Solving Recurrences: The master theorem

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Adapted from lectures of

Prof. Charles Leiserson, MIT &

CLRC textbook 3rd ed ch4, pages 88 ~ 97

Outline

- Solving Recurrences
 - The master method

The master method

The master method applies to recurrences of the form

$$T(n) = a T(n/b) + f(n) ,$$

where $a \ge 1$, b > 1, and f is asymptotically positive.

$$T(n) = a T(n/b) + f(n) ,$$

- Which term has the high rate of growth?
- There are three possibilities!
 - Case I: Rate of growth of term 1 is faster
 - Case II: Rate of 1 and 2 is the same
 - Case III: Rate of growth of term 2 is faster

Term 1 is determined by calculating

$$n^{\log_{b}a}$$

Case "I"

Compare f(n) with $n^{\log_b a}$:

- 1. $f(n) = O(n^{\log_b a \varepsilon})$ for some constant $\varepsilon > 0$.
 - f(n) grows polynomially slower than $n^{\log b^a}$ (by an n^{ϵ} factor).

Solution: $T(n) = \Theta(n^{\log_b a})$.

Ex.
$$T(n) = 4T(n/2) + n$$

 $a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n.$
Case 1: $f(n) = O(n^{2-\epsilon})$ for $\epsilon = 1$.
 $\therefore T(n) = \Theta(n^2).$

Case II

Compare f(n) with $n^{\log_b a}$:

- 2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \ge 0$.
 - f(n) and $n^{\log ba}$ grow at similar rates.

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Solution: T(n) = \Theta(n^{\log_b a} \lg^{k+1} n).
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Ex.
$$T(n) = 4T(n/2) + n^2$$

 $a = 4, b = 2 \Rightarrow n^{\log ba} = n^2; f(n) = n^2.$
Case 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.
 $T(n) = \Theta(n^2 \lg n)$.

Case III

Compare f(n) with $n^{\log_b a}$:

- 3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
 - f(n) grows polynomially faster than $n^{\log b^a}$ (by an n^{ϵ} factor),

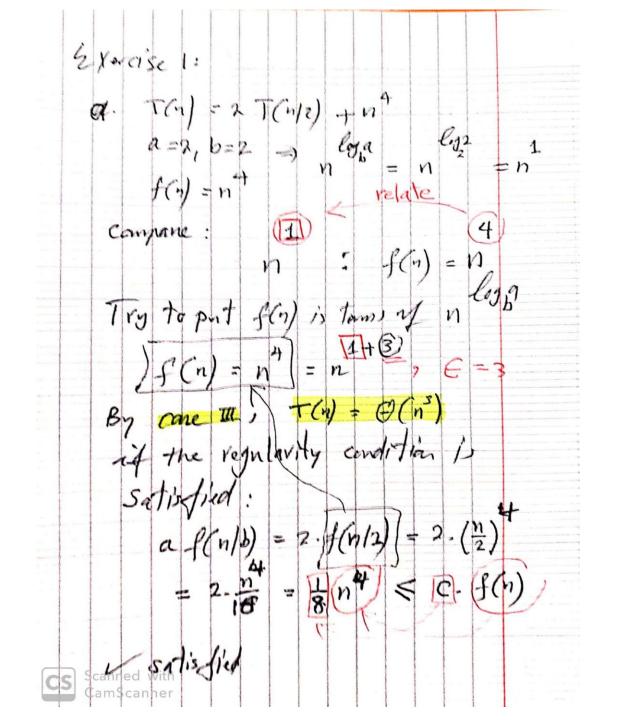
and f(n) satisfies the regularity condition that $af(n/b) \le cf(n)$ for some constant c < 1.

Solution: $T(n) = \Theta(f(n))$.

Ex. $T(n) = 4T(n/2) + n^3$ $a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3.$ Case 3: $f(n) = \Omega(n^{2+\epsilon})$ for $\epsilon = 1$ and $4(n/2)^3 \le cn^3$ (reg. cond.) for c = 1/2. $\therefore T(n) = \Theta(n^3).$

Not Applicable!

Ex. $T(n) = 4T(n/2) + n^2/\lg n$ $a = 4, b = 2 \Rightarrow n^{\log b^a} = n^2; f(n) = n^2/\lg n.$ Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.



b.
$$T(n) = T(7n/10) + n$$

$$= 1 \cdot T(n/(1017)) + n$$
So, $a = 1$, $b = 10/7$, $f(n) = n$
Calculate $n = n$
Compare $n = n$
Compare $n = n$

$$= n = n$$
Compare $n = n$

$$= n = n$$
Compare $n = n$

$$= n = n$$

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Compare $n = n$

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Compare $n = n$

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