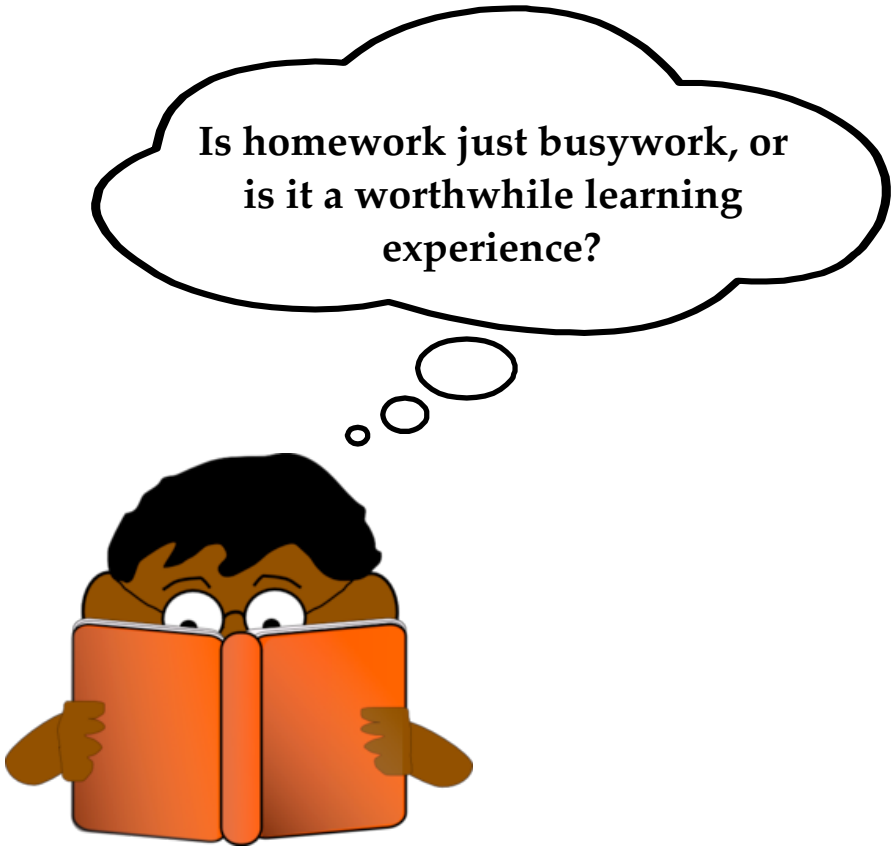
# Simple Linear Regression




**Data**

- Eighth graders were asked to consider their mathematics homework over the last month, and respond to the survey question of approximately *how many hours of time they spend doing mathematics homework per week*.
- Scores on a standardized mathematics achievement test were also collected. (The test has a national mean score of 50 and a standard deviation of 10.)

Keith, T. Z. (2006). *Multiple regression and beyond*. Boston: Pearson.

# Read in Data and Load Libraries

```
# Load the data (homework-achievement.csv)
> math = read.csv("EPSY-8262/data/homework-achievement.csv")

# Load libraries; Note: you may need to install them first
> library(ggplot2)
> library(psych)
> library(sm)

# The ggplot2 library will let us use the ggplot() function
# The psych library will let us use the describe() function
# The sm library will let us use the sm.density() function
```

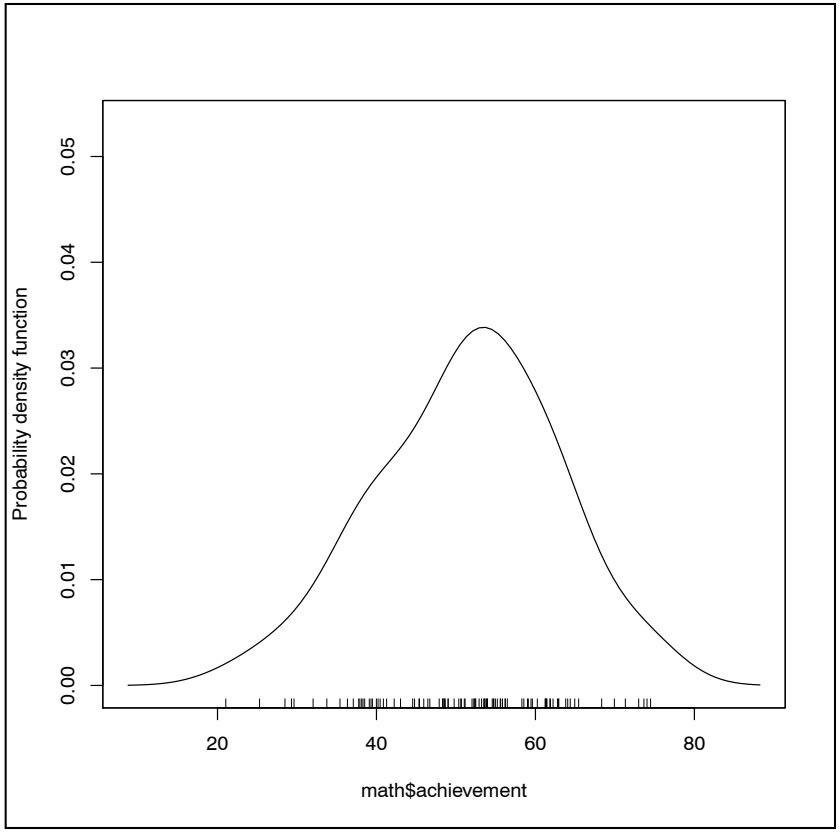
# Examine the Data

```
> head(math)

  homework achievement
1        2          54
2        0          53
3        4          53
4        0          56
5        2          59
6        0          30

> tail(math)

    homework achievement
95         6          50
96         4          48
97         3          58
98         2          39
99         2          41
100        1          51
```

# Examine the Outcome

```
> sm.density(math$achievement)

> describe(math$achievement)

  vars   n  mean    sd median trimmed   mad min max range  skew kurtosis   se
1    1 100 51.41 11.29     53   51.65 11.86  22  75    53 -0.22     -0.3 1.13
```

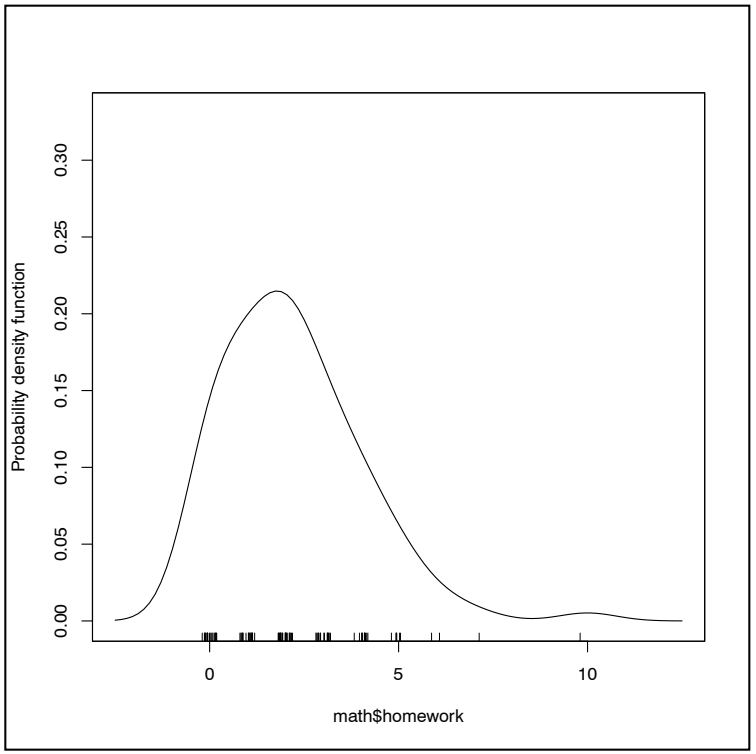
The marginal distribution of mathematics achievement scores is unimodal with a mean of 51. There is variation in these scores ($SD = 11$).



# Examine the Predictor

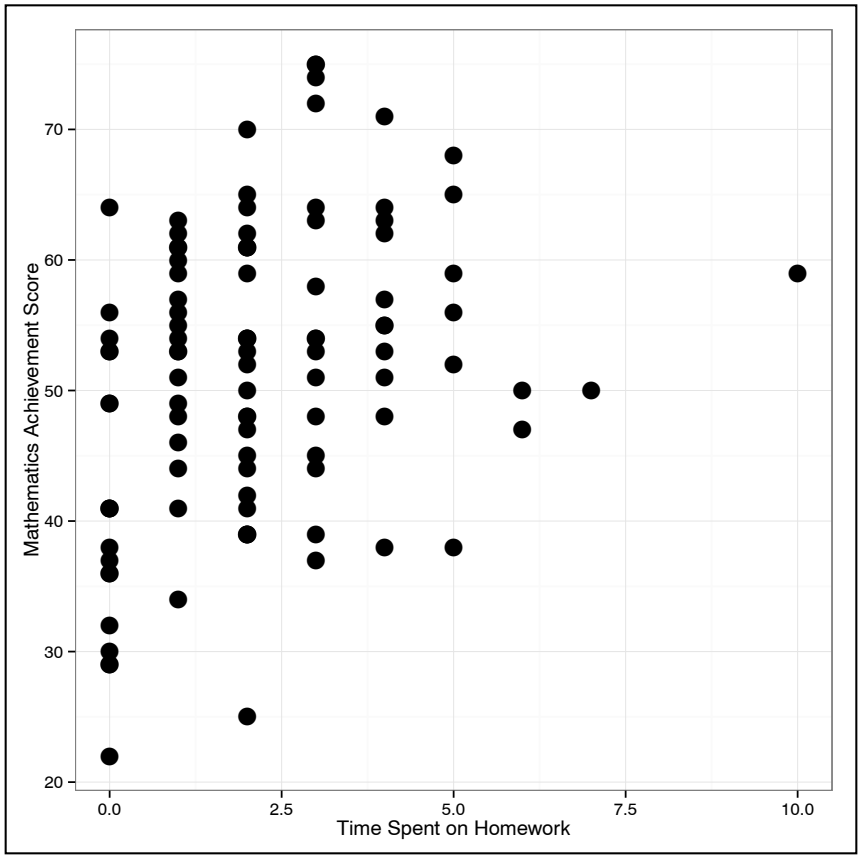
```
> sm.density(math$homework)

> describe(math$homework)

  vars   n mean   sd median trimmed  mad min max range skew kurtosis   se
1    1 100  2.2 1.81      2    2.01 1.48   0  10    10 1.11     2.19 0.18
```

The marginal distribution of time spent on homework is right-skewed with a mean of 2.2. There is a great deal of variation in the measurements ($SD = 1.8$).



# Examining the Distribution of the Outcome Conditioned on the Predictor

```
> ggplot(data = math, aes(x = homework, y = achievement)) +
    geom_point() +
    xlab("Time Spent on Homework") +
    ylab("Mathematics Achievement Score") +
    theme_bw()
```




The plot suggests a relationship (in the *sample*) between time spent on mathematics homework and mathematics achievement scores.

- Functional form of the relationship?
- Direction?
- Strength?
- Weird observations?

# Correlation

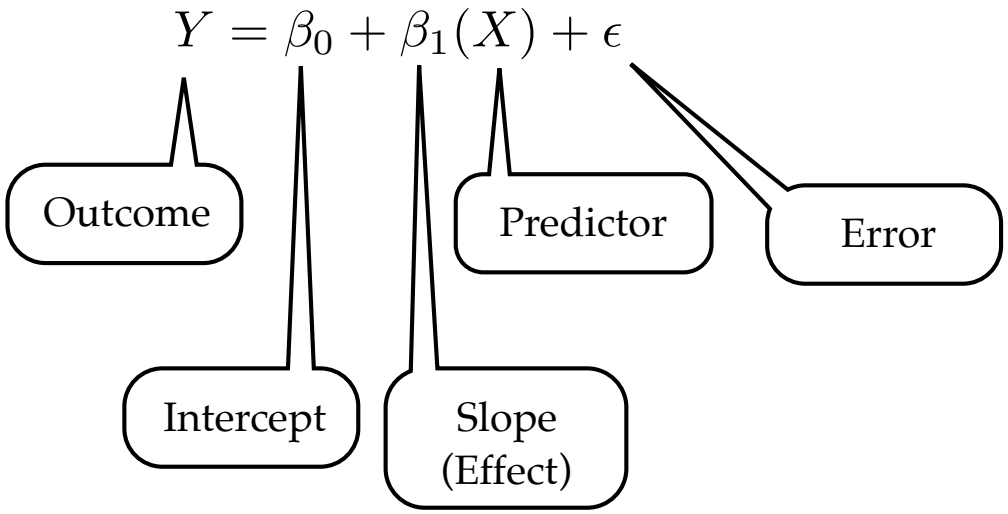
We use the `cor()` function to find the correlation. We give it an indexed data frame, `math[`*rows, columns*`]`, where *rows* is empty (all rows) and *columns* gives the names of the variables we want the correlation between.

```
> cor(math[ , c("homework", "achievement")]

             homework achievement
homework    1.0000000   0.3199936
achievement 0.3199936   1.0000000
```

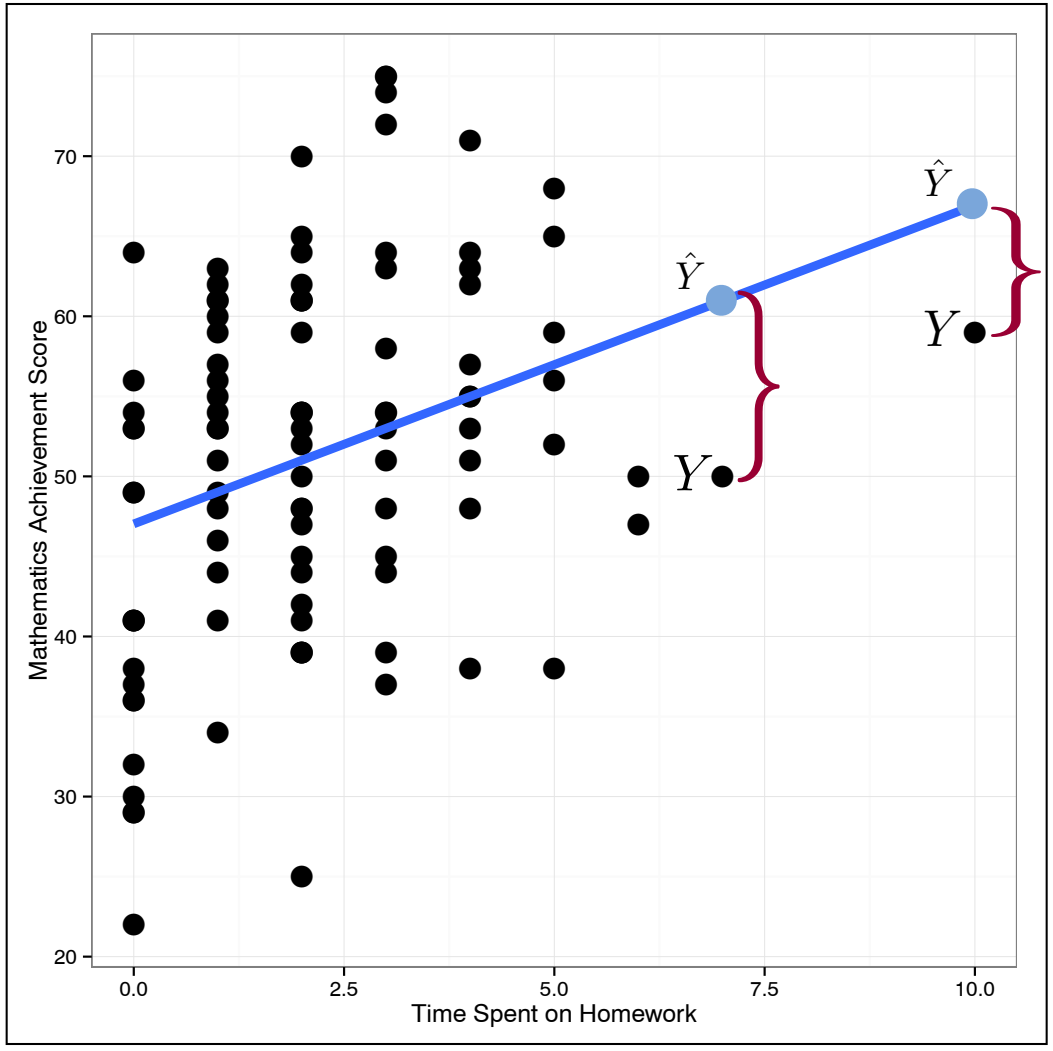
The Pearson correlation between time spent on mathematics homework and mathematics achievement suggests a moderate relationship between the variables, $r = 0.32$.

# Fit a Model to the Data

Since the relationship's functional form seems reasonably linear, we will fit a linear model to the data.



# Why an Error Term?




We use a *single line* to describe the relationship between homework and achievement for *all* of the observations in the sample.

- The error allows for discrepancy between the line (predicted $Y$) and the observed $Y$
- For some observations the discrepancy is smaller than for others.

# Regression (Fitted) Equation

The regression equation is the *systematic* part of the model that is fixed (the same) for all observations.

$$Y = \beta_0 + \beta_1(X) + \epsilon$$

Systematic (fixed) — Random (stochastic)

$$Y = \hat{Y} + \epsilon$$

$$\hat{Y} = \beta_0 + \beta_1(X)$$

One goal of regression analysis is to estimate the values of the regression parameters (i.e., intercept and slope).

# Sample Estimates

The regression equation describes the linear relationship *in the population*.

$$\hat{Y} = \beta_0 + \beta_1(X)$$

We will use a *sample of data* (not the entire population) to approximate the parameters in this equation. The values we get for the intercept and slope are *estimates*.

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1(X)$$

*Hats* are used to denote that the value is an estimate. Synonymously, a hat means *predicted value*.

# Fitting the Regression Model Using R

```
> lm.1 = lm(achievement ~ homework, data = math)
> lm.1

Call:
lm(formula = achievement ~ homework, data = math)

Coefficients:
(Intercept)     homework
      47.03         1.99
```

$$\hat{\beta}_0 = 47.03$$

$$\hat{\beta}_1 = 1.99$$

$$\hat{\text{achievement}} = 47.03 + 1.99(\text{homework})$$

# Interpreting the Intercept

$$\widehat{\text{achievement}} = 47.03 + 1.99(\text{homework})$$

$$\hat{\beta}_0 = 47.03$$

The $y$-intercept gives the $y$-value where the line passes through the $y$-axis. It gives the predicted value of $y$ when $x = 0$.

$$\widehat{\text{achievement}} = 47.03 + 1.99(0)$$

$$\widehat{\text{achievement}} = 47.03$$

If a student spends 0 hours per week on mathematics homework, we would predict that student to have a mathematics achievement score of 47.03.

# Interpreting the Slope

$$\widehat{\text{achievement}} = 47.03 + 1.99(\text{homework})$$

$$\hat{\beta}_1 = 1.99$$

The *slope* describes the *predicted* change in $y$ relative to the change in $x$.

$$\frac{\Delta \hat{Y}}{\Delta X} = \frac{1.99}{1}$$

Each one-unit difference in $x$ is associated with a 1.99-unit predicted difference in $y$.

For each additional hour spent on mathematics homework per week, we predict, *on average*, a 1.99-point difference in mathematics achievement score.

Consider three students…one who spends 2 hours per week on mathematics homework, one who spends 3 hours per week on mathematics homework, and another who spends 4 hours per week on mathematics homework.

2-hours per week

$$\widehat{\text{achievement}} = 47.03 + 1.99(2)$$
$$= 51.01$$

3-hours per week

$$\widehat{\text{achievement}} = 47.03 + 1.99(3)$$
$$= 53$$

Each student's $X$-value differs by 1. The difference in predicted $Y$ is 1.99.

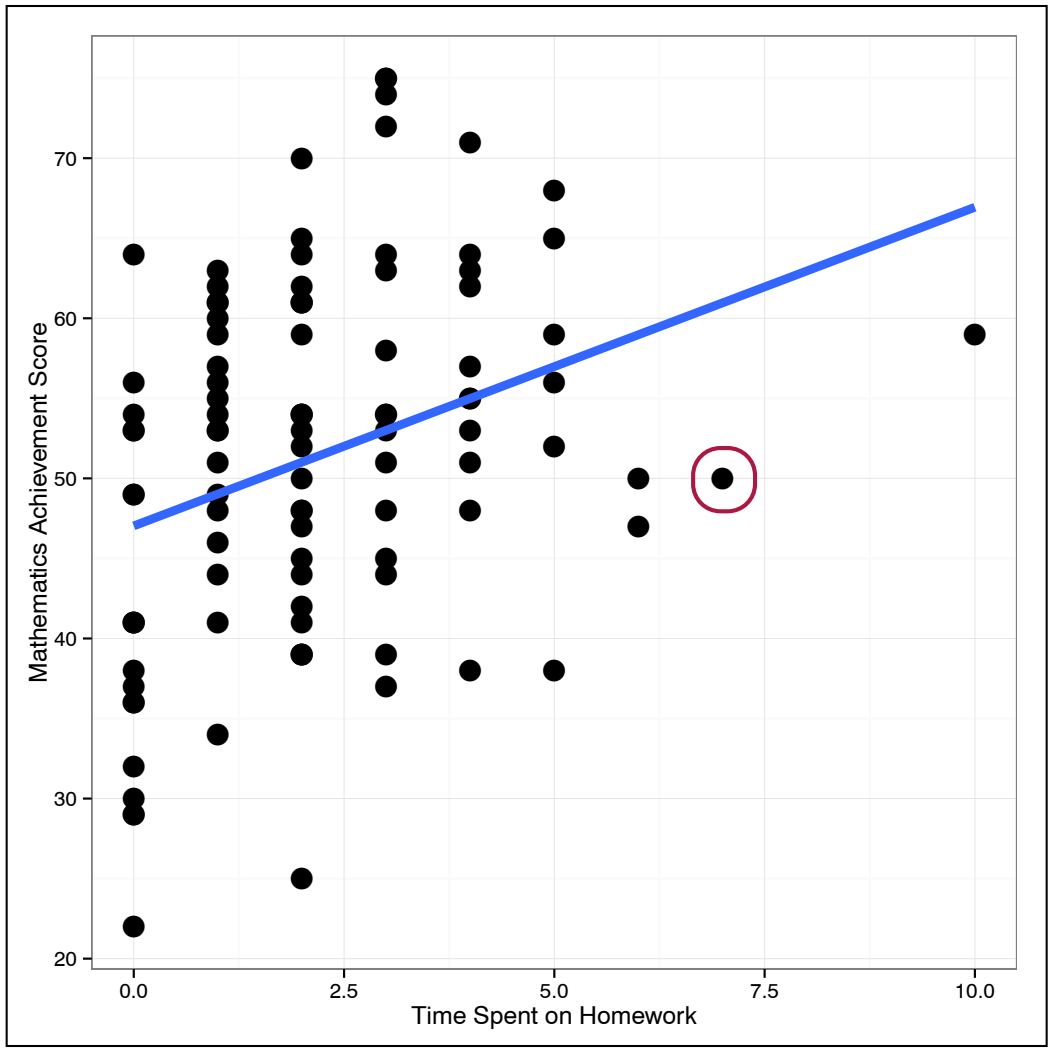
4-hours per week

$$\widehat{\text{achievement}} = 47.03 + 1.99(4)$$
$$= 54.99$$

Each difference of one hour spent on mathematics homework per week, is associated with a two-point difference in mathematics achievement score, *on average*.

# Observation, Prediction, and Error

$$\hat{\text{achievement}} = 47.03 + 1.99(\text{homework})$$




**Observation 12**

$$x = 7$$
$$y = 50$$

The student's observed mathematics achievement score is 50.

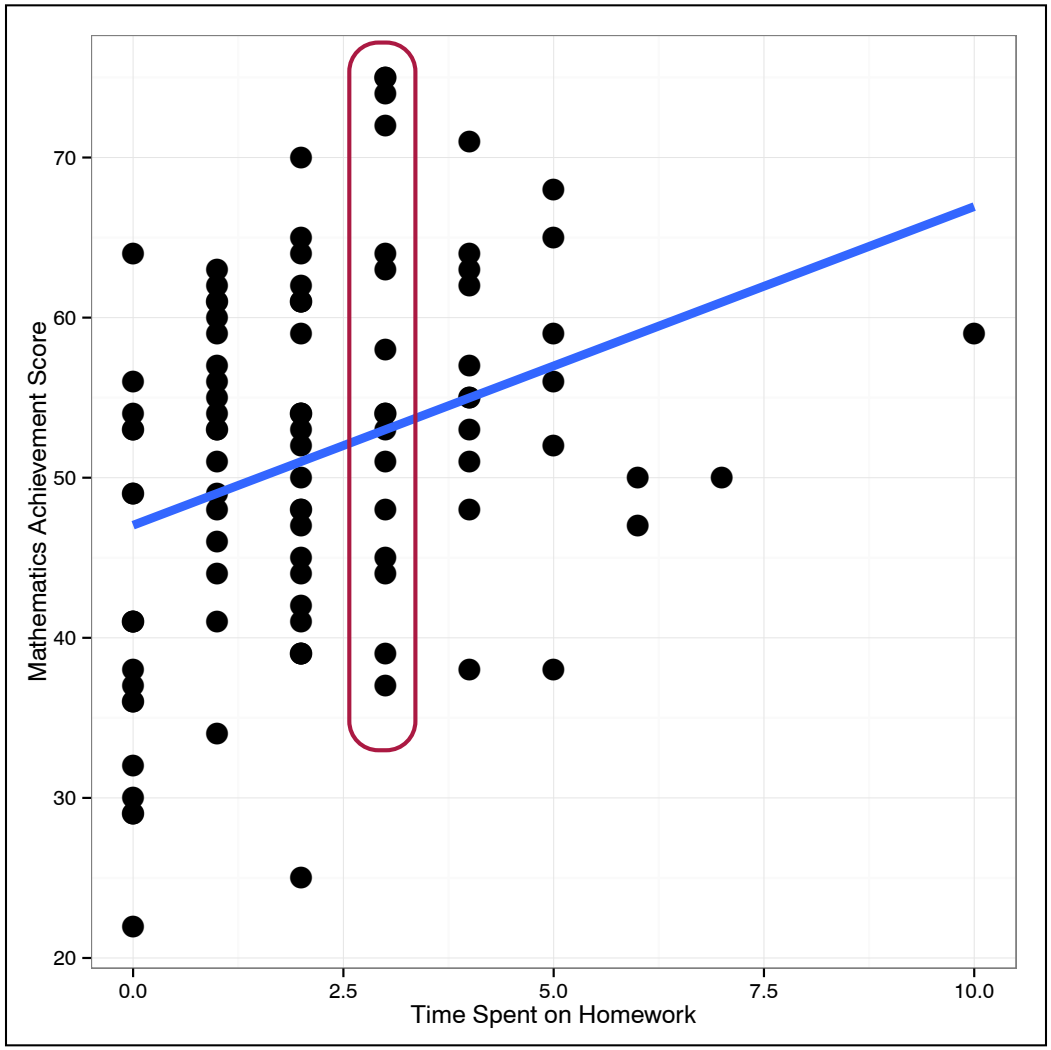
$$\hat{\text{achievement}} = 47.03 + 1.99(7)$$
$$= 60.96$$

The student's predicted mathematics achievement score is 60.96.

$$\hat{\epsilon} = 50 - 60.96$$
$$= -10.96$$

The student's observed mathematics achievement is 10.96 points lower than his/her predicted mathematics achievement score.

$$\hat{\text{achievement}} = 47.03 + 1.99(\text{homework})$$




All of these student's have the same $X$-value ($x = 3$)

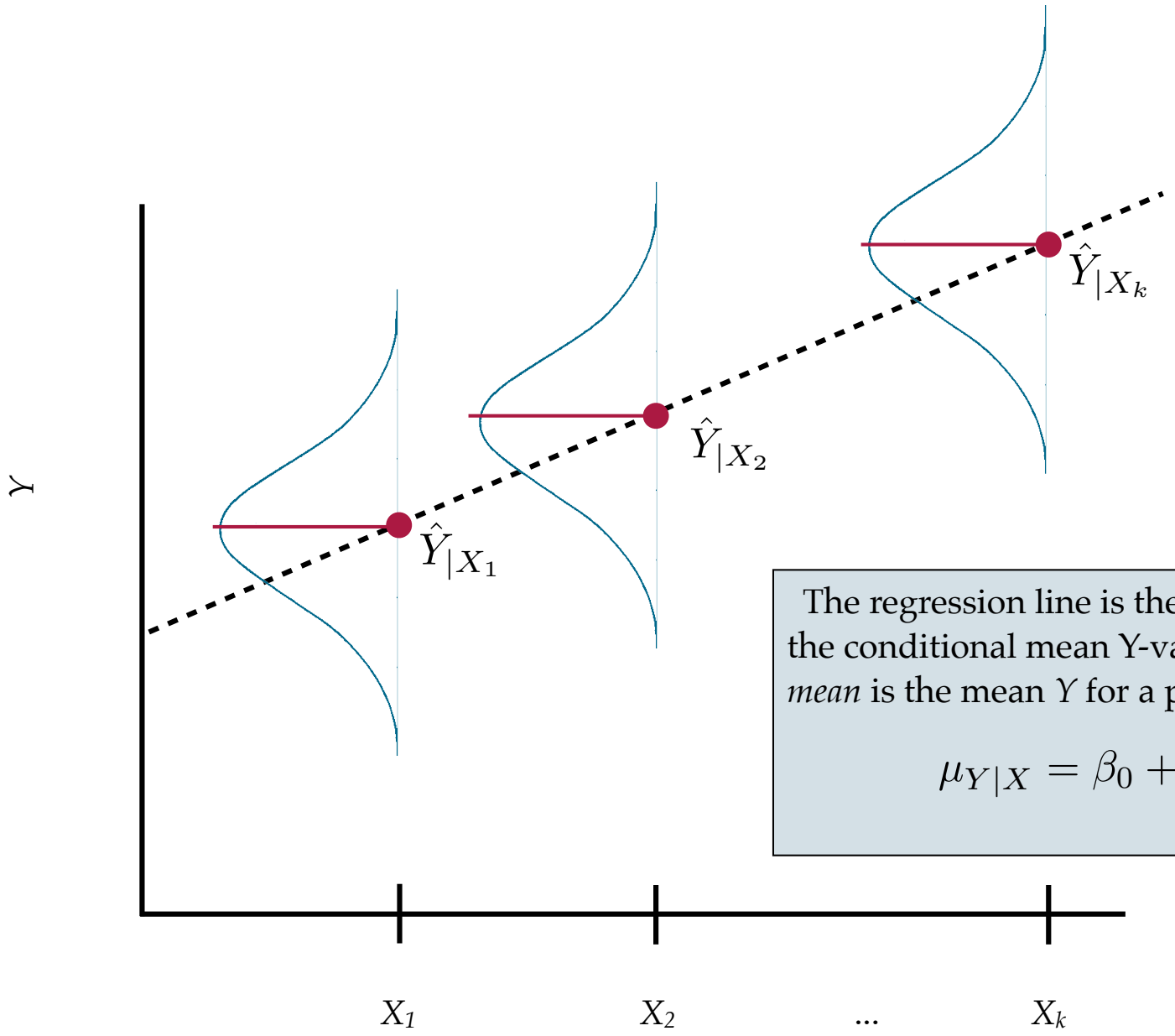
$$\begin{aligned}\hat{\text{achievement}} &= 47.03 + 1.99(3) \\ &= 53\end{aligned}$$

All of these student's have the same predicted mathematics achievement score of 53.

Since their observed $Y$ varies, their error term will also vary.

Observations that have the same $X$-value will have the same predicted (fitted) value, despite possibly having different observed $Y$-values.

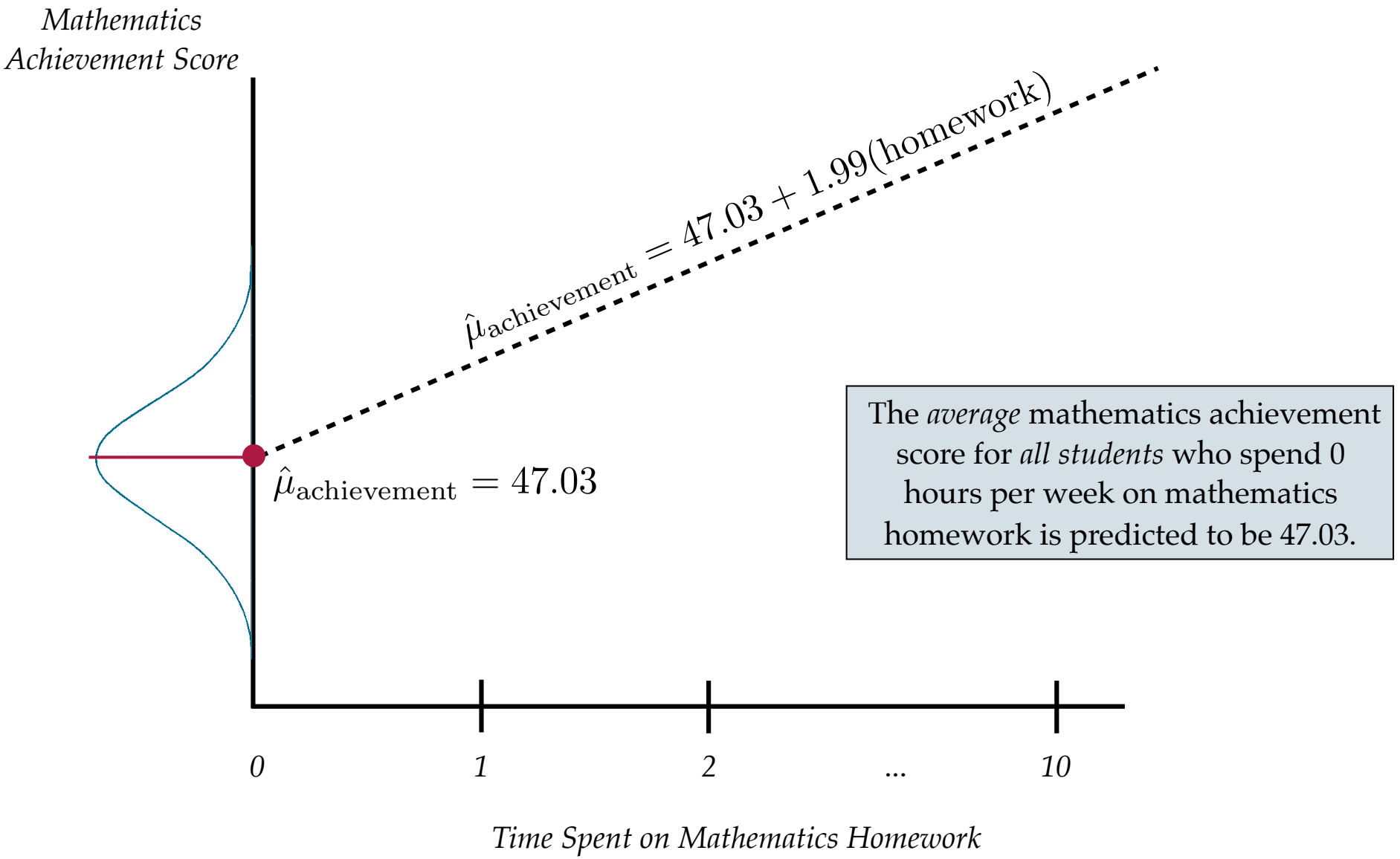
# Predicted Values are Means




The regression line is the lie that connects the conditional mean Y-values. A *conditional mean* is the mean *Y* for a particular *X*-value.

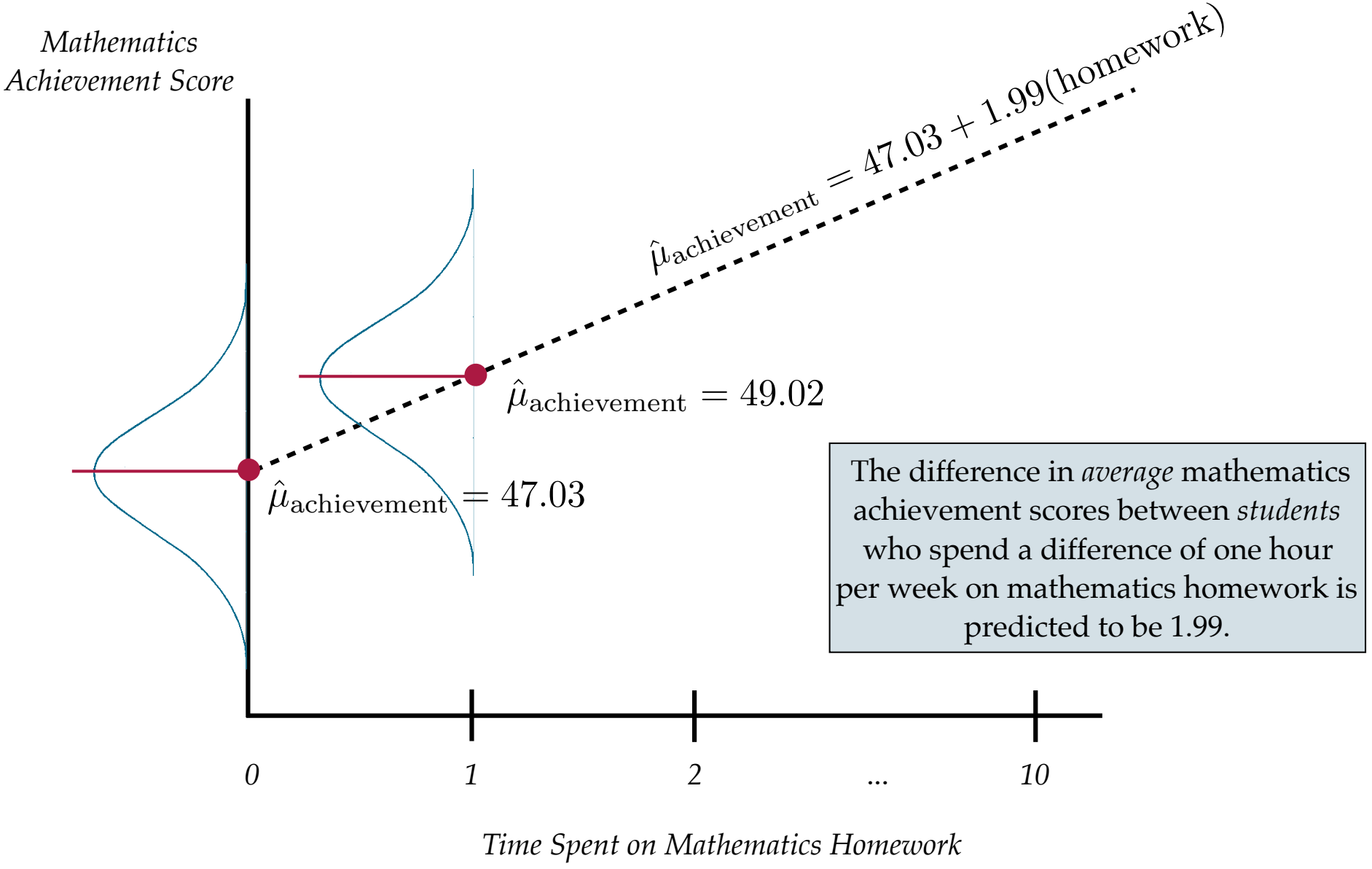
$$\mu_{Y|X} = \beta_0 + \beta_1(X)$$

# Interpretation of the Intercept (Revisited)




The *average* mathematics achievement score for *all students* who spend 0 hours per week on mathematics homework is predicted to be 47.03.

# Interpretation of the Slope (Revisited)




The difference in *average* mathematics achievement scores between *students* who spend a difference of one hour per week on mathematics homework is predicted to be 1.99.

# Least Squares Estimation

$$\hat{\text{achievement}} = 47.03 + 1.99(\text{homework})$$

How do we get the values of 47.03 and 1.99 for the intercept and slope?

These values are based on an estimation method called *Least Squares*. Every estimation method requires two things:

- **Quantification of Model Fit:** We quantify how well (or not well) the estimated equation fits the data; and
- **Optimization:** We find the "best" equation based on that quantification. (this boils down to finding the equation that produces the biggest or smallest measure of fit.)

For most statistical models we quantify the model fit by examining the errors. Error is a measure of model misfit (i.e., bigger errors = worse fitting model).

**Model A**

$$\hat{\text{achievement}} = 30 + 1(\text{homework})$$

**Model B**

$$\hat{\text{achievement}} = 20 - 2(\text{homework})$$

| Homework (*X*) | Achievement (*Y*) | Predicted Achievement | Error |
|---|---|---|---|
| 3 | 63 | | |
| 1 | 44 | | |
| 3 | 64 | | |
| 5 | 68 | | |
| 2 | 25 | | |

## Model A

$$\hat{\text{achievement}} = 30 + 1(\text{homework})$$

| Homework (*X*) | Achievement (*Y*) | Predicted Achievement | Error |
|---|---|---|---|
| 3 | 63 | | |
| 1 | 44 | | |
| 3 | 64 | | |
| 5 | 68 | | |
| 2 | 25 | | |

## Model B

$$\hat{\text{achievement}} = 20 - 2(\text{homework})$$

| Homework (*X*) | Achievement (*Y*) | Predicted Achievement | Error |
|---|---|---|---|
| 3 | 63 | | |
| 1 | 44 | | |
| 3 | 64 | | |
| 5 | 68 | | |
| 2 | 25 | | |

# Sum of Squared Error

$$\hat{\text{achievement}} = 47.03 + 1.99(\text{homework})$$

$$\text{SS}_{\text{Error}} = \sum (y_i - \hat{y}_i)^2$$

```
# Compute residuals
> yhat = 47.03 + 1.99 * math$homework
> res.1 = math$achievement - yhat
> head(res.1)

[1]   2.99   5.97  -1.99   8.97   7.99 -17.03


# SSE
> SSE.1 = sum(math$residual)
> SSE.1

[1] 11318.96
```

| Homework ($X$) | Achievement ($Y$) | Predicted Achievement | Error | Squared Error |
|---|---|---|---|---|
| 2 | 54 | 51.01 | 2.99 | 8.94 |
| 0 | 53 | 47.03 | 5.97 | 35.64 |
| 4 | 53 | 54.99 | –1.99 | 3.96 |
| 0 | 56 | 47.03 | 8.97 | 80.46 |
| 2 | 59 | 51.01 | 7.99 | 63.84 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

**SSE = 11318.96**

The SSE represents the variation in $Y$ that remains unexplained after fitting the regression model.

# Intercept-Only Model

Although the SSE seems large, in this case SSE = 11319, we cannot interpret the magnitude of this value in absolute terms.

We, can however, compare this to the SSE from a baseline model. The baseline model we use is the intercept-only model.

$$\hat{Y}_i = \beta_0 + \epsilon_i$$

Intercept-only model: $\hat{Y}_i = \beta_0 + \epsilon_i$

Simple regression model: $\hat{Y}_i = \beta_0 + \beta_1(X_i) + \epsilon_i$

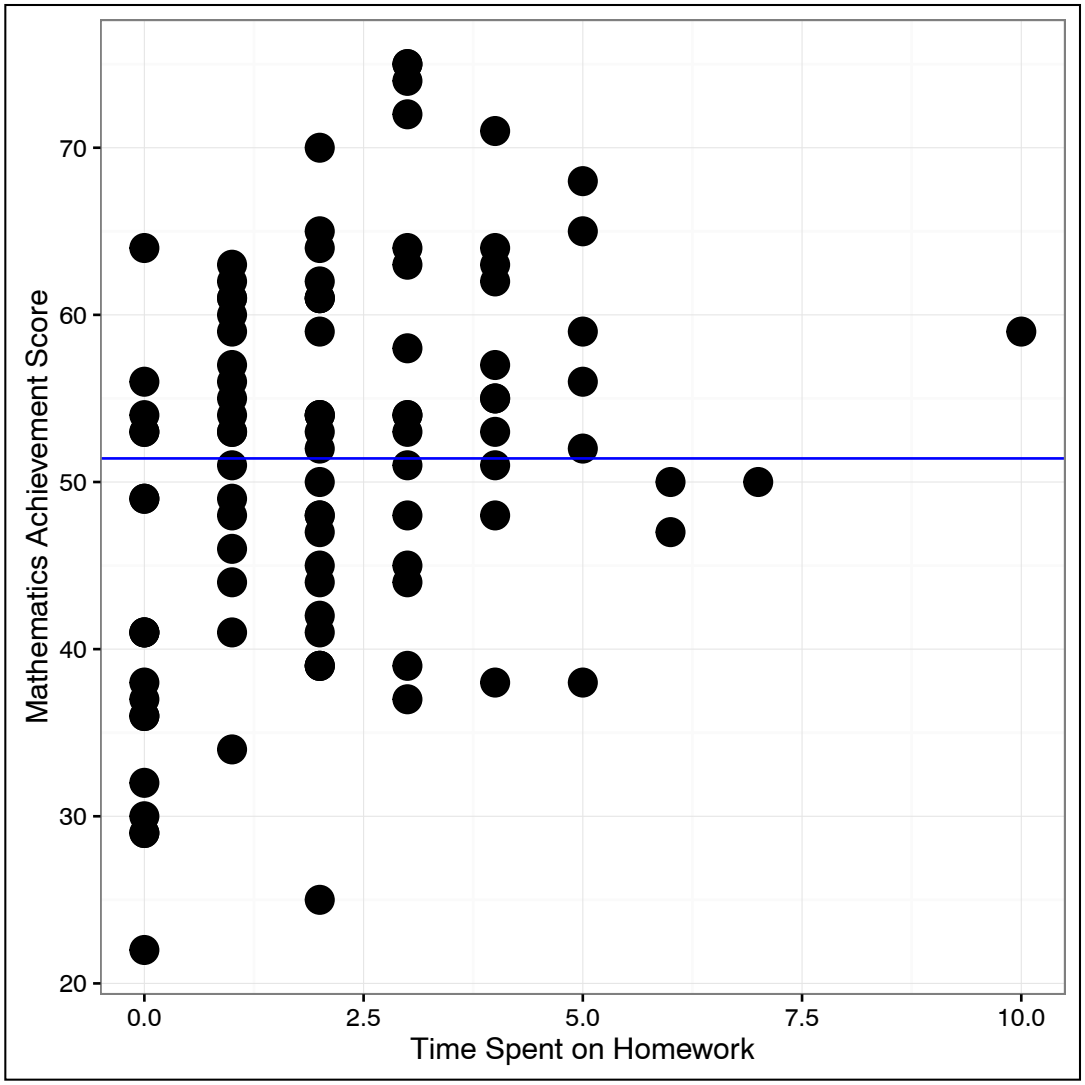
Notice that if the effect of $X$ is 0, the simple regression model reduces to the intercept-only model. This is also the model with no predictors. Thus the SSE for the intercept-only model is the unexplained variation in $Y$ when there are no predictors.

To fit the baseline model we again use the `lm()` function.

```
# Fit intercept-only (baseline) model
> lm.0 = lm(achievement ~ 1, data = math)
> lm.0

Call:
lm(formula = achievement ~ 1, data = math)

Coefficients:
(Intercept)
      51.41
```

$$\hat{\text{achievement}} = 51.41$$

This model predicts that no matter how much time a student spends on mathematics homework, that student's predicted achievement score would be 51.41.




$$\hat{\text{achievement}} = 51.41$$

Here the fitted line is a flat, line having a $y$-intercept of 51.41.

# Residuals for the Fitted Intercept-Only Model

$$\hat{\text{achievement}} = 51.41$$

$$\text{SS}_{\text{Error}} = \sum (y_i - \hat{y}_i)^2$$

```
# Compute residuals
> res.0 = math$achievement - 51.41
> head(res.0)

[1]   2.59   1.59   1.59   4.59   7.59 -21.41


# Compute SSE
> SSE.0 = sum(res.0 ^ 2)
> SSE.0

[1] 12610.19
```

| | | Simple regression model | | | Intercept-only model | | |
|---|---|---|---|---|---|---|---|
| Homework (*X*) | Achievement (*Y*) | Predicted Achievement | Error | Squared Error | Predicted Achievement | Error | Squared Error |
| 2 | 54 | 51.01 | 2.99 | 8.94 | 51.41 | 2.59 | 6.71 |
| 0 | 53 | 47.03 | 5.97 | 35.64 | 51.41 | 1.59 | 2.53 |
| 4 | 53 | 54.99 | –1.99 | 3.96 | 51.41 | 1.59 | 2.53 |
| 0 | 56 | 47.03 | 8.97 | 80.46 | 51.41 | 4.59 | 21.07 |
| 2 | 59 | 51.01 | 7.99 | 63.84 | 51.41 | 7.59 | 57.61 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| | | | | **SSE = 11318.96** | | | **SSE = 12610.19** |

The SSE for the simple regression model is smaller than the SSE for the intercept-only model. This suggests that the **homework predictor explains some of the variation in achievement**.

# Explained Variation ($R^2$)

How much of the unexplained variation was explained by adding the homework predictor to the model? Another way of asking this question is: How much did the unexplained variation reduce from the intercept-only model to the simple regression model?

```
# Difference in the unexplained variation
> SSE.0 - SSE.1

[1] 1291.231


> 1291.231 / sse.0

[1] 0.1023958
```

10% of the initial unexplained variation in achievement scores can be explained by differences in homework (i.e., including homework as a predictor). This is referred to as the $R^2$ of the model.

# Partitioning of Variation

The total amount of unexplained variation (SSE from the intercept-only model) is partitioned into two parts: that which is explained by homework, and that which is not.

| Variation source | SS | |
|---|---|---|
| Homework | 1291.23 | Model Sum of Squares |
| Residuals | 11318.96 | Residual Sum of Squares |
| Total | 12610.19 | Total Sum of Squares |

$$\text{SS}_{\text{Total}} = \text{SS}_{\text{Model}} + \text{SS}_{\text{Residuals}}$$

$$R^2 = \frac{\text{SS}_{\text{Model}}}{\text{SS}_{\text{Total}}}$$