* By CLT, we know that $\bar{x}$ converges to normal but what about $\max(x_i)$, $\min(x_i)$, range, median, ... ?

* Consider a random sample $X_1, \dots, X_n$ from a population iid with CDF F. We can order the observations;

$$\min(X_i) = X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)} = \max(X_i) \quad \text{discrete}$$

$$X_{(1)} < X_{(2)} < \dots < X_{(n)} \quad \text{continuous}$$

Notation : $X_{(r)}$ is the r^{th} order statistic of the sample

1) Distribution of $X_{(1)}$:

$$\begin{aligned}
F_{X_{(1)}}(x) &= \mathbb{P}(X_{(1)} \leq x) \\
&= 1 - \mathbb{P}(X_{(1)} \geq x) \\
&= 1 - \mathbb{P}(X_1 \geq x, \dots, X_n \geq x) \\
&= 1 - (\mathbb{P}(X_1 \geq x))^n \\
&= 1 - (F(x))^n
\end{aligned}$$

$$f_{X(1)}(x) = n\,(F(x))^{n-1} \cdot f(x)$$

2) Distribution of $X_{(n)}$:

$$\begin{aligned}
F_{(X_n)}(x) &= \mathbb{P}(X_{(n)} \leq x) \\
&= \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) \\
&= (1 - F(x))^n
\end{aligned}$$

$$f_{X_n}(x) = n\,(1 - F(x))^{n-1} \cdot f(x)$$

3) Distribution of the general case $X_{(r)}$:

$$
\begin{aligned}
F_{X_{(r)}}(x) &= \mathbb{P}(X_{(r)} \leq x) \\
&= \mathbb{P}(\text{at least } r \text{ of the } n \text{ samples are} \leq x) \\
&= \sum_{k=r}^{n} \binom{n}{k} (F(x))^k (1-F(x))^{n-k} \\
&= 1 - \sum_{k=0}^{r-1} \binom{n}{k} F(x)^k (1-F(x))^{n-k} \\
&= 1 - \frac{1}{\beta(n-r+1, r)} \int_0^{1-F(x)} z^{n-r} (1-z)^{r-1} \, dz
\end{aligned}
$$

$$
f_{X_{(r)}}(y) = \frac{n!}{(r-1)!\,(n-r)!} (F(x))^{r-1} (1-F(x))^{n-r} f(x)
$$

Suppose $X_1, \dots, X_n \overset{iid}{\sim} f_\theta(x)$, $f_\theta(x_1, \dots, x_n) = \prod_{i=1}^{n} f_\theta(x_i)$
where $f_\theta(x_1, \dots, x_n)$ is the joint pdf.

Let $y_i = X_{(i)}$, $g(y_1, \dots, y_n) = n!\, f_\theta(x_1, \dots, x_n)$; $x_1 < \dots < x_n$

4) Joint Distribution of $X_{(1)}$ and $X_{(n)}$:

$$
\begin{aligned}
F_{X_{(1)}, X_{(n)}}(x, y) &= \mathbb{P}(X_{(1)} \leq x, X_{(n)} \leq y) \\
&= \mathbb{P}(X_{(n)} \leq y) - \mathbb{P}(X_{(1)} > x, X_{(n)} < y) \\
&= \mathbb{P}(X_1, \dots, X_n \leq y) - \mathbb{P}(x < X_1, \dots, X_n < y) \\
&= [F(y)]^n - [F(y) - F(x)]^n
\end{aligned}
$$

$$
\begin{aligned}
f_{X_{(1)}, X_{(n)}}(x, y) &= \frac{\partial^2}{\partial x \partial y} F_{X_{(1)}, X_{(n)}}(x, y) \\
&= \frac{\partial}{\partial x} \left(n [F(y)]^{n-1} \cdot f(y) - n [F(y) - F(x)]^{n-1} f(y) \right) \\
&= n(n-1) [F(y) - F(x)]^{n-2} f(y) f(x)
\end{aligned}
$$

5) Distribution of Range :

$R = X_{(n)} - X_{(1)}$,

let $(x, y) = (X_1, X_{(n)}) \longrightarrow (x, r) = (x, y-x)$

$$|J| = \begin{array}{c|c} & \begin{matrix} x & y \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array} \qquad (x, y) \longrightarrow (x, \underbrace{y-x}_{r})$$

so $|J| = 1$

$$f_{X_{(1)}, X_{(n)}}(x, y) = n(n-1)\left[F(y) - F(x)\right]^{n-2} f(y) f(x)$$

$$f_{X_{(1)}, R}(x, y) = n(n-1)\left[F(r+x) - F(x)\right]^{n-2} f(r+x) f(x)$$

6) Distribution of Median :

let $(X_{(1)}, X_{(n)}) \longrightarrow \left(X_{(1)}, \frac{X_{(1)} + X_{(n)}}{2}\right)$ $\quad (x, y) \longmapsto (x, \frac{x+y}{2})$

$$J = \begin{array}{c|c} & \begin{matrix} x & y \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \end{array} \Rightarrow |J| = 1/2$$

$$f_{X_{(1)}, X_{(n)}}(x, y) = n(n-1)\left[F(y) - F(x)\right]^{n-2} f(x) f(y)$$

$$f_{X_{(1)}, M}(x, m) = \frac{n(n-1)}{2}\left[F(2m-x) - F(x)\right]^{n-2} f(x) f(2m-x)$$

* Problems:

1) $X \sim U[0, \theta]$

$$f_{X_{(1)}}(x) = \frac{n}{\theta}\left(1 - \frac{x}{\theta}\right)^{n-1} \mathbb{1}(0 \leq x \leq \theta) \qquad f_X(x) = \frac{\mathbb{1}(0 \leq x \leq \theta)}{\theta}$$

$$f_{X_{(n)}}(x) = \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1} \mathbb{1}(0 \leq x \leq \theta)$$

2) $X \sim Exp(\lambda)$

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = \lambda\left[\frac{e^{-\lambda t}}{-\lambda}\right]_0^x = 1 - e^{-\lambda x}$$

$$f_{X_{(1)}}(x) = n\left[1 - (1 - e^{-\lambda x})\right]^{n-1} (\lambda e^{-\lambda x}) \mathbb{1}(x > 0)$$
$$= \lambda n e^{-\lambda n x} \quad \mathbb{1}(x > 0)$$

$$f_{X_{(n)}}(x) = n\left[1 - e^{-\lambda x}\right]^{n-1} (\lambda e^{-\lambda x}) \mathbb{1}(x > 0)$$

3) $X \sim Unif[0, 1]$

$$f_X(x) = \mathbb{1}(0 \leq x \leq 1)$$
$$F_X(x) = x$$

$$f_{X_1, R}(x, r) = n(n-1) r^{n-2} \mathbb{1}(0 \leq x \leq 1) \mathbb{1}(0 \leq r + x \leq 1)$$
$$= n(n-1) r^{n-2} \mathbb{1}(0 \leq x \leq 1 - r) \mathbb{1}(0 \leq r \leq 1)$$

$$f_R(r) = n(n-1) r^{n-2} (1 - r) \mathbb{1}(0 \leq r \leq 1)$$

$$f_{X, M}(x, m) = \frac{n(n-1)}{2} [2m - 2x]^{n-2} \mathbb{1}(0 \leq x \leq 1) \mathbb{1}(0 \leq 2m - x \leq 1)$$

$$= 2^{n-1} n(n-1) (m - x)^{n-2} \quad \mathbb{1}(2m - 1 \leq x \leq 2m) \mathbb{1}(0 \leq m \leq \tfrac{1}{2})$$

$$f_M(m) = \int^{2m} (m - x)^{n-2} dx \; (2^{n-1} \cdot n(n-1) \mathbb{1}(0 \leq m \leq \tfrac{1}{2}))$$

$$\int_{2m-1}$$

$$= \int_{-m}^{1-m} u^{n-2}\, du \left(2^{n-1} \cdot n(n-1)\, \mathbb{1}\left(0 \leq m \leq \tfrac{1}{2}\right)\right)$$

$$= 2^{n-1}\, n \cdot \left((1-m)^{n-2} - (-m)^{n-2}\right) \mathbb{1}\left(0 \leq m \leq \tfrac{1}{2}\right)$$