Solving Recurrences: The master theorem

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Adapted from lectures of

Prof. Charles Leiserson , MIT &

CLRC textbook 3rd ed ch4, pages 88 ~ 97

Outline

- Solving Recurrences
  - The master method

The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

$$\overset{}{T(n) = a\, \overset{1}{T(n/b)} + \overset{2}{f(n)}}\ ,$$

- Which term has the high rate of growth?
- There are three possibilities!
  - Case I: Rate of growth of term 1 is faster
  - Case II: Rate of 1 and 2 is the same
  - Case III: Rate of growth of term 2 is faster

- Term 1 is determined by calculating

$$n^{\log_b a}$$

Case "I"

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

Ex. $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2)$.

Case II

Compare $f(n)$ with $n^{\log_b a}$:

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2.$

Case 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore T(n) = \Theta(n^2 \lg n)$.

Case III

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \leq c f(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3).$

Not Applicable!

Ex. $T(n) = 4T(n/2) + n^2/\lg n$
$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2/\lg n$.
Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\lg n)$.

Exercise 1:

a. $T(n) = 2T(n/2) + n^4$

$a=2,\ b=2 \Rightarrow n^{\log_b a} = n^{\log_2 2} = n^1$

$f(n) = n^4$

relate

Compare: $\boxed{1}$ $\quad$ 4

$n \ : \ f(n) = n$

Try to put $f(n)$ in terms of $n^{\log_b a}$

$\boxed{f(n) = n^4} = n^{\boxed{1}+3}$, $\epsilon = 3$

By case III, $T(n) = \Theta(n^3)$

if the regularity condition is satisfied:

$a f(n/b) = 2 \cdot \boxed{f(n/2)} = 2 \cdot \left(\frac{n}{2}\right)^4$

$= 2 \cdot \frac{n^4}{16} = \boxed{\frac{1}{8}} n^4 \leq \boxed{c} \cdot f(n)$

✓ satisfied

b. $T(n) = T(7n/10) + n$

$$= 1 \cdot T(n/(10/7)) + n$$

So, $a = 1$, $b = 10/7$, $f(n) = \boxed{n^1}$

calculate $n^{\log_b a} = n^{\log_{10/7} 1} = \boxed{n^0}$

Compare n^0 , $n^1 = n^{0+1}$, $\epsilon = 1$

check reg. condition (Case III)

$$a \cdot f(n/b) = 1 \cdot f(n/(10/7))$$

$$= 1 \cdot \frac{n}{10/7} = \frac{7}{10} n \leq c \cdot f(n)$$

So, by case III :

$$T(n) = \Theta(f(n))$$

$$= \Theta(n)$$

C. $T(n) = T(n-2) + n^2$

n/b ✗

Master method cannot be applied!

d. $T(n) = 16\,T(n/4) + n^2$

$a = 16,\; b = 4,\; f(n) = n^2$

$n^{\log_b a} = n^{\log_4 16} = n^{2}$

$f(n) = n^{2}$

both with the same rate

So, We think in case II.

$f(n) = n^2 \cdot \log^{k} n \;,\; k = 0$

By case II,

$\therefore T(n) = \Theta(n^2 \log^{k+1} n)$

$= \Theta(n^2 \lg n)$

e- $T(n) = 2T(n/2) + \sqrt{n}$

$a = b = 2$

$n^{\log_b a} = n^{\log_2 2} = n^{1}$ / $f(n) = \sqrt{n} = n^{1/2}$

it is clear that, $f(n) = n^{1/2} = n^{1-\frac{1}{2}}$, $\epsilon = \frac{1}{2}$

By case I,

$$T(n) = \Theta(n^{\log_b a})$$

$$= \Theta(n)$$