Local false sign rate and point mass

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We study the property of lfsr when the true model has point mass while the fitted model does not include it. We focus on the cases where the observed statistic is small or large(for example, a z score around 0.1 or 3).

Notation: the pdf and cdf of a normal distribution $N(\mu, \sigma^2)$ are denoted as $\phi_{\mu,\sigma^2}(\cdot)$ and $\Phi_{\mu,\sigma^2}(\cdot)$ respectively. For standard normal, we drop the script μ, σ^2.

Consider generating z from the model

$$\begin{aligned} &z \sim N(\mu, 1), \\ &\mu \sim \pi_0\delta_0 + \pi_1 N(0, \sigma_1^2), \end{aligned} \tag{1}$$

where δ_0 is a point mass at 0.

The posterior of μ is

$$\mu|z \sim \tilde{\pi}_0\delta_0 + \tilde{\pi}_1 N(\frac{\sigma_1^2}{\sigma_1^2+1}z, \frac{\sigma_1^2}{\sigma_1^2+1}), \tag{2}$$

where

$$\tilde{\pi}_0 = \frac{\pi_0 N(z;0,1)}{\pi_0 N(z;0,1) + \pi_1 N(z;0,1+\sigma_1^2)}, \tag{3}$$

and $\tilde{\pi}_1 = 1 - \tilde{\pi}_0$.

The (true) local false sign rate(lfsr) of μ is $\min\{p(\mu \leq 0|z), p(\mu \geq 0|z)\}$. WLOG, we assume the observed z is positive, and in this case $lfsr = p(\mu \leq 0|z)$, where

$$p(\mu \leq 0|z) = \tilde{\pi}_0 + \tilde{\pi}_1 \Phi\left(-\sqrt{\frac{\sigma_1^2}{\sigma_1^2+1}}z\right). \tag{4}$$

Assume in practice, we use the following prior on μ,

$$\mu \sim \pi_0 N(0, \sigma_0^2) + \pi_1 N(0, \sigma_1^2), \tag{5}$$

where σ_0 is a small postive number. For simplicity, we assume the same π_0, π_1 and σ_1^2 as the ones in true model.

The posterior of μ is

$$\mu|z \sim \hat{\tilde{\pi}}_0 N(\frac{\sigma_0^2}{\sigma_0^2+1}z, \frac{\sigma_0^2}{\sigma_1^2+1}) + \hat{\tilde{\pi}}_1 N(\frac{\sigma_1^2}{\sigma_1^2+1}z, \frac{\sigma_1^2}{\sigma_1^2+1}), \tag{6}$$

where

$$\hat{\tilde{\pi}}_0 = \frac{\pi_0 N(z;0,1+\sigma_0^2)}{\pi_0 N(z;0,1+\sigma_0^2) + \pi_1 N(z;0,1+\sigma_1^2)}. \tag{7}$$

The estimated lfsr is

$$\widehat{lfsr} = \hat{\tilde{\pi}}_0 \Phi\left(-\sqrt{\frac{\sigma_0^2}{\sigma_0^2+1}}z\right) + \hat{\tilde{\pi}}_1 \Phi\left(-\sqrt{\frac{\sigma_1^2}{\sigma_1^2+1}}z\right). \tag{8}$$

Assume z is small(close to 0). Taking the log ratio of $\tilde{\pi}_0$ and $\hat{\tilde{\pi}}_0$ gives

$$\begin{aligned}
&\log\left(\frac{\tilde{\pi}_0}{\hat{\tilde{\pi}}_0}\right) \\
=&\log\phi(z) - \log\phi_{0,1+\sigma_0^2}(z) + \log\left(\pi_0\phi_{0,1+\sigma_0^2}(z) + \pi_1\phi_{0,1+\sigma_1^2}(z)\right) - \log\left(\pi_0\phi(z) + \pi_1\phi_{0,1+\sigma_1^2}(z)\right) \\
=&\log\left(1 + \frac{\pi_1\phi_{0,1+\sigma_1^2}(z)}{\pi_0\phi_{0,1+\sigma_0^2}(z)}\right) - \log\left(1 + \frac{\pi_1\phi_{0,1+\sigma_1^2}(z)}{\pi_0\phi(z)}\right) \\
\approx&\frac{\pi_1\phi_{0,1+\sigma_1^2}(z)}{\pi_0\phi_{0,1+\sigma_0^2}(z)} - \frac{\pi_1\phi_{0,1+\sigma_1^2}(z)}{\pi_0\phi(z)} \\
=&\frac{\pi_1}{\pi_0}\frac{\phi_{0,1+\sigma_1^2}(z)}{\phi(z)\phi_{0,1+\sigma_0^2}(z)}\left(\phi(z) - \phi_{0,1+\sigma_0^2}(z)\right) \\
=&\frac{\pi_1}{\pi_0}\left(\frac{1+\sigma_0^2}{1+\sigma_1^2}\right)^{1/2}\exp\left(\frac{1}{2}\left(1 + \frac{1}{1+\sigma_0^2} - \frac{1}{1+\sigma_1^2}\right)z^2\right)\left(\exp\left(-\frac{z^2}{2}\right) - (1+\sigma^2)^{-1/2}\exp\left(-\frac{z^2}{2(1+\sigma_0^2)}\right)\right) \\
\approx&\frac{\pi_1}{\pi_0}\left(\frac{1+\sigma_0^2}{1+\sigma_1^2}\right)^{1/2}\left(\frac{3}{2} + \frac{1}{2}\left(\frac{1}{1+\sigma_0^2} - \frac{1}{1+\sigma_1^2}\right)z^2\right)\left(1 - (1+\sigma_0^2)^{-1/2} + \frac{1}{2}\left((1+\sigma_0^2)^{-3/2} - 1\right)z^2\right) \\
\approx&\frac{3}{2}\frac{\pi_1}{\pi_0}\left(\frac{1+\sigma_0^2}{1+\sigma_1^2}\right)^{1/2}\left(1 - (1+\sigma_0^2)^{-1/2}\right) \\
=&\frac{3}{2}\frac{\pi_1}{\pi_0}\frac{(1+\sigma_0^2)^{1/2} - 1}{(1+\sigma_1^2)^{1/2}} \\
\approx&\frac{3}{2}\frac{\pi_1}{\pi_0}\frac{\sigma_0^2}{2}(1+\sigma_1^2)^{-1/2} \\
=&\frac{3}{4}\frac{\pi_1}{\pi_0}\frac{\sigma_0^2}{\sqrt{1+\sigma_1^2}},
\end{aligned} \tag{9}$$

where we have omitted terms of order $O(z^2), O(\sigma_0^4)$. (Approxmations: $\log(1+x) \approx 1+x$, $\exp(cx) \approx 1+cx$, $(1+x)^{1/2} \approx 1 + x/2$, when x is close to 0).

We can further compare $\widehat{lfsr}$ to $lfsr$,

$$\begin{aligned}
&\widehat{lfsr} - lfsr \\
=&\left(\left(1 - e^{\left(-\frac{3}{4}\frac{\pi_1}{\pi_0}\frac{\sigma_0^2}{\sqrt{1+\sigma_1^2}}\right)}\right)\Phi\left(-\sqrt{\frac{\sigma_1^2}{1+\sigma_1^2}}z\right) + e^{\left(-\frac{3}{4}\frac{\pi_1}{\pi_0}\frac{\sigma_0^2}{\sqrt{1+\sigma_1^2}}\right)}\Phi\left(-\sqrt{\frac{\sigma_0^2}{1+\sigma_0^2}}z\right) - 1\right)\tilde{\pi}_0.
\end{aligned} \tag{10}$$

When σ_0^2 is small(and in practice $\frac{\pi_1}{\pi_0}$ is small), we have $\exp\left(-\frac{3}{4}\frac{\pi_1}{\pi_0}\frac{\sigma_0^2}{\sqrt{1+\sigma_1^2}}\right) \approx 1$ and $\Phi\left(-\sqrt{\frac{\sigma_0^2}{1+\sigma_0^2}}z\right) \approx 0.5$, so

$$\widehat{lfsr} - lfsr \approx -\frac{1}{2}\tilde{\pi}_0. \tag{11}$$

The conclusion is for a small z generated from two group model with point mass, the estimated lfsr underestimates its true lfsr by approxmately $\frac{1}{2}\tilde{\pi}_0$, if fitting a two group model without point mass. The main reason is that the estimated posterior null probability is close to the true one but it's multiplied by 0.5.

For a large z, we have

$$\begin{aligned}
&\log\left(\frac{\tilde{\pi}_0}{\hat{\tilde{\pi}}_0}\right) \\
=&\log\left(1+\frac{\pi_1\phi_{0,1+\sigma_1^2}(z)}{\pi_0\phi_{0,1+\sigma_0^2}(z)}\right) - \log\left(1+\frac{\pi_1\phi_{0,1+\sigma_1^2}(z)}{\pi_0\phi(z)}\right) \\
=&\log\left(1+\frac{\pi_1}{\pi_0}\left(\frac{1+\sigma_0^2}{1+\sigma_1^2}\right)^{\frac{1}{2}}\exp\left(\frac{\sigma_1^2-\sigma_0^2}{2(1+\sigma_0^2)(1+\sigma_1^2)}z^2\right)\right) - \log\left(1+\frac{\pi_1}{\pi_0}\left(\frac{1}{1+\sigma_1^2}\right)^{\frac{1}{2}}\exp\left(\frac{\sigma_1^2}{2(1+\sigma_1^2)}z^2\right)\right) \\
=&\frac{\sigma_1^2-\sigma_0^2}{2(1+\sigma_0^2)(1+\sigma_1^2)}z^2 + \log\left(\exp\left(-\frac{\sigma_1^2-\sigma_0^2}{2(1+\sigma_0^2)(1+\sigma_1^2)}z^2\right)+\frac{\pi_1}{\pi_0}\left(\frac{1+\sigma_0^2}{1+\sigma_1^2}\right)^{\frac{1}{2}}\right) \\
&-\frac{\sigma_1^2}{2(1+\sigma_1^2)}z^2 - \log\left(\exp\left(-\frac{\sigma_1^2}{2(1+\sigma_1^2)}z^2\right)+\frac{\pi_1}{\pi_0}\left(\frac{1}{1+\sigma_1^2}\right)^{\frac{1}{2}}\right) \\
\approx&-\frac{\sigma_0^2}{2(1+\sigma_0^2)}z^2 + \frac{1}{2}\log(1+\sigma_0^2) + \frac{\pi_0}{\pi_1}(1+\sigma_1^2)^{\frac{1}{2}}\exp\left(-\frac{1}{2}\frac{\sigma_1^2}{1+\sigma_1^2}z^2\right)\left((1+\sigma_0^2)^{-\frac{1}{2}}\exp(\sigma_0^2z^2/2)-1\right) \\
\approx&\frac{\sigma_0^2}{2}(1-z^2)\left(1-\frac{\pi_0}{\pi_1}\sqrt{1+\sigma_1^2}\exp\left(-\frac{1}{2}\frac{\sigma_1^2}{1+\sigma_1^2}z^2\right)\right)
\end{aligned} \tag{12}$$

Note that for really large z and small σ_0^2, $\frac{\sigma_0^2}{2}(1-z^2)$ is already a good approximation. For example, $z=4$ and $\sigma_0^2=0.01$.

For large z, $\hat{\tilde{\pi}}_0$ overestiamte the true $\tilde{\pi}_0$. But for large z, the true $\tilde{\pi}_0$ is very close to 0, so the $\widehat{lfsr}$ does not differ much from $lfsr$.