Robust social planner with a single capital stock: notes*

April 27, 2018

1 Adjustment cost model

Let K_t be capital and I_t be investment and C_t be capital. Suppose that

$$K_{t+1} = K_t \exp\left[\alpha_k + \left(\frac{I_t}{K_t}\right) - \phi\left(\frac{I_t}{K_t}\right)^2\right] \exp\left(\beta Z_t + \sigma_k \cdot W_{t+1} - \frac{1}{2}|\sigma_k|^2\right)$$

and

$$C_t + I_t = AK_t$$

where A is a fixed parameter. The process Z_t is an AR1 normalized to have mean zero:

$$Z_{t+1} = \alpha_z + \exp(-\xi) Z_t + \sigma_z \cdot W_{t+1}$$

where $|\exp(\xi)| < 1$.

Taking logs gives

$$\log K_{t+1} = \log K_t + \alpha_k + \left(\frac{I_t}{K_t}\right) - \phi\left(\frac{I_t}{K_t}\right)^2 + \beta Z_t + -\frac{1}{2}|\sigma_k|^2 + \sigma_k \cdot W_{t+1} \tag{1}$$

and dividing through by K_t gives

$$\frac{C_t}{K_t} + \frac{I_t}{K_t} = A \tag{2}$$

We assume the period utility function

$$(1-\beta)\log C_t = (1-\beta)(\log C_t - \log K_t) + (1-\beta)\log K_t$$

*We thank Dongchen Zou for helping with the computations.

where $0 < \beta < 1$. We scale by $1 - \beta$ for convenience. Guess a date t continuation value

$$V_t = \log K_t + f(Z_t)$$

that under expected utility and optimal decisions for $\frac{C_t}{K_t}$ and $\frac{I_t}{K_t}$ subject to constraints (1) and (2) satisfies

$$V_t = (1 - \beta) \log C_t + \beta E \left(V_{t+1} | \mathcal{F}_t \right).$$

We set the following parameters:

$$\begin{aligned} \alpha_y &= .373 & \beta &= 1 \\ \alpha_z &= 0 & \xi &= .017 \end{aligned}$$

$$\sigma = \begin{bmatrix} (\sigma_y)' \\ (\sigma_z)' \end{bmatrix} = \begin{bmatrix} .481 & 0 \\ .012 & .027 \end{bmatrix} \tag{3}$$

where $\sigma_y = \sigma_k$ and the implied growth rate for consumption net of the contribution from Z is

$$\alpha_y = \alpha_k + \left(\frac{I_t}{K_t} \right) - \phi \left(\frac{I_t}{K_t} \right)^2 - \frac{1}{2} |\sigma_k|^2.$$

(The $\frac{I}{K}$ ratio turns out to be constant in this model.) We construct the matrix σ so that one of the shocks has permanent consequences and the other is temporary. The contribution f to the value function is affine in the realized value z of Z_t.

Now we execute analogous calculation with a risk-sensitivity or robustness adjustment of the continuation value. We want to study impacts of risk sensitivity or robustness concern on $\frac{C_t}{K_t}$ and $\frac{I_t}{K_t}$. The continuation value under robustness is:

$$V_t = (1 - \beta) \log C_t - \beta \theta \log E \left[\exp \left(-\frac{1}{\theta} V_{t+1} \right) | \mathcal{F}_t \right]$$

Lars XXXXX: Lars and Tom agree that this is an interesting calculation to explore and compare with no robustness version above. On to-do list.

2 Permanent income model

We use a model of a type that Hall and Flavin used to capture a version of Friedman's permanent income model of consumption. Let $\{Y_t\}$ be the logarithm of an exogenous nonfinancial income process that is governed by an additive functional

$$Y_{t+1} - Y_t = \mathbb{D}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1} + \nu$$

where

$$X_{t+1} = \mathbb{A}_x X_t + \mathbb{B}_x W_{t+1}$$

and A is a stable matrix. Define $Y_t^0 = \bar{y}_0 + t\nu$ and

$$Y_{t+1}^1 - Y_t^1 = \mathbb{D}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1}.$$

Let $\widehat{K}_t$ be an asset stock that can be negative, meaning that we allow indebtedness. Combine an exogenous return $\exp(\rho)$ on the asset and a time t exogenous nonfinancial income to deduce that the asset stock evolves according to

$$\widehat{K}_{t+1} + \exp\left(\widehat{C}_t\right) = \exp(\rho)\widehat{K}_t + \exp(Y_t), \tag{4}$$

where $\widehat{C}_t$ is the logarithm of consumption at date t and ρ parameterizes an exogenous constant rate of return on assets. It is convenient to scale variables by nonfinancial income, so we define $C_t = \widehat{C}_t - Y_t$ and $K_t = \frac{\widehat{K}_t}{\exp(Y_t)}$. Divide both sides of equation (4) by $\exp(Y_t)$ to obtain

$$K_{t+1} \exp\left(Y_{t+1} - Y_t\right) + \exp\left(C_t\right) = \exp(\rho) K_t + 1. \tag{5}$$

A representative consumer with time separable preferences, logarithmic one period utility of consumption, and subjective discount rate δ optimizes by choosing a consumption process that satisfies the Euler equation

$$\exp(-\delta + \rho) E\left[\exp(\widehat{C}_t - \widehat{C}_{t+1}) | X_t\right] = 1$$

or equivalently

$$\exp(-\delta + \rho) E\left[\exp(C_t - C_{t+1} + Y_t - Y_{t+1}) | X_t\right] = 1.$$

We assume that

$$\exp(-\delta + \rho - \nu) = 1,$$

a restriction that supports steady states in which the log consumption-log income ratio equals $\bar{c}$.

2.1 Steady state

Steady state means of asymptotically stationary components of (C_t, K_t) must satisfy

$$\bar{k} \exp(\nu) + \exp(\bar{c}) = \exp(\rho)\bar{k} + 1$$

or equivalently

$$\exp(\bar{c}) = [\exp(\rho) - \exp(\nu)]\, \bar{k} + 1,$$

where we assume that $\rho > \nu$. Notice that we are free to set $\bar{k}$. For convenience, we assume that $\bar{k} = 0$ and hence that $\exp(\bar{c}) = 1$.

2.2 First-order approximation

We take a first-order small-noise approximation that scales W_{t+1} by q and let q tend to zero. We use the following notation. Processes with superscripts 1 are effectively first-order derivatives of corresponding original processes with respect to q evaluated at $\mathsf{q} = 1$:

$$\begin{aligned} K^1_{t+1} \exp(\nu) + \bar{k} \exp(\nu) \left(Y^1_{t+1} - Y^1_t\right) + \exp(\bar{c}) C^1_t \\ = \exp(\rho) K^1_t \end{aligned}$$

or

$$K^1_{t+1} = \exp(\rho - \nu) K^1_t - \bar{k} \left(Y^1_{t+1} - Y^1_t\right) - \exp(\bar{c} - \nu) \exp(\bar{c}) C^1_t. \tag{6}$$

The restrictions $exp(\bar{c}) = 1$ and $\bar{k} = 0$ make equation (6) become

$$K^1_{t+1} = \exp(\rho - \nu) K^1_t - \exp(-\nu) C^1_t \tag{7}$$

To derive an approximate decision rule for consumption that solves a representative agent planning problem, we first solve equation (6) forward and take conditional expectations:[1]

$$
\begin{aligned}
\exp(\nu)K_t^1 &= \sum_{j=0}^{\infty} \lambda^{j+1} E\left(C_{t+j}^1 + Y_{t+j}^1 \mid \mathcal{F}_t\right) - \sum_{j=0}^{\infty} \lambda^{j+1} E\left(Y_{t+j}^1 \mid \mathcal{F}_t\right) \\
&= \left(\frac{\lambda}{1-\lambda}\right)(C_t^1 + Y_t^1) - \left(\frac{\lambda}{1-\lambda}\right)\sum_{j=1}^{\infty} \lambda^j E\left(Y_{t+j}^1 - Y_{t+j-1}^1 \mid \mathcal{F}_t\right) - \left(\frac{\lambda}{1-\lambda}\right) Y_t^1
\end{aligned} \tag{8}
$$

A first-order approximation to the Euler equation that expresses first-order conditions for optimzation of the planning problem is

$$
E\left[C_{t+1}^1 + Y_{t+1}^1 | \mathcal{F}_t\right] = C_t^1 + Y_t^1.
$$

Substituting this approximation to the Euler equation repeatedly in equation (8) gives

$$
\exp(\nu)K_t^1 = \left(\frac{\lambda}{1-\lambda}\right) C_t^1 - \left(\frac{\lambda}{1-\lambda}\right)\sum_{j=1}^{\infty} \lambda^j E\left(Y_{t+j}^1 - Y_{t+j-1}^1 \mid \mathcal{F}_t\right). \tag{9}
$$

where $\lambda = \exp(\nu - \rho)$. The approximating decision rule (8) implies that C_t^1 and K_t^1 are cointegrated with cointegrating vector $\begin{bmatrix} 1 & [\exp(\nu) - \exp(\rho)] \end{bmatrix}$.

From equation (8), we obtain the approximation

$$
C_t^1 = \frac{\exp(-\nu)(1-\lambda)}{\lambda} K_t^1 + \exp(-\nu) \sum_{j=1}^{\infty} \lambda^j E\left(Y_{t+j}^1 - Y_{t+j-1}^1 \mid \mathcal{F}_t\right) \tag{10}
$$

This links the log consumption-income ratio to the planner's two income sources: financial income and non-financial income. The non-financial income contribution to the log consumption-income ratio is

$$
\sum_{j=1}^{\infty} \lambda^j E\left(Y_{t+j}^1 - Y_{t+j-1}^1 \mid \mathcal{F}_t\right) = \lambda A \left(I - \lambda A\right)^{-1} X_t
$$

Approximate decision rule (8) corresponds is a first-order approximation to a solution of the planner's problem in the Hansen et al. (1999) economy without robustness (i.e., the problem obtained by setting their $\sigma = 0$).

[1] With or without taking conditional expectations of all time indexed variables, We can solve equation (6) forward.Cite a reference to one of many places where we have done this in other publications.

2.3 Impulse responses

We evaluate first-order approximate decision rules for consumption and assets at a nonfinancial income process adapted from Hansen et al. (1999), They assumed two components of nonfinancial income, one more persistent than the other. To construct the first component, let

$$X^1_{1,t+1} = .704 X_{1,t} + \begin{bmatrix} .144 & 0 \end{bmatrix} W_{t+1}$$

where $Y^1_{1,t+1} = Y^1_t + X^1_{1,t+1}$. To construct the second component, let

$$X^1_{2,t+1} = X^1_{2,t} - .154 X^1_{2,t-1} + \begin{bmatrix} 0 & .206 \end{bmatrix} W_{t+1}$$

and construct $Y^1_{2,t+1} = X^1_{2,t+1}$. Let $Y^1_{t+1} = (.01) Y^1_{1,t+1} + (.01) Y^2_{2,t+1}$.[2] We represent this $\{Y_t\}$ process as an additive functional. Set $\rho = .00663$ and $\nu = .00373$.

[2] We take income numbers from the first column of Table 2 of Hansen et al. (1999) with two modifications. In Hansen et al. (1999), both income processes are stationary but one has an autoregressive root of .998. We set this to one here. This has a nontrivial impact on the consumption volatility, which Hansen et al. estimated in levels. We scale both innovation standard deviations by 1.33 to achieve a consumption growth rate volatility of .482 expressed as a percent (log differences multiplied by 100).

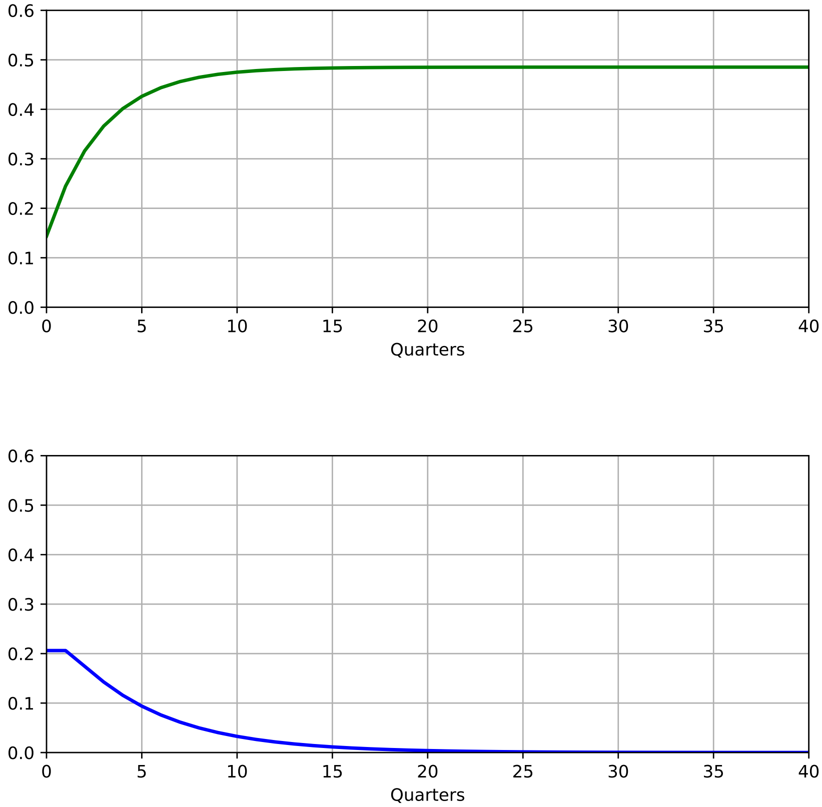

Figure 1: Impulse responses of log income to the two shock processes. Top panel: permanent shock. Bottom panel: transitory shock. Parameter settings from Hansen et al. (1999).

As lag length on the horizontal axis becomes large, the positive limit of the impulse response coefficients for the first shock in figure 1 tell us that the first shock has permanent effects; the zero limit of the impulse response coefficients for the second shock tells us that it has only transitory effects. The planner cannot self-insure against the permanent shock via a form of precautionary savings; but the planner can self-insure against the transitory shock. Under the time separable preferences being used here, the responses of consumption to shocks are both constant across horizons; so impulse response functions to both shocks are just step functions. This is an implication of the consumption smoothing built into the model. The two consumption responses for the two shocks are:

$$\text{permanent shock} = .482$$
$$\text{transitory shock} = .00383,$$

(when multiplied by 100 Lars XXXX: please clarify why you added this qualification about 100.) which shows the dramatic difference induced by the endogenous consumption savings responses.

Present values of impulse response functions of consumption and non-financial income to both shocks display a feature described by Hansen et al. (1991): discounted by λ, the present value of the impulse response function of the "deficit" defined as consumption minus non-financial income is zero for each shock. Since C_t^1 is the first-order approximation to the log consumption/income ratio, the infinite discounted sum of the response of C_t^1 to both shocks should be zero. Figure 2 computes the discounted sum over the alternative horizons in order to study how quickly the sum converges to zero. From the time scale, we see that convergence is slow.

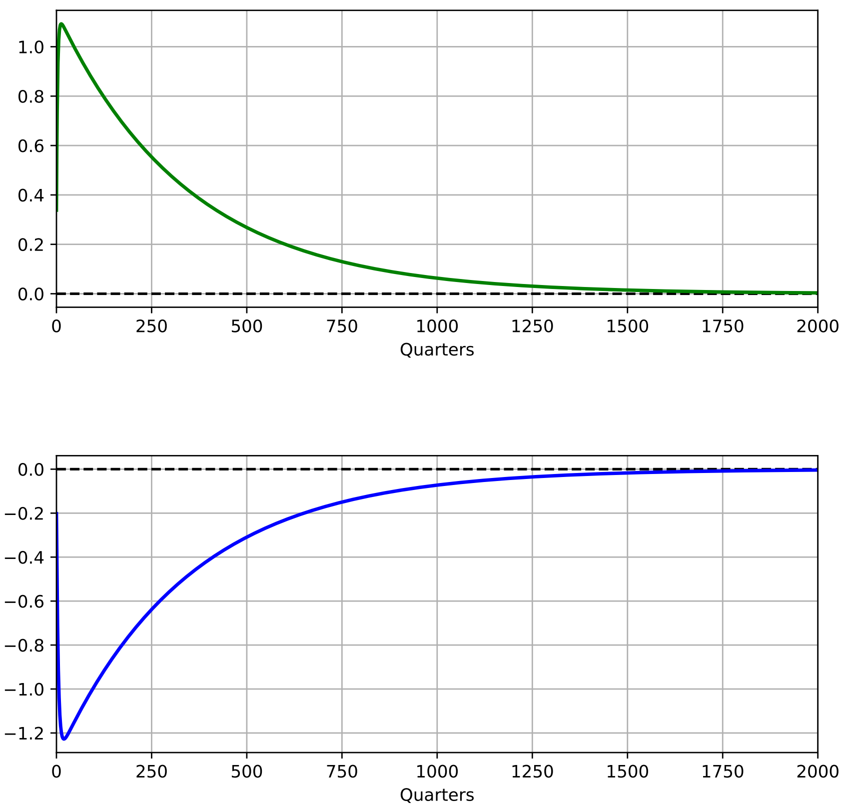

Figure 2: Present value responses for the two shock processes. Top panel: permanent shock. Bottom panel: transitory shock. Parameter settings from Hansen et al. (1999).

2.4 Robustness

Following Hansen et al. (1999), we now attribute a concern about robustness to the planner by using the utility recursion:

$$V_t = [1 - \exp(-\delta)](C_t + Y_t) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right)|\mathcal{F}_t\right]$$

where V_t is a date t continuation value and $\xi \leqslant +\infty$. Setting $\xi = \infty$ eliminates concerns about robustness and returns us to time-separable log utility. The parameter ξ is the inverse of the risk sensitivity parameter of Hansen et al. (1999). The first-order approximation to a continuation value process is of the form

$$V_t^1 = [1 - \exp(-\delta)](C_t^1 + Y_t^1) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)|\mathcal{F}_t\right]$$

where $F_c \cdot W_{t+1}$ gives the response of C_{t+1}^1 to W_{t+1}. To construct an observational equivalence like that obtained by Hansen et al. (1999), we posit that the evolution for $C_{t+1}^1 + Y_{t+1}^1$ is:

$$C_{t+1}^1 + Y_{t+1}^1 = C_t^1 + Y_t^1 + (F_c + F_y) \cdot W_{t+1}. \tag{11}$$

Then

$$V_t^1 = C_t^1 + Y_t^1 - \frac{|F_c + F_y|^2}{2\xi}$$

and so

$$V_{t+1}^1 = C_t^1 + Y_t^1 - \frac{|F_c + F_y|^2}{2\xi} + (F_c + F_y) \cdot W_{t+1}.$$

In effect, the planner's concern about robustness induces him to construct V_t^1 by changing the measure of W_{t+1} from normal with mean 0 and covariance matrix I to become normally distributed with mean $-\frac{1}{\xi}(F_c + F_y)$ and covariance matrix I. This equivalent with multiplying the conditional density of W_{t+1} under the baseline normal $(0, I)$ model with the following positive random variable that acts as a likelihood ratio for constructing the robustness-induced distribution:

$$M_{t+1}^0 = \frac{\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)}{E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1\right)|\mathcal{F}_t\right]}.$$

Because a robust planner acts as if he is an ordinary planner who simply evaluates conditional expectations with the distorted probability distribution instead of the benchmark distribution for W_{t+1}, it follows that the consumption Euler equation of the robust planner is

$$\exp(-\delta + \rho - \nu)E\left[M_{t+1}^0\left(C_{t+1}^1 + Y_{t+1}^1\right)|\mathcal{F}_t\right] = C_t^1 + Y_t^1$$

Under an implied worst-case model associated with the probability distortion M_{t+1}^0, the expected growth rate in consumption is lowered by:

$$-\frac{|F_c + F_y|^2}{\xi}$$

This affects the risk-free interest rate.

To obtain a characterization of the equivalence between the effects of discounting through δ and robustness through $\frac{1}{\xi}$, we follow Hansen et al. (1999) by fixing a target interest rate and then adjusting δ to hit that target. To be consistent with evolution equation (11), we therefore assume:

$$\delta = \rho - \nu - \frac{|F_c + F_y|^2}{\xi}$$

This gives an affine in $\frac{1}{\xi}$ counterpart to formula (28) in Hansen et al. (1999) with slope coefficient $-\frac{|F_c+F_y|^2}{2}$ as is depicted in Figure 3.

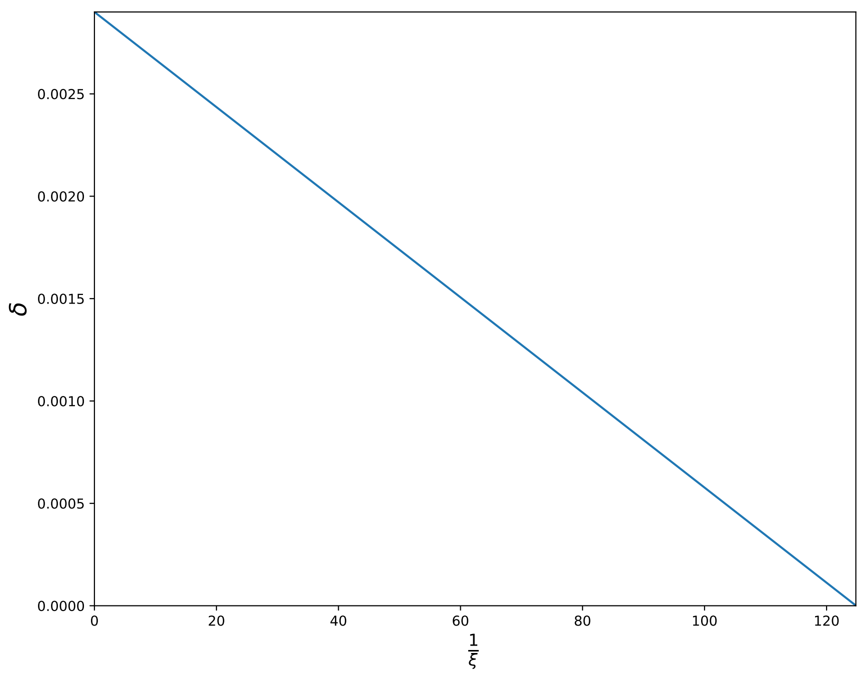

Figure 3: Subjective discount rates and robustness. This plot shows how to adjust the subjective discount rate δ for a given value of $\frac{1}{\xi}$ while leaving the implied riskless rate fixed. Lars XXXXX: we should ask the RA's to use a larger font for the axis labels.

Tom: write some things about the next very nice calculations.

The uncertainty price vector for the two shocks is:

$$-\frac{1}{\xi}\left(F_c + F_y\right) = \frac{.01}{\xi}\begin{bmatrix} .482 \\ .00383 \end{bmatrix}$$

The exposure to the shock with permanent consequences requires much larger compensations because the robust planner fears the misspecification of that so much more.

3 Habit persistence

We aim now to construct a multiplicative-functional counterpart to HST's specification with habit persistence. To accomplish this, we change the planner's preferences to

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}\right)|\mathcal{F}_t\right]$$

where $0 \leqslant \psi < 1$ and

$$\exp(H_{t+1} + Y_{t+1}) = \exp(-\psi)\left[\exp(H_t + Y_t)\right] + \left[1 - \exp(-\psi)\right]\left[\exp(C_t + Y_t)\right] \tag{12}$$

and

$$\exp(U_t + Y_t) = \upsilon[\exp(C_t + Y_t), \exp(H_t + Y_t)], \tag{13}$$

where υ is the CES function

$$\upsilon(c, h) = \left[(1 - \alpha)c^{1-\eta} + \alpha h^{1-\eta}\right]^{\frac{1}{1-\eta}}.$$

With appropriate parameter settings, this υ can capture either durability of consumption goods or habit persistence or both.

3.1 H dynamics

After dividing both sides by $\exp(Y_t)$, rewrite the H dynamics equation (**??**) as

$$\exp(H_{t+1})\exp(Y_{t+1} - Y_t) = \exp(-\psi)\exp(H_t) + [1 - \exp(-\psi)]\exp(C_t).$$

A steady state counterpart to this equation, namely,

$$\exp(\bar{h})\exp(\nu) = \exp(-\psi)\exp(\bar{h}) + [1 - \exp(-\psi)]\exp(\bar{c})$$

determines $\bar{h}$. The first-order approximation is

$$\exp(\nu)\exp(\bar{h})\left[H^1_{t+1} + (Y^1_{t+1} - Y^1_t)\right] = \exp(-\psi)\exp(\bar{h})H^1_t + [1 - \exp(-\psi)]\exp(\bar{c})C^1_t.$$

or

$$H^1_{t+1} = \exp(-\nu - \psi)H^1_t + [\exp(-\nu) - \exp(-\psi - \nu)]\left[\frac{\exp(\bar{c})}{\exp(\bar{h})}\right]C^1_t - Y^1_{t+1} + Y^1_t,$$

which after simplification becomes

$$H^1_{t+1} = \exp(-\nu - \psi)H^1_t + [1 - \exp(-\nu - \psi)]C^1_t - Y_{t+1} + Y_t. \tag{14}$$

3.1.1 CES algebra

For constructing approximations to (13), it is useful compute the first derivatives:

$$\begin{aligned} mc &= (1 - \alpha)u^\eta c^{-\eta} \\ mh &= \alpha u^\eta h^{-\eta} \end{aligned}$$

Define

$$\bar{u} = \frac{1}{1 - \eta}\log\left((1 - \alpha)\exp[(1 - \eta)\bar{c}] + \alpha\exp[(1 - \eta)\bar{h}]\right),$$

which is the steady-state version of U_t. The first-order approximation of U_t is:

$$U^1_t = (1 - \alpha)\exp\left[(\eta - 1)(\bar{u} - \bar{c})\right]C^1_t + \alpha\exp\left[(\eta - 1)(\bar{u} - \bar{h})\right]H^1_t \tag{15}$$

3.1.2 Co-state evolution

There are two co-state equations and one set of first-order conditions for consumption:

$$\alpha\exp\left[(\eta - 1)(U_t + Y_t) - \eta(H_t + Y_t)\right] - \exp\left(\widehat{MH}_t\right) + \exp(-\delta - \psi)E\left[\exp\left(\widehat{MH}_{t+1}\right) \mid \mathcal{F}_t\right] = 0$$

$$
\begin{aligned}
&(1-\alpha)\exp\left[(\eta-1)(U_t+Y_t)-\eta(C_t+Y_t)\right] \\
&+\exp(-\delta)[1-\exp(-\psi)]E\left[\exp\left(\widehat{MH}_{t+1}\right)\mid\mathcal{F}_t\right]-\exp(-\delta)E\left[\exp\left(\widehat{MK}_{t+1}\right)\mid\mathcal{F}_t\right]=0
\end{aligned}
$$

$$
\exp(-\delta+\rho)E\left[\exp\left(\widehat{MK}_{t+1}\right)\mid\mathcal{F}_t\right]-\exp\left(\widehat{MK}_t\right)=0.
$$

Multiply each of these by $\exp(Y_t)$ to get:

$$
\exp(-\delta-\psi)E\left[\exp\left(MH_{t+1}+Y_t-Y_{t+1}\right)\mid\mathcal{F}_t\right]=\exp\left(MH_t\right)-\alpha\exp\left[(\eta-1)U_t-\eta H_t\right]
$$

$$
\begin{aligned}
(1-\alpha)\exp\left[(\eta-1)U_t-\eta C_t\right]=&\exp(-\delta)E\left[\exp\left(MK_{t+1}+Y_t-Y_{t+1}\right)\mid\mathcal{F}_t\right]\\
&-\exp(-\delta)[1-\exp(-\psi)]E\left[\exp\left(MH_{t+1}+Y_t-Y_{t+1}\right)\mid\mathcal{F}_t\right]
\end{aligned}
$$

$$
\exp(-\delta+\rho)E\left[\exp\left(MK_{t+1}+Y_t-Y_{t+1}\right)\mid\mathcal{F}_t\right]=\exp\left(MK_t\right)
$$

3.1.3 Additional steady state calculations

$$
\exp(-\delta-\psi-\nu)\exp\left(\overline{mh}\right)=\exp\left(\overline{mh}\right)-\alpha\exp\left[(\eta-1)\bar{u}-\eta\bar{h}\right]
$$

$$
(1-\alpha)\exp\left[(\eta-1)\bar{u}-\eta\bar{c}\right]=\exp(-\delta-\nu)\exp\left(\overline{mk}\right)-\exp(-\delta-\nu)[1-\exp(-\psi)]\exp\left(\overline{mh}\right)
$$

3.1.4 First-order approximation

$$
\begin{aligned}
&\exp\left(-\delta-\psi-\nu+\overline{mh}\right)E\left[MH^1_{t+1}+Y^1_t-Y^1_{t+1}\mid\mathcal{F}_t\right]\\
&=\exp\left(\overline{mh}\right)MH^1_t-\alpha\exp\left[(\eta-1)\bar{u}-\eta\bar{h}\right]\left[(\eta-1)U^1_t-\eta H^1_t\right]
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&(1-\alpha)\exp\left[(\eta-1)\bar{u}-\eta\bar{c}\right]\left[(\eta-1)U^1_t-\eta C^1_t\right]\\
&=\exp(-\delta-\nu)E\left[\exp(\overline{mk})MK^1_{t+1}-[1-\exp(-\psi)]\exp(\overline{mh})MH^1_{t+1}\mid\mathcal{F}_t\right)\\
&+\exp(-\delta-\nu)\left[\exp(\overline{mk})-[1-\exp(-\psi)]\exp(\overline{mh})\right]E\left(Y^1_t-Y^1_{t+1}|\mathcal{F}_t\right).
\end{aligned}
\tag{17}
$$

$$
\exp(-\delta+\rho-\nu)E\left[MK^1_{t+1}+(Y^1_t-Y^1_{t+1})|\mathcal{F}_t\right]=MK^1_t.
\tag{18}
$$

3.1.5 Solution strategy

One approach is to use the deflating subspace calculations described in Hansen and Sargent (2008, ch. 4).

1. Construct
$$Z_t^1 = \begin{bmatrix} MK_t^1 \\ MH_t^1 \\ K_t^1 \\ H_t^1 \\ X_t \end{bmatrix}$$

2. Take equations (15) and (17) and solve for U_t^1 and C_t^1 in terms of Z_t^1 and Z_{t+1}^1.

3. Use equations (16), (17), (6), and (14) after substituting for U_t^1, C_t^1 and $E\left(Y_{t+1}^1 - Y_t^1 \mid \mathcal{F}_1\right) = D \cdot X_t$ and form the system:
$$\mathbb{L} Z_{t+1}^1 = \mathbb{J} Z_t^1$$
where we initially zero out the shocks and use $X_{t+1} = AX_t$.

4. Consider a solution of the co-states in terms of the states of the form:
$$\begin{bmatrix} MK_t^1 \\ MH_t^1 \end{bmatrix} = \mathbb{N}_{11} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} + \mathbb{N}_{12} X_t.$$
Substituting this into the system dynamics gives:
$$\mathbb{L} \begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} K_{t+1}^1 \\ H_{t+1}^1 \end{bmatrix} \\ X_{t+1} \end{bmatrix} = \mathbb{J} \begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix}$$
To compute $\mathbb{N}_{11}$ and $\mathbb{N}_{12}$ we will require that the dynamics for
$$\begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix}$$
be weakly stable. We accomplish this by first forming a generalized Schur decomposition. There will be a total of seven eigenvalues, three of which are associated with

the exogenous dynamics. These three are all stable. There will be four eigenvalues associated with the endogenous dynamics, two of which are stable and two of which are unstable. One of the "endogenous" eigenvalues will be unity, and we will count this as (weakly) stable. This leads us to form:

$$\begin{bmatrix} \mathbb{N}_{1,1} & \mathbb{N}_{1,2} \\ I & 0 \\ 0 & I \end{bmatrix}$$

by taking linear combinations of the five stable generalized eigenvalues. See Hansen and Sargent (2008, ch. 4) for elaboration.[3] To check the calculation verify that the eigenvalues of the resulting state dynamics are indeed weakly stable.

5. Perform the following check. I think but have not verified that $\mathbb{L}$ is nonsingular. Compute:

$$\mathbb{L}^{-1}\mathbb{J}$$

and thus

$$Z^1_{t+1} = \mathbb{L}^{-1}\mathbb{J}Z^1_t$$

We know that

$$\begin{bmatrix} I & -\mathbb{N}_{11} & -\mathbb{N}_{12} \end{bmatrix} Z_{t+1} = 0.$$

Thus

$$\begin{bmatrix} I & -\mathbb{N}_{11} & -\mathbb{N}_{12} \end{bmatrix} \mathbb{L}^{-1}\mathbb{J} \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ I & 0 \\ 0 & I \end{bmatrix} = 0.$$

6. Compute the eigenvalues of the matrix:

$$\mathbb{A} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathbb{L}^{-1}\mathbb{J} \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ I & 0 \\ 0 & I \end{bmatrix}$$

and check that they coincide with the weakly stable eigenvalues.

[3] This will involve employing an ordered Schur decomposition. Evan Anderson wrote one in Matlab. We can translate it to Julia. Evan's program is described in Hansen and Sargent (2008, ch. 4) and available at the website for the Matlab programs for that book.

7. Add the shocks back to the X evolution equation to get:

$$\begin{bmatrix} MK_t^1 \\ MH_t^1 \end{bmatrix} = \mathbb{N}_{11} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} + \mathbb{N}_{12} X_t,$$

and

$$\begin{bmatrix} \begin{bmatrix} K_{t+1}^1 \\ H_{t+1}^1 \end{bmatrix} \\ X_{t+1} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \begin{bmatrix} K_t^1 \\ H_t^1 \end{bmatrix} \\ X_t \end{bmatrix} + \mathbb{B} W_{t+1} \tag{19}$$

where

$$\mathbb{B} = \begin{bmatrix} 0 \\ \mathbb{B}_x \end{bmatrix}$$

The matrix $\mathbb{A}$ should be block triangular with $\mathbb{A}_x$ in the lower block.

The parameter η introduces a form of intertemporal complementarity into preferences; it grows as ρ becomes larger. The next two graphs plot the impulse responses for log consumption. Figure 4 investigates how the choice η alters the responses. For all these relatively large values of η, for the permanent shock the immediate response is muted relative to the long-term response as the response increases with the horizon. Larger values of η apparently induce a more sluggish consumption response. The qualitative nature of these responses looks very similar to those posed in long-run risk models with recursive utility. Here the consumption response is endogenous.

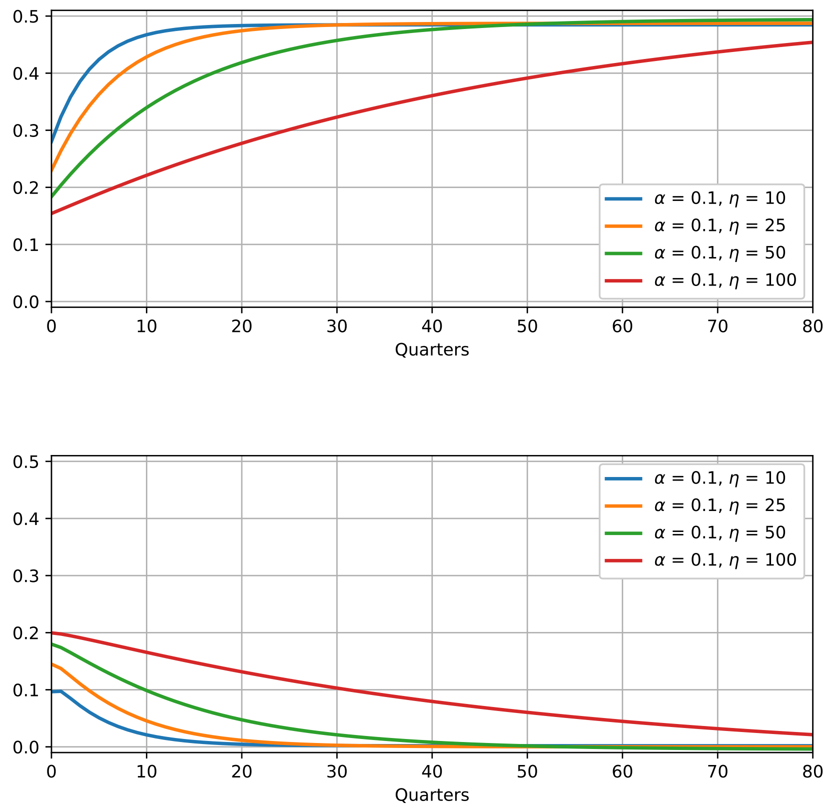

Figure 4: Consumption responses for the two shock processes for habit persistent preferences for $\alpha = .1$, $\psi = .4$ and alternative choice for η. Top panel: permanent shock. Bottom panel: transitory shock.

Figure 5 shows how changing α alters the impulse response for the logarithm of consumption for fixed values $\psi = .3$ and $\eta = 40$. While preserving the same qualitative response patterns for consumption, increasing α from .3 to .7 has little impact on consumption. At the more extreme $\alpha = .1$ and .9 there is substantially more curvature in the initial part of the responses and convergence occurs faster.

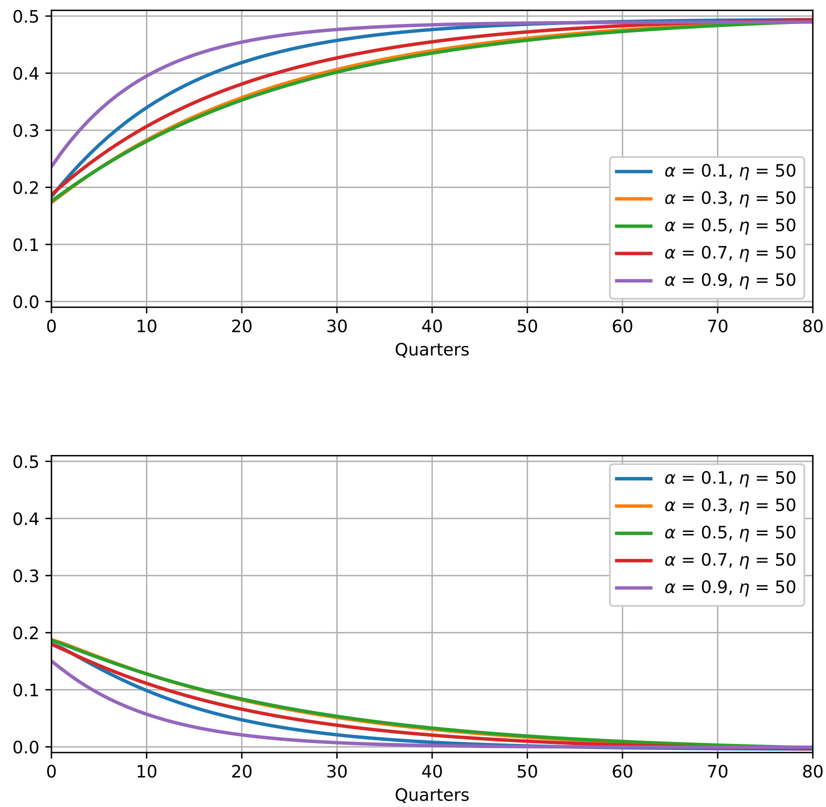

Figure 5: Consumption responses for the two shock processes for habit persistent preferences for $\eta = 50$, $\psi = .4$ and alternative choice for α. Top panel: permanent shock. Bottom panel: transitory shock.

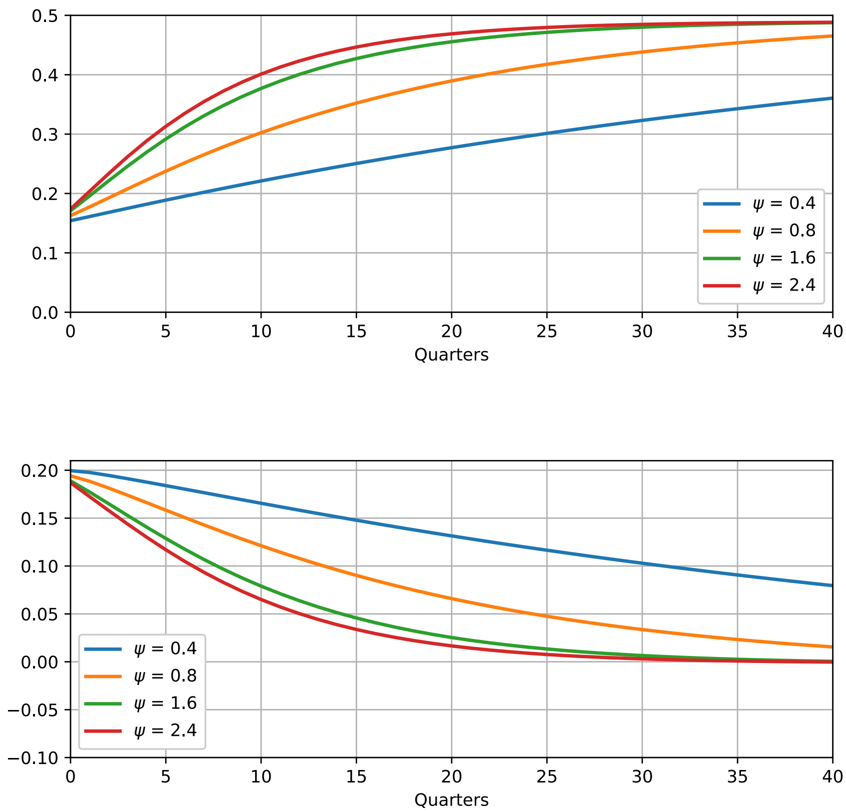

Figure 6: Consumption responses for the two shock processes for habit persistent preferences for $\alpha = .1$, $\eta = 100$, $\psi = .4, .8, 1.6, 2.4$. Top panel: permanent shock. Bottom panel: transitory shock.

Figure 6, show how the consumption impulse response changes as we alter the rate of depreciation in the household capital ψ. We are particularly, interested in the responses to the permanent shock. As expected, the responses approximate their limiting value more quickly when we increase ψ. Tom Notice the similarity to the long-run risk consumption response except here we generate endogenously.

Next we add in a concern about robustness as in Hansen et al. (1999). This requires that we compute the first-order term for the continuation value process.

$$V_t = [1 - \exp(-\delta)](U_t + Y_t) - \exp(-\delta)\xi \log E \left[\exp\left(-\frac{1}{\xi} V_{t+1} \right) | \mathcal{F}_t \right]$$

Thus

$$V_t^1 - Y_t^1 = [1 - \exp(-\delta)]U_t^1 - \exp(-\delta)\xi \log E\left[\exp\left(-\frac{1}{\xi}V_{t+1}^1 - Y_t^1\right)|\mathcal{F}_t\right]$$

Represent:

$$\begin{aligned} Y_{t+1}^1 - Y_t^1 &= \mathbb{S}_y \cdot X_t + \mathbb{F}_y \cdot W_{t+1} \\ U_t^1 &= \mathbb{S}_u \cdot \begin{bmatrix} K_t^1 \\ H_t^1 \\ X_t \end{bmatrix} \\ V_t^1 - Y_t^1 &= \mathbb{S}_v \cdot \begin{bmatrix} K_t^1 \\ H_t^1 \\ X_t \end{bmatrix} + \mathsf{s}_v \end{aligned}$$

where $\mathbb{S}_u$ comes from the model solution using formula (15) and $\mathbb{S}_v$ and s_v are to be computed as in Hansen et al. (2008). In particular,

$$(\mathbb{S}_v)' = [1 - \exp(-\delta)](\mathbb{S}_u)' + \exp(-\delta)\left[(\mathbb{S}_v)'\mathbb{A} + \begin{bmatrix} 0 & 0 & (\mathbb{S}_y)' \end{bmatrix}\right],$$

and

$$\mathsf{s}_v = \exp(-\delta)\left[\mathsf{s}_v - \frac{\xi}{2}\left|(\mathbb{S}_v)'\mathbb{B} + (\mathbb{S}_y)'\mathbb{B}_x\right|^2\right]$$

The first equation is affine in $\mathbb{S}_v$ and can be solved prior to the second equation. Given $\mathbb{S}_v$, the second equation is affine in v and may be solved easily as well. We compute uncertainty prices given by the two-dimensional vector:

$$\frac{1}{\xi}\left[(\mathbb{S}_v)'\mathbb{B} + F_y\right] = \frac{1}{\xi}\begin{bmatrix} .482 \\ .000394 \end{bmatrix}$$

for $(\alpha, \eta, \psi) = (.1, 100, 1.6)$.

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