Some stochastic growth models*

May 30, 2018

1 Adjustment cost model

Let K_t be capital, I_t be investment, C_t be capital, and Z_t be a shock process governed by

$$Z_{t+1} = \alpha_z + \exp\left(-\xi\right) Z_t + \sigma_z \cdot W_{t+1}$$

where $|\exp(-\xi)| < 1$, $\{W_{t+1}\}$ is an i.i.d. process whose time t component that is distributed according to a normal distribution with mean 0 and variance 1, and

$$\frac{\alpha_z}{1 - \exp(-\xi)} = 1.$$

The feasibility constraint restricting C_t, I_t, K_t is

$$C_t + I_t = A K_t$$

where A is a fixed parameter. Dividing the above equation by K_t gives

$$\frac{C_t}{K_t} + \frac{I_t}{K_t} = A. \tag{1}$$

The capital accumulation technology is

$$K_{t+1} = K_t \exp\left[\alpha_k + \left(\frac{I_t}{K_t}\right) - \phi\left(\frac{I_t}{K_t}\right)^2\right] \exp\left(\beta Z_t + \sigma_k \cdot W_{t+1} - \frac{1}{2}|\sigma_k|^2\right).$$

Taking logarithms of both sides of the capital accumulation technology gives

$$\log K_{t+1} = \log K_t + \alpha_k + \left(\frac{I_t}{K_t}\right) - \phi\left(\frac{I_t}{K_t}\right)^2 + \beta Z_t + -\frac{1}{2}|\sigma_k|^2 + \sigma_k \cdot W_{t+1}. \qquad (2)$$

We impute to a planner the period utility function $(1 - \exp(-\delta))\log C_t$, where $0 < \exp(-\delta) < 1$. We find it convenient to represent this one-period utility function as

$$(1 - \exp(-\delta))\log C_t = (1 - \exp(-\delta))(\log C_t - \log K_t) + (1 - \exp(-\delta))\log K_t.$$

A sequence $\{V_t\}$ of continuation values for discounted expected utilities satisfies

$$V_t = (1 - \exp(-\delta))\log C_t + \exp(-\delta)E(V_{t+1}|\mathcal{F}_t).$$

The planner maximizes V_0 by choosing $\frac{C_t}{K_t}$ and $\frac{I_t}{K_t}$ for $t \geqslant 0$ subject to constraints (2) and (1) and initial conditions for K_0 and Z_0.

We can find the optimal value sequence $\{V_t\}$ and the decision rules that attain it by applying a guess and verify method as follows. Let f be the affine function

$$f(Z) = f_0 + f_z Z.$$

and guess that the date t continuation value for the planner is

$$V_t = \log K_t + f(Z_t).$$

1.1 Extend and finish as described below

Tom and Lars XXXXX: some calculations are to be filled in and described here. LPH this can and should be done, but given the AK nature this example basically gives us a production based interpretation of the typical long run risks specification where the Z dynamic will determine the consumption and capital dynamics almost one for one.

We set the following parameters:

$$\begin{array}{rclcrcl}
\alpha_y & = & .373 & \quad & \beta & = & 1 \\
\alpha_z & = & 0 & \quad & \xi & = & .017
\end{array}$$

$$\sigma = \begin{bmatrix} (\sigma_y)' \\ (\sigma_z)' \end{bmatrix} = \begin{bmatrix} .481 & 0 \\ .012 & .027 \end{bmatrix} \tag{3}$$

where $\sigma_y = \sigma_k$ and the implied growth rate for consumption net of the contribution from Z is

$$\alpha_y = \alpha_k + \left(\frac{I_t}{K_t} \right) - \phi \left(\frac{I_t}{K_t} \right)^2 - \frac{1}{2} |\sigma_k|^2.$$

(The $\frac{I}{K}$ ratio turns out to be constant in this model.) We construct the matrix σ so that one of the shocks has permanent consequences and the other is temporary.

Now we execute analogous calculation with a risk-sensitivity or robustness adjustment of the continuation value. We want to study impacts of risk sensitivity or robustness concern on $\frac{C_t}{K_t}$ and $\frac{I_t}{K_t}$. The continuation value under robustness is:

$$V_t = (1 - \exp(-\delta)) \log C_t - \exp(-\delta)\theta \log E \left[\exp \left(-\frac{1}{\theta} V_{t+1} \right) | \mathcal{F}_t \right]$$

Lars XXXXX: Lars and Tom agree that this is an interesting calculation to explore and compare with no robustness version above. Maybe change the θ to $\exp(-\xi)$ as below. On to-do list.

2 Permanent income model

We modify members of a class of models described by Hansen and Sargent (2013, ch. 11) to capture rational expectations versions of Milton Friedman's permanent income model of consumption. Our key extension is to assume that nonfinancial income, the key exogenous driving process is a multiplicative functionals instead of the covariance stationary process usually assumed.

Let $W_{t+1} \sim \mathcal{N}(0, I)$ be an i.i.d. vector process and let $\{Y_t\}$ be the logarithm of an exogenous nonfinancial income process that is governed by an additive functional

$$Y_{t+1} - Y_t = \mathbb{D}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1} + \nu$$

where

$$X_{t+1} = \mathbb{A}_x X_t + \mathbb{B}_x W_{t+1}$$

and A is a stable matrix. Define

$$Y_t = Y_t^0 + Y_t^1$$

where $Y_t^0 = \bar{y}_0 + t\nu$ and

$$Y_{t+1}^1 - Y_t^1 = \mathbb{D}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1}.$$

Let $\widehat{C}_t$ be the logarithm of consumption at date t and let $\widehat{K}_t$ be the stock of a risk-free asset (or a debt, if it is negative). The asset bears a fixed risk-free one-period return equal to $\exp(\rho)$. Feasibility at time t requires

$$\widehat{K}_{t+1} + \exp\left(\widehat{C}_t\right) = \exp(\rho)\widehat{K}_t + \exp(Y_t). \tag{4}$$

Stochastic growth in the multiplicative functional $\{Y_t\}$ makes it convenient to scale variables by Y_t, so we define $C_t = \widehat{C}_t - Y_t$ and $K_t = \frac{\widehat{K}_t}{\exp(Y_t)}$. Dividing both sides of equation (4) by $\exp(Y_t)$ yields

$$K_{t+1}\exp\left(Y_{t+1} - Y_t\right) + \exp\left(C_t\right) = \exp(\rho)K_t + 1. \tag{5}$$

A representative consumer with logarithmic one period utility of consumption, discounted time separable preferences, and subjective discount rate δ optimally chooses a consumption process that satisfies the Euler equation

$$\exp(-\delta + \rho)E\left[\exp(\widehat{C}_t - \widehat{C}_{t+1})|X_t\right] = 1.$$

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This Euler equation can also be expressed as

$$\exp(-\delta + \rho)E\left[\exp(C_t - C_{t+1} + Y_t - Y_{t+1})|X_t\right] = 1.$$

2.1 Steady state

We assume that

$$\exp(-\delta + \rho - \nu) = 1, \tag{6}$$

where recall that δ is the discount rate in preferences, ρ is the risk-free rate of return, and ν is the rate of growth in the deterministic part of nonfinancial income Y_t^0. Restriction (6) implies a steady state in which the log consumption-log income ratio equals $\bar{c}$. Steady state means of asymptotically stationary components of (C_t, K_t) must satisfy

$$\bar{k}\exp(\nu) + \exp(\bar{c}) = \exp(\rho)\bar{k} + 1$$

or equivalently

$$\exp(\bar{c}) = \left[\exp(\rho) - \exp(\nu)\right]\bar{k} + 1,$$

where we assume that $\rho > \nu$. Notice that we are free to set $\bar{k}$. For convenience, we assume that $\bar{k} = 0$ and hence that $\exp(\bar{c}) = 1$. Tom XXXXX: say more here.

2.2 First-order approximation

We scale W_{t+1} by q, let q tend to zero, and obtain a first-order small-noise approximation around a steady state. We define some processes that we use to construct approximations in terms of the following notation. Processes with superscripts 1 are first-order derivatives of corresponding original processes with respect to q evaluated at $\mathsf{q} = 1$. A first-order approximation of the feasibility restriction is

$$K_{t+1}^1 \exp(\nu) + \bar{k}\exp(\nu)\left(Y_{t+1}^1 - Y_t^1\right) + \exp(\bar{c})C_t^1$$
$$= \exp(\rho)K_t^1$$

or

$$K_{t+1}^1 = \exp(\rho - \nu)K_t^1 - \bar{k}\left(Y_{t+1}^1 - Y_t^1\right) - \exp(\bar{c} - \nu)\exp(\bar{c})C_t^1. \tag{7}$$

The restrictions $exp(\bar{c}) = 1$ and $\bar{k} = 0$ make equation (7) become

$$K_{t+1}^1 = \exp(\rho - \nu)K_t^1 - \exp(-\nu)C_t^1 \tag{8}$$

Lars and Tom: Should we make a decision for the entire section about what we describe as an individual representative agent problem and what we call a planning problem? We can certainly do this and even give the individual arrow prices if we like. To derive a first-order approximation to an optimal decision rule for consumption, we solve equation (7) forward, take conditional expectations, and impose consumption Euler equations.[1] To begin, solve equation (7) forward and then take conditional expectations of future random variables to get

$$\exp(\nu)K_t^1 = \sum_{j=0}^{\infty} \lambda^{j+1} E\left(C_{t+j}^1 + Y_{t+j}^1 \mid \mathcal{F}_t\right) - \sum_{j=0}^{\infty} \lambda^{j+1} E\left(Y_{t+j}^1 \mid \mathcal{F}_t\right). \tag{9}$$

A first-order approximation to the Euler equation for the planning problem is

$$E\left[C_{t+1}^1 + Y_{t+1}^1 \mid \mathcal{F}_t\right] = C_t^1 + Y_t^1.$$

Substituting this approximation to the Euler equation repeatedly into equation (9) gives

$$\exp(\nu)K_t^1 = \left(\frac{\lambda}{1-\lambda}\right)(C_t^1 + Y_t^1) - \left(\frac{\lambda}{1-\lambda}\right)\sum_{j=1}^{\infty} \lambda^j E\left(Y_{t+j}^1 - Y_{t+j-1}^1 \mid \mathcal{F}_t\right) - \left(\frac{\lambda}{1-\lambda}\right)Y_t^1$$

or

$$\exp(\nu)K_t^1 = \left(\frac{\lambda}{1-\lambda}\right)C_t^1 - \left(\frac{\lambda}{1-\lambda}\right)\sum_{j=1}^{\infty} \lambda^j E\left(Y_{t+j}^1 - Y_{t+j-1}^1 \mid \mathcal{F}_t\right),$$

where $\lambda = \exp(\nu - \rho)$. Solve the above equation to obtain the approximation

$$C_t^1 = \frac{\exp(\nu)(1-\lambda)}{\lambda}K_t^1 + \sum_{j=1}^{\infty} \lambda^j E(Y_{t+j}^1 - Y_{t+j-1}^1 \mid \mathcal{F}_t) \tag{10}$$

that links an optimal log consumption-income ratio to two financial income (as a function of K_t^1) and nonfinancial income. The approximating decision rule (10) implies that C_t^1 and K_t^1 are cointegrated with cointegrating vector $\begin{bmatrix} 1 & [\exp(\nu) - \exp(\rho)] \end{bmatrix}$ and that non-

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[1]With or without taking conditional expectations of all time indexed variables, We can solve equation (7) forward. Cite one of many places where we have done this.

financial income's contribution to the log consumption-income ratio is

$$\sum_{j=1}^{\infty} \lambda^j \mathbb{E}(Y_{t+j}^1 - Y_{t+j-1}^1 | \mathcal{F}_t) = \lambda \mathbb{D}_y^T (I - \lambda A_x)^{-1} X_t = M X_t$$

Decision rule (10) is a first-order approximation to an optimal decision rule similar to the one in a Hansen et al. (1999) economy without robustness (i.e., the problem that is obtained by setting their $\sigma = 0$).

2.3 Impulse responses

As an example, we evaluate first-order approximate decision rules for consumption and assets at a nonfinancial income process adapted from Hansen et al. (1999). They assumed two components of nonfinancial income, one more persistent than the other. To construct the first component, let

$$X_{1,t+1}^1 = .704 X_{1,t} + \begin{bmatrix} .144 & 0 \end{bmatrix} W_{t+1}$$

where $Y_{1,t+1}^1 = Y_t^1 + X_{1,t+1}^1$. To construct the second component, let

$$X_{2,t+1}^1 = X_{2,t}^1 - .154 X_{2,t-1}^1 + \begin{bmatrix} 0 & .206 \end{bmatrix} W_{t+1}$$

and construct $Y_{2,t+1}^1 = X_{2,t+1}^1$. Let $Y_{t+1}^1 = (.01)Y_{1,t+1}^1 + (.01)Y_{2,t+1}^2$.[2] We represent this $\{Y_t\}$ process as an additive functional. Set $\rho = .00663$ and $\nu = .00373$.

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[2]We take income numbers from the first column of Table 2 of Hansen et al. (1999) with two modifications. In Hansen et al. (1999), both income processes are stationary but one has an autoregressive root of .998. We set this to one here. This has a nontrivial impact on the consumption volatility, which Hansen et al. estimated in levels. We scale both innovation standard deviations by 1.33 to achieve a consumption growth rate volatility of .482 expressed as a percent (log differences multiplied by 100).
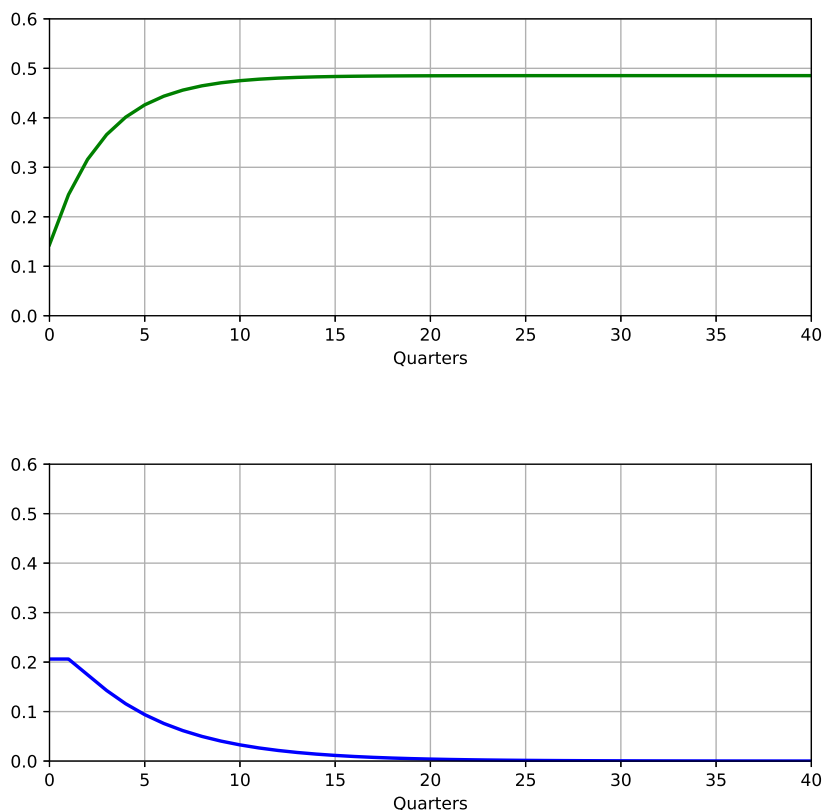
Figure 1: Impulse responses of log income to the two shock processes. Top panel: permanent shock. Bottom panel: transitory shock. Parameter settings from Hansen et al. (1999).

As lag length on the horizontal axis becomes large, the positive limit of the impulse response coefficients for the first shock in figure 1 tell us that the first shock has permanent effects on nonfinancial income; the zero limit of the impulse response coefficients for the second shock tells us that it has only transitory effects. The planner can use savings partially to self-insure against the transitory shock via precautionary savings but cannot use savings to insure against the permanent shock. Under the time separable preferences being used here, the responses of consumption to shocks are both constant across horizons; so impulse response functions to both shocks are just step functions. This is an implication of the consumption smoothing built into the model. One hundred times the response of

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consumption responses to the two shocks are:[3]

$$\text{permanent shock} = .482$$

$$\text{transitory shock} = .00383,$$

which shows the dramatic differences induced by the endogenous consumption-savings responses.

Present values of impulse response functions of consumption and non-financial income to both shocks display a tell-tale feature described by Hansen et al. (1991). Define the consumer's time t "deficit" as consumption minus non-financial income. Then Hansen et al. note that discounted by λ the present value of the impulse response function of the deficit is zero for each shock. In particular, temporarily let $\{w_j\}_{j=0}^{\infty}$ be an impulse response of the log consumption/non-financial income ratio to a shock. Then form the sequence $\{s_j\}_{j=0}^{\infty}$ whose jth element is

$$s_j = \sum_{m=0}^{j} \lambda^m w_m.$$

We plot this object for both shocks in figure 2. For each shock, Hansen et al. tell us to expect that $\lim_{j \to +\infty} s_j = 0$, which evidently prevails. Figure 2 plots these discounted sums over the alternative horizons in order to study how quickly they converge to zero, albeit slowly.

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[3]We multiplied these objects by 100 in order to drop two zeros from the reported numbers.
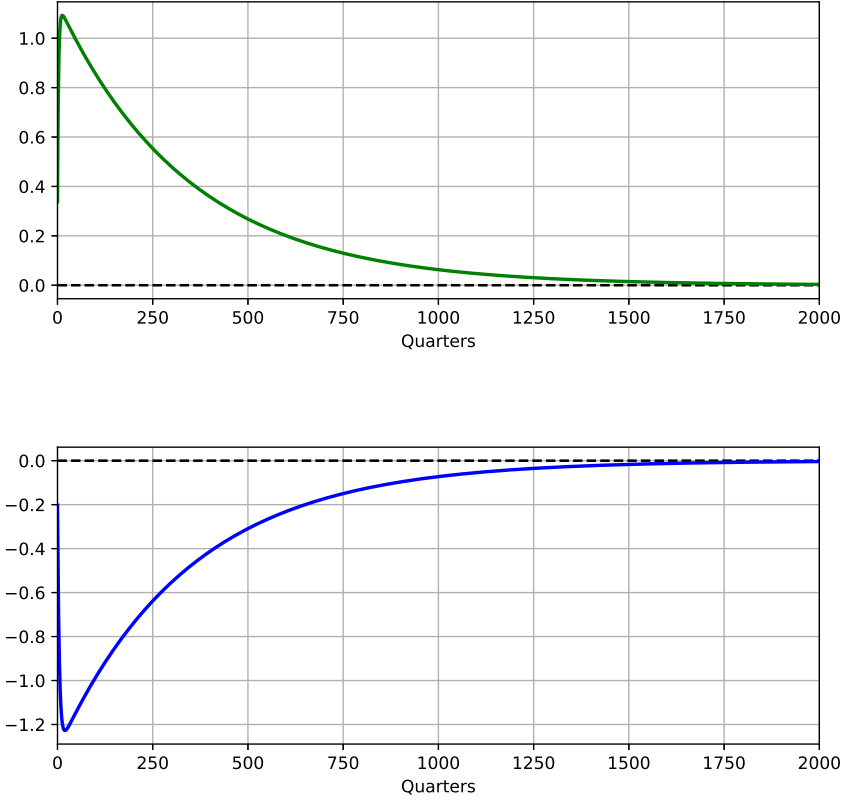
Figure 2: Responses of $s_j = \sum_{m=0}^{j} \lambda^m w_m$ for the "deficit" for two shock processes. Top panel: permanent shock. Bottom panel: transitory shock. Parameter settings from Hansen et al. (1999). Notice how the permanent shock leads to an immediate deficit while a temporary shock leads to an immediate surplus; each of these ultimately completely evaporates.

2.4   Robustness

Following Hansen et al. (1999), we now attribute a concern about robustness to the planner by changing the utility recursion to:

$$V_t = [1 - \exp(-\delta)](C_t + Y_t) - \exp(-\delta)\xi \log E \left[\exp\left(-\frac{1}{\xi} V_{t+1} \right) | \mathcal{F}_t \right]$$

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where V_t is a date t continuation value and $\xi \leqslant +\infty$. Setting $\xi = \infty$ eliminates concerns about robustness and returns us to time-separable log utility. The parameter ξ is the inverse of the risk sensitivity parameter of Hansen et al. (1999). The first-order approximation to a continuation value process is of the form

$$V_t^1 = [1 - \exp(-\delta)](C_t^1 + Y_t^1) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)|\mathcal{F}_t\right]$$

where $F_c \cdot W_{t+1}$ gives the response of C_{t+1}^1 to W_{t+1}. To obtain an observational equivalence finding like that of Hansen et al. (1999), we guess that the evolution of $C_{t+1}^1 + Y_{t+1}^1$ satisfies:

$$C_{t+1}^1 + Y_{t+1}^1 = C_t^1 + Y_t^1 + (F_c + F_y) \cdot W_{t+1}. \tag{11}$$

Then

$$V_t^1 = C_t^1 + Y_t^1 - \frac{|F_c + F_y|^2}{2\xi}$$

and

$$V_{t+1}^1 = C_t^1 + Y_t^1 - \frac{|F_c + F_y|^2}{2\xi} + (F_c + F_y) \cdot W_{t+1}.$$

In effect, the planner's concern about robustness induces him to construct V_t^1 by changing the measure of W_{t+1} from normal with mean 0 and covariance matrix I to normal with mean $-\frac{1}{\xi}(F_c + F_y)$ and covariance matrix I. This change of measure can be accomplished by multiplying the conditional density of W_{t+1} under a baseline normal $(0, I)$ model with the following positive random variable:

$$M_{t+1}^0 = \frac{\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)}{E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)|\mathcal{F}_t\right]}. \tag{12}$$

The random variable M_{t+1}^0 can serve as a likelihood ratio because (a) it is nonnegative, and (b) its conditional expectation equals one. Because a robust planner acts as if he is an ordinary planner who evaluates conditional expectations with a distorted probability distribution instead of the benchmark distribution for W_{t+1}, it follows that the consumption Euler equation of a robust planner is

$$\exp(-\delta + \rho - \nu)E\left[M_{t+1}^0\left(C_{t+1}^1 + Y_{t+1}^1\right)|\mathcal{F}_t\right] = C_t^1 + Y_t^1.$$

Under the model implied by the likelihood ratio M_{t+1}^0 defined in (12), the expected growth rate in consumption is lowered by[4]

$$-\frac{|F_c + F_y|^2}{\xi},$$

which affects the risk-free interest rate. Tom and Lars: give the formula for the risk-free rate here.

To obtain a characterization of the equivalence between the effects of discounting that operate through δ and the effects of concerns about robustness that operate through $\frac{1}{\xi}$, we follow Hansen et al. (1999) by fixing a target interest rate and then adjusting δ to hit that target. To be consistent with evolution equation (11), we therefore assume:

$$\delta = \rho - \nu - \frac{|F_c + F_y|^2}{\xi}$$

This is an affine-in-$\frac{1}{\xi}$ counterpart to formula (28) in Hansen et al. (1999) with slope coefficient[5] $-\frac{|F_c + F_y|^2}{2}$ as depicted in Figure 3.

Tom: add sentence here. Also point to formula in Robustness 2008.

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[4]This perturbed model is a worst-case model that emerges from the minimization problem associated with the planner's robust control problem.

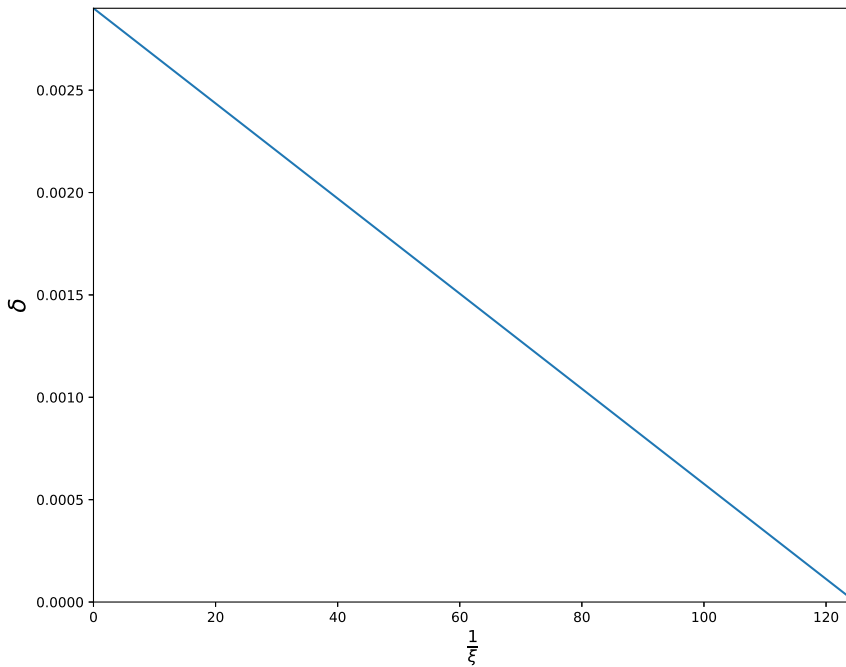[5]Compare with formula (8.3.18) on Hansen and Sargent (2013, p. 231).

Figure 3: Subjective discount rates and robustness. This plot shows how to adjust the subjective discount rate δ for a given value of $\frac{1}{\xi}$ while leaving the implied riskless rate fixed. Lars XXXXX: we should ask the RA's to use a larger font for the axis labels.

Tom: write some things about the next very nice calculations.

The uncertainty price vector for the two shocks is:

$$-\frac{1}{\xi}\left(F_c + F_y\right) = \frac{.01}{\xi}\begin{bmatrix} .482 \\ .00383 \end{bmatrix}$$

The exposure to the shock with permanent consequences requires much larger compensations because the robust planner fears the misspecification of that so much more.

3 Habit persistence

We aim now to construct a multiplicative-functional counterpart to Hansen et al.'s specification with habit persistence. To accomplish this without concerns about robustness we

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change the planner's preferences to the standard time separable discounted expected utility value function[6]

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) + \exp(-\delta)E\left[V_{t+1}|\mathcal{F}_t\right].$$

and period utility U_t has the form

$$\exp(U_t) = \upsilon[\exp(C_t), \exp(H_t)], \tag{13}$$

where υ is the CES function

$$\upsilon(c, h) = \left[(1 - \alpha)c^{1-\eta} + \alpha h^{1-\eta}\right]^{\frac{1}{1-\eta}}.$$

and the stock of consumer habits or durable goods H_t follows the law of motion

$$\exp(H_{t+1} + Y_{t+1}) = \exp(-\psi)\left[\exp(H_t + Y_t)\right] + [1 - \exp(-\psi)]\left[\exp(C_t + Y_t)\right] \tag{14}$$

where $0 \leqslant \exp(-\psi) < 1$. With appropriate parameters, the CES function υ can capture either durability of consumption goods or habit persistence or both.

3.0.1 Useful CES algebra

For constructing first-order approximations to (13), it is useful compute the first derivatives:

$$mc = (1 - \alpha)\upsilon^{\eta}c^{-\eta}$$
$$mh = \alpha\upsilon^{\eta}h^{-\eta}$$

Define

$$\bar{u} = \frac{1}{1 - \eta}\log\left((1 - \alpha)\exp[(1 - \eta)\bar{c}] + \alpha\exp[(1 - \eta)\bar{h}]\right),$$

which is the steady-state version of U_t. The first-order approximation of U_t is:

$$U_t^1 = (1 - \alpha)\exp\left[(\eta - 1)(\bar{u} - \bar{c})\right]C_t^1 + \alpha\exp\left[(\eta - 1)(\bar{u} - \bar{h})\right]H_t^1 \tag{15}$$

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[6]We shall activate concerns about robustness later by using the risk-sensitive recursion

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) - \exp(-\delta)\xi\log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right)|\mathcal{F}_t\right].$$

3.0.2   First-order conditions

Let $\{\widehat{MK}_t\}$ be a stochastic process of Lagrange multipliers on the law of motion of the financial asset and let $\{\widehat{MH}_t\}$ be a stochastic process of Lagrange multipliers on the law of motion (14) or (16) of the stock H_t of consumer habits or durables. It is convenient also to define

$$MK_t = \widehat{MK}_t + Y_t$$
$$MH_t = \widehat{MH}_t + Y_t.$$

First-order necessary conditions for the planner's problem with respect to H, K, and c are a co-state equation for MH_t

$$\alpha \exp\left[(\eta - 1)(U_t + Y_t) - \eta(H_t + Y_t)\right] - \exp\left(\widehat{MH}_t\right) + \exp(-\delta - \psi)E\left[\exp\left(\widehat{MH}_{t+1}\right) \mid \mathcal{F}_t\right] = 0,$$

a co-state equation for MK_t,

$$(1 - \alpha)\exp\left[(\eta - 1)(U_t + Y_t) - \eta(C_t + Y_t)\right]$$
$$+ \exp(-\delta)[1 - \exp(-\psi)]E\left[\exp\left(\widehat{MH}_{t+1}\right) \mid \mathcal{F}_t\right] - \exp(-\delta)E\left[\exp\left(\widehat{MK}_{t+1}\right) \mid \mathcal{F}_t\right] = 0$$

and an Euler equation for consumption:

$$\exp(-\delta + \rho)E\left[\exp\left(\widehat{MK}_{t+1}\right) \mid \mathcal{F}_t\right] - \exp\left(\widehat{MK}_t\right) = 0.$$

It is convenient to multiply each of these by $\exp(Y_t)$ to get:

$$\exp(-\delta - \psi)E\left[\exp\left(MH_{t+1} + Y_t - Y_{t+1}\right) \mid \mathcal{F}_t\right] = \exp\left(MH_t\right) - \alpha \exp\left[(\eta - 1)U_t - \eta H_t\right]$$

$$(1 - \alpha)\exp\left[(\eta - 1)U_t - \eta C_t\right] = \exp(-\delta)E\left[\exp\left(MK_{t+1} + Y_t - Y_{t+1}\right) \mid \mathcal{F}_t\right]$$
$$- \exp(-\delta)[1 - \exp(-\psi)]E\left[\exp\left(MH_{t+1} + Y_t - Y_{t+1}\right) \mid \mathcal{F}_t\right]$$

$$\exp(-\delta + \rho)E\left[\exp\left(MK_{t+1} + Y_t - Y_{t+1}\right) \mid \mathcal{F}_t\right] = \exp\left(MK_t\right)$$

3.1 H dynamics

After dividing both sides of the H dynamics equation (14) by $\exp(Y_t)$, we obtain

$$\exp(H_{t+1})\exp(Y_{t+1} - Y_t) = \exp(-\psi)\exp(H_t) + [1 - \exp(-\psi)]\exp(C_t). \qquad (16)$$

The following steady state counterpart to this equation

$$\exp(\bar{h})\exp(\nu) = \exp(-\psi)\exp(\bar{h}) + [1 - \exp(-\psi)]\exp(\bar{c})$$

determines $\bar{h}$. The first-order approximation to the H-dynamics is

$$\exp(\nu)\exp(\bar{h})\left[H_{t+1}^1 + (Y_{t+1}^1 - Y_t^1)\right] = \exp(-\psi)\exp(\bar{h})H_t^1 + [1 - \exp(-\psi)]\exp(\bar{c})C_t^1.$$

or

$$H_{t+1}^1 = \exp(-\nu - \psi)H_t^1 + [\exp(-\nu) - \exp(-\psi - \nu)]\left[\frac{\exp(\bar{c})}{\exp(\bar{h})}\right]C_t^1 - Y_{t+1}^1 + Y_t^1,$$

which after simplification becomes

$$H_{t+1}^1 = \exp(-\nu - \psi)H_t^1 + [1 - \exp(-\nu - \psi)]C_t^1 - Y_{t+1} + Y_t. \qquad (17)$$

3.1.1 Additional steady state calculations

$$\exp(-\delta - \psi - \nu)\exp\left(\overline{mh}\right) = \exp\left(\overline{mh}\right) - \alpha\exp\left[(\eta - 1)\bar{u} - \eta\bar{h}\right]$$

$$(1 - \alpha)\exp\left[(\eta - 1)\bar{u} - \eta\bar{c}\right] = \exp(-\delta - \nu)\exp\left(\overline{mk}\right) - \exp(-\delta - \nu)[1 - \exp(-\psi)]\exp\left(\overline{mh}\right)$$

3.1.2 First-order approximation

$$\exp\left(-\delta - \psi - \nu + \overline{mh}\right)E\left[MH_{t+1}^1 + Y_t^1 - Y_{t+1}^1 \mid \mathcal{F}_t\right]$$
$$= \exp\left(\overline{mh}\right)MH_t^1 - \alpha\exp\left[(\eta - 1)\bar{u} - \eta\bar{h}\right]\left[(\eta - 1)U_t^1 - \eta H_t^1\right] \qquad (18)$$

$$(1 - \alpha)\exp\left[(\eta - 1)\bar{u} - \eta\bar{c}\right]\left[(\eta - 1)U_t^1 - \eta C_t^1\right]$$
$$= \exp(-\delta - \nu)E\left[\exp(\overline{mk})MK_{t+1}^1 - [1 - \exp(-\psi)]\exp(\overline{mh})MH_{t+1}^1 \mid \mathcal{F}_t\right)$$
$$+ \exp(-\delta - \nu)\left[\exp(\overline{mk}) - [1 - \exp(-\psi)]\exp(\overline{mh})\right]E\left(Y_t^1 - Y_{t+1}^1 | \mathcal{F}_t\right). \qquad (19)$$

$$\exp(-\delta + \rho - \nu)E\left[MK^1_{t+1} + (Y^1_t - Y^1_{t+1})|\mathcal{F}_t\right] = MK^1_t. \tag{20}$$

3.1.3 Solution strategy

One approach is to use the deflating subspace calculations described in Hansen and Sargent (2008, ch. 4).

1. Construct

$$Z^1_t = \begin{bmatrix} MK^1_t \\ MH^1_t \\ K^1_t \\ H^1_t \\ X_t \end{bmatrix}$$

2. Take equations (15) and (19) and solve for U^1_t and C^1_t in terms of Z^1_t and Z^1_{t+1}.

3. Use equations (18), (19), (7), and (17) after substituting for U^1_t, C^1_t and $E\left(Y^1_{t+1} - Y^1_t \mid \mathcal{F}_1\right) = D \cdot X_t$ and form the system:[7]

$$\mathbb{L}Z^1_{t+1} = \mathbb{J}Z^1_t$$

where we initially zero out the shocks and use $X_{t+1} = AX_t$.

4. Consider a solution of the co-states in terms of the states of the form:

$$\begin{bmatrix} MK^1_t \\ MH^1_t \end{bmatrix} = \mathbb{N}_{11}\begin{bmatrix} K^1_t \\ H^1_t \end{bmatrix} + \mathbb{N}_{12}X_t.$$

Substituting this into the system dynamics gives:

$$\mathbb{L}\begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \begin{bmatrix} K^1_{t+1} \\ H^1_{t+1} \end{bmatrix} \\ X_{t+1} \end{bmatrix} = \mathbb{J}\begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \begin{bmatrix} K^1_t \\ H^1_t \end{bmatrix} \\ X_t \end{bmatrix}$$

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[7]We implicitly appeal to a certainly-equivalence property to allow us to replace $E[Z^1_{t+1}|\mathcal{F}_t]$ with Z^1_{t+1} on the left side of this equation.

To compute $\mathbb{N}_{11}$ and $\mathbb{N}_{12}$ we will require that the dynamics for

$$\left[\begin{array}{c} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{array}\right]$$

be weakly stable. We accomplish this by first forming a generalized Schur decomposition. There will be a total of seven eigenvalues, three of which are associated with the exogenous dynamics. These three are all stable. There will be four eigenvalues associated with the endogenous dynamics, two of which are stable and two of which are unstable. One of the "endogenous" eigenvalues will be unity, and we will count this as (weakly) stable. This leads us to form:

$$\begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix}$$

by taking linear combinations of the five stable generalized eigenvalues. See Hansen and Sargent (2008, ch. 4) for elaboration.[8] To check the calculation verify that the eigenvalues of the resulting state dynamics are indeed weakly stable.

5. Perform the following check. I think but have not verified that $\mathbb{L}$ is nonsingular. Compute:
$$\mathbb{L}^{-1}\mathbb{J}$$

and thus
$$Z_{t+1}^1 = \mathbb{L}^{-1}\mathbb{J}Z_t^1$$

We know that
$$\begin{bmatrix} I & -\mathbb{N}_{11} & -\mathbb{N}_{12} \end{bmatrix} Z_{t+1}^1 = 0.$$

Thus
$$\begin{bmatrix} I & -\mathbb{N}_{11} & -\mathbb{N}_{12} \end{bmatrix} \mathbb{L}^{-1}\mathbb{J} \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ I & 0 \\ 0 & I \end{bmatrix} = 0.$$

---

[8]This will involve employing an ordered Schur decomposition. Evan Anderson wrote one in Matlab. We can translate it to Julia. Evan's program is described in Hansen and Sargent (2008, ch. 4) and available at the website for the Matlab programs for that book.

6. Compute the eigenvalues of the matrix:

$$\mathbb{A} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathbb{L}^{-1} \mathbb{J} \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ I & 0 \\ 0 & I \end{bmatrix}$$

and check that they coincide with the weakly stable eigenvalues.

7. Add the shocks back to the X evolution equation to get:

$$\begin{bmatrix} MK_t^1 \\ MH_t^1 \end{bmatrix} = \mathbb{N}_{11} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} + \mathbb{N}_{12} X_t,$$

and

$$\begin{bmatrix} \begin{bmatrix} K_{t+1}^1 \\ H_{t+1}^1 \end{bmatrix} \\ X_{t+1} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix} + \mathbb{B} W_{t+1} \qquad (21)$$

where

$$\mathbb{B} = \begin{bmatrix} 0 \\ \mathbb{B}_x \end{bmatrix}$$

The matrix $\mathbb{A}$ should be block triangular with $\mathbb{A}_x$ in the lower block.

The parameter η introduces a form of intertemporal complementarity into preferences; it grows as ρ becomes larger. The next two graphs plot the impulse responses for log consumption. Figure 4 investigates how the choice η alters the responses. For all these relatively large values of η, for the permanent shock the immediate response is muted relative to the long-term response as the response increases with the horizon. Larger values of η apparently induce a more sluggish consumption response. The qualitative nature of these responses looks very similar to those posed in long-run risk models with recursive utility. Here the consumption response is endogenous.
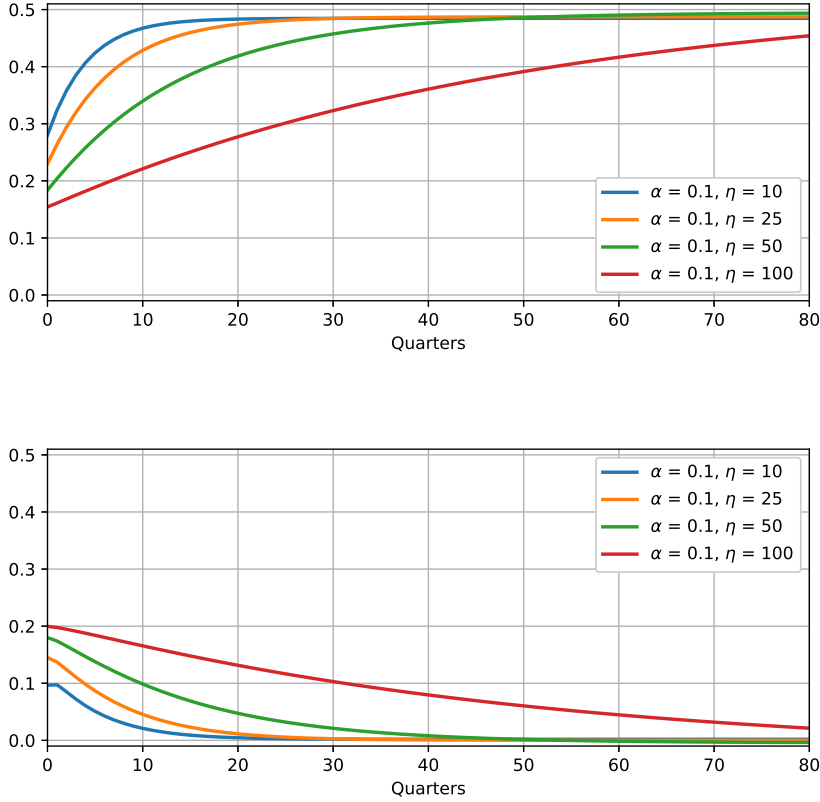
Figure 4: Consumption responses for the two shock processes for habit persistent preferences for $\alpha = .1$, $\psi = .4$ and alternative choice for η. Top panel: permanent shock. Bottom panel: transitory shock.

Figure 5 shows how changing α alters the impulse response for the logarithm of consumption for fixed values $\psi = .4$ and $\eta = 50$. While preserving the same qualitative response patterns for consumption, increasing α from .3 to .7 has little impact on consumption. At the more extreme $\alpha = .1$ and .9 there is substantially more curvature in the initial part of the responses and convergence occurs faster.
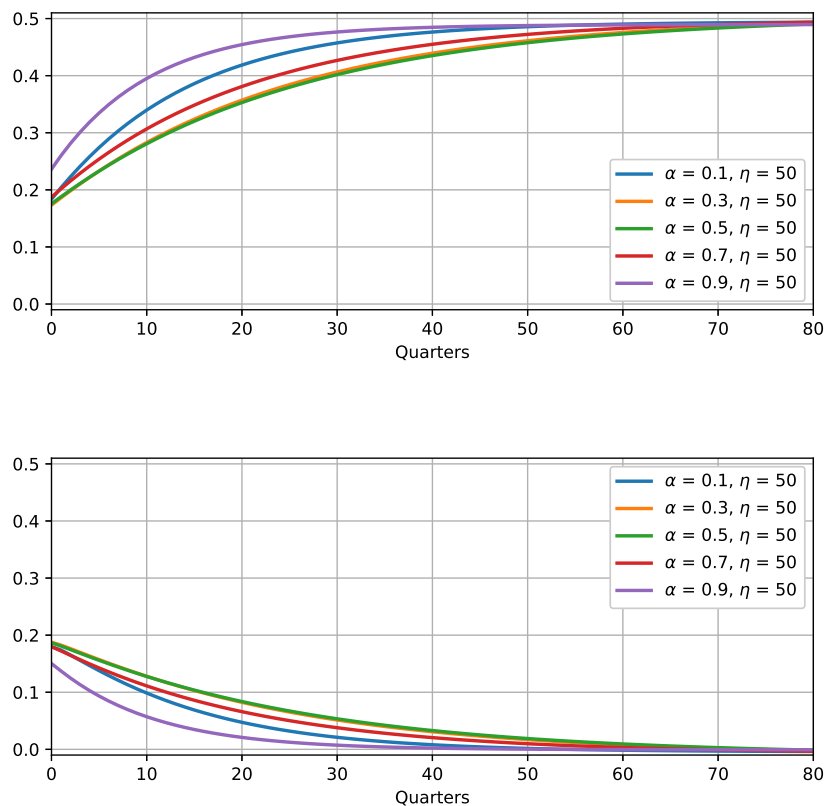
Figure 5: Consumption responses for the two shock processes for habit persistent preferences for $\eta = 50$, $\psi = .4$ and alternative choice for α. Top panel: permanent shock. Bottom panel: transitory shock.
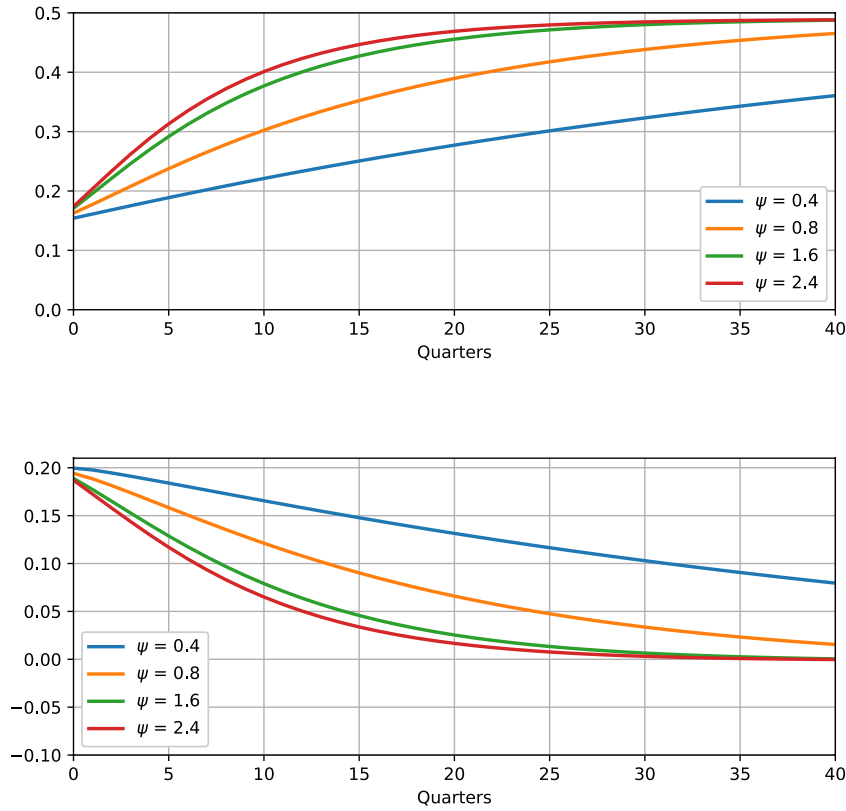
Figure 6: Consumption responses for the two shock processes for habit persistent preferences for $\alpha = .1$, $\eta = 100$, $\psi = .4, .8, 1.6, 2.4$. Top panel: permanent shock. Bottom panel: transitory shock.

Figure 6 shows how the consumption impulse response changes when we alter the rate of depreciation ψ of household capital. We are particularly interested in the responses to the permanent shock. As expected, the responses approximate their limiting value more quickly when we increase ψ. Lars XXXXX: could the scale on the y (coordinate) axis in the lower panel be improved to facilitate comparison? Tom Notice the similarity to the long-run risk consumption response except here we generate endogenously.

Next we add in a concern about robustness as in Hansen et al. (1999). This requires

that we compute the first-order term for the continuation value process.

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right) |\mathcal{F}_t\right]$$

Thus

$$V_t^1 - Y_t^1 = [1 - \exp(-\delta)]U_t^1 - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1 - Y_t^1\right) |\mathcal{F}_t\right]$$

Represent:

$$Y_{t+1}^1 - Y_t^1 = \mathbb{S}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1}$$

$$U_t^1 = \mathbb{S}_u \cdot \begin{bmatrix} K_t^1 \\ H_t^1 \\ X_t \end{bmatrix}$$

$$V_t^1 - Y_t^1 = \mathbb{S}_v \cdot \begin{bmatrix} K_t^1 \\ H_t^1 \\ X_t \end{bmatrix} + \mathsf{s}_v$$

where $\mathbb{S}_u$ comes from the model solution using formula (15) and $\mathbb{S}_v$ and s_v are to be computed as in Hansen et al. (2008). In particular,

$$(\mathbb{S}_v)' = [1 - \exp(-\delta)](\mathbb{S}_u)' + \exp(-\delta)\left[(\mathbb{S}_v)'\mathbb{A} + \begin{bmatrix} 0 & 0 & (\mathbb{S}_y)' \end{bmatrix}\right],$$

and

$$\mathsf{s}_v = \exp(-\delta)\left[\mathsf{s}_v - \frac{\xi}{2}|(\mathbb{S}_v)'\mathbb{B} + (\mathbb{S}_y)'\mathbb{B}_x|^2\right]$$

The first equation is affine in $\mathbb{S}_v$ and can be solved prior to the second equation. Given $\mathbb{S}_v$, the second equation is affine in v and may be solved easily as well. We compute uncertainty prices given by the two-dimensional vector:

$$\frac{1}{\xi}[(\mathbb{S}_v)'\mathbb{B} + F_y] = \frac{1}{\xi}\begin{bmatrix} .482 \\ .000394 \end{bmatrix}$$

for $(\alpha, \eta, \psi) = (.1, 100, 1.6)$.

References

Hansen, Lars Peter and Thomas J. Sargent. 2008. *Robustness*. Princeton, New Jersey: Princeton University Press.

———. 2013. *Recursive Models of Dynamic Linear Economies*. Princeton, New Jersey: Princeton University Press.

Hansen, Lars Peter, William Roberds, and Thomas J. Sargent. 1991. Time Series Implications of Present Value Budget Balance and of Martingale Models of Consumption and Taxes. In *Rational Expectations Econometrics*, edited by Lars Peter Hansen and Thomas J. Sargent, chap. 5. Boulder, Colorado: Westview Press.

Hansen, Lars Peter, Thomas Sargent, and Thomas Tallarini. 1999. Robust Permanent Income and Pricing. *Review of Economic Studies* 66 (4):873–907.

Hansen, Lars Peter, John C. Heaton, and Nan Li. 2008. Consumption Strikes Back?: Measuring Long Run Risk. *Journal of Political Economy* .