of a constant term. Here we use that the log of the expectation of a log-normally distributed random variable is the log of the expectation plus one half the conditional variance.

Given the lognormality of the first-order approximation, the term $\log \mathcal{R}(\cdot)$ can be expressed as a conditional mean plus one half the conditional variance where the conditional variance is constant. We will use this in the computation. The presence of the constant term in the first-order approximation to the continuation value will induce constant terms in the first-order approximations for other variables as functions of the state. This constant term can be computed as follows.

i) Use the first-order approximations to the evolution of the forward-looking variable, $\log V_t - \log K_t$ along with exogenous state Z_t with the conditional expectation $\mathcal{E}\left(\cdot \mid \mathfrak{F}_t\right)$ in place of $\log \mathcal{R}_t(\cdot)$ as in (7) for the first-order approximation for the continuation value. This allows us to target the dependence of $\log V_t^1 - \log K_t$ and Z_t and the $\log K_{t+1} - \log K_t$ on Z_t and W_{t+1}.

ii) Use outcomes of the previous step to compute the constant risk adjustment that is induced by $\log \mathcal{R}_t(\cdot)$. Write

$$V_{t+1}^1 - \log K_t^1 = \left(V_{t+1}^1 - \log K_{t+1}^1\right) + \left(\log K_{t+1}^1 - \log K_t^1\right)$$

Then the constant for the linearized recursion is:

$$\frac{(1-\gamma)}{2} |\mathbb{J}_v + \mathbb{J}_k|^2$$

where $\mathbb{J}_v \cdot W_{t+1}$ is the innovation in $\left(V_{t+1}^1 - \log K_{t+1}^1\right)$ and $\mathbb{J}_k \cdot W_{t+1}$ is the innovation in $\log K_{t+1} - \log K_t$ implied by the step ii computation.

iii) Compute remaining constants as follows. We compute constant terms for $\log C_t^1 - \log K_t^1 = c^1$, $\log K_{t+1}^1 - \log K_{t+1}^1 = g^1$, $V_t - \log K_t^1 = v^1$, as follows. Take linearized continuation value equation, the linearized capital evolution equation and the linearized consumption capital equation, plug in the unknown constant terms and include the computed constant term for the linearized continuation value updating equation. Solve for the first-order constants: c^1, g^1, v^1.

1.5 Log stochastic discount factor

$$\log S_{t+1} - \log S_t = -\delta - \rho \left(\log C_{t+1} - \log C_t\right) + (\rho - \gamma)\left[V_{t+1} - \mathcal{R}_t\left(V_{t+1}; \mathsf{q}\right)\right]$$

Order zero approximation:

$$\log S_{t+1}^0 - \log S_t^0 = -\delta - \rho g$$

Order one approximation for the stochastic discount factor (net of constant terms)

$$\begin{aligned}\log S_{t+1}^1 - \log S_t^1 = & -\rho\left(\log C_{t+1}^1 - \log K_{t+1}^1 - \log C_t^1 + \log K_t^1\right)\\ & + (\rho - \gamma)\left[V_{t+1}^1 - \log K_t^1 - \log \mathcal{E}\left(V_{t+1}^1 - \log K_t^1 | \mathfrak{F}_t\right) - \frac{(1-\gamma)}{2}|\mathbb{J}_v + \mathbb{J}_k|^2\right]\\ & - \rho\left(\log K_{t+1}^1 - \log K_t^1\right)\end{aligned}$$

1.6 Calibration

Lars XXXXX: this section is cryptic. So you want to add a few words? Tom XXXXXX We could add some empirical targets.

Free parameters $\mathbb{A}$, ϕ, α_k and δ along with equations (**??**) and (**??**) along with the risk free rate:

$$r^f = \delta + \rho g$$

and the investment capital ratio

$$\mathbb{A} - c = i$$

We use observations on i, c, r^f and g for a given value of ρ.

1.7 Transformed shocks

We can transform shocks and the matrix σ so that one shock has permanent consequences and the other is temporary. We do this by first computing the limiting impulse response vector

$$\tilde{\varsigma}_1 = \sigma_k + \frac{1}{1 - \exp(-\kappa)}\sigma_z.$$

Then we construct two vectors, the first we call ς_1. It is proportional to $\tilde{\varsigma}_1$ but rescaled to have unit norm. The second is orthogonal to ς_1 and also scaled to have unit norm. Call this second vector ς_2. Then form

$$\sigma^* = \sigma\varsigma$$

where

$$\varsigma = \begin{bmatrix}\varsigma_1 & \varsigma_2\end{bmatrix}$$

and note that

$$\sigma^*(\sigma^*)' = \sigma\sigma'.$$

and that consequently

$$\sigma W_t = \varsigma^* W_t^*$$

where $\varsigma^* = \sigma^* \varsigma^{-1}$. The limiting impulse response of log consumption and log capital to the second transformed shock W_1^* should be zero.

RA request: Please implement these proposed calculations with the following parameter values:

$$\begin{aligned} \alpha_c &= .373 \times .01 \\ \alpha_z &= 0 \qquad \kappa = .017 \end{aligned}$$

$$\sigma = \begin{bmatrix} (\sigma_k)' \\ (\sigma_z)' \end{bmatrix} = .01 \times \begin{bmatrix} .481 & 0 \\ .012 & .027 \end{bmatrix} \tag{8}$$

2 Permanent income model

We modify members of a class of models described by **?**, ch. 11 to capture rational expectations versions of Milton Friedman's permanent income model of consumption. Our key extension is to assume that nonfinancial income, the key exogenous driving process is a multiplicative functionals instead of the covariance stationary process usually assumed.

Let $W_{t+1} \sim \mathcal{N}(0, I)$ be an i.i.d. vector process and let $\{Y_t\}$ be the logarithm of an exogenous nonfinancial income process that is governed by an additive functional

$$Y_{t+1} - Y_t = \mathbb{D}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1} + \nu$$

where

$$X_{t+1} = \mathbb{A}_x X_t + \mathbb{B}_x W_{t+1}$$

and A is a stable matrix. Define

$$Y_t = Y_t^0 + Y_t^1$$

where $Y_t^0 = \bar{y}_0 + t\nu$ and

$$Y_{t+1}^1 - Y_t^1 = \mathbb{D}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1}.$$

Let $\widehat{C}_t$ be the logarithm of consumption at date t and let $\widehat{K}_t$ be the stock of a risk-free asset (or a debt, if it is negative). The asset bears a fixed risk-free one-period return equal to $\exp(\rho)$. Feasibility at time t requires

$$\widehat{K}_{t+1} + \exp\left(\widehat{C}_t\right) = \exp(\rho)\widehat{K}_t + \exp(Y_t). \tag{9}$$

Stochastic growth in the multiplicative functional $\{Y_t\}$ makes it convenient to scale variables by Y_t, so we define $C_t = \widehat{C}_t - Y_t$ and $K_t = \frac{\widehat{K}_t}{\exp(Y_t)}$. Dividing both sides of equation (9) by $\exp(Y_t)$ yields

$$K_{t+1} \exp\left(Y_{t+1} - Y_t\right) + \exp\left(C_t\right) = \exp(\rho)K_t + 1. \tag{10}$$

A representative consumer with logarithmic one period utility of consumption, discounted time separable preferences, and subjective discount rate δ optimally chooses a consumption process that satisfies the Euler equation

$$\exp(-\delta + \rho)E\left[\exp(\widehat{C}_t - \widehat{C}_{t+1})|X_t\right] = 1.$$

This Euler equation can also be expressed as

$$\exp(-\delta + \rho) E \left[\exp(C_t - C_{t+1} + Y_t - Y_{t+1}) | X_t \right] = 1.$$

2.1 Steady state

We assume that

$$\exp(-\delta + \rho - \nu) = 1, \tag{11}$$

where recall that δ is the discount rate in preferences, ρ is the risk-free rate of return, and ν is the rate of growth in the deterministic part of nonfinancial income Y_t^0. Restriction (11) implies a steady state in which the log consumption-log income ratio equals $\bar{c}$. Steady state means of asymptotically stationary components of (C_t, K_t) must satisfy

$$\bar{k} \exp(\nu) + \exp(\bar{c}) = \exp(\rho) \bar{k} + 1$$

or equivalently

$$\exp(\bar{c}) = \left[\exp(\rho) - \exp(\nu) \right] \bar{k} + 1,$$

where we assume that $\rho > \nu$. Notice that we are free to set $\bar{k}$. For convenience, we assume that $\bar{k} = 0$ and hence that $\exp(\bar{c}) = 1$. Tom XXXXX: say more here.

2.2 First-order approximation

We scale W_{t+1} by q, let q tend to zero, and obtain a first-order small-noise approximation around a steady state. We define some processes that we use to construct approximations in terms of the following notation. Processes with superscripts 1 are first-order derivatives of corresponding original processes with respect to q evaluated at $\mathsf{q} = 1$. A first-order approximation of the feasibility restriction is

$$\begin{aligned} K_{t+1}^1 \exp(\nu) + \bar{k} \exp(\nu) \left(Y_{t+1}^1 - Y_t^1 \right) + \exp(\bar{c}) C_t^1 \\ = \exp(\rho) K_t^1 \end{aligned}$$

or

$$K_{t+1}^1 = \exp(\rho - \nu) K_t^1 - \bar{k} \left(Y_{t+1}^1 - Y_t^1 \right) - \exp(\bar{c} - \nu) \exp(\bar{c}) C_t^1. \tag{12}$$

The restrictions $exp(\bar{c}) = 1$ and $\bar{k} = 0$ make equation (12) become

$$K^1_{t+1} = \exp(\rho - \nu) K^1_t - \exp(-\nu) C^1_t \tag{13}$$

Lars and Tom: Should we make a decision for the entire section about what we describe as an individual representative agent problem and what we call a planning problem? We can certainly do this and even give the individual arrow prices if we like. To derive a first-order approximation to an optimal decision rule for consumption, we solve equation (12) forward, take conditional expectations, and impose consumption Euler equations.[1] To begin, solve equation (12) forward and then take conditional expectations of future random variables to get

$$\exp(\nu) K^1_t = \sum_{j=0}^{\infty} \lambda^{j+1} E\left(C^1_{t+j} + Y^1_{t+j} \mid \mathcal{F}_t\right) - \sum_{j=0}^{\infty} \lambda^{j+1} E\left(Y^1_{t+j} \mid \mathcal{F}_t\right). \tag{14}$$

A first-order approximation to the Euler equation for the planning problem is

$$E\left[C^1_{t+1} + Y^1_{t+1} | \mathcal{F}_t\right] = C^1_t + Y^1_t.$$

Substituting this approximation to the Euler equation repeatedly into equation (14) gives

$$\exp(\nu) K^1_t = \left(\frac{\lambda}{1-\lambda}\right)(C^1_t + Y^1_t) - \left(\frac{\lambda}{1-\lambda}\right) \sum_{j=1}^{\infty} \lambda^j E\left(Y^1_{t+j} - Y^1_{t+j-1} \mid \mathcal{F}_t\right) - \left(\frac{\lambda}{1-\lambda}\right) Y^1_t$$

or

$$\exp(\nu) K^1_t = \left(\frac{\lambda}{1-\lambda}\right) C^1_t - \left(\frac{\lambda}{1-\lambda}\right) \sum_{j=1}^{\infty} \lambda^j E\left(Y^1_{t+j} - Y^1_{t+j-1} \mid \mathcal{F}_t\right),$$

where $\lambda = \exp(\nu - \rho)$. Solve the above equation to obtain the approximation

$$C^1_t = \frac{\exp(\nu)(1-\lambda)}{\lambda} K^1_t + \sum_{j=1}^{\infty} \lambda^j E(Y^1_{t+j} - Y^1_{t+j-1} | \mathcal{F}_t) \tag{15}$$

that links an optimal log consumption-income ratio to two financial income (as a function of K^1_t) and nonfinancial income. The approximating decision rule (15) implies that C^1_t and K^1_t are cointegrated with cointegrating vector $\begin{bmatrix} 1 & [\exp(\nu) - \exp(\rho)] \end{bmatrix}$ and that non-

[1] With or without taking conditional expectations of all time indexed variables, We can solve equation (12) forward. Cite one of many places where we have done this.

financial income's contribution to the log consumption-income ratio is

$$\sum_{j=1}^{\infty} \lambda^j \mathbb{E}(Y_{t+j}^1 - Y_{t+j-1}^1 | \mathcal{F}_t) = \lambda \mathbb{D}_y^T (I - \lambda A_x)^{-1} X_t$$

Decision rule (15) is a first-order approximation to an optimal decision rule similar to the one in a **?** economy without robustness (i.e., the problem that is obtained by setting their $\sigma = 0$).

2.3 Impulse responses

As an example, we evaluate first-order approximate decision rules for consumption and assets at a nonfinancial income process adapted from **?**. They assumed two components of nonfinancial income, one more persistent than the other. To construct the first component, let

$$X_{1,t+1}^1 = .704 X_{1,t} + \begin{bmatrix} .144 & 0 \end{bmatrix} W_{t+1}$$

where $Y_{1,t+1}^1 = Y_t^1 + X_{1,t+1}^1$. To construct the second component, let

$$X_{2,t+1}^1 = X_{2,t}^1 - .154 X_{2,t-1}^1 + \begin{bmatrix} 0 & .206 \end{bmatrix} W_{t+1}$$

and construct $Y_{2,t+1}^1 = X_{2,t+1}^1$. Let $Y_{t+1}^1 = (.01) Y_{1,t+1}^1 + (.01) Y_{2,t+1}^2$.[2] We represent this $\{Y_t\}$ process as an additive functional. Set $\rho = .00663$ and $\nu = .00373$.

[2] We take income numbers from the first column of Table 2 of **?** with two modifications. In **?**, both income processes are stationary but one has an autoregressive root of .998. We set this to one here. This has a nontrivial impact on the consumption volatility, which **?** estimated in levels. We scale both innovation standard deviations by 1.33 to achieve a consumption growth rate volatility of .482 expressed as a percent (log differences multiplied by 100).

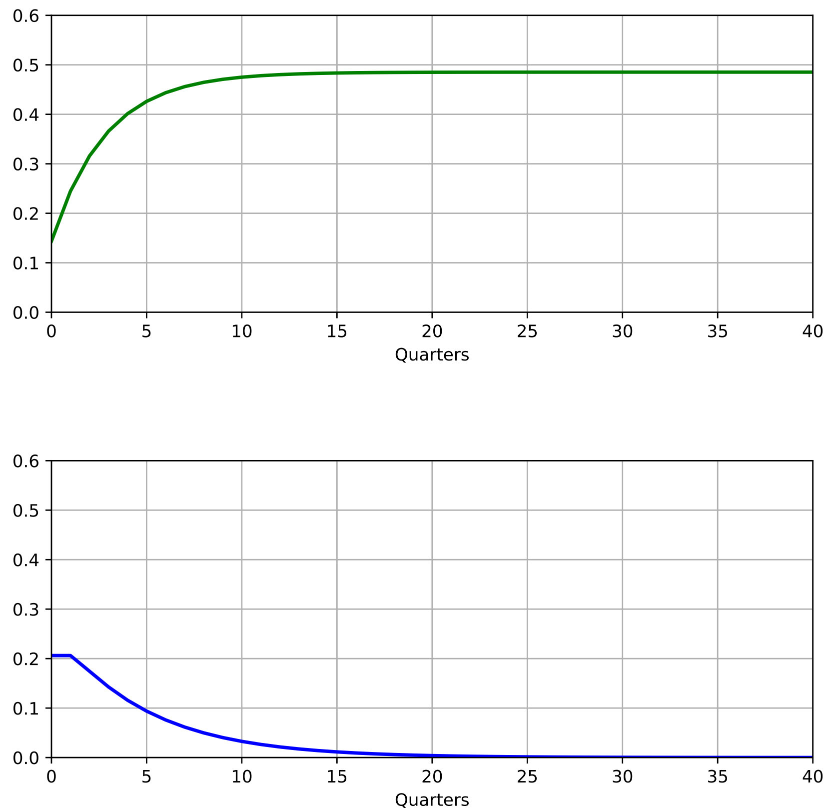

Figure 1: Impulse responses of log income to the two shock processes. Top panel: permanent shock. Bottom panel: transitory shock. Parameter settings from **?**.

As lag length on the horizontal axis becomes large, the positive limit of the impulse response coefficients for the first shock in figure 1 tell us that the first shock has permanent effects on nonfinancial income; the zero limit of the impulse response coefficients for the second shock tells us that it has only transitory effects. The planner can use savings partially to self-insure against the transitory shock via precautionary savings but cannot use savings to insure against the permanent shock. Under the time separable preferences being used here, the responses of consumption to shocks are both constant across horizons; so impulse response functions to both shocks are just step functions. This is an implication of the consumption smoothing built into the model. One hundred times the response of consumption responses to the two shocks are:[3]

$$\text{permanent shock} = .482$$
$$\text{transitory shock} = .00383,$$

which shows the dramatic differences induced by the endogenous consumption-savings responses.

Present values of impulse response functions of consumption and non-financial income to both shocks display a tell-tale feature described by **?**. Define the consumer's time t "deficit" as consumption minus non-financial income. Then **?** note that discounted by λ the present value of the impulse response function of the deficit is zero for each shock. In particular, temporarily let $\{w_j\}_{j=0}^{\infty}$ be an impulse response of the log consumption/non-financial income ratio to a shock. Then form the sequence $\{s_j\}_{j=0}^{\infty}$ whose jth element is

$$s_j = \sum_{m=0}^{j} \lambda^m w_m.$$

We plot this object for both shocks in figure 2. For each shock, **?** tell us to expect that $\lim_{j\to+\infty} s_j = 0$, which evidently prevails. Figure 2 plots these discounted sums over the alternative horizons in order to study how quickly they converge to zero, albeit slowly.

[3] We multiplied these objects by 100 in order to drop two zeros from the reported numbers.

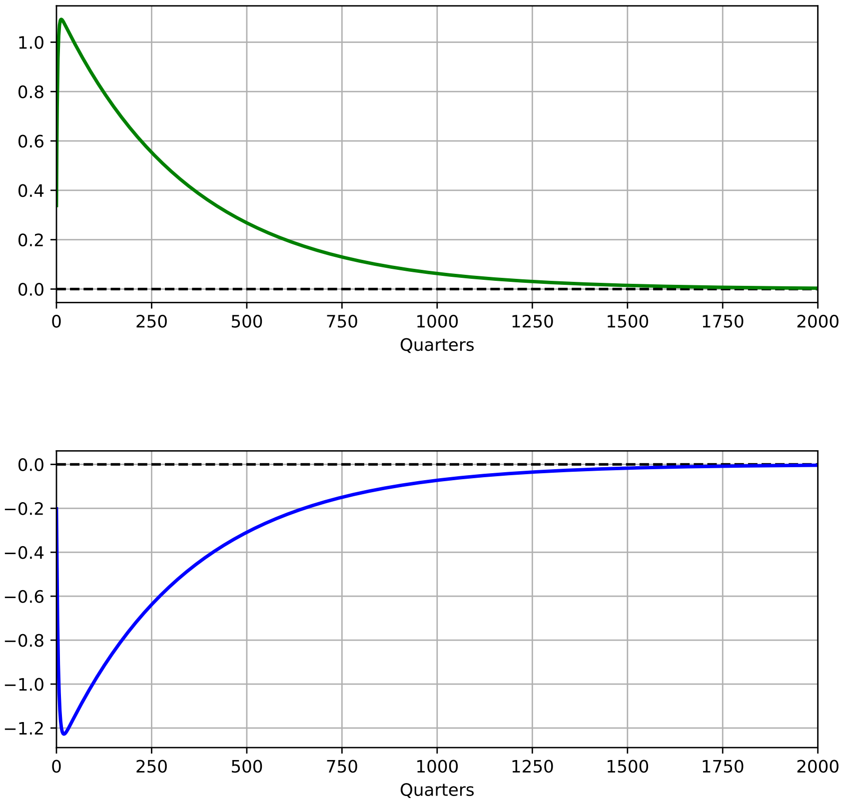

Figure 2: Responses of $s_j = \sum_{m=0}^{j} \lambda^m w_m$ for the "deficit" for two shock processes. Top panel: permanent shock. Bottom panel: transitory shock. Parameter settings from **?**. Notice how the permanent shock leads to an immediate deficit while a temporary shock leads to an immediate surplus; each of these ultimately completely evaporates.

2.4 Robustness

Following **?**, we now attribute a concern about robustness to the planner by changing the utility recursion to:

$$V_t = [1 - \exp(-\delta)](C_t + Y_t) - \exp(-\delta)\theta \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right)|\mathcal{F}_t\right]$$

where V_t is a date t continuation value and $\theta \leqslant +\infty$. Setting $\theta = \infty$ eliminates concerns about robustness and returns us to time-separable log utility. The parameter ξ is the inverse of the risk sensitivity parameter of **?**. The first-order approximation to a continuation value process is of the form

$$V_t^1 = [1 - \exp(-\delta)](C_t^1 + Y_t^1) - \exp(-\delta)\theta \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)|\mathcal{F}_t\right]$$

where $F_c \cdot W_{t+1}$ gives the response of C_{t+1}^1 to W_{t+1}. To obtain an observational equivalence finding like that of **?**, we guess that the evolution of $C_{t+1}^1 + Y_{t+1}^1$ satisfies:

$$C_{t+1}^1 + Y_{t+1}^1 = C_t^1 + Y_t^1 + (F_c + F_y) \cdot W_{t+1}. \tag{16}$$

Then

$$V_t^1 = C_t^1 + Y_t^1 - \frac{|F_c + F_y|^2}{2\xi}$$

and

$$V_{t+1}^1 = C_t^1 + Y_t^1 - \frac{|F_c + F_y|^2}{2\xi} + (F_c + F_y) \cdot W_{t+1}.$$

In effect, the planner's concern about robustness induces him to construct V_t^1 by changing the measure of W_{t+1} from normal with mean 0 and covariance matrix I to normal with mean $-\frac{1}{\xi}(F_c + F_y)$ and covariance matrix I. This change of measure can be accomplished by multiplying the conditional density of W_{t+1} under a baseline normal $(0, I)$ model with the following positive random variable:

$$M_{t+1}^0 = \frac{\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)}{E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)|\mathcal{F}_t\right]}. \tag{17}$$

The random variable M_{t+1}^0 can serve as a likelihood ratio because (a) it is nonnegative, and (b) its conditional expectation equals one. Because a robust planner acts as if he is an ordinary planner who evaluates conditional expectations with a distorted probability distribution instead of the benchmark distribution for W_{t+1}, it follows that the consumption Euler equation of a robust planner is

$$\exp(-\delta + \rho - \nu)E\left[M_{t+1}^0\left(C_{t+1}^1 + Y_{t+1}^1\right)|\mathcal{F}_t\right] = C_t^1 + Y_t^1.$$

Under the model implied by the likelihood ratio M_{t+1}^0 defined in (17), the expected growth rate in consumption is lowered by[4]

$$-\frac{|F_c + F_y|^2}{\theta},$$

which affects the risk-free interest rate. Tom and Lars: give the formula for the risk-free rate here.

To obtain a characterization of the equivalence between the effects of discounting that operate through δ and the effects of concerns about robustness that operate through $\frac{1}{\xi}$, we follow **?** by fixing a target interest rate and then adjusting δ to hit that target. To be consistent with evolution equation (16), we therefore assume:

$$\delta = \rho - \nu - \frac{|F_c + F_y|^2}{\xi}$$

This is an affine-in-$\frac{1}{\xi}$ counterpart to formula (28) in **?** with slope coefficient[5] $-\frac{|F_c+F_y|^2}{2}$ as depicted in Figure 3.

Tom: add sentence here. Also point to formula in Robustness 2008.

[4]This perturbed model is a worst-case model that emerges from the minimization problem associated with the planner's robust control problem.

[5]Compare with formula (8.3.18) on **?**, p. 231.

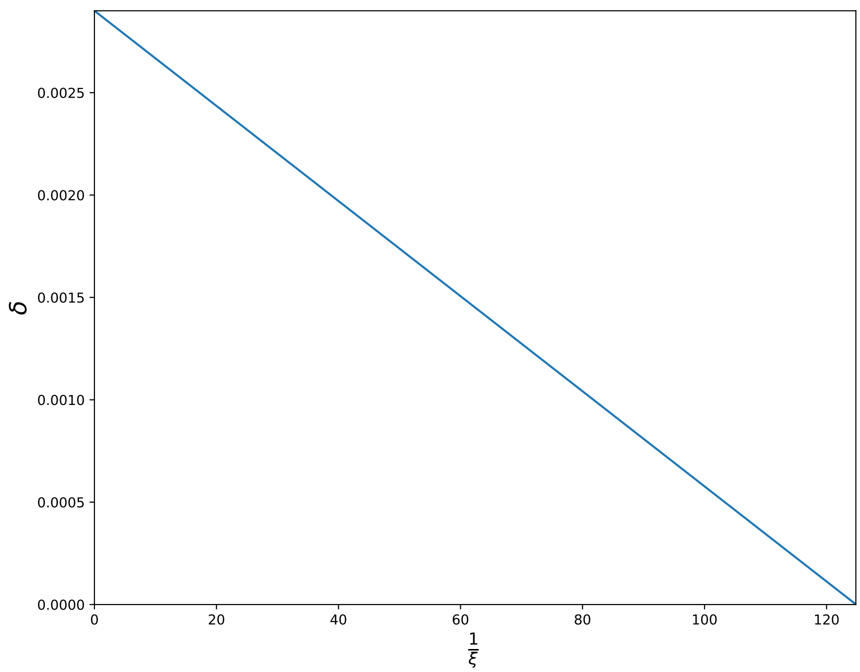

Figure 3: Subjective discount rates and robustness. This plot shows how to adjust the subjective discount rate δ for a given value of $\frac{1}{\xi}$ while leaving the implied riskless rate fixed. Lars XXXXX: we should ask the RA's to use a larger font for the axis labels.

Tom: write some things about the next very nice calculations.

The uncertainty price vector for the two shocks is:

$$-\frac{1}{\xi}\left(F_c + F_y\right) = \frac{.01}{\xi}\begin{bmatrix} .482 \\ .00383 \end{bmatrix}$$

The exposure to the shock with permanent consequences requires much larger compensations because the robust planner fears the misspecification of that so much more.

3 Habit persistence

We aim now to construct a multiplicative-functional counterpart to **?**'s specification with habit persistence. To accomplish this without concerns about robustness we change the planner's preferences to the standard time separable discounted expected utility value function[6]

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) + \exp(-\delta) E \left[V_{t+1} | \mathcal{F}_t \right].$$

and period utility U_t has the form

$$\exp(U_t) = \upsilon[\exp(C_t), \exp(H_t)], \tag{18}$$

where υ is the CES function

$$\upsilon(c, h) = \left[(1 - \alpha) c^{1-\eta} + \alpha h^{1-\eta} \right]^{\frac{1}{1-\eta}}.$$

and the stock of consumer habits or durable goods H_t follows the law of motion

$$\exp(H_{t+1} + Y_{t+1}) = \exp(-\psi) \left[\exp(H_t + Y_t) \right] + [1 - \exp(-\psi)] \left[\exp(C_t + Y_t) \right] \tag{19}$$

where $0 \leqslant \exp(-\psi) < 1$. With appropriate parameters, the CES function υ can capture either durability of consumption goods or habit persistence or both.

3.0.1 Useful CES algebra

For constructing first-order approximations to (18), it is useful compute the first derivatives:

$$\begin{aligned} mc &= (1 - \alpha) \upsilon^{\eta} c^{-\eta} \\ mh &= \alpha \upsilon^{\eta} h^{-\eta} \end{aligned}$$

Define

$$\bar{u} = \frac{1}{1 - \eta} \log \left((1 - \alpha) \exp[(1 - \eta) \bar{c}] + \alpha \exp[(1 - \eta) \bar{h}] \right),$$

which is the steady-state version of U_t. The first-order approximation of U_t is:

$$U_t^1 = (1 - \alpha) \exp \left[(\eta - 1)(\bar{u} - \bar{c}) \right] C_t^1 + \alpha \exp \left[(\eta - 1)(\bar{u} - \bar{h}) \right] H_t^1 \tag{20}$$

[6]We shall activate concerns about robustness later by using the risk-sensitive recursion

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) - \exp(-\delta) \xi \log E \left[\exp \left(-\frac{1}{\xi} V_{t+1} \right) | \mathcal{F}_t \right].$$

3.0.2 First-order conditions

Let $\{\widehat{MK}_t\}$ be a stochastic process of logarithms of Lagrange multipliers on the law of motion of the financial asset and let $\{\widehat{MH}_t\}$ be a stochastic process of logarithms of Lagrange multipliers on the law of motion (19) or (21) of the stock H_t of consumer habits or durables. It is convenient also to define

$$\begin{aligned} MK_t &= \widehat{MK}_t + Y_t \\ MH_t &= \widehat{MH}_t + Y_t. \end{aligned}$$

First-order necessary conditions for the planner's problem with respect to H and K give a co-state equation for MH_t

$$\begin{aligned} &\exp(-\delta)\alpha E\left(\exp\left[(\eta-1)U_{t+1} - \eta H_{t+1} - Y_{t+1}\right] \mid \mathcal{F}_t\right) \\ &- \exp\left(\widehat{MH}_t\right) + \exp(-\delta-\psi) E\left[\exp\left(\widehat{MH}_{t+1}\right) \mid \mathcal{F}_t\right] = 0, \end{aligned}$$

a co-state equation for MK_t,

$$\exp(-\delta+\rho) E\left[\exp\left(\widehat{MK}_{t+1}\right) \mid \mathcal{F}_t\right] - \exp\left(\widehat{MK}_t\right) = 0.$$

The first-order conditions for consumption give:

$$\begin{aligned} &(1-\alpha)\exp\left[(\eta-1)(U_t+Y_t) - \eta(C_t+Y_t)\right] \\ &+ \left[1-\exp(-\psi)\right]\exp\left(\widehat{MH}_t\right) - \exp\left(\widehat{MK}_t\right) = 0. \end{aligned}$$

It is convenient to multiply each of these by $\exp(Y_t)$ to get:

$$\begin{aligned} &\exp(-\delta)\alpha E\left(\exp\left[(\eta-1)U_{t+1} - \eta H_{t+1} + Y_t - Y_{t+1}\right] \mid \mathcal{F}_t\right) \\ &+ \exp(-\delta-\psi) E\left[\exp\left(MH_{t+1} + Y_t - Y_{t+1}\right) \mid \mathcal{F}_t\right] = \exp\left(MH_t\right) \end{aligned}$$

$$\exp(-\delta+\rho) E\left[\exp\left(MK_{t+1} + Y_t - Y_{t+1}\right) \mid \mathcal{F}_t\right] = \exp\left(MK_t\right)$$

$$(1-\alpha)\exp\left[(\eta-1)U_t - \eta C_t\right] = \exp\left(MK_t\right) - \left[1-\exp(-\psi)\right]\exp\left(MH_t\right)$$

3.1 H dynamics

After dividing both sides of the H dynamics equation (19) by $\exp(Y_t)$, we obtain

$$\exp(H_{t+1})\exp(Y_{t+1}-Y_t) = \exp(-\psi)\exp(H_t) + [1-\exp(-\psi)]\exp(C_t). \tag{21}$$

The following steady state counterpart to this equation

$$\exp(\bar{h})\exp(\nu) = \exp(-\psi)\exp(\bar{h}) + [1-\exp(-\psi)]\exp(\bar{c})$$

determines $\bar{h}$. The first-order approximation to the H-dynamics is

$$\exp(\nu)\exp(\bar{h})\left[H^1_{t+1} + (Y^1_{t+1} - Y^1_t)\right] = \exp(-\psi)\exp(\bar{h})H^1_t + [1-\exp(-\psi)]\exp(\bar{c})C^1_t.$$

or

$$H^1_{t+1} = \exp(-\nu-\psi)H^1_t + [\exp(-\nu) - \exp(-\psi-\nu)]\left[\frac{\exp(\bar{c})}{\exp(\bar{h})}\right]C^1_t - Y^1_{t+1} + Y^1_t,$$

which after simplification becomes

$$H^1_{t+1} = \exp(-\nu-\psi)H^1_t + [1-\exp(-\nu-\psi)]C^1_t - Y_{t+1} + Y_t. \tag{22}$$

3.1.1 Additional steady state calculations

$$\exp(-\delta-\nu-\psi)\exp\left(\overline{mh}\right) = \exp\left(\overline{mh}\right) - \alpha\exp(-\delta-\nu)\exp\left[(\eta-1)\bar{u} - \eta\bar{h}\right]$$
$$(1-\alpha)\exp\left[(\eta-1)\bar{u} - \eta\bar{c}\right] = \exp\left(\overline{mk}\right) - [1-\exp(-\psi)]\exp\left(\overline{mh}\right)$$

3.1.2 First-order approximation

$$\begin{aligned}
&\exp\left(-\delta-\nu-\psi+\overline{mh}\right)E\left(MH^1_{t+1} + Y^1_t - Y^1_{t+1} \mid \mathcal{F}_t\right)\\
&+\alpha\exp\left[-\delta-\nu+(\eta-1)\bar{u}-\eta\bar{h}\right]E\left[(\eta-1)U^1_{t+1} - \eta H^1_{t+1} + Y^1_t - Y^1_{t+1} \mid \mathcal{F}_t\right]\\
&= \exp\left(\overline{mh}\right)MH^1_t
\end{aligned} \tag{23}$$

$$\begin{aligned}
&(1-\alpha)\exp\left[(\eta-1)\bar{u}-\eta\bar{c}\right]\left[(\eta-1)U^1_t - \eta C^1_t\right]\\
&= \exp(\overline{mk})MK^1_t - [1-\exp(-\psi)]\exp(\overline{mh})MH^1_t.
\end{aligned} \tag{24}$$

$$\exp(-\delta + \rho - \nu) E\left[MK_{t+1}^1 + (Y_t^1 - Y_{t+1}^1)|\mathcal{F}_t\right] = MK_t^1. \tag{25}$$

3.1.3 Solution strategy

One approach is to use the deflating subspace calculations described in **?**, ch. 4.

1. Construct
$$Z_t^1 = \begin{bmatrix} MK_t^1 \\ MH_t^1 \\ K_t^1 \\ H_t^1 \\ X_t \end{bmatrix}$$

2. Take equations (20) and (24) and solve for U_t^1 and C_t^1 in terms of Z_t^1 and Z_{t+1}^1.

3. Use equations (23), (24), (12), and (22) after substituting for U_t^1, C_t^1 and $E\left(Y_{t+1}^1 - Y_t^1 \mid \mathcal{F}_1\right) = D \cdot X_t$ and form the system:[7]
$$\mathbb{L} Z_{t+1}^1 = \mathbb{J} Z_t^1$$
where we initially zero out the shocks and use $X_{t+1} = AX_t$.

4. Consider a solution of the co-states in terms of the states of the form:
$$\begin{bmatrix} MK_t^1 \\ MH_t^1 \end{bmatrix} = \mathbb{N}_{11} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} + \mathbb{N}_{12} X_t.$$
Substituting this into the system dynamics gives:
$$\mathbb{L} \begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} K_{t+1}^1 \\ H_{t+1}^1 \end{bmatrix} \\ X_{t+1} \end{bmatrix} = \mathbb{J} \begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix}$$
To compute $\mathbb{N}_{11}$ and $\mathbb{N}_{12}$ we will require that the dynamics for
$$\begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix}$$

[7] We implicitly appeal to a certainly-equivalence property to allow us to replace $E[Z_{t+1}^1|\mathcal{F}_t]$ with Z_{t+1}^1 on the left side of this equation.

be weakly stable. We accomplish this by first forming a generalized Schur decomposition. There will be a total of seven eigenvalues, three of which are associated with the exogenous dynamics. These three are all stable. There will be four eigenvalues associated with the endogenous dynamics, two of which are stable and two of which are unstable. One of the "endogenous" eigenvalues will be unity, and we will count this as (weakly) stable. This leads us to form:

$$\begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix}$$

by taking linear combinations of the five stable generalized eigenvalues. See **?**, ch. 4 for elaboration.[8] To check the calculation verify that the eigenvalues of the resulting state dynamics are indeed weakly stable.

5. Perform the following check. I think but have not verified that $\mathbb{L}$ is nonsingular. Compute:

$$\mathbb{L}^{-1}\mathbb{J}$$

and thus

$$Z^1_{t+1} = \mathbb{L}^{-1}\mathbb{J}Z^1_t$$

We know that

$$\begin{bmatrix} I & -\mathbb{N}_{11} & -\mathbb{N}_{12} \end{bmatrix} Z^1_{t+1} = 0.$$

Thus

$$\begin{bmatrix} I & -\mathbb{N}_{11} & -\mathbb{N}_{12} \end{bmatrix} \mathbb{L}^{-1}\mathbb{J} \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ I & 0 \\ 0 & I \end{bmatrix} = 0.$$

6. Compute the eigenvalues of the matrix:

$$\mathbb{A} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathbb{L}^{-1}\mathbb{J} \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ I & 0 \\ 0 & I \end{bmatrix}$$

[8]This will involve employing an ordered Schur decomposition. Evan Anderson wrote one in Matlab. We can translate it to Julia. Evan's program is described in **?**, ch. 4 and available at the website for the Matlab programs for that book.

and check that they coincide with the weakly stable eigenvalues.

7. Add the shocks back to the X evolution equation to get:

$$\begin{bmatrix} MK_t^1 \\ MH_t^1 \end{bmatrix} = \mathbb{N}_{11} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} + \mathbb{N}_{12} X_t,$$

and

$$\begin{bmatrix} \begin{bmatrix} K_{t+1}^1 \\ H_{t+1}^1 \end{bmatrix} \\ X_{t+1} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix} + \mathbb{B} W_{t+1} \tag{26}$$

where

$$\mathbb{B} = \begin{bmatrix} \begin{bmatrix} \mathbb{B}_k \\ \mathbb{B}_h \end{bmatrix} \\ \mathbb{B}_x \end{bmatrix}$$

The row vector $\mathbb{B}_k$ is given by :

$$\mathbb{B}_k = -.01\bar{k} \begin{bmatrix} .144 & .206 \end{bmatrix}$$

and the row vector $\mathbb{B}_h$ is given by:

$$\mathbb{B}_h = -.01 \begin{bmatrix} .144 & .206 \end{bmatrix}$$

The matrix $\mathbb{A}$ should be block triangular with $\mathbb{A}_x$ in the lower block.

The parameter η introduces a form of intertemporal complementarity into preferences; it grows as ρ becomes larger. The next two graphs plot the impulse responses for log consumption. Figure 4 investigates how the choice η alters the responses. For all these relatively large values of η, for the permanent shock the immediate response is muted relative to the long-term response as the response increases with the horizon. Larger values of η apparently induce a more sluggish consumption response. The qualitative nature of these responses looks very similar to those posed in long-run risk models with recursive utility. Here the consumption response is endogenous.

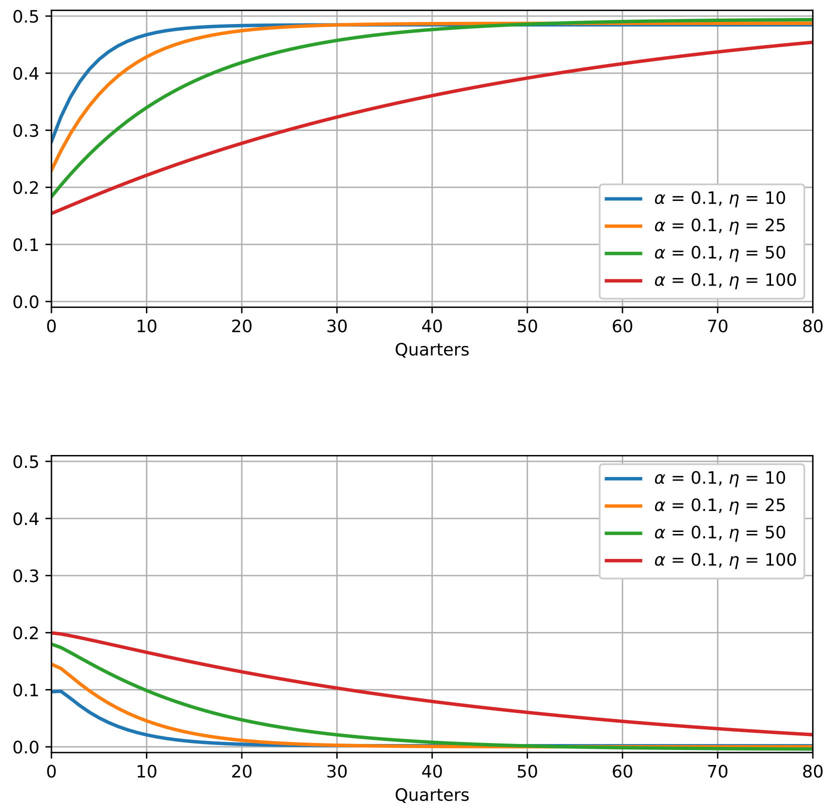

Figure 4: Consumption responses for the two shock processes for habit persistent preferences for $\alpha = .1$, $\psi = .4$ and alternative choice for η. Top panel: permanent shock. Bottom panel: transitory shock.

Figure 5 shows how changing α alters the impulse response for the logarithm of consumption for fixed values $\psi = .4$ and $\eta = 50$. While preserving the same qualitative response patterns for consumption, increasing α from .3 to .7 has little impact on consumption. At the more extreme $\alpha = .1$ and .9 there is substantially more curvature in the initial part of the responses and convergence occurs faster.

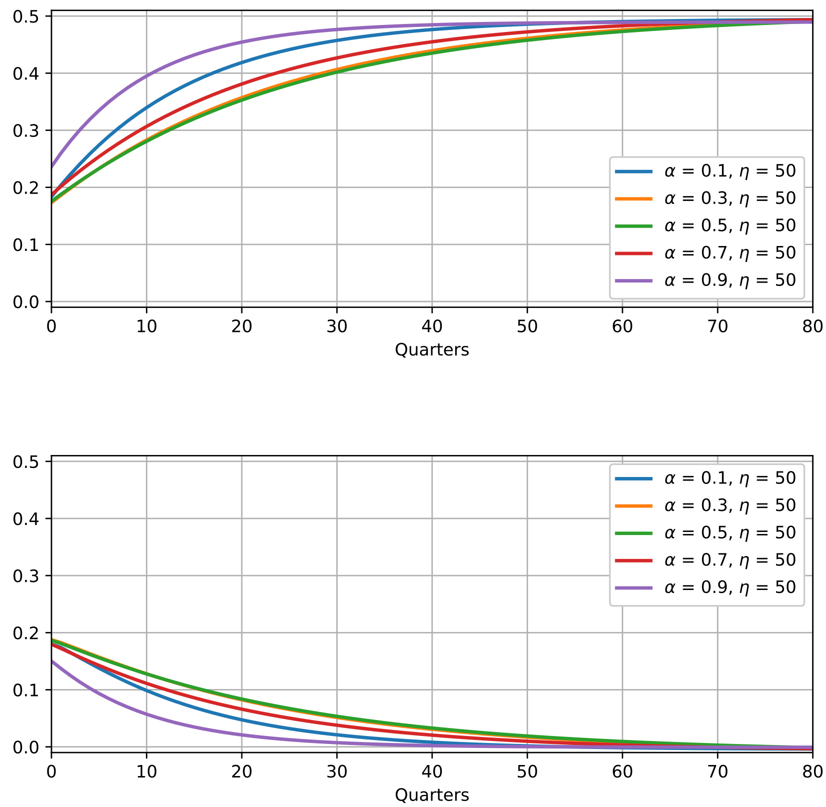

Figure 5: Consumption responses for the two shock processes for habit persistent preferences for $\eta = 50$, $\psi = .4$ and alternative choice for α. Top panel: permanent shock. Bottom panel: transitory shock.

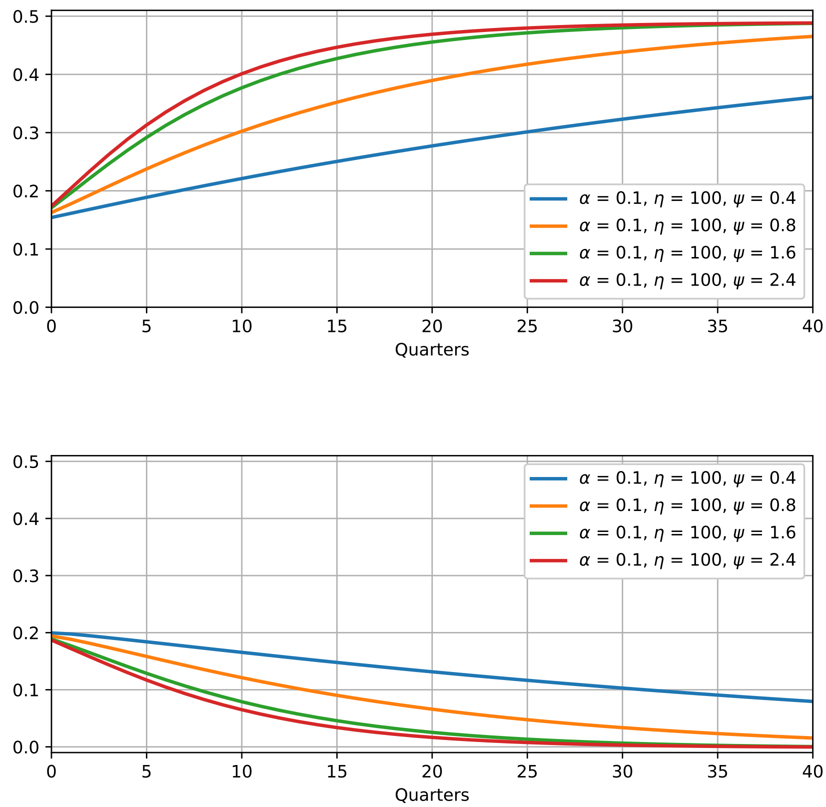

Figure 6: Consumption responses for the two shock processes for habit persistent preferences for $\alpha = .1$, $\eta = 100$, $\psi = .4, .8, 1.6, 2.4$. Top panel: permanent shock. Bottom panel: transitory shock.

Figure 6 shows how the consumption impulse response changes when we alter the rate of depreciation ψ of household capital. We are particularly interested in the responses to the permanent shock. As expected, the responses approximate their limiting value more quickly when we increase ψ. Lars XXXXX: could the scale on the y (coordinate) axis in the lower panel be improved to facilitate comparison? Tom Notice the similarity to the long-run risk consumption response except here we generate endogenously.

Next we add in a concern about robustness as in **?**. This requires that we compute the first-order term for the continuation value process.

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right)|\mathcal{F}_t\right]$$

Thus

$$V_t^1 - Y_t^1 = [1 - \exp(-\delta)]U_t^1 - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1 - Y_t^1\right)|\mathcal{F}_t\right]$$

Represent:

$$\begin{aligned}
Y_{t+1}^1 - Y_t^1 &= \mathbb{S}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1} \\
U_t^1 &= \mathbb{S}_u \cdot \begin{bmatrix} K_t^1 \\ H_t^1 \\ X_t \end{bmatrix} \\
V_t^1 - Y_t^1 &= \mathbb{S}_v \cdot \begin{bmatrix} K_t^1 \\ H_t^1 \\ X_t \end{bmatrix} + \mathsf{s}_v
\end{aligned}$$

where $\mathbb{S}_u$ comes from the model solution using formula (20) and $\mathbb{S}_v$ and s_v are to be computed as in **?**. In particular,

$$(\mathbb{S}_v)' = [1 - \exp(-\delta)](\mathbb{S}_u)' + \exp(-\delta)\left[(\mathbb{S}_v)'\mathbb{A} + \begin{bmatrix} 0 & 0 & (\mathbb{S}_y)' \end{bmatrix}\right],$$

and

$$\mathsf{s}_v = \exp(-\delta)\left[\mathsf{s}_v - \frac{\xi}{2}\left|(\mathbb{S}_v)'\mathbb{B} + (\mathbb{S}_y)'\mathbb{B}_x\right|^2\right]$$

The first equation is affine in $\mathbb{S}_v$ and can be solved prior to the second equation. Given $\mathbb{S}_v$, the second equation is affine in v and may be solved easily as well. We compute uncertainty prices given by the two-dimensional vector:

$$\frac{1}{\xi}\left[(\mathbb{S}_v)'\mathbb{B} + F_y\right] = \frac{1}{\xi}\begin{bmatrix} .482 \\ .00394 \end{bmatrix}$$

for $(\alpha, \eta, \psi) = (.1, 100, 1.6)$.