$\frac{8}{8} + \frac{11}{12} = \frac{20}{20}$ $\boxed{\frac{3}{3}}$

[redacted]

Hand In Homework

2.5: 8, 10, 20, 26, 35
2.6: 3a, 8a, 16

8. $f(u,v,w) = (e^{u-w}, \cos(v+u) + \sin(u+v+w))$

$g(x,y) = (e^x, \cos(y-x), e^{-y}) = g(a)$

$a = (e^x, \cos(y-x), e^{-y})$

$f \circ g = f(a) = (e^{e^x - e^{-y}}, \cos(\cos(y-x) + e^x) + \sin(e^x + \cos(y-x) + e^{-y}$

$= (e^{e^x - e^{-y}}, \cos^2(y-x) + \cos(e^x) + \sin(e^x) + \sin(\cos(y-x)) + \sin(e$

$Df(u,v,w) = \begin{bmatrix} df_1/du & df_1/dv & df_1/dw \\ df_2/du & df_2/dv & df_2/dw \end{bmatrix} =$

$\begin{bmatrix} e^{u-w} & 0 & -e^{u-w} \\ -\sin(v+u) + \cos(u+v+w) & -\sin(v+u) + \sin(u+v+w) & \cos(u+v+w \end{bmatrix}$

$Dg(x,y) = \begin{bmatrix} dg_1/dx & dg_1/dy \\ dg_2/dx & dg_2/dy \\ dg_3/dx & dg_3/dy \end{bmatrix} = \begin{bmatrix} e^x & 0 \\ \sin(y-x) & -\sin(y-x) \\ 0 & -e^{-y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$

$Df(g(0,0))\, Dg(0,0)$

$g(0,0) = (1,1,1)$

$Df(g(0,0)) = \begin{bmatrix} 1 & 0 & -1 \\ -\sin(2) + \cos(3) & -\sin(2) + \sin(3) & \cos(3) \end{bmatrix}$

$Df(g(0,0))\, Dg(0,0) = \begin{bmatrix} 1 & 1 \\ -\sin(2) + \cos(3) & -\cos 3 \end{bmatrix}$

10. $T(x,y,z) = x^2 + y^2 + z^2$

$\sigma(t) = (\cos t, \sin t, t)$

$T(t)$ = temperature at time t

a. $T(\sigma(t)) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$

$T'(\sigma(t)) = 2t$

b. $t = (\pi/2) + .01$

$\sigma(t) = (\cos(\pi/2), \sin(\pi/2), \pi/2) = (0, 1, \pi/2)$

$L(t) = T(t) + dT/dx(t)(.01) + dT/dy(t)(.01) + dT/dz(t)(.01)$

$L(\frac{\pi}{2}) = 2t + 2x(\frac{\pi}{2})(.01) + 2y(\frac{\pi}{2})(.01) + 2z(\frac{\pi}{2})(.01)$

3a. $f(x,y) = x^y$

$(x_0, y_0) = (e, e)$

$dd = \nabla f \cdot v$

$\nabla f = (df/dx \quad df/dy) = (e y^{e-1} \quad e^e) = (e^e \quad e^e)$

$v = (5 \quad 12)$

$\nabla f \cdot v = (e^e \quad e^e) \cdot (5 \quad 12) = 17e^e$

8a. $\nabla f = (2x + 3z \quad -4y \quad 3x) = (3 \quad 8 \quad 3)$

$(x-1, y-2, z-1/3) \cdot (3 \quad 8 \quad 3)$

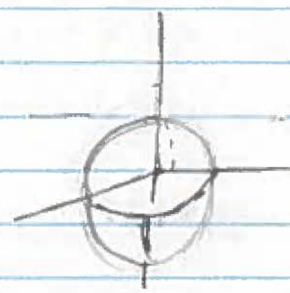
$3x - 3 + 8y - 16 + 3z - 1 = 0$

$3x + 8y + 3z = 20$

16. $f(x,y,z) = -\sqrt{1 - x^2 - y^2}$

$f : \{(x,y) : x^2 + y^2 < 1\} \to \mathbb{R}$

$Df = \left(\frac{x}{\sqrt{1 - x^2 + y^2}} \quad \frac{y}{\sqrt{1 - x^2 + y^2}} \right.$



$z = \sqrt{1 - x^2 + y^2}$

$z^2 = 1 - x^2 - y^2$

$\{(x,y,z) : 1 = x^2 + y^2 + z^2 \text{ and } z < 0\}$

$v = \begin{pmatrix} x_0 \\ y_0 \\ -\sqrt{1 - x_0^2 - y_0^2} \end{pmatrix}$ = radius $\perp$ plane

$0 = \frac{df}{dx}(x - x_0) + \frac{df}{dy}(y - y_0) + (-1)(z - z_0)$

$\vec{n} = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} & -1 \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{1 - x_0^2 - y_1^3}} & \frac{y}{\sqrt{1 - x_0 - y_1^3}} & -1 \end{pmatrix}$

elucidate on your reasoning here.

11/12

[redacted]

Math Supplementary Questions B

1a. $F(\vec{x},\vec{y}) = \vec{x}\times\vec{y}$

$F(\vec{x_0}+\vec{h}, \vec{y_0}+\vec{k}) = \vec{x_0}+h\times\vec{y_0}+\vec{k}$

Cross Product (only works in $\mathbb{R}^3$)

$\|U\times V\| = \|U\|\|V\|\sin\theta$

$U=(U_1,U_2,U_3)^T$

$V=(V_1,V_2,V_3)^T$

$U\times V = \det\begin{pmatrix}\hat{i}&\hat{j}&\hat{k}\\U_1&U_2&U_3\\V_1&V_2&V_2\end{pmatrix} = (U_2V_3-U_3V_2)^T$

nice.

$F(x,y) = x\times y$

$F(x+h, y+k) - F(x,y)$

$(x+h)\times(y+k) - x\times y$

$x\times(y+k) + h\times(y+k) - x\times y$

$\cancel{x\times y} + x\times k + h\times y + h\times k \cancel{- x\times y}$

$x\times k + h\times y + h\times k = T\left(\binom{h}{k}\right)$

$$\frac{F\left(\binom{x}{y}+\binom{h}{k}\right) - F\binom{x}{y} - T\binom{h}{k}}{\left\|\binom{h}{k}\right\|} = \frac{h\times k}{\left\|\binom{h}{k}\right\|}$$

To show that T is linear you need to verify tha

$T\left(\binom{h}{k}+\binom{h'}{k'}\right) = T\binom{h}{k}+T\binom{h'}{k'}$

And for $a\in\mathbb{R}$

$T\left(a\binom{h}{k}\right) = aT\binom{h}{k}$

See back of paper to see the properties I use to solve this

[redacted]

09/07/14
7^{th} Sept. 14.

HW2

2. I. One example of a function f on a set X that maps to $\mathbb{R}$, which is not $\mathcal{F}$-measurable, but has the properties that $|f|$ and f^2 are $\mathcal{F}$-measurable is:

$f : [0,1] \Rightarrow \mathbb{R}$ s.t. $\forall x \in [0,1]$, and $\mathcal{F}$ is the σ-alg of Lebesgue measurable sets on X.

$$f(x) = \begin{cases} 1 & x \in A \\ -1. & x \notin A, \end{cases}$$

where A is some non-measurable subset of X ($\to [0,1]$) (for example a Vitali set), i.e. $A \notin \mathcal{F}$. A is not in the σ-algebra $\mathcal{F}$.

To see that f is not $\mathcal{F}$-measurable consider the case where $\alpha = 0$.
Then
$$\{x \in X : f(x) > \alpha\} = A.$$

But $A \notin \mathcal{F}$, $\therefore f$ is not $\mathcal{F}$-measurable

Both $|f|$ and f^2 are given by

$$|f| = 1 \qquad \forall x \in [0,1]$$
$$f^2 = 1 \qquad \forall x \in [0,1]$$

These are both measurable as if $\alpha \geq 1$,

$$\{x \in X : f(x) > \alpha\} = \emptyset \in \mathcal{F}.$$

or if $\alpha < 1$.
$$\{x \in X : f(x) > \alpha\} = X \in \mathcal{F}.$$

2.L. $\varphi_n(x)$ is defined in Lemma 2.11 as follows,

$$\varphi_n(x) = \frac{K}{2^n} \quad \text{for } x \in E_{kn}$$

where,

$$E_{kn} = \left\{ x \in X : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\} \quad \text{if } k = 0, 1, 2, \ldots \, n2^{n-1}$$

$$E_{kn} = \{ x \in X : f(x) \geq n \} \quad \text{if } K = n2^n.$$

Since $f(x) \leq K \quad \forall x \in X$, there must exist a smallest integer N_0, s.t $K \leq N_0$ and

$$x \in \bigcup_{k=0}^{N_0 2^{N_0}} E_{kN_0} \quad \forall x \in X.$$

as $f(x) < N_0 \quad \forall x$ and $\forall$

$\forall n \geq N_0$ the set $E_{n2^n, n}$ will be empty, and so $\forall x \in X$, $\exists k \in \mathbb{N}$ s.t

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$$\frac{K}{2^n} \leq f(x) < \frac{k+1}{2^n}$$

$$\Rightarrow 0 \leq f(x) - \frac{k}{2^n} < \frac{1}{2^n} \quad \Rightarrow \quad \left| f(x) - \frac{k}{2^n} \right| < \frac{1}{2^n} \quad \forall n > N_0$$

~~But~~

$$\Rightarrow |f(x) - \varphi_n(x)| < \frac{1}{2^n} \quad \forall n > N_0, \ \forall x \in X.$$

Given an $\varepsilon > 0$, we can always choose an $n = N_1 > N$ large enough to ensure that $\frac{1}{2^n} < \varepsilon \quad \forall n \geq N_1$.

$\therefore \varphi_n(x)$ converges uniformly to $f(x)$ on X.

2.M. $f : X \to Y$

$E, F \subseteq Y.$ and $f^{-1}(E) = \{x \in X : f(x) \in E\}$

$f^{-1}(\phi) = \{x \in X : f(x) \in \phi\} = \phi.$ $\because$ there are no elements of the empty set, so $f(x) \notin \phi$

$f^{-1}(Y) = \{x \in X : f(x) \in Y\} = X.$ by definition of f.

$$\begin{aligned} f^{-1}(E \setminus F) &= \{x \in X : f(x) \in E \setminus F\} \\ &= \{x \in X : f(x) \in E\} \setminus \{x \in X : f(x) \in F\} \\ &= f^{-1}(E) \setminus f^{-1}(F). \end{aligned}$$

If $\{E_\alpha\}$ is a nonempty collection of subsets of Y then,

$$f^{-1}\left(\bigcup_\alpha E_\alpha\right) = \{x \in X : f(x) \in \bigcup_\alpha E_\alpha\} = \bigcup_\alpha \{x \in X : f(x) \in E_\alpha\} = \bigcup_\alpha f^{-1}(E_\alpha)$$

~~Because~~ so each $x \in f^{-1}\left(\bigcup_\alpha E_\alpha\right)$ lies in at least one set $\{x \in X : f(x) \in E_\alpha\}$.

also

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$f^{-1}\left(\bigcap_\alpha E_\alpha\right) = \{x \in X : f(x) \in \bigcap_\alpha E_\alpha\}$, so $x \in f^{-1}\left(\bigcap_\alpha E_\alpha\right)$ lies in all sets $\{x \in X : f(x) \in E_\alpha\}$

$$\therefore f^{-1}\left(\bigcap_\alpha E_\alpha\right) = \bigcap_\alpha \{x \in X : f(x) \in E_\alpha\} = \bigcap_\alpha f^{-1}(E_\alpha).$$

✓

It follows then that if $\mathcal{F}_Y$ is a σ-algebra of subsets of Y that $\mathcal{F}_X = \{f^{-1}(E) : E \in \mathcal{F}_Y\}$ is a σ-algebra of subsets of X.

To see this notice that

(i) $\phi \in \mathcal{F}_X$ $\because$ $\phi \in \mathcal{F}_Y$ and $f^{-1}(\phi) = \phi$.

(ii) $X \in \mathcal{F}_X$ $\because$ $Y \in \mathcal{F}_Y$ and $f^{-1}(Y) = X$.

(iii) $A \in \mathcal{F}_X \Rightarrow A^c \in \mathcal{F}_X$ $\because$ if $E \in \mathcal{F}_Y$ then $E^c \in \mathcal{F}_Y$ and so $A = f^{-1}(E) \in \mathcal{F}_X$ and $B = f^{-1}(E^c) \in \mathcal{F}_X$.

But $B = \{x \in X : f(x) \in E^c\} = \{x \in X : f(x) \in E\}^c = A^c$.

ctd $\rightarrow$

2.N.
ctd.

(iii) suppose $f^{-1}(E) \in \mathcal{F}_x$. Then we know that $(f^{-1}(E))^c \in \mathcal{F}_x$ as $\mathcal{F}_x$ is a σ-algebra.

But,

$$(f^{-1}(E))^c = \{x \in X : f(x) \in E\}^c = \{x \in X : f(x) \in E^c\} = f^{-1}(E^c).$$

So, both $f^{-1}(E)$ and $f^{-1}(E^c)$ are in $\mathcal{F}_x$. &

By definition of $\mathcal{F}_y$, if $f^{-1}(E)$ and $f^{-1}(E^c) \in \mathcal{F}_x$, then E and $E^c \in \mathcal{F}_y$

(iv) Suppose $(f^{-1}(E_n))$ is a sequence on $\mathcal{F}_x$, then $\bigcup_{n=1}^{\infty} f^{-1}(E_n) \in \mathcal{F}_x$ (σ-alg).

$\therefore$ $f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) \in \mathcal{F}_x$; using the result from problem 2.M.

if $f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) \in \mathcal{F}_x$, then by definition of $\mathcal{F}_y$, $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}_y$.

As these four properties hold we can conclude that $\mathcal{F}_y$ is a σ-algebra if $\mathcal{F}_x$ is.

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3B. $\mu_1, \ldots, \mu_n$ are measures on $\mathcal{F}$. $a_1, \ldots, a_n$ are nonnegative real numbers.
Show that
$$\lambda(E) = \sum_{j=1}^{n} a_j \mu_j(E),$$
is a measure on X, where $E \in \mathcal{F}$.

Need to show that: (i) $\lambda(\emptyset) = 0$
(ii) $\lambda(E) \geq 0 \quad \forall E \in \mathcal{F}$
(iii) $\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \lambda(E_k)$, for disjoint seq. (E_k) in $\mathcal{F}$

(i) μ_i are measures, so $\mu_i(\emptyset) = 0$, $\forall i \in \mathbb{N}$
$$\therefore \sum_{g=1}^{n} a_j \mu_j(\emptyset) = 0,$$
$$\Rightarrow \lambda(\emptyset) = 0.$$

(ii) We know that $\mu_i(E) \geq 0$, $\forall i \in \mathbb{N}, \forall E \in \mathcal{F}$.

Since all the numers $a_1, \ldots, a_n$ are nonnegative we get that
$$\sum_{j=1}^{n} a_j \mu_j(E) \geq 0 \Rightarrow \lambda(E) \geq 0.$$

(iii) let (E_k) be a disjoint sequence on $\mathcal{F}$.

We have that $\mu_i\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu_i(E_k)$, $\forall i \in \mathbb{N}$.
$$\Rightarrow a_i \mu_i\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} a_i \mu_i(E_k), \quad \forall i \in \mathbb{N}.$$
$$\Rightarrow \sum_{i=1}^{n} a_i \mu_i\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} a_i \mu_i(E_k).$$

[redacted]

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HW3

3D $X = \mathbb{N}$, $\mathcal{F} = \mathbb{P}(\mathbb{N})$.

(a_n) is a sequence of nonnegative real numbers.

Show that if $\mu(\emptyset)=0$ and $\mu(E) = \sum_{n\in E} a_n$, $E \neq \emptyset$,

then μ is a measure on $\mathcal{F}$.

μ is a measure on $\mathcal{F}$ if (i) $\mu(\emptyset)=0$ automatically true.

(ii) $\mu(E) \geq 0 \quad \forall E \in \mathcal{F}$.

(iii) $\mu\left(\coprod_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ for disjoint E_n in $\mathcal{F}$.

(ii) ~~This holds~~

$\mu(E) = \sum_{n\in E} a_n$ but $a_n \geq 0 \quad \forall n \in \mathbb{N}$.

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$\therefore \mu(E) \geq 0 \quad \forall E \in \mathcal{F}$.

(iii) $\mu\left(\coprod_{\alpha=1}^{\infty} E_\alpha\right) = \sum_{n \in \coprod_{\alpha=1}^{\infty} E_\alpha} a_n = \sum_{\alpha=1}^{\infty} \sum_{n\in E_\alpha} a_n = \sum_{\alpha=1}^{\infty} \mu(E_\alpha)$.

In the case where $\mu\left(\coprod_{\alpha=1}^{\infty} E_\alpha\right)$ is infinite, $\sum_{n\in \coprod_{\alpha=1} E_\alpha} a_n$ is also $=\infty$ by definition.

This sum must $=\infty$ no matter how we order the sum, and so our rearrangement of the sum will still yield the value $+\infty$ and the statement is true.

So μ is a measure on $\mathcal{F}$.

For every measure μ on $\mathcal{F}$. $\mu\left(\coprod_{\alpha=1}^{\infty} E_\alpha\right) = \sum_{\alpha=1}^{\infty} \mu(E_\alpha)$

Because μ is a measure and E_α are subsets of $\mathbb{N}$, we can ~~break thes~~ always break these sets down to be unions of sets containing just one natural number.

~~So define $(a_n) = \mu(\{n\})$ ... the set $\{n\}$ ... $E_n =$~~

~~then $\mu\left(\coprod_{\alpha=1} E_\alpha\right) = \sum \mu\left(\coprod \{n\}\right) = \sum_n a_n = \sum \mu(n)$~~

3H From Lemma 3.4(b) we get that, for a decreasing sequence (F_n),

$$\mu(F_1) - \lim \mu(F_n) = \mu(F_1) - \mu\left(\bigcap_{n=1}^{\infty} F_n\right).$$

In the case that $\mu(F_1)$ is not restricted to be finite and $= \infty$, we cannot simply subract infinity from either side of this equality and so in general it may not be true.
To show that it may fail we will give an example of a decreasing sequence (F_n) s.t $\mu(F_1) = \infty$ and

$$\lim \mu(F_n) \neq \mu\left(\bigcap_{n=1}^{\infty} F_n\right).$$

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Let μ be the Lebesgue measure on the borel σ-algebra of open intervals in $\mathbb{R}$.
let $(F_n) = [n, \infty)$
so $F_1 = [1, \infty)$
$\mu(F_1) = \infty$ (using the result shown later in 3.R.

So. $\lim \mu(F_n) = \infty$,

~~Then.~~ $\cancel{\mu\left(\bigcap_{n=1}^{\infty} F_n\right)}$ take $x \in \bigcap_{n=1}^{\infty} F_n$

so $x \in \bigcap_{n=1}^{\infty} [n, \infty)$. Take N to be the smallest Integer so that $N > x$.

Then $x \notin [N, \infty)$, $\therefore x \notin \bigcap_{n=1}^{\infty} [n, \infty)$.

so $\bigcap_{n=1}^{\infty} F_n = \emptyset$ and so $\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = 0$.

$\therefore$ if the requirement that $\mu(F_1) < \infty$ is dropped, Lemma 3.4(b) does not necesarily hold.

3.0. Show that Lemma 3.4 holds if μ is a signed measure,
i.e. μ can take any real value, $\mu(\emptyset)=0$, and there is countable additivity over disjoint sets.

so not $\pm\infty$.

3.4 (a) for a charge.

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n) \quad \text{for an increasing sequence } E_n$$

Let $A_1 = E_1$ and $A_n = E_n \setminus E_{n-1}$ for $n>1$.
A_n is disjoint s.t.

$$E_n = \bigcup_{j=1}^{n} A_j, \quad \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n.$$

Since we still have countable additivity

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty}\mu(A_n) = \lim_{m\to\infty}\sum_{n=1}^{m}\mu(A_n).$$

At this point we can't use Lemma 3.3 immediately as μ is not a measure.
However $\mu(A_n) = \mu(E_n \setminus E_{n-1})$ still.

Since E_n is increasing $E_{n-1} \subseteq E_n$.

$$\therefore E_n = \underbrace{E_{n-1} \cup (E_n \setminus E_{n-1})}_{\text{disjoint}}$$

We have additivity of signed measures of disjoint sets.

$\Rightarrow \mu(E_n) = \mu(E_{n-1}) + \mu(E_n \setminus E_{n-1})$ and μ can never take $\pm\infty$ as a value

So $\mu(E_n \setminus E_{n-1}) = \mu(E_n) - \mu(E_{n-1})$, so the series $\sum_{n=1}^{m}\mu(E_n \setminus E_{n-1})$ telescopes and $\sum_{n=1}^{m}\mu(A_n) = \mu(E_m)$, So $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty}\mu(E_n)$.

3.R. $(\mathbb{R}, \mathbb{B}, \lambda)$

(a) E is a singleton set.
Then $E = \{a\}$ can be written as

$$E = \{a\} = \left[(-\infty, a) \cup (a, \infty)\right]^c \equiv A^c \quad \text{for some } a \in \mathbb{R}.$$

Since A is a union of two open intervals (in $\mathbb{B}$), and $\mathbb{B}$ is a σ-algebra, $A \in \mathbb{B}$.
And so it follows that $A^c = E \in \mathbb{B}$.

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$E = \{a\} = [a, a]$ can also be written as

$$E = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a + \frac{1}{n}\right) \quad \text{(using the result from P. 2.A)}$$

Define $\left(a - \frac{1}{n}, a + \frac{1}{n}\right) := (E_n)$ a decreasing sequence.

So we can use Lemma 3.4 (b) to write that:

$$\lambda(E) = \lambda\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n) = \lim_{n\to\infty} \left[a + \frac{1}{n} - \left(a - \frac{1}{n}\right)\right] = \lim_{n\to\infty} \frac{2}{n} = 0.$$

$\therefore \lambda(E) = 0.$

(b) Since E is countable it can be written as countable disjoint union of singleton sets E_n.
i.e. $E = \bigcup_{n=1}^{\infty} E_n$.

$\therefore \lambda(E) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n)$, but E_n are singletons, so $\lambda(E_n) = 0$ $\forall n$.

$\therefore \lambda(E) = 0$ if E is countable.

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3.5.

(ii) Show that if K is a compact subset of $\mathbb{R}$ then $\lambda(K) < +\infty$.

If K is compact then it is closed and bounded.

If K is bounded then $~~M \le~~ x \le M$ for some M (finite) $\forall x \in K$.

Let $\bar{K}$ be the closed interval $[-M, M]$.
$\lambda(\bar{K}) = 2M$.

But K is contained in $\bar{K}$, so using Lemma 3.3 we get that

$$\lambda(K) \le \lambda(\bar{K}) = 2M < +\infty.$$

✓

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4. A

Let φ be a simple function in $M^+(X, \mathcal{F})$ that is not necessarily in the standard representation.

0

$$\varphi = \sum_{k=1}^{m} b_k \chi_{F_k}.$$

If φ is in the standard representation, then $\int \varphi \, d\mu = \sum_{k=1}^{m} b_k \mu(F_k)$, and we are done.

If not, then
$$\varphi = b_1 \chi_{F_1} + \dots + b_m \chi_{F_m}$$
so $\int \varphi \, d\mu = \int b_1 \chi_{F_1} + \dots + b_m \chi_{F_m} \, d\mu$.

Because each $b_k \chi_{F_k}$ is a simple function itself (in standard representati
we can use Lemma 4.3 to obtain; This lemma does not directly apply since $b_k \chi_{F_k} \notin M^+$ is a possibility

×

$$\int \varphi \, d\mu = \int b_1 \chi_{F_1} \, d\mu + \dots + \int b_m \chi_{F_m} \, d\mu$$, the using def. 4.2

$$\int \varphi \, d\mu = b_1 \mu(F_1) + \dots + b_m \mu(F_m) = \sum_{k=1}^{m} b_k \mu(F_k).$$ ∎

If one assumes linearity of

[redacted]

MATH 6210 HW 1 [redacted]

2A. (i) Show that $[a,b] = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})$

Firsty. If $x \in [a,b]$, then $x \in (a-\frac{1}{2}, b+\frac{1}{n})$ $\forall n \in \mathbb{N}$.

So we only need to show that if $x \in \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})$ then $x \in [a,b]$.

If $x \in \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})$, then $a-\frac{1}{n} < x < b+\frac{1}{n}$ $\forall n \in \mathbb{N}$.

$\Rightarrow$ $a-x < \frac{1}{n}$ and $x-b < \frac{1}{n}$, $\forall n \in \mathbb{N}$.

Now, suppose $a-x > 0$

There must then exist an $n \in \mathbb{N}$ s.t $a-x > \frac{1}{n}$.

This contradicts the fact that $a-x \not> \frac{1}{n}$.

$\Rightarrow a-x \leq 0$

$\Rightarrow x \geq a$.

Similarly, suppose that $x-b > 0$.

$\exists n \in \mathbb{N}$ s.t $x-b \geq \frac{1}{n}$ which is also a contradiction and implies that $x \leq b$

So if $x \in \bigcap_{n=1}(a-\frac{1}{n}, b+\frac{1}{2})$ then $a \leq x \leq b$ $\Rightarrow x \in [a,b]$.

[redacted]

Because $\bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n}) = \left(\bigcup_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})^c \right)^c$

and $\bigcup_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})^c$ is in the σ algebra containing all open intervals

Its complement is ~~therefore~~ also in the σ algebra and so $\bigcap_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n}) = [a,b]$ is too.

Therefore all closed intervals are contained in the σ algebra containing all open intervals.

2C. (A_n) is a sequence of subsets of a set X.

$E_0 = \emptyset$

for $n \in \mathbb{N}$

$$E_n = \bigcup_{k=1}^{n} A_k \ , \quad F_n = A_n \setminus E_{n-1}.$$

(i) $E_n = \bigcup_{k=1}^{n} A_k = \left(\bigcup_{k=1}^{n-1} A_k\right) \cup A_n = E_{n-1} \cup A_n$

$\therefore$ E_n is monotonically increasing.

(ii) $F_n = A_n \setminus E_{n-1}$. To show that F_n is a disjoint sequence sonsider $n \neq m$ and $n > m$ w.l.o.g.

$$F_n \cap F_m = (A_n \setminus E_{n-1}) \cap (A_m \setminus E_{m-1})$$
$$= \left(A_n \setminus \bigcup_{k=1}^{n-1} A_k\right) \cap \left(A_m \setminus \bigcup_{k=1}^{m-1} A_k\right)$$
$$= (A_n \setminus (A_1 \cup A_2 \cup \ldots \cup A_m \ldots \cup A_{n-1}) \cap (A_m \setminus (A_1 \cup A_2 \ldots \cup A_{m-1})$$

$= \emptyset$ as F_n excludes all of A_m and F_m ~~consists~~ lies entirely in A_m.

$\therefore$ F_n is a disjoint sequence.

(iii) $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty}\left(\bigcup_{k=1}^{n} A_k\right) = \bigcup_{n=1}^{\infty} A_n$.

Also, we note that $\bigcup_{k=1}^{1} F_k = A_1 = \bigcup_{k=1}^{1} A_k$.

We assume that $\bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} A_k$ and show that this implies that $\bigcup_{k=1}^{n+1} F_k = \bigcup_{k=1}^{n+1} A_k$.

$\Rightarrow F_{k+1} \cup \left(\bigcup_{k=1}^{n} F_k\right) = F_{k+1} \cup \left(\bigcup_{k=1}^{n} A_k\right)$ $\longrightarrow$

2D (A_n) is a sequence of subsets on a set X.
A consists of all points $x \in X$, such that x is in infinitely many sets A_n.

Let x be a general point in A.
Then,
$$x \in \bigcup_{n=m}^{\infty} A_n \quad \forall m \in \mathbb{N}.$$

This is because, if one can find an m for which this is not true, then x must only be in finitely many of the sets A_n.

If $x \in \bigcup_{n=m}^{\infty} A_n \ \forall m \in \mathbb{N}$ then it must be in the intersection of these unions, i.e.,
$$x \in \bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} A_n\right).$$

We must now show that if $x \in \bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} A_n\right)$ then x is in infinitely many sets A_n. i.e. $x \in A$

Suppose $x \in \bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} A_n\right)$ and $x \notin A$.

If this were the case, we could always choose m large enough so that $x \notin \bigcup_{n=m}^{\infty} A_n$. (because it is only in finitely many of A

And so if $x \notin \bigcup_{n=m}^{\infty} A_n \ \forall m \in \mathbb{N}$, then $x \notin \bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} A_n\right)$.

We have arrived at a contradiction and can conclude that $x \in A$.

$$\therefore \quad A = \bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} A_n\right)$$

∎

2 E (E_n) is monotone increasing sequence of subsets of a set X.
i.e. $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots\cdots$

We can use the definitions of the lim sup E_n and the lim inf E_n from problems 2D and 2E to show that

$$\limsup E_n = \bigcup_{n=1}^{\infty} E_n = \liminf E_n$$

Define E, the limit set of (E_n) so that $E = \bigcup_{n=1}^{\infty} E_n$.

$$\limsup E_n = \bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} E_n\right).$$

But $\bigcup_{n=m}^{\infty} E_n = \bigcup_{n=1}^{\infty} E_n = E \because E_n$ is monotonically increasing.

$$\therefore \limsup E_n = \bigcap_{m=1}^{\infty} E = E = \bigcup_{n=1}^{\infty} E_n.$$

$$\liminf E_n = \bigcup_{m=1}^{\infty}\left(\bigcap_{n=m}^{\infty} E_n\right) = \bigcup_{m=1}^{\infty}(E_m)$$ $\because \bigcap_{n=m}^{\infty} E_n$ is just the smallest E_n, i.e. E_m (monotonicity).

$$\therefore \liminf E_n = \bigcup_{m=1}^{\infty} E_n.$$

and $\limsup E_n = \liminf E_n = \bigcup_{n=1}^{\infty} E_n$. ■

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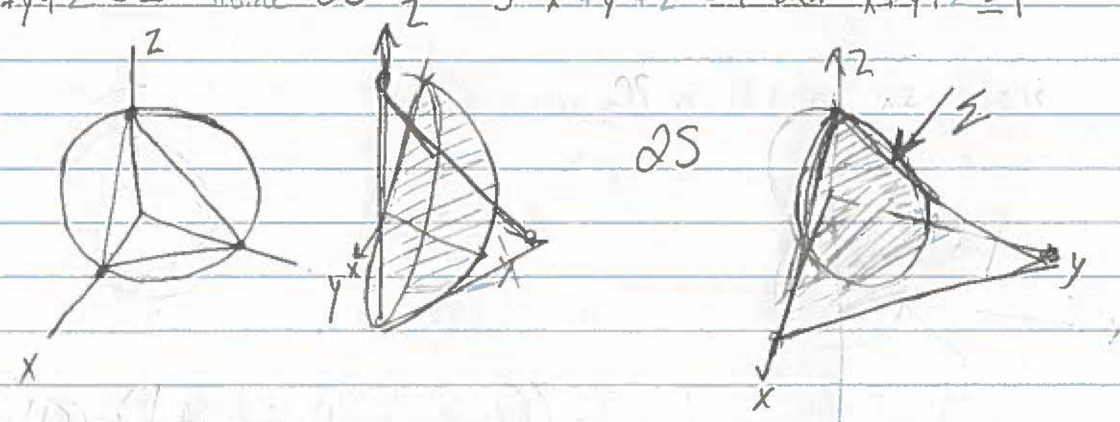
12/4/14

HW#10

8.2#16,19,24,26a

16) Stokes' Theorem says that $\iint_S (\nabla \times F)\cdot dS = \int_{\partial S} F\cdot ds$ for F a C^1 vector field on the oriented surface S. This means that only ∂S, the boundary of S, matters in calculating $\iint_S (\nabla\times F)\cdot dS$. As long as $\partial\Sigma = \partial S$, $\int_{\partial\Sigma} F\cdot ds$ can be used to solve the problem for any such surface Σ whose boundary is the same as S.

Σ: $x+y+z=1$ inside ∂S S: $x^2+y^2+z^2=1$ and $x+y+z\geq 1$



$$F = \begin{vmatrix} i & j & k \\ x & y & z \\ i & j & k \end{vmatrix} = i(y\hat{k}-z\hat{j}) - j(x\hat{k}-z\hat{i}) + k(x\hat{j}-y\hat{i})$$

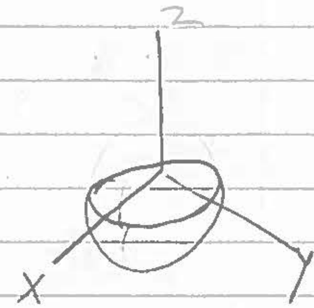
$$\hat{n} \text{ to plane} = \frac{1\hat{i}+1\hat{j}+1\hat{k}}{\sqrt{1+1+1}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\text{curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\hat{k}-z\hat{j} & z\hat{i}-x\hat{k} & x\hat{j}-y\hat{i} \end{vmatrix} = i(-1-1) - j(1+1) + k(-1-1)$$

$$= (-2,-2,-2)$$

19) $F = y\hat{i} - x\hat{j} + zx^3y^2\hat{k}$

$\iint_S (\nabla \times F) \cdot n \, dA \qquad S: x^2+y^2+z^2=1, \quad z \le 0$



$\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds$

$\partial S: c(t) = (\cos t, \sin t, 0)$

$c'(t) = (-\sin t, \cos t, 0)$

$F(c(t)) = \sin t\,\hat{i} - \cos t\,\hat{j} + 0\,\hat{k}$

$$\begin{bmatrix} \sin t \\ -\cos t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix} = -\sin^2 t - \cos^2 t = -1$$

$$-\int_0^{2\pi} 1 \, dt = \boxed{-2\pi}$$

24) $r(x,y,z) = (x,y,z)$

Let $\vec{F} = v \times r$

$v = (v_1, v_2, v_3)$

$$\vec{F} = \begin{vmatrix} i & j & k \\ v & v_2 & v_3 \\ x & y & z \end{vmatrix}$$

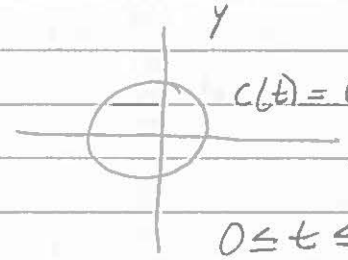
$$= i(zv_2 - yv_3) - j(zv_1 - xv_3) + k(yv_1 - xv_2)$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zv_2 - yv_3 & xv_3 - zv_1 & yv_1 - xv_2 \end{vmatrix} = i(v_1 + v_1) - j(-v_2 - v_2) + k(v_3 + v_3)$$

$$= (2v_1, 2v_2, 2v_3) = 2\vec{v}$$

8.3 #14, 16, 17d, 19a

14) $\vec{F}(x,y,z) = xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k}$



$c(t) = (\cos t, \sin t, 0)$

$0 \le t \le 2\pi$

$F(c(t)) = (\cos t \sin t, 0, 0)$

$c'(t) = (-\sin t, \cos t, 0)$

$$\int_0^{2\pi} F(c(t)) \cdot c'(t)\,dt = \int_0^{2\pi} -\sin^2 t \cos t\,dt = \int_0^0 -u^2\,du = \boxed{0}$$

$u = \sin t \qquad u_1 = \sin(0) = 0$

$du = \cos t\,dt \qquad u_2 = \sin(2\pi) = 0$

A circulation of zero means that the vector field is irrotational and that it is the gradiant of some function.

16) (a) $\int_C (x\,dy - y\,dx)/(x^2+y^2)$ $\qquad c(t) = (\cos t, \sin t)$

$0 \le t \le 2\pi$

$$\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \qquad F(c(t)) = (-\sin t, \cos t)$$

$c'(t) = (-\sin t, \cos t)$

$F(c(t)) \cdot c'(t) = 1$

$$\int_0^{2\pi} 1\,dt = \boxed{2\pi}$$

(b) $\vec{F}$ is not a conservative vector field because $\int F \cdot ds \ne 0$

(c) $$\frac{\partial P}{\partial y} = (-y)\left(-(x^2+y^2)^{-2} \cdot 2y\right) + (x^2+y^2)^{-1} \cdot -1 = \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = (x)\left(-(x^2+y^2)^{-2} \cdot 2x\right) + (x^2+y^2)^{-1} = \frac{-2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} \longrightarrow$$

$$\text{①} \int_0^1 \left((4t - 4t^2 + t^3) + 6t^2 - 3t^3 \right) \hat{i} + 2t^2 \hat{j} \cdot (1,-1)\, dt$$

$$= \int_0^1 \left(4t + 2t^2 - 2t^3\right) \hat{i} + 2t^2 \hat{j} \cdot (1,-1)\, dt = \int_0^1 4t + 2t^2 - 2t^3 - 2t^2 \, dt$$

$$= \int_0^1 (4t - 2t^3)\, dt = 2t^2 - \frac{t^4}{2} \Big]_0^1 = 2 - \frac{1}{2} = \frac{3}{2} \cdot -1 = \frac{-3}{2}$$ ← correct orientation

$$\text{②} \int_0^3 \left(\left(4t - \frac{8}{3}t^2 + \frac{4}{9}t^3\right) + 6t^2 - 2t^3 \right) \hat{i} + \left(2t^2 + \frac{t^3}{3}\right) \hat{j} \cdot \left(1, -\frac{2}{3}\right) dt$$

$$= \int_0^3 \left(-\frac{14}{9}t^3 + \frac{10}{3}t^2 + 4t \right) \hat{i} + \left(2t^2 + \frac{t^3}{3}\right) \hat{j} \cdot \left(1, -\frac{2}{3}\right) dt$$

$$= \int_0^3 \left(-\frac{14}{9}t^3 + \frac{10}{3}t^2 + 4t \right) - \left(\frac{4}{3}t^2 + \frac{2t^3}{9} \right) dt$$

$$= \int_0^3 -\frac{16}{9}t^3 + 2t^2 + 4t \, dt = -\frac{4}{9}t^4 + \frac{2}{3}t^3 + 2t^2 \Big]_0^3 = -36 + 18 + 18 = 0$$

$$-\frac{3}{2} + 0 = \boxed{\frac{-3}{2}}$$

8.4 # 6, 8, 16, 17

6) $\vec{F} = x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}$ S: $x^2+y^2+z^2=1$

$x(\rho,\theta,\phi) = \rho\cos\theta\sin\phi$ $\quad 0 < \rho \le 1$
$y(\rho,\theta,\phi) = \rho\sin\theta\sin\phi$ $\quad 0 \le \theta < 2\pi$
$z(\rho,\theta,\phi) = \rho\cos\phi$ $\quad 0 \le \phi \le \pi$

$\iint_S (F\cdot n)\, dS = \iiint_W (\text{div } F)\, dW$

$\text{div } \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$

$\text{div } \vec{F}(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi)) = 3(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi)$

$= 3\rho^2$

$$3\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2\, d\rho\, d\theta\, d\phi = 3\int_0^{\pi}\int_0^{2\pi} \frac{\rho^3}{3}\Big]_0^1 d\theta\, d\phi = \int_0^{\pi} \theta\Big]_0^{2\pi} d\phi = \boxed{2\pi^2}$$

8) $\iint_{\partial W} F\cdot dS$ $\quad W$ is unit cube

$X = u$
$Y = v$
$Z = w$

Face #	Equation	normal	Parametrization	
1	$x=0$	$-\hat{i}$	$(0, v, w)$	$0 \le u \le 1$
2	$x=1$	$\hat{i}$	$(1, v, w)$	$0 \le v \le 1$
3	$y=0$	$-\hat{j}$	$(u, 0, w)$	$0 \le w \le 1$
4	$y=1$	$\hat{j}$	$(u, 1, w)$	
5	$z=0$	$-\hat{k}$	$(u, v, 0)$	
6	$z=1$	$\hat{k}$	$(u, v, 1)$	

$$\iint_{\partial W_1} F\cdot(-\hat{i})\, dv\, dw + \iint_{\partial W_2} F\cdot(\hat{i})\, dv\, dw + \iint_{\partial W_3} F\cdot(-\hat{j})\, du\, dw + \iint_{\partial W_4} F\cdot\hat{j}\, du\, dw$$

$$+ \iint_{\partial W_5} F\cdot(-\hat{k})\, du\, dv + \iint_{\partial W_6} F\cdot\hat{k}\, du\, dv$$

(2) Divergence theorem

$$\iint_{\partial W} F\cdot dS = \iiint_{W} (\operatorname{div}\vec{F})\cdot dV \qquad \operatorname{div}\vec{F} = 2x + 2z = 2(x+z)$$

$0 \le x \le 1$
$0 \le y \le 1$
$0 \le z \le 1$

$$2\int_0^1\int_0^1\int_0^1 (x+z)\,dx\,dy\,dz = 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + xz\right]_0^1 dy\,dz$$

$$= 2\int_0^1\int_0^1 \left(\frac{1}{2}+z\right) dy\,dz = 2\int_0^1 \left[\frac{y}{2} + yz\right]_0^1 dz = 2\int_0^1 \left(\frac{1}{2}+z\right) dz$$

$$= 2\left(\frac{z}{2} + \frac{z^2}{2}\right)\Big]_0^1 = 2\left(\frac{1}{2}+\frac{1}{2}\right) = \boxed{2}$$

(6) $\iint_{\partial S} F\cdot n\, dA \qquad F(x,y,z) = \hat{\imath} + \hat{\jmath} + z(x^2+y^2)^2\hat{k}$

$\partial S: \; x^2+y^2 \le 1$
$0 \le z \le 1$

$$\iint_{\partial S} F\cdot n\, dA = \iiint_{W} (\operatorname{div}\vec{F})\, dV$$

6/6

$x(r,\theta,z) = r\cos\theta \qquad 0 \le r \le 1$
$y(r,\theta,z) = r\sin\theta \qquad 0 \le \theta \le 2\pi$
$z(r,\theta,z) = z \qquad 0 \le z \le 1$

$\operatorname{div}\vec{F} = (x^2+y^2)^2 \qquad \operatorname{div}\vec{F}\,(x(r,\theta,z), y(r,\theta,z), z(r,\theta,z)) = r^4$

$$\int_0^1\int_0^{2\pi}\int_0^1 r^4\, r\, dr\,d\theta\,dz = \int_0^1\int_0^{2\pi} \left[\frac{r^{\cancel{5}\,6}}{\cancel{5}\,6}\right]_0^1 d\theta\,dz = \int_0^1 \left[\frac{\theta}{5}\right]_0^{2\pi} dz$$

$$= \frac{2\pi z}{5}\Big]_0^1 = \boxed{\frac{2\pi}{\cancel{5}\,6}} = \pi/3 \leftarrow \text{careful!}$$

$$= x^2 \overbrace{dx\wedge dy dz}^{dx\,dy\,dz} + z^2 dz\wedge dx dy + y^2 dy\wedge dz\, dx$$

$dz\wedge dx\, dy = (dz\wedge dx)\wedge dy = (-1)(dx\wedge dz)\wedge dy \overset{(-1)}{=} dx\wedge(dz\wedge dy) = dx\wedge(dy\wedge dz$

so $dz\wedge dx\, dy = dx\,dy\,dz$

$dy\wedge dz\, dx = dy\wedge(dz\wedge dx) = (-1)\, dy\wedge(dx\wedge dz) = (-1)(dy\wedge dx)\wedge dz$

$= (dx\wedge dy)\wedge dz = dx\,dy\,dz$

$$\boxed{(x^2+y^2+z^2)\, dx\,dy\,dz}$$

(e) $\omega = e^{xyz} dx\,dy$ $\qquad \omega\wedge\eta = (e^{xyz} dx\,dy)\wedge(e^{-xyz} dz)$

$\eta = e^{-xyz} dz$ $\qquad = dx\,dy\wedge dz$

$= \boxed{dx\,dy\,dz}$

3) (a) $\omega = x^2y + y^3$

$d\omega = d(x^2y + y^3) = \boxed{2xy\, dx + (x^2+3y^2)\, dy}$

(c) $\omega = xy\, dy + (x+y)^2 dx$

$d\omega = d(xy\,dy + (x+y)^2 dx) = (d(xy)\wedge dy) + (d(x+y)^2\wedge dx)$

$= (y\,dx + x\,dy)\wedge dy + (2(x+y)dx + 2(x+y)dy)\wedge dx$

$= y\,dx\wedge dy + x\,dy\wedge dy + 2(x+y)dx\wedge dx + 2(x+y)dy\wedge dx$

$= 2(x+y)dy\wedge dx - y\,dy\wedge dx = \boxed{(2x+y)\, dy\,dx}$

(d) $\omega = x\,dx\,dy + z\,dy\,dz + y\,dz\,dx$

$d\omega = d(x\,dx\,dy + z\,dy\,dz + y\,dz\,dx)$

$= dx\wedge dx\,dy + dz\wedge dy\,dz + dy\wedge dz\,dx = 0 + 0 + \boxed{dx\,dy\,dz}$

$= dx\,dy\,dz$

from above

exercise 1.6

② Evaluate using Stokes' Theorem

$\int_{\partial S} \omega = \iint_S d\omega$

$d\omega = d(x+y) \wedge dz + d(y+z) \wedge dx + d(x+z) \wedge dy$

$= (dx+dy) \wedge dz + (dy+dz) \wedge dx + (dx+dz) \wedge dy$

$\iint_S d\omega$

$= \underbrace{dx \wedge dz + dy \wedge dz + dy \wedge dx + dz \wedge dx + dx \wedge dy + dz \wedge dy}_{\text{everything cancels out}}$

$\iint_S 0 = \boxed{0} \longleftarrow d\omega = 0$

12) $\omega = z\,dx\,dy + x\,dy\,dz + y\,dz\,dx$

$\iint_S \omega$

① Evaluate directly

$\Phi(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) \qquad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$

$$\int_0^{\pi}\int_0^{2\pi} (\cos\phi) \cdot \frac{\partial(x,y)}{\partial(\theta,\phi)} + (\cos\theta\sin\phi) \cdot \frac{\partial(y,z)}{\partial(\theta,\phi)} + (\sin\theta\sin\phi) \cdot \frac{\partial(z,x)}{\partial(\theta,\phi)} \, d\theta\, d\phi$$

$$\frac{\partial(x,y)}{\partial(\theta,\phi)} = \begin{vmatrix} -\sin\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \end{vmatrix} = -\sin^2\theta\cos\phi\sin\phi - \cos^2\theta\cos\phi\sin\phi$$

$$= -\sin\phi\cos\phi \xrightarrow[\text{orientation}]{\text{correct}} \sin\phi\cos\phi$$

$$\frac{\partial(y,z)}{\partial(\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi & \sin\theta\cos\phi \\ 0 & -\sin\phi \end{vmatrix} = -\sin^2\phi\cos\theta \longrightarrow \sin^2\phi\cos\theta$$

$$\frac{\partial(z,x)}{\partial(\theta,\phi)} = \begin{vmatrix} 0 & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\cos\phi \end{vmatrix} = -\sin^2\phi\sin\theta \rightarrow \sin^2\phi\sin\theta$$

$$3\int_0^{\pi}\int_0^{2\pi}\int_0^{1}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\int_0^{2\pi}\rho^3\sin\phi\Big]_0^1 d\theta\,d\phi = \int_0^{\pi}\int_0^{2\pi}\sin\phi\,d\theta\,d\phi$$

$$= 2\pi\int_0^{\pi}\sin\phi\,d\phi = -2\pi\cos\phi\Big]_0^{\pi} = 2\pi-(-2\pi) = \boxed{4\pi}$$

Supplement G

1) $\vec{F}(x,y) = (P(x,y), Q(x,y))$

(a) $\int_{C2} -Q(x,y)dx + P(x,y)dy = 3$ $\quad$ $D2: x^2+y^2=4$ $\quad$ $D1: x^2+y^2=1$ $\quad$ 3/3

by Green's Theorem, $\int_{C2} -Q(x,y)dx + P(x,y)dy = \iint_{D2} \underbrace{\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)}_{=2} dx\,dy$ $\quad$ $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y} = 2$

$$= \iint_{D2} 2r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} 2r\,dr\,d\theta$$

$$\int_{C1} -Q(x,y)dx + P(x,y)dy = \int_0^{2\pi}\int_0^{1} 2r\,dr\,d\theta = \frac{1}{2}\int_{C2} -Q(x,y)dx + P(x,y)dy = \boxed{\frac{3}{2}}$$

(b) If the vector field were C^1 on the entire plane,

$\int_{C2} -Q(x,y)dx + P(x,y)dy$ could not $= 3$ because

$\therefore \iint_{D2} 2\,dx\,dy \neq 3.$

Is D_2 the disk or the circle.

Problem Set 5

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2-10 For any topological space M, let $C(M)$ denote the algebra of continuous functions $f: M \to \mathbb{R}$. Given a continuous map $F: M \to N$, define $F^*: C(N) \to C(M)$ by $F^*(f) = f \circ F$.

a) Show that F^* is linear.

Let $\alpha, \beta \in \mathbb{R}$, $f, g \in C(M)$. Observe that

$$F^*(\alpha f + \beta g) = (\alpha f + \beta g) \circ F \underset{\substack{\uparrow \\ \text{Composition} \\ \text{is linear}}}{=} (\alpha f) \circ F + (\beta g) \circ F = \alpha(f \circ F) + \beta(g \circ F)$$
$$= \alpha F^*(f) + \beta F^*(g). //$$

b) Suppose that M and N are smooth Manifolds. Show that $F: M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

Proof: First assume $F: M \to N$ is smooth. Then for any $f \in C^\infty(N)$,

$F^*(f) = f \circ F: M \to \mathbb{R}$ is a composition of smoth map and is therefore smooth. Hence $F^*(f) \in C^\infty(M)$.

Conversely assume that $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Let $p \in M$ and let (V, ψ) be a chart near $F(p)$. Since F is continuous $F^{-1}(V)$ is an open set containg p and we may therefor find a chart near p (U, φ) s.t. $U \subseteq F^{-1}(V)$ and so $(U, \varphi), (V, \psi)$ are adapted to F. Now observe that each coordinate map $\psi^i: N \to \mathbb{R}$, is $C^\infty(N)$ and so by hypothesis

$$F^*(\psi^i) = \psi^i \circ F \in C^\infty(M) \quad \text{consequently}$$

$$\psi \circ F = (\psi^1 \circ F, \psi^2 \circ F, \ldots \psi^{(n)} \circ F) \text{ is smooth.}$$

and since compositions of smooth maps are smooth

$$(\psi \circ F) \circ \varphi^{-1} = \psi \circ F \circ \varphi^{-1} \text{ is smooth. This however}$$

is precisely the presentation of F at p and since Smoothness is a local property we may conclude that F is smooth. //

C) Suppose that $F: M \to N$ is a homeomorphism between smooth manifolds.

Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N) \to C^\infty(M)$.

Proof: Let us first state the following Fact:

Fact: Let $F: M \to N$, $G: N \to P$ be continuous then for any $f \in C^\infty(P)$

i) $(G \circ F)^*(f) = G^* \circ F^*(f)$.

ii) $Id_M^* = Id_{C^\infty(M)}$

The proof of this fact is completely analogous to that given for the similar fact concerning pushforwards.

Let us assume that $F: M \to N$ is a diffeomorphism. We have already shown in A that F^* is linear. To see it is injective suppose that

$$F^*(f) = F^*(g) \quad \text{for } f, g \in C^\infty(N).$$

Then by b $F^*(f), F^*(g) \in C^\infty(M)$ and so we may apply F^{-1*} to both sides. This gives

$F^{-1*} \circ F^*(f) = F^{-1*} \circ F^*(g)$, but by i) above we have

$(F^{-1} \circ F)^*(f) = (F^{-1} \circ F)^*(g)$ and since $F^{-1} \circ F = id_{C(N)}$ by ii) we

Conclude $f = g$.

To see our map is surjective simply notice that for $g \in C^\infty(M)$

$F^{-1*}(g) \in C^\infty(N)$ by part b and the fact that F^{-1} is smooth.

Consequently

$F^* F^{-1*}(g) = (F \circ F^{-1})^*(g) = Id_{C^\infty(M)}(g) = g$ where we have used i), ii). Thus F^* is an isomorphism with inverse F^{-1*}.

C continued.

Conversely assume that $F^*: C^\infty(N) \to C^\infty(M)$ is an isomorphism.

By part b we know that F is a smooth map.

Also observe that by our assumption

$$F^*(C^\infty(N)) = C^\infty(M)$$

we have

$$F^{-1*} \circ F^*(C^\infty(M)) = F^{-1*} C^\infty(M).$$

Moreover by i, ii we have

$$F^{-1*} C^\infty(M) = (F \circ F^{-1})^* C^\infty(M) = C^\infty(M)$$

and so by part b we conclude that

F^{-1} is smooth showing F is a bijective function

with smooth inverse. Hence F is a diffeomorphism. //

2-14 Suppose A, B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \le f(x) \le 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(A) = B$.

Lemma 1: Let A, B, M be as above. Then there is $g \in C^\infty(M)$ such that $0 \le g \le 1$

$$A = g^{-1}(0) \text{ and } B \subseteq g^{-1}(1).$$

Proof: Let $\mathcal{O}$ be an open set such that $A \subseteq \mathcal{O}$, $\mathcal{O} \cap B = \emptyset$. (We know M is T^4).
Theorem 2.29, actually the proof of the theorem, guarantees a function $f_\alpha \in C^\infty(M)$ s.t. $f_\alpha^{-1}(0) = A$ and $0 \le f_\alpha(x) \le 1$ for all $x \in M$. We remark that the second property is not given in the statement of the theorem however it was guaranteed in the proof.
We observe that $\mathcal{O}$, $M \setminus A$ is an open cover of M. We may select a subordinate partition of unity λ_α and λ_B resp.

5/5

Define $g: M \to \mathbb{R}$ by

$$g(x) = f_\alpha(x)\,\lambda_\alpha(x) + \lambda_B(x).$$

It is clear that $0 \le g(x)$ for all $x \in M$. Observe that because $f_\alpha \le 1$

$$g(x) \le 1\,\lambda_\alpha(x) + \lambda_B(x) = 1 \text{ and so } 0 \le g(x) \le 1 \text{ for all } x \in M.$$

Note that for any $x \in B$, $x \notin \mathcal{O}$ and so

$$g(x) = f_\alpha(x) \cdot 0 + \lambda_B(x) = \lambda_B(x) = 1. \text{ Thus } B \subseteq g^{-1}(1).$$

Finally observe that $g(x) = 0$ only when $f_\alpha(x)\,\lambda_\alpha(x) = 0 = \lambda_B(x)$ but then $\lambda_\alpha(x) = 1$ implying $f_\alpha(x) = 0$ which happens only when $x \in A$. This proves the Lemma.

Lemma 2: Let A, B be as above. Then there is $h \in C^\infty(M)$ s.t.

$$\tfrac{1}{2} \le h(x) \le 1 \text{ and } h^{-1}(1) = B.$$

Proof: Take f_b as guaranteed by Theorem 2.29, i.e. $f^{-1}(0) = B$ and $0 \le f_b(x) \le 1$ for all $x \in M$. Set $h(x) = 1 - \tfrac{1}{2} f_b(x)$. It is clear that this does the trick. //

2-14 Continued.

Now we prove the result. Take $g \in C^\infty(M)$ as guaranteed in Lemma 1 and $h \in C^\infty(M)$ from Lemma 2. Since $C^\infty(M)$ is a ring $f: M \to \mathbb{R}$ defined by $f(x) = g(x)h(x)$ $x \in M$ is smooth. Moreover since both $0 \leq g(x), h(x) \leq 1$ for all $x \in M$ we have $0 \leq f(x) \leq 1$ for all $x \in M$.

Observe that

$$f(x) = 1 \text{ if and only if } g(x) \text{ and } h(x) = 1,$$

but this happens precisely when $x \in B$ and so

$$f^{-1}(1) = B.$$

Now since $h(x) \neq 0$ for all $x \in M$, $f(x) = 0$ exactly when $g(x) = 0$. By lemma 1,

$$f^{-1}(0) = g^{-1}(0) = A.$$

Thus we have shown that f is our desired map. //

+-2 Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary and $F: M \to N$ is smooth. Show that if $p \in M$ is a point such that dF_p is non-singular, then $F(p) \in \operatorname{Int} N$.

proof: It is clear that it suffices to prove the claim in the case where $M = \mathbb{R}^m$ and $N = \mathbb{H}^n$ for otherwise it is the case that, upon choice of appropriate charts, the property that $d\hat{F}_p$ is non-singular is equivalent to the property that dF_p is non singular. ($\hat{F}$ is the presentation of F).

So let us suppose that $F: \mathbb{R}^m \to \mathbb{H}^n \subseteq \mathbb{R}^n$ and that for some $p \in \mathbb{R}^m$, $F(p) \in \partial \mathbb{H}^n$. Now computing dF_p we have

$$D_F = \begin{pmatrix} \partial_1 F_1, & \cdots & \partial_1 F_n \\ \partial_2 F_1, & \cdots & \partial_2 F_n \\ \vdots & & \\ \partial_m F_1 & \cdots & \partial_m F_n \end{pmatrix}.$$

We know however that $F_n(x) \geq 0$ for all $x \in \mathbb{R}^m$ because $F(\mathbb{R}^m) \subseteq \mathbb{H}^n$. If $\partial_i F_n(p) \neq 0$ for any i, we could replace p by $x = p + h e_i$ for some small h and therefor produce $x \in \mathbb{R}^n$ s.t. $F(x) < 0$; this would contradict the previous sentence. We must therefore conclude that $\partial_i F_n(p) = 0$ for all i.

So

$$D_F(p) = \begin{pmatrix} \partial_1 F_1 & \cdots & \partial_1 F_{n-1} & 0 \\ \vdots & & \partial_2 F_{n-1} & 0 \\ & & \vdots & 0 \\ & & & 0 \\ \partial_m F_1 & & \partial_m F_{n-1} & 0 \end{pmatrix}$$ which is surjective and so

dF_p is singular.

This proves the claim. //

4-6 Let M be a non-empty compact manifold. Show that there is no smooth submersion $F: M \to \mathbb{R}^k$ for any $k > 0$.

Proof: Let us assume to reach a contradiction that $F: M \to \mathbb{R}^k$ is a smooth submersion for some $k > 0$. Since F is smooth, it is continuous and therefore

$$F(M) \subseteq \mathbb{R}^k \text{ is a non-empty compact set.}$$

By Heine-Borel's Theorem, it is closed. Appealing to Proposition 4.28 F is also an open map. We may therefore conclude that $F(M)$ is open, because it is the image of an open set under F.

So $F(M)$ is a non-empty clopen subset of $\mathbb{R}^k$. Since $\mathbb{R}^k$ is connected we must have $F(M) = \mathbb{R}^k$. This however cannot be true because $\mathbb{R}^k$ is not compact. //

5/5.

4.8 Let $\pi: \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x,y) = xy$. Show that π is surjective and smooth and for each smooth manifold P, a map, $F: \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

proof:

Remark: When checking smoothness in the context of $\mathbb{R}$ or $\mathbb{R}^2$, we note that we need not concern ourselves with charts, for the identity chart will always suffice and smoothness is the standard notion of $\mathbb{R}^n$ smoothness.

It is clear that $\pi: \mathbb{R}^2 \to \mathbb{R}$ is surjective; for any $x \in \mathbb{R}$ $\pi(x,1) = x$. As per our remark, $\pi(x,y)$ is a polynomial and is therefore a smooth map.

Let us now assume that F is smooth. We know that the composition of smooth maps is smooth and therefore $F \circ \pi$ is smooth. Conversely assume that $F \circ \pi$ is smooth. Observe that the function $x \xmapsto{\gamma} (x,1)$ is also smooth. but then we have $F(x) = F(x \cdot 1) = F \circ \pi(x,1) = (F \circ \pi) \circ \gamma(x)$ and so F is the composition of smooth maps. It is therefore smooth.

Finally we see π is not a smooth submersion. To see this simply observe that with respect to the identity charts on $\mathbb{R}^2$ and $\mathbb{R}$)

$$D\pi_{(0,0)} = (y, x)\Big|_{(0,0)} = (0,0) : \mathbb{R}^2 \to \mathbb{R} \text{ is}$$

not surjective and thus $d\pi_{(0,0)}$ is not surjective so π is not a submersion. //