1) Gamma : $f_X(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}(x>0)$, $\alpha, \beta > 0$

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{\mathbb{R}} e^{tx} f_X(x)\, dx$$

$$= \int_0^{+\infty} e^{tx} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx$$

$u = (\beta - t)x$

$du = (\beta - t)\, dx$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} e^{-(\beta-t)x}\, dx$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \frac{1}{\beta-t} \int_0^{+\infty} \left(\frac{u}{\beta-t} \right)^{\alpha-1} e^{-u}\, du$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \frac{1}{(\beta-t)^{\alpha}} \int_0^{+\infty} u^{\alpha-1} e^{-u}\, du$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \frac{1}{(\beta-t)^{\alpha}} \cdot \Gamma(\alpha)$$

$$= \left(\frac{\beta}{\beta-t} \right)^{\alpha} , \quad t \neq \beta$$

$$M_X'(t) = \alpha \left(\frac{\beta}{\beta-t} \right)^{\alpha-1} \left(\frac{\beta}{(\beta-t)^2} \right)$$

$$= \frac{\alpha}{\beta-t} \left(\frac{\beta}{\beta-t} \right)^{\alpha}$$

$$\mathbb{E}(X) = M_X'(0) = \frac{\alpha}{\beta}$$

$$M_X''(t) = \frac{\alpha}{(\beta-t)^2} \left(\frac{\beta}{\beta-t} \right)^{\alpha} + \frac{\alpha^2}{\beta-t} \left(\frac{\beta}{\beta-t} \right)^{\alpha-1} \cdot \frac{\beta}{(\beta-t)^2}$$

$$= \alpha \left(\beta \right)^{\alpha} \left(\alpha \right)^2 \left(\beta \right)^{\alpha}$$

$$\frac{\quad}{(\beta-t)^2}\left(\frac{\quad}{\beta-t}\right)+\left(\frac{\alpha}{\beta-t}\right)\left(\frac{\quad}{\beta-t}\right)$$

$$\mathbb{E}(X^2) = M_X''(0) = \frac{\alpha}{\beta^2}+\frac{\alpha^2}{\beta^2}$$

$$Var(X) = \mathbb{E}(X^2)-\mathbb{E}(X)^2 = \frac{\alpha}{\beta^2}$$

2) Beta : $f_X(x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} x^{a-1}(1-x)^{b-1}\mathbb{1}(0<x<1)$ $a,b>0$

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{\mathbb{R}} e^{tx} f_X(x)\,dx$$

$$= \int_0^1 e^{tx}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{1}{B(a,b)}\int_0^1 \sum_{k\geq 0}\frac{(tx)^k}{k!} x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{1}{B(a,b)}\sum_{k\geq 0}\frac{t^k}{k!}\int_0^1 x^{k+a-1}(1-x)^{b-1}\,dx$$

$$= \frac{1}{B(a,b)}\sum_{k\geq 0}\frac{t^k}{k!} B(a+k,b)$$

$$= \sum_{k\geq 0}\frac{t^k}{k!}\frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a+b+k)}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$= \sum_{k\geq 0}\frac{t^k}{k!}\frac{(a+k-1)\cdots(a+1)}{(a+b+k-1)\cdots(a+b+1)}$$

$$= \sum_{k\geq 0}\frac{t^k}{k!}\left(\prod_{j=1}^{k-1}\frac{a+j}{a+b+j}\right)$$

$$\mathbb{E}(X) = \int_0^1 \frac{x}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{1}{\quad}\int^1 x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$B(\alpha,\beta) \quad {}_0$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{\alpha!\,(\alpha+\beta-1)!}{(\alpha+\beta)!\,(\alpha-1)!}$$

$$= \frac{\alpha}{\alpha+\beta}$$

$$\mathbb{E}(X^2) = \int_0^1 \frac{x^2}{B(\alpha,\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\, dx$$

$$= \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{(\alpha+1)!\,(\alpha+\beta-1)!}{(\alpha+\beta+1)!\,(\alpha-1)!}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\mathrm{Var}(X) = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$= \frac{\alpha^3+\alpha^2\beta+\alpha^2+\alpha\beta-\alpha^3-\alpha^2\beta-\alpha^2}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

* Normal : $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\mu \in \mathbb{R}$ $\sigma > 0$

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{\mathbb{R}} e^{tx} f_X(x)\, dx$$
$$= \int_{\mathbb{R}} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(\mu^2 - 2x\mu + x^2)} dx$$
$$= \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} \exp\left(\frac{\mu^2 - 2x(\mu + t\sigma^2) + x^2}{-2\sigma^2}\right) dx$$

$$(x - (\mu + t\sigma^2))^2 = x^2 - 2x(\mu + t\sigma^2) + (\mu + t\sigma^2)^2$$
$$= x^2 - 2x(\mu + t\sigma^2) + \mu^2 + 2\mu t\sigma^2 + t^2\sigma^4$$

$$= \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} \exp\left(\frac{(x - (\mu - t\sigma^2))^2}{-2\sigma^2}\right) e^{\frac{-2\mu t\sigma^2 - t^2\sigma^4}{-2\sigma^2}} dx$$

$$= \exp\left(\frac{t^2\sigma^2}{2} + \mu t\right)$$

$$\mathbb{E}(X) = (t\sigma^2 + \mu) e^{\frac{t^2+\sigma^2}{2} + \mu t} \Big|_{t=0} = \mu$$

$$\mathbb{E}(X^2) = \sigma^2 e^{\frac{t^2\sigma^2}{2} + \mu t} + (t\sigma^2 + \mu)^2 e^{\frac{t^2\sigma^2}{2} + \mu t} = \sigma^2 + \mu^2$$

$$Var(X) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

* Exponential : $f_X(x) = \lambda e^{-\lambda x} \mathbb{1}(x > 0)$, $\lambda > 0$

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{\mathbb{R}} e^{tx} f_X(x)\, dx$$

$$= \int_0^{+\infty} \lambda e^{-(\lambda-t)x} \, dx \qquad u = (\lambda - t)x$$

$$du = (\lambda - t)\, dx$$

$$= \frac{\lambda}{\lambda - t} \int_0^{+\infty} e^{-u} \, du$$

$$= \frac{\lambda}{\lambda - t} \quad ; \quad \lambda \neq t$$

$$\mathbb{E}(X) = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{1}{\lambda}$$

$$\mathbb{E}(X^2) = \frac{2\lambda(\lambda-t)}{(\lambda-t)^4} = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{1}{\lambda^2}$$

* Logistic: $f_X(x) = \dfrac{e^{-\frac{(x-\mu)}{\sigma}}}{\sigma\left(1+e^{-\frac{(x-\mu)}{\sigma}}\right)^2}$, $\mu \in \mathbb{R}$ $\sigma > 0$

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{\mathbb{R}} e^{tx} f_X(x) \, dx$$

$$= \int_{\mathbb{R}} \frac{e^{tx} \, e^{-\frac{(x-\mu)}{\sigma}}}{\sigma\left(1+e^{-(x-\mu)/\sigma}\right)^2} \, dx \qquad u = \frac{1}{1+e^{-\left(\frac{x-\mu}{\sigma}\right)}}$$

$$du = \frac{(1/\sigma)\, e^{-\left(\frac{x-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right)^2} dx$$

$$= \int_0^1 \exp\left(t\mu - t\sigma \log\left(\frac{1-u}{u}\right)\right) du$$

$$= \int_0^1 \exp\left(t\mu + \log\left(\frac{1-u}{u}\right)^{-t\sigma}\right) du$$

$$= e^{t\mu} \int_0^1 \left(\frac{1-u}{u}\right)^{-t\sigma} du$$

$$= e^{t\mu} \int_0^1 u^{t\sigma} (1-u)^{-t\sigma} \, du$$

$$= e^{t\mu} \, \mathcal{B}(t\sigma + 1, 1 - t\sigma)$$

$$\mathbb{E}(X) = \int_{\mathbb{R}} x \frac{e^{-\left(\frac{x-\mu}{2}\right)}}{\sigma\left(1+e^{-(x-\mu)/\sigma}\right)^2}\, dx$$

$$= \int_0^1 \mu - \sigma \log\left(\frac{1-u}{u}\right) du$$

$$= \mu u \Big|_0^1 - \sigma \int_0^1 \log\left(\frac{1-u}{u}\right) du$$

$$a = \log\left(1-\frac{1}{u}\right) \to a' = \frac{1}{u(u-1)}$$

$$b' = 1 \longrightarrow b = u$$

$$= \mu - \sigma \int_0^1 \log\left(1-\frac{1}{u}\right) du$$

$$= \mu - \sigma \left(u \log\left(1-\frac{1}{u}\right) \Big|_0^1 - \int_0^1 \frac{1}{u-1}\, du \right)$$

$$= \mu - \sigma \left(0 - \lim_{u\to 0} u \log\left(1-\frac{1}{u}\right) - \log|u-1| \Big|_0^1 \right)$$

* $-1 \leq \rho(X,Y) \leq 1$

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big)}{\sqrt{\mathbb{E}((X-\mathbb{E}(X))^2)\,\mathbb{E}((Y-\mathbb{E}(Y))^2}}$$

$$\leq \frac{\sqrt{\mathbb{E}((X-\mathbb{E}(X)^2)\,\mathbb{E}((Y-\mathbb{E}(Y)^2)}}{\sqrt{\mathbb{E}((X-\mathbb{E}(X)^2)\,\mathbb{E}((Y-\mathbb{E}(Y)^2)}}$$

$$= 1$$

* A bivariate normal variable (X,Y) is said to have a bivariate normal distribution if the pdf of (X,Y) is of the following form :

$$f(x,y) = \frac{1}{2\pi\,\sigma_1\,\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu}{\sigma_1}\right)\left(\frac{y-\mu_2}{\ }\right) + \left(\frac{y-\mu_2}{\ }\right)^2\right\}} \quad ; (x,y) \in \mathbb{R}^2$$

$$\left(\frac{\quad}{\sigma_2}\right) + \left(\frac{\quad}{\sigma_2}\right)\Bigg\}, \ (x,y) \in \mathbb{R}$$

Show that :
1) $X \sim N(\mu_1, \sigma_1^2)$
2) $Y \sim N(\mu_2, \sigma_2^2)$
3) $X \mid Y = y \sim N(\quad , \quad)$

1) $$f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\, dy$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-\mu}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)\right\} dy$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma_1^2} + \frac{-1}{2(1-\rho^2)}\left(\frac{\rho^2}{\sigma_1^2}(x-\mu_1)^2 - \frac{2\rho}{\sigma_1\sigma_2}(x-\mu_1)(y-\mu_2) + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right\} dy$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma_1^2} + \frac{-1}{2(1-\rho^2)}\left(\frac{\rho}{\sigma_1}(x-\mu_1) - \frac{(y-\mu_2)}{\sigma_2}\right)^2\right\} dy$$

$$= \frac{\exp\left\{\frac{-(x-\mu)^2}{2\sigma_1^2}\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{\mathbb{R}} \exp\left\{\frac{-1}{2\sigma_2^2(1-\rho^2)}\left(y-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right)^2\right\} dy$$

Let $\mu' = \mu^2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$ and $\sigma' = \sigma_2\sqrt{1-\rho^2}$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{\frac{-(x-\mu_1)^2}{2\sigma_1^2}\right\} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_2^2(1-\rho^2)^2}} \exp\left\{\frac{-1}{2\sigma'^2}(y-\mu')^2\right\} dy}_{=1}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}$$

2) $$f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\, dx$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)\right\} dx$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-(y-\mu_2)^2}{2\sigma_2^2} - \frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{\rho^2(y-\mu_2)^2}{\sigma_2^2}\right)\right\} dx$$

$$= \frac{\exp\left\{-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}}{2\pi\sigma_2\sigma_1\sqrt{1-\rho^2}} \int_{\mathbb{R}} \exp\left\{\frac{-1}{2\sigma_1^2(1-\rho^2)}\left(x-\mu_1-\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)^2\right\} dx$$

let $\mu' = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)^2$ and $\sigma' = \sigma_1\sqrt{1-\rho^2}$

$$= \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{\frac{-(y-\mu_2)^2}{2\sigma_2^2}\right\} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma'^2}} \exp\left(\frac{-(x-\mu')^2}{2\sigma'^2}\right) dx}_{=1}$$

$$= \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{\frac{-(y-\mu_2)^2}{2\sigma_2^2}\right\}$$

3) $f_{X|Y=y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$

$$= \frac{\sqrt{2\pi\sigma_2^2}\exp\left\{\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\exp\left\{-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2(1-\rho^2)}} \exp\left\{\frac{-1}{2(1-\rho^2)\sigma_1^2}\left((x-\mu_1)^2 - 2\rho\frac{\sigma_1}{\sigma_2}(x-\mu_1)(y-\mu_2) + \frac{\sigma_1^2}{\sigma_2^2}(y-\mu_2)^2 - \frac{\sigma_1}{\sigma_2}(1-\rho^2)(y-\mu_2)^2\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2(1-\rho^2)}} \exp\left\{\frac{1}{2(1-\rho^2)\sigma_1^2}\left((x-\mu_1)^2 - 2\rho\frac{\sigma_1}{\sigma_2}(x-\mu_1)(y-\mu_2) + \rho^2\frac{\sigma_1^2}{\sigma_2^2}(y-\mu_2)^2\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2(1-\rho^2)}} \exp\left\{\frac{1}{2(1-\rho^2)\sigma_1^2}\left(x-\mu_1-\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)^2\right\}$$

$$\sim N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \sigma_1^2(1-\rho^2)\right)$$

* Suppose $\forall 1 \leq i \leq n$ $X_i \sim N(0,1)$. Show that $X_i^2 \sim \chi_1^2$

$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and let $Y_i = X_i^2$

$$F_{Y_i}(y) = \mathbb{P}(Y_i \leq y)$$
$$= \mathbb{P}(X_i^2 \leq y)$$
$$= \mathbb{P}(-\sqrt{y} \leq X_i \leq \sqrt{y})$$

$= F_{X_i}(\sqrt{y}) - F_{X_i}(-\sqrt{y})$

$$f_{Y_i}(y) = \frac{1}{2\sqrt{y}} f_{X_i}(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_{X_i}(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}} \left(\frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \right) \qquad \Gamma(1/2) = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2} \sim \chi^2_1$$

$$= \frac{(1/2)^{1/2}}{\Gamma(1/2)} (y)^{1/2 - 1} e^{-y/2} \sim \text{Gamma}(1/2, 1/2)$$

Suppose $Z \sim \chi^2_n$

$$f_Z(z) = \frac{z^{n/2 - 1} e^{-z/2}}{2^{n/2} \Gamma(n/2)} = \frac{(1/2)^{n/2}}{\Gamma(n/2)} z^{n/2 - 1} e^{-z/2} \sim \text{Gamma}(n/2, 1/2)$$

Now, we want to find the distribution of $Z = \sum_{i=1}^n X_i^2$

$$M_Z(t) = \mathbb{E}(e^{tZ})$$

$$= \mathbb{E}(e^{t(X_1^2 + \dots + X_n^2)})$$

$$= \mathbb{E}(e^{tX_1^2}) \cdot \mathbb{E}(e^{tX_2^2}) \dots \mathbb{E}(e^{tX_n^2}) \quad \text{as } X_i\text{'s are independent}$$

$$= \prod_{i=1}^n \mathbb{E}(e^{tX_i^2})$$

$$= (\mathbb{E}(e^{tX_1}))^n \quad \text{as } X_i\text{'s are iid}$$

$$\mathbb{E}(e^{tX_1}) = \int_{\mathbb{R}} e^{tx} \cdot f_X(x)\, dx$$

$$= \int_0^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi x}} e^{-x/2}\, dx$$

$$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} (x)^{-1/2} e^{-x/2(1-2t)}\, dx \qquad u = \frac{x(1-2t)}{2}, \quad du = \frac{1-2t}{2} dx$$

$$= \int_0^{+\infty} \sqrt{\frac{2}{\dots}} \left(\frac{2u}{\dots} \right)^{-1/2} e^{-u} \frac{1}{\dots} du$$

$$\int_0 \sqrt{\pi} \quad \left(1-2t \right) \qquad (1-2t)$$

$$= \frac{1}{\sqrt{1-2t}} \int_0^{+\infty} \frac{1}{\Gamma(1/2)} u^{1-1/2} e^{-u} \, du$$

$$= \frac{1}{\sqrt{1-2t}} \cdot \frac{1}{\Gamma(1/2)} \cdot \Gamma(1/2)$$

$$= \frac{1}{\sqrt{1-2t}} \, , \quad t < \frac{1}{2}$$

$$M_Z(t) = (1-2t)^{-n/2}$$

$$\mathbb{E}\left(\sum_{i=1}^{n} x_i^2\right) = M_Z'(t)\Big|_{t=0}$$

$$= \frac{-n}{2} (1-2t)^{-n/2-1} (-2) \Big|_{t=0}$$

$$= n$$

$$\mathbb{E}\left(\left(\sum_{i=1}^{n} x_i^2\right)^2\right) = M_Z''(t)\Big|_{t=0}$$

$$= n\left(-\frac{n}{2} - 1\right) (1-2t)^{-n/2-2} (-2) \Big|_{t=0}$$

$$= n^2 + 2n$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} x_i^2\right) = \mathbb{E}\left(\left(\sum_{i=1}^{n} x_i\right)^2\right) - \mathbb{E}\left(\sum_{i=1}^{n} x_i^2\right)^2$$

$$= n^2 + 2n - n^2$$

$$= 2n$$

* Alternatively, $M_{X_1^2}(t) = \mathbb{E}(e^{tX_1^2})$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{tx^2} \cdot e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2(1-2t)}\, dx, \quad t < \frac{1}{2}$$

$$= \frac{1}{\sqrt{1-2t}} \int_{\mathbb{R}} \sqrt{\frac{1-2t}{2\pi}}\, e^{-x^2/2(1-2t)}\, dx$$

$$= (1-2t)^{-1/2}$$

$$M_{X_1^2+\dots+X_n^2}(t) = (1-2t)^{-n/2}$$

$$M_Y(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha}, \quad Y \sim \text{Gamma}(\alpha, \beta)$$

choose $Y \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$ $M_Y(t) = (1-2t)^{-n/2}$

* Sampling distribution:

1) Chi-squared : $X \sim N(0,1)$, $Y = X^2 \sim \chi^2_1$
2) Student-t : $Y \sim \chi^2_n$, $Z = \frac{X}{\sqrt{Y/n}} \sim t_n$
3) F-distribution: $X \sim \chi^2_m$, $Z = \frac{X/m}{Y/n} \sim F_{m,n}$

- F-distribution is for multiple populations, whereas the rest are for a single population.

- F - distribution is the ratio of two chi-squared so it compares two samples.

* Degrees of Freedom : S^2 has $n-1$ degrees of freedom since there are n variables & a single restriction $\sum_{i=1}^{n}(X_i - \bar{X}) = 0$

\# of quantities involved in the expression − # of linear restrictions

* All sampling distributions are probability distributions but only standard normal is both a sampling distribution and probability distribution.

* Median :

1) Exponential : $\frac{1}{2} = F_X(m) = 1 - e^{-\lambda m} \Leftrightarrow$

$$e^{-\lambda m} = \frac{1}{2} \Leftrightarrow$$

$$-\lambda m = -\ln(2) \Leftrightarrow$$

$$m = \frac{\ln(2)}{\lambda}$$

2) Binomial : $\mathbb{P}(X \leq m) = \frac{1}{2}$ & $\mathbb{P}(X \geq m) = \frac{1}{2}$

$$\frac{1}{2} = \sum_{n=0}^{n} \binom{n}{n} p^n (1-p)^n$$

3) Normal : Mean = Median = μ

4) Poisson : $\frac{1}{2} = \sum_{n \geq 0} \frac{e^{-\lambda} \lambda^n}{n!}$

* **Mode** :

1) Exponential : $Mode(X) = \max(f_X(n)) = 0$
because $f(0) = \lambda > 0$ & $0 < e^{-\lambda n} < 1$
so $\lambda e^{-\lambda n} \leq \lambda$ so zero is the most occuring.

2) Normal : $Mode(X) = Mean(X) = \mu$

* **Examples** :

1) Binomial : # of heads we get when tossing a coin n times.

2) Poisson : # of SUAD students arriving to the cafeteria in an hour where only a single student can enter at a time.

3) Normal : Average height of SUAD students.

4) Exponential : The amount of time until a natural disaster occurs.

disaster occurs.

5) Gamma : The amount of time a product will last.