Chapter 15 Finite word length effects In practice, the samples of discrete-time signals, trigonometric numbers in Fourier transform computations, and filter coefficients are represented with finite precision (that is, by using a finite number of bits). Furthermore, all computations are performed with finite accuracy. Chapter 15 is devoted to the study of finite precision effects on digital signal processing operations.

Learning summary

- Signals are physical quantities that carry information in their patterns of variation. Continuous-time signals are continuous functions of time, while discrete-time signals are sequences of real numbers. If the values of a sequence are chosen from a finite set of numbers, the sequence is known as a digital signal. Continuous-time, continuous-amplitude signals are also known as analog signals.
- A system is a transformation or operator that maps an input signal to an output signal. If the input and output signals belong to the same class, the system carries the name of the signal class. Thus, we have continuous-time, discrete-time, analog, and digital systems. Systems with input and output signals from different classes are known as interface systems or converters from one signal type to another.
- Signal processing is concerned with the acquisition, representation, manipulation, transformation, and extraction of information from signals. In analog signal processing these operations are implemented using analog electronic circuits. Digital signal processing involves the conversion of analog signals into digital, processing the obtained sequence of finite precision numbers using a digital signal processor or general purpose computer, and, if necessary, converting the resulting sequence back into analog form.

TERMS AND CONCEPTS

Analog representation The physical representation of a continuous-time signal by a voltage or current proportional to its amplitude.

Analog-to-digital converter (ADC) A device used to convert analog signals into digital signals.

Analog signal Continuous-time signals are also called analog signals because their amplitude is "analogous" (that is, proportional) to the physical quantity they represent.

Analog signal processing (ASP) The conversion of analog signals into electrical signals by special transducers or sensors and their processing by analog electrical and electronic circuits.

Analog system See continuous-time system.

Binary code A group of bits (zeros and ones) representing a quantized numerical quantity.

Continuous-time signal A signal whose value $s(t)$ (amplitude) is defined for every value of the independent variable t (time).

Continuous-time system A system which transforms a continuous-time input signal into a continuous-time output signal.

Deterministic signal A signal whose future values can be predicted exactly from past values.

Digital representation The physical representation of a digital signal by a combination of ON/OFF pulses corresponding to the digits of a binary number.

Digital signal A discrete-time signal whose amplitude $s[n]$ takes values from a finite set of real numbers.

Digital signal processing (DSP) The representation of analog signals by sequences of numbers, the processing of these sequences by numerical computation techniques, and the conversion of such sequences into analog signals.

Digital-to-analog converter (DAC) A device used to convert digital signals into analog signals.

Discrete-time signal A signal whose value $s[n]$ is defined only at a discrete set of values of the independent variable n (usually the set of integers).

Discrete-time system A system which transforms a discrete-time input signal into a discrete-time output signal.

Digital system A system which transforms a digital input signal into a digital output signal.

Random signal A signal whose future values cannot be predicted exactly from past values.

Quantization The process of representing the samples of a discrete-time signal using binary numbers with a finite number of bits (that is, with finite accuracy).

Sampling The process of taking instantaneous measurements (samples) of the amplitude of a continuous-time signal at regular intervals of time.

Sampling period The time interval between consecutive samples of a discrete-time signal.

Sampling rate The number of samples per second obtained during periodic sampling.

Signal Any physical quantity that varies as a function of time, space, or any other variable or variables.

Signal processing A discipline concerned with the acquisition, representation, manipulation, and transformation of signals.

System An interconnection of elements and devices for a desired purpose.

FURTHER READING

1. A more detailed introduction to signals and systems can be found in many books, including Oppenheim *et al.* (1997) and Haykin and Van Veen (2003).
2. More advanced and broader treatments of discrete-time signal processing can be found in many textbooks. Oppenheim and Schafer (2010) and Proakis and Manolakis (2007) are closer to the approach followed in this book.
3. A detailed treatment of practical digital signal processors is provided in Kuo and Gan (2005), Kuo *et al.* (2006), and Welch *et al.* (2006).
4. A variety of digital signal processing applications are discussed in the following texts: image processing in Gonzalez and Woods (2008) and Pratt (2007), digital communication in Rice (2009), digital control in Dorf and Bishop (2008), digital audio and video in Zölder (2008) and Fischer (2008), computer music in Moore (1990), and radar in Richards (2005).

Review questions

1. What is a signal and how does it convey information?
2. Describe various different ways a signal can be classified.
3. What is the difference between a mathematical and physical representation of a signal?

2.1 Discrete-time signals

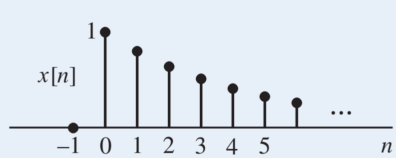
A discrete-time signal $x[n]$ is a sequence of numbers defined for every value of the integer variable n. We will use the notation $x[n]$ to represent the nth sample of the sequence, $\{x[n]\}_{N_1}^{N_2}$ to represent the samples in the range $N_1 \leq n \leq N_2$, and $\{x[n]\}$ to represent the entire sequence. When the meaning is clear from the context, we use $x[n]$ to represent either the nth sample or the entire sequence. A discrete-time signal is *not* defined for noninteger values of n. For example, the value of $x[3/2]$ is not zero, just undefined. In this book, we use the terms *discrete-time signal* and *sequence* interchangeably.

When $x[n]$ is obtained by sampling a continuous-time signal $x(t)$, the interval T between two successive samples is known as the *sampling period* or *sampling interval*. The quantity $F_s = 1/T$, called the *sampling frequency* or *sampling rate*, equals the number of samples per unit of time. If T is measured in seconds, the units of F_s are number of samples per second (sampling rate) or Hz (sampling frequency).

Signal representation There are several ways to represent a discrete-time signal. The more widely used representations are illustrated in Table 2.1 by means of a simple example. Figure 2.1 also shows a pictorial representation of a sampled signal using index n as well as sampling instances $t = nT$. We will use one of the two representations as appropriate in a given situation.

The *duration* or *length* L_x of a discrete-time signal $x[n]$ is the number of samples from the first nonzero sample $x[n_1]$ to the last nonzero sample $x[n_2]$, that is $L_x = n_2 - n_1 + 1$. The range $n_1 \leq n \leq n_2$ is denoted by $[n_1, n_2]$ and it is called the *support* of the sequence.

Table 2.1 Discrete-time signal representations.

Representation	Example
Functional	$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$
Tabular	$\begin{array}{c\|cccccccc} n & \ldots & -2 & -1 & 0 & 1 & 2 & 3 & \ldots \\ \hline x[n] & \ldots & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \ldots \end{array}$
Sequence	$x[n] = \{ \ \ldots \ 0 \ \underset{\uparrow}{1} \ \frac{1}{2} \ \frac{1}{4} \ \frac{1}{8} \ \ldots \ \}$
Pictorial	

[1] The symbol $\uparrow$ denotes the index $n = 0$; it is omitted when the table starts at $n = 0$.

Table 2.2 Summary of discrete-time system properties.

Property	Input		Output
	$x[n]$	$\overset{\mathcal{H}}{\longmapsto}$	$y[n]$
	$x_k[n]$	$\overset{\mathcal{H}}{\longmapsto}$	$y_k[n]$
Linearity	$\sum_k c_k x_k[n]$	$\overset{\mathcal{H}}{\longmapsto}$	$\sum_k c_k y_k[n]$
Time-invariance	$x[n-n_0]$	$\overset{\mathcal{H}}{\longmapsto}$	$y[n-n_0]$
Stability	$\lvert x[n]\rvert \le M_x < \infty$	$\overset{\mathcal{H}}{\longmapsto}$	$\lvert y[n]\rvert \le M_y < \infty$
Causality	$x[n] = 0$ for $n \le n_0$	$\overset{\mathcal{H}}{\longmapsto}$	$y[n] = 0$ for $n \le n_0$

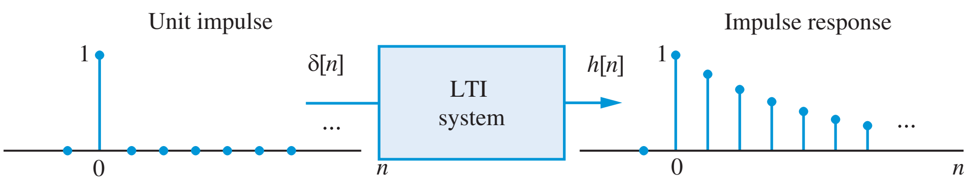


Figure 2.9 The impulse response of a linear time-invariant system.

effect and evaluate the performance of the system. The specification of the desired "effects" in precise mathematical terms requires a deep understanding of signal properties and is the subject of signal analysis. Understanding and predicting the effect of a general system upon the input signal is almost impossible. To develop a meaningful and feasible analysis, we limit our attention to systems that possess the properties of linearity and time-invariance.

The main premise of this section is that the response of a linear time-invariant (LTI) system to any input can be determined from its response $h[n]$ to the unit sample sequence $\delta[n]$ (see Figure 2.9), using a formula known as convolution summation. The sequence $h[n]$, which is known as *impulse response*, can also be used to infer all properties of a linear time-invariant system. Without linearity, we can only catalog the system output for each possible input.

A fundamental implication of linearity is that individual signals which have to be summed at the input are processed independently inside the system, that is, they superimpose and do not interact with each other. The superposition property greatly simplifies the analysis of linear systems, because if we express an input $x[n]$ as a sum of simpler sequences

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \cdots, \tag{2.27}$$

then the response $y[n]$ is given by

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \cdots, \tag{2.28}$$

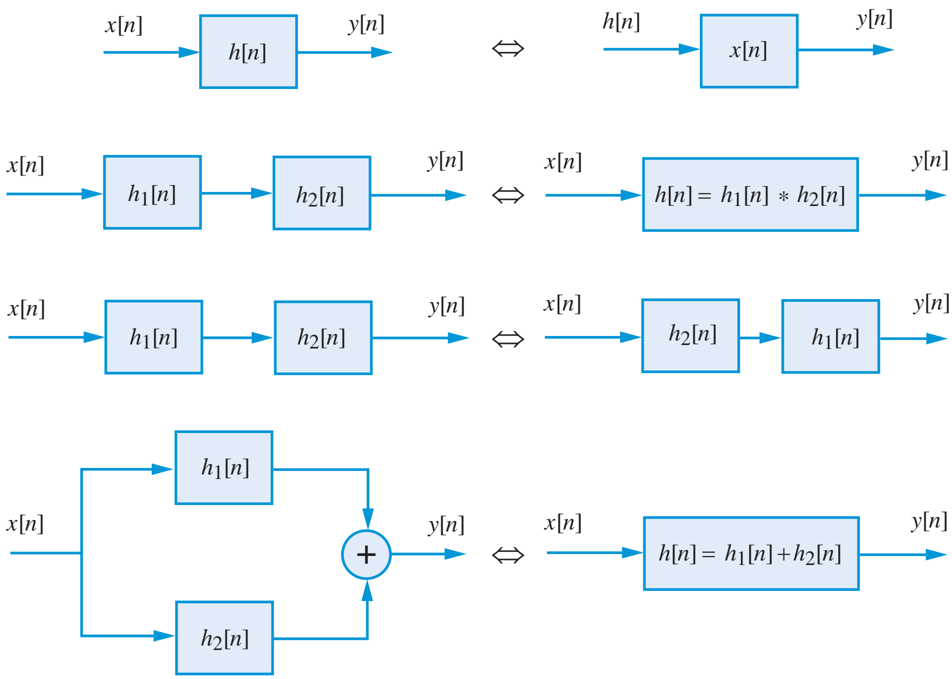


Figure 2.15 Convolution properties in the context of linear time-invariant systems. Systems on the same row are equivalent.

This can be shown by changing the summation variable k by $m = n - k$ in (2.36) as follows

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n]. \tag{2.45}$$

Therefore, a linear time-invariant system with input $x[n]$ and impulse response $h[n]$ will have the same output as a system having impulse response $x[n]$ and input $h[n]$.

Now consider the *cascade interconnection* of two linear time-invariant systems, where the output of the first system is input to the second system (see Figure 2.15). The outputs of these systems are

$$v[n] = \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] \quad \text{and} \quad y[n] = \sum_{m=-\infty}^{\infty} h_2[m]v[n-m]. \tag{2.46}$$

Substituting the first equation into the second and interchanging the order of the summations, we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h_2[m]h_1[(n-k)-m]. \tag{2.47}$$

Table 2.3 Summary of convolution properties.

Property	Formula
Identity	$x[n] * \delta[n] = x[n]$
Delay	$x[n] * \delta[n-n_0] = x[n-n_0]$
Commutative	$x[n] * h[n] = h[n] * x[n]$
Associative	$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$
Distributive	$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

We can easily see that the last summation is the convolution of $h_1[n]$ and $h_2[n]$ evaluated at $n-k$. If we define the sequence $h[n] \triangleq h_1[n] * h_2[n]$, then from (2.47) we obtain

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]. \tag{2.48}$$

Hence, the impulse response of two linear time-invariant systems connected in cascade is the convolution of the impulse responses of the individual systems.

If we consider the *parallel interconnection* of two linear time-invariant systems (see Figure 2.15) it is easy to show that

$$y[n] = h_1[n] * x[n] + h_2[n] * x[n] = (h_1[n] + h_2[n]) * x[n] \triangleq h[n] * x[n], \tag{2.49}$$

where $h[n] \triangleq h_1[n] + h_2[n]$. Therefore, the impulse response of two systems connected in parallel is the sum of the individual impulse responses.

The properties of convolution are summarized in Table 2.3 whereas their implications for system interconnections are illustrated in Figure 2.15.

2.5.2 Causality and stability

Since a linear time-invariant system is completely characterized by its impulse response sequence $h[n]$, we can use $h[n]$ to check whether the system is causal and stable.

Result 2.5.1 A linear time-invariant system with impulse response $h[n]$ is causal if

$$h[n] = 0 \quad \text{for} \quad n < 0. \tag{2.50}$$

Proof. If we write the convolution sum (2.36) in expanded form as

$$y[n] = \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \cdots, \tag{2.51}$$

Table 2.4 Response of linear time-invariant systems to some test sequences.

Type of response	Input sequence		Output sequence
Impulse	$x[n]=\delta[n]$	$\stackrel{\mathcal{H}}{\longmapsto}$	$y[n]=h[n]$
Step	$x[n]=u[n]$	$\stackrel{\mathcal{H}}{\longmapsto}$	$y[n]=s[n]=\sum_{k=-\infty}^{n} h[k]$
Exponential	$x[n]=a^n$, all n	$\stackrel{\mathcal{H}}{\longmapsto}$	$y[n]=H(a)a^n$, all n
Complex sinusoidal	$x[n]=\mathrm{e}^{\mathrm{j}\omega n}$, all n	$\stackrel{\mathcal{H}}{\longmapsto}$	$y[n]=H\left(\mathrm{e}^{\mathrm{j}\omega}\right)\mathrm{e}^{\mathrm{j}\omega n}$, all n

$H(a)=\sum_{-\infty}^{\infty} h[n]a^{-n}$

We start by drawing, just for clarity, the envelopes of the two sequences; the shape of the envelopes is not important. Figure 2.16(a) shows the sequences $x[k]$ and $h[n-k]$ as a function of the summation index k. The sequence $h[n-k]$ is obtained by folding $h[k]$ to obtain $h[-k]$ and then shifting, by n samples, to get $h[n-k]$. Note that the sample $h[M_1]$ is now located at $k=n-M_1$ and the sample $h[M_2]$ at $k=n-M_2$. Since $M_1 \leq M_2$ this reflects the time-reversal (flipping) of the sequence $h[k]$. For illustration purposes, without loss of generality, we choose n to position $h[n-k]$ on the left of $x[k]$. Changing the parameter n will shift $h[n-k]$ to a different position along the k-axis. Careful inspection of Figure 2.16 shows that, depending on the overlap between the sequences $x[k]$ and $h[n-k]$, there are three distinct limits of summation for the convolution sum. These limits are indicated by the beginning and the end of the shaded intervals. Clearly, the convolution sum is zero when $n-M_1<N_1$ or $n-M_2>N_2$ because the sequences $x[k]$ and $h[n-k]$ do not overlap. Therefore, $y[n]$ is nonzero in the range $L_1=M_1+N_1 \leq n \leq L_2=M_2+N_2$. The three distinct ranges for the convolution sum are defined as follows.

Partial overlap (left) The range of summation, as shown in Figure 2.16(b), is from $k=N_1$ to $k=n-M_1$. This range is valid as long as $n-M_1 \geq N_1$ or $n \geq M_1+N_1$ and $n-M_2 \leq N_1$ or $n \leq M_2+N_1$. Hence, we have

$$y[n]=\sum_{k=N_1}^{n-M_1} x[k]h[n-k], \quad \text{for } N_1+M_1 \leq n \leq N_1+M_2.$$

Full overlap The range of summation, as shown in Figure 2.16(c), is from $k=n-M_2$ to $k=n-M_1$. This range is valid as long as $n-M_2>N_1$ or $n>N_1+M_2$ and $n-M_1<N_2$ or $n<M_1+N_2$. Hence,

$$y[n]=\sum_{k=n-M_2}^{n-M_1} x[k]h[n-k], \quad \text{for } N_1+M_2<n<M_1+N_2.$$

Partial overlap (right) The range of summation, as shown in Figure 2.16(d), is from $k=n-M_2$ to $k=N_2$. This range is valid as long as $n-M_1 \geq N_2$ or $n \geq M_1+N_2$ and $n-M_2 \leq N_2$ or $n \leq M_2+N_2$. Hence,

Learning summary

- Linearity makes it possible to characterize a system in terms of the responses $h_k[n]$ to the shifted impulses $\delta[n-k]$ for all k, whereas time-invariance implies that $h_k[n] = h[n-k]$. The combination of linearity and time-invariance allows the complete characterization of a system by its impulse response $h[n]$.
- The impulse response $h[n]$ of an LTI system can be used to compute the output of the system for any input via the convolution sum and check whether the system is causal and stable.
  - Input-output description: $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
  - Stability: $\sum_{n=-\infty}^{\infty} |h[k]| < \infty$
  - Causality: $h[n] = 0$ for $n < 0$.
- The subclass of linear time-invariant systems, which are realizable in practice, is described by linear constant-coefficient difference equations

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

  If all feedback coefficients a_k are zero, we have a system with a finite duration impulse response (FIR), which is usually implemented nonrecursively. If at least one of a_k are nonzero, we have a recursive system with an infinite duration impulse response (IIR). In most signal processing applications, we assume that systems described by difference equations are initially at rest, that is, the initial conditions $y[-1], \ldots, y[-N]$ are set to zero. In the next chapter, we introduce a new tool, the z-transform, and use it to analyze linear time-invariant systems.
- Continuous-time LTI systems are completely characterized, like discrete-time systems, by their impulse response $h(t)$. Simply, the convolution sum is replaced by the convolution integral and the conditions for stability and causality are modified in an obvious manner. Practically realizable continuous-time LTI systems are described by linear constant coefficient differential equations.

TERMS AND CONCEPTS

Additivity property A property of a system in which a sum of input produces the corresponding sum of outputs, that is, $\mathcal{H}\{x_1[n] + x_2[n]\} = \mathcal{H}\{x_1[n]\} + \mathcal{H}\{x_2[n]\}$.

Bounded signal A signal $x[n]$ is bounded if there exists a positive constant M such that $|x[n]| \leq M$ for all n.

Causal system A system whose present value of its output does not depend on future values of its input. An LTI system is causal if its impulse response is zero for $n < 0$.

Convolution An operation that produces the output of an LTI system to any arbitrary input using system impulse response. For discrete-time systems, it is given by a summation operation and for continuous-time systems, it is given by an integral operation.

Discrete-time signal A signal whose value $x[n]$ is defined for every value of the integer variable n, also called a sequence.

Discrete-time system A system which transforms a discrete-time input signal $x[n]$ into a discrete-time output signal $y[n]$. Mathematically, it is described by $y[n] = \mathcal{H}\{x[n]\}$.

Dynamic system A system whose output $y[n]$ for every n depends on its inputs and outputs at other times.

Elementary signals Simple signals like unit sample, unit step, etc., that are useful in representation and analysis.

Energy of a signal The quantity $\sum_{-\infty}^{\infty} |x[n]|^2$ is defined as the signal energy and denoted by $\mathcal{E}_x$.

FIR system An LTI system characterized by a finite (duration) impulse response.

Fundamental period The smallest value N with respect to which a periodic signals repeats itself.

Homogeneity property A property of a system in which a scaled input produces the corresponding scaled output, that is, $\mathcal{H}\{a\,x[n]\} = a\mathcal{H}\{x[n]\}$.

Impulse response Response of an LTI system to the unit sample signal. It is denoted by $h[n]$.

IIR system An LTI system characterized by an infinite (duration) impulse response.

LCCDE A linear constant-coefficient difference equation relating a linear combination of the present and past outputs to a linear combination of the present and past inputs. An LTI system can be described as an LCCDE.

Linear system A system that satisfies the properties of homogeneity and additivity, that is, the principle of superposition.

LTI system A system that is both linear and time invariant. It is completely characterized by its impulse response.

Memoryless system A system whose output $y[n]$ for every n depends only on its input $x[n]$ at the same time.

Noncausal system A system whose output depends on future values of its input.

Nonrecursive system A system whose output at each n cannot be computed from its previously computed output values. Nonrecursive systems are FIR systems.

Periodic signal A signal $x[n] = x[n+N]$ that repeats every $N > 0$ samples for all n.

Power of a signal The quantity $\lim_{L\to\infty} \frac{\mathcal{E}_x}{L}$ is defined as the signal power and denoted by $\mathcal{P}_x$.

Principle of superposition A property of a system in which a linear combination of inputs produces a corresponding linear combination of outputs, that is, $\mathcal{H}\{a_1x_1[n] + a_2x_2[n]\} = a_1\mathcal{H}\{x_1[n]\} + a_2\mathcal{H}\{x_2[n]\}$.

Practically realizable system A discrete-time system is practically realizable if its practical implementation requires a finite amount of memory and a finite number of arithmetic operations.

Recursive system A system whose output at each n can be computed from its previously computed output values. Recursive systems are IIR systems.

Sampling period or interval The time interval between consecutive samples of a discrete-time signal.

Sampling rate or frequency The number of samples per second obtained during periodic sampling.

(BIBO) Stable system A system that produces bounded output for every bounded input. An LTI system is BIBO stable if its impulse response is absolutely summable.

State of a system The relevant information at $n = n_0$, concerning the past history of the system, which is required to determine the output to any input for $n \geq n_0$.

Steady-state response A response of a stable LTI system that continues or persists as $n \to \infty$. It is either a constant or sinusoidal in nature.

Step response Response of an LTI system to the unit step signal.

Time invariant (or fixed) system A system whose input/output pairs are invariant to a shift in time, that is, a time-shifted input produces a corresponding time-shifted output.

Transient response A response of an LTI system that decays to zero as $n \to \infty$.

Zero-input response A response of an LTI system due to initial conditions when no input has been applied.

Zero-state response A response of an LTI system due to an applied input when no initial conditions are present.

MATLAB functions and scripts

Name	Description	Page
conv	Computation of convolution sequence	56
conv0*	Compute convolution and its support	57
conv2	Convolution of 2D sequences	61
convmtx	Convolution matrix	55
convser	Serial computation of convolution	57
convvec	Vector computation of convolution	56
delta*	Generate unit sample sequence	28
filter	Implementation of a difference equation	67
filter2	Implementation of 2D FIR spatial filter	61
firstream*	Real-time FIR filter simulation	59
fold*	Fold or flip a sequence	29
impz	Computation of impulse response	68
persegen*	Generate periodic sequence	28
plot	General plotting function	30
pulsetrain	Generate a pulse train	80
shift*	Shift a sequence by n_0 samples	29
stem	Plot a sequence	30
stepz	Computation of step response	68
sound	Playing of audio signals	30
timealign*	Create sequences with the same support	29
unitpulse*	Generate unit pulse sequence	28
unitstep*	Generate unit step sequence	28
wavread	Read a wave audio file	30
wavwrite	Write a wave audio file	30

*Part of the MATLAB toolbox accompanying the book.

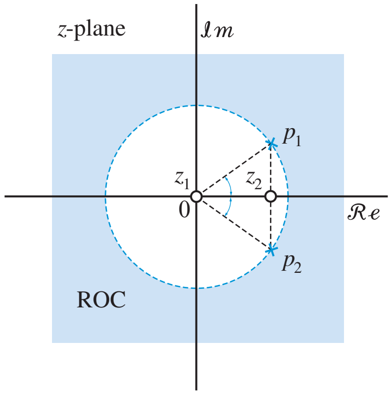


Figure 3.5 Pole-zero plot and region of convergence for Example 3.7.

Table 3.1 Some common z-transform pairs

	Sequence $x[n]$	z-Transform $X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3.	$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4.	$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
5.	$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
6.	$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7.	$(\cos\omega_0 n)u[n]$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
8.	$(\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
9.	$(r^n\cos\omega_0 n)u[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
10.	$(r^n\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

- The ROC *cannot* include any poles.
- The ROC is a connected (that is, a single contiguous) region.
- For finite duration sequences the ROC is the entire z-plane, with the possible exception of $z = 0$ or $z = \infty$.

Table 3.2 Some z-transform properties.

	Property	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
2.	Time shifting	$x[n-k]$	$z^{-k}X(z)$	R_x except $z = 0$ or ∞
3.	Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\|a\|R_x$
4.	Differentation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5.	Conjugation	$x^*[n]$	$X^*(z^*)$	R_x
6.	Real-part	$\text{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	At least R_x
7.	Imaginary part	$\text{Im}\{x[n]\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	At least R_x
8.	Folding	$x[-n]$	$X(1/z)$	$1/R_x$
9.	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
10.	Initial-value theorem	$x[n] = 0$ for $n < 0$	$x[0] = \lim\limits_{z\to\infty} X(z)$	

Time reversal The time reversal or folding property is expressed as

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z). \quad \text{ROC} = \frac{1}{R_x} \tag{3.60}$$

The proof is easily obtained by conjugating both sides of the definition of the z-transform (3.9). The notation $\text{ROC} = 1/R_x$ means that R_x is inverted; that is, if $R_x = \{r_1 < |z| < r_2\}$, then $1/R_x = \{1/r_2 < |z| < 1/r_1\}$.

Initial-value theorem If $x[n]$ is a causal sequence, that is, $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z), \tag{3.61}$$

which is obtained by considering the limit of each term in the z-transform summation.

Summary of properties For convenience, the properties of the z-transforms are summarized in Table 3.2.

3.5 System function of LTI systems

In Section 2.4 we showed that every LTI can be completely characterized in the time domain by its impulse response $h[n]$. In this respect, using the impulse response $h[n]$, we can compute the output of the system for any input via the convolution summation

Learning summary

- Any sequence $x[n]$ can be uniquely characterized by its z-transform: a complex function $X(z)$, of the complex variable z, accompanied by a given ROC.
- The z-transform converts convolution equations and linear constant coefficient difference equations (LCCDEs) into algebraic equations, which are easier to manipulate analytically. Figure 3.12 graphically shows relationships between difference equation, system function, and impulse response.
- In the z-domain, a LTI system is uniquely described by its system function

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \frac{Y(z)}{X(z)}.$$

- Systems described by the linear constant coefficient difference equation

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

  have a rational system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})},$$

  with M zeros z_k, $1 \leq k \leq M$ and N poles p_k, $1 \leq k \leq N$. The poles of the system determine its stability and the time-domain behavior of its impulse response:
  - If all poles are inside the unit circle, that is, $|p_k| < 1$ for all k, the system is stable. In practice, unstable systems lead to numerical overflow.
  - Real poles contribute exponentially decaying components in the impulse response. The distance of poles from the origin determines the speed of decay.
  - Complex-conjugate poles contribute exponentially decaying sinusoidal components in the impulse response. The distance of poles from the origin determines the decay of the envelop and the angle with the real axis of the frequency of the oscillation.

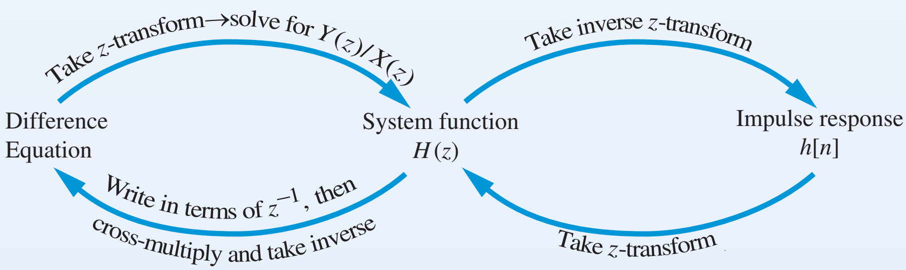


Figure 3.12 System representations and their relationships in graphical form.

- The z-transform allows the decomposition of systems with high-order rational system functions into first-order systems with real poles and second-order systems with complex-conjugate poles.
- The two major contributions of z-transforms to the study of LTI systems are:
  - The location of the poles determines whether the system is stable or not.
  - We can construct systems, whose impulse response has a desired shape in the time domain, by properly placing poles in the complex plane.
- The major application of one-sided z-transforms is in the solution of LCCDEs with nonzero initial conditions. Most DSP applications involve LCCDEs with zero initial conditions.

TERMS AND CONCEPTS

All-pole system An LTI system whose system function has only poles (and trivial zeros at the origin).

All-zero system An LTI system whose system function has only zeros (and trivial poles at the origin).

Anticausal sequence A sequence that is zero for positive n, i.e. $n > 0$. Also called a left-sided sequence.

Causal sequence A sequence that is zero for negative n, i.e. $n < 0$. Also called a right-sided sequence.

FIR system An LTI system characterized by a finite(-duration) impulse response.

IIR system An LTI system characterized by an infinite(-duration) impulse response.

Impulse response Response of an LTI system to an impulse sequence, denoted by $h[n]$.

Left-sided sequence A sequence that is zero for positive n, i.e. $n > 0$. Also called an anti-causal sequence.

Noncausal sequence A sequence that is nonzero for positive as well as negative values of n. Also called a two-sided sequence.

Partial fraction expansion (PFE) A decomposition of a higher degree rational function into a sum of first-order rational functions.

Pole of a system function A value of z at which the system function has a singularity (or becomes infinite).

Region of convergence (ROC) A set of values of z for which the series (3.9) converges. It is always bounded by a circle.

Residue A complex number that describes the behavior of the inverse-z-transform of a function around its pole singularity. For a rational function, it is needed in the partial fraction expansion method.

Right-sided sequence A sequence that is zero for negative n, i.e. $n < 0$. Also called a causal sequence.

System function The z-transform of the impulse response $h[n]$ of an LTI system, denoted by $H(z)$. Also called the transfer function.

Transfer function The z-transform of the impulse response $h[n]$ of an LTI system, denoted by $H(z)$. Also called the system function.

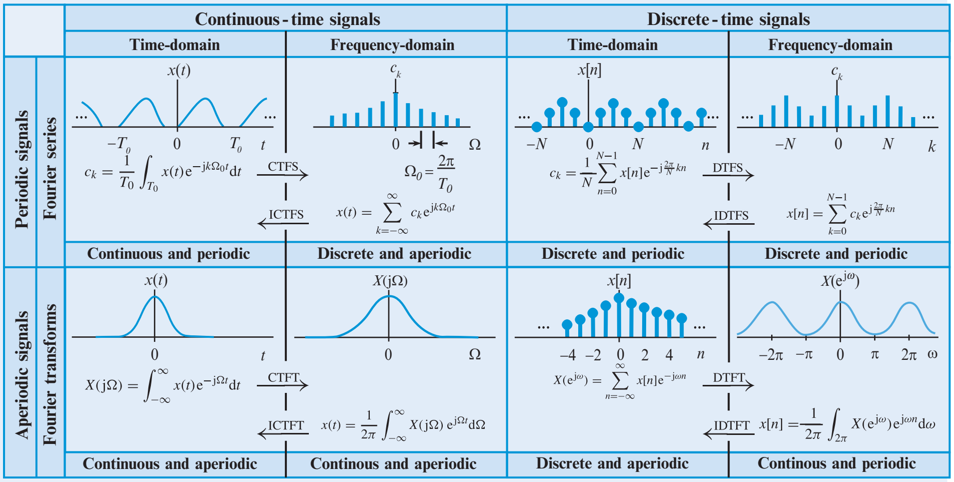
Two-sided sequence A sequence that is nonzero for positive as well as for negative values of n. Also called a noncausal sequence.

Zero of a system function A value of z at which the system function becomes zero.

z-transform (one-sided) A mapping of a positive-time sequence $x[n]$, $n \geq 0$, into a complex-valued function $X(z)$ of a complex variable z, given by the series in (3.100).

z-transform (two-sided) A mapping of a sequence $x[n]$ into a complex-valued function $X(z)$ of a complex variable z, given by the series in (3.9).

Table 4.1 Summary of Fourier representation of signals.



are known as the magnitude spectrum and phase spectrum of the signal. The magnitude and phase spectra are collectively called the Fourier spectrum or simply the spectrum of a signal. The exact form of the mathematical formulas used to determine the spectrum from the signal (Fourier analysis equation) and the signal from its spectrum (Fourier synthesis equation) depend on whether the time is continuous or discrete and whether the signal is periodic or aperiodic. This leads to the four different Fourier representations summarized in Table 4.1. Careful inspection of this table leads to the following conclusions:

- Continuous-time periodic signals are represented by an infinite Fourier series of harmonically related complex exponentials. Therefore, the spectrum exists only at $F = 0, \pm F_0, \pm 2F_0, \ldots$, that is, at discrete values of F. The spacing between the lines of this discrete or line spectrum is $F_0 = 1/T_0$, that is the reciprocal of the fundamental period.
- Continuous-time aperiodic signals are represented by a Fourier integral of complex exponentials over the entire frequency axis. Therefore, the spectrum exists for all F, $-\infty < F < \infty$. Knowledge of $X(j2\pi F)$ for $-\infty < F < \infty$ is needed to represent $x(t)$ for $-\infty < t < \infty$.
- Discrete-time periodic signals are represented by a finite Fourier series of harmonically related complex exponentials. The spacing between the lines of the resulting discrete spectrum is $\Delta\omega = 2\pi/N$, where N is the fundamental period. The DTFS coefficients of a periodic signal are periodic and the analysis equation involves a finite sum over a range of 2π.
- Discrete-time aperiodic signals are represented by a Fourier integral of complex exponentials over any frequency range of length 2π radians. Knowledge of the periodic DTFT function $X(e^{j\omega})$ over any interval of length 2π is needed to recover $x[n]$ for $-\infty < n < \infty$.

Table 4.2 Special cases of the DTFT for real signals.

Signal	Fourier transform
Real and even	real and even
Real and odd	imaginary and odd
Imaginary and even	imaginary and even
Imaginary and odd	real and odd

Thus, real signals with even symmetry have real spectra with even symmetry. These four special cases are summarized in Table 4.2.

Real and odd signals If $x[n]$ is real and odd, that is, $x[-n] = -x[n]$, then $x[n]\cos\omega n$ is an odd and $x[n]\sin\omega n$ is an even function of n. Therefore, from (4.110) and (4.119) we obtain

$$X_{\mathrm{R}}\left(\mathrm{e}^{\mathrm{j}\omega}\right) = 0, \tag{4.123}$$

$$X_{\mathrm{I}}\left(\mathrm{e}^{\mathrm{j}\omega}\right) = -2\sum_{n=1}^{\infty} x[n]\sin(\omega n), \qquad \text{(odd symmetry)} \tag{4.124}$$

$$x[n] = -\frac{1}{\pi}\int_0^{\pi} X_{\mathrm{I}}\left(\mathrm{e}^{\mathrm{j}\omega}\right)\sin(\omega n)\,\mathrm{d}\omega. \qquad \text{(odd symmetry)} \tag{4.125}$$

Thus, real signals with odd symmetry have purely imaginary spectra with odd symmetry.

The symmetry properties of the DTFT are summarized in Table 4.3. We shall illustrate these properties with some examples.

Example 4.12 Causal exponential sequence

Consider the sequence $x[n] = a^n u[n]$. For $|a| < 1$, the sequence is absolutely summable, that is

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty. \tag{4.126}$$

Therefore, the DTFT exists and is given by

$$\begin{aligned} X(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=0}^{\infty} a^n \mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{\infty} (a\mathrm{e}^{-\mathrm{j}\omega})^n \\ &= \frac{1}{1-a\mathrm{e}^{-\mathrm{j}\omega}}. \qquad \text{if } |a\mathrm{e}^{-\mathrm{j}\omega}| < 1 \text{ or } |a| < 1 \end{aligned} \tag{4.127}$$

Table 4.3 Symmetry properties of the DTFT.

Sequence $x[n]$	Transform $X(e^{j\omega})$
	Complex signals
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x_R[n]$	$X_e(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$
$jx_I[n]$	$X_o(e^{j\omega}) \triangleq \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$
$x_e[n] \triangleq \frac{1}{2}(x[n] + x^*[-n])$	$X_R(e^{j\omega})$
$x_o[n] \triangleq \frac{1}{2}(x[n] - x^*[-n])$	$jX_I(e^{j\omega})$
	Real signals
Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
$x_e[n] = \frac{1}{2}(x[n] + x[-n])$ Even part of $x[n]$	$X_R(e^{j\omega})$ real part of $X(e^{j\omega})$ (even)
$x_o[n] = \frac{1}{2}(x[n] - x[-n])$ Odd part of $x[n]$	$jX_I(e^{j\omega})$ imaginary part of $X(e^{j\omega})$ (odd)

If $x[n]$ is real $(-1 < a < 1)$, using the properties of complex numbers, we obtain

$$X_R(e^{j\omega}) = \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} = X_R(e^{-j\omega}), \qquad \text{(even)} \qquad (4.128a)$$

$$X_I(e^{j\omega}) = \frac{-a\sin(\omega)}{1 - 2a\cos(\omega) + a^2} = -X_I(e^{-j\omega}), \qquad \text{(odd)} \qquad (4.128b)$$

$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos(\omega) + a^2}} = \left|X(e^{-j\omega})\right|, \qquad \text{(even)} \qquad (4.128c)$$

$$\angle X(e^{j\omega}) = \tan^{-1}\frac{-a\sin(\omega)}{1 - a\cos(\omega)} = -\angle X(e^{-j\omega}). \qquad \text{(odd)} \qquad (4.128d)$$

These functions are plotted in Figure 4.27 for a lowpass sequence $(0 < a < 1)$ and a highpass sequence $(-1 < a < 0)$. ■

Starting with the right hand side of (4.153) we have

$$\begin{aligned}\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega &= \frac{1}{2\pi}\int_{2\pi}\left[\sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n}\right]X_2^*(e^{j\omega})d\omega \\ &= \sum_{n=-\infty}^{\infty} x_1[n]\left[\frac{1}{2\pi}\int_{2\pi} X_2^*(e^{j\omega})e^{-j\omega n}d\omega\right] \\ &= \sum_{n=-\infty}^{\infty} x_1[n]\left[\frac{1}{2\pi}\int_{2\pi} X_2^*(e^{-j\omega})e^{j\omega n}d\omega\right] \\ &= \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]. \quad \text{(using (4.148))}\end{aligned}$$

For $x_1[n] = x_2[n] = x[n]$, we obtain Parseval's relation (4.94).

Summary of DTFT properties For easy reference, the operational properties of the DTFT are summarized in Table 4.4.

Table 4.4 Operational properties of the DTFT.

	Property	Sequence	Transform
		$x[n]$	$\mathcal{F}\{x[n]\}$
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})$
2.	Time shifting	$x[n-k]$	$e^{-jk\omega}X(e^{j\omega})$
3.	Frequency shifting	$e^{j\omega_0 n}x[n]$	$X[e^{j(\omega-\omega_0)}]$
4.	Modulation	$x[n]\cos\omega_0 n$	$\frac{1}{2}X[e^{j(\omega+\omega_0)}] + \frac{1}{2}X[e^{j(\omega-\omega_0)}]$
5.	Folding	$x[-n]$	$X(e^{-j\omega})$
6.	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
7.	Differentiation	$nx[n]$	$-j\dfrac{dX(e^{j\omega})}{d\omega}$
8.	Convolution	$x[n] * h[n]$	$X(e^{j\omega})H(e^{j\omega})$
9.	Windowing	$x[n]w[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})W[e^{j(\omega-\theta)}]d\theta$
10.	Parseval's theorem	$\displaystyle\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega$	
11.	Parseval's relation	$\displaystyle\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

4.5.5 Signals with poles on the unit circle

The DTFT of a sequence $x[n]$ can be determined by evaluating its z-transform on the unit circle, provided that the ROC includes the unit circle (see Section 3.2). However, there are some useful aperiodic sequences with poles on the unit circle. For example, the z-transform of the unit step sequence

$$X(z) = \frac{1}{1 - z^{-1}}, \quad \text{ROC: } |z| > 1 \tag{4.165}$$

has a pole at $z = 1$. The DTFT is finite if $z = e^{j\omega} \neq 1$ or $\omega \neq 2\pi k$, k integer.

Similarly, the z-transform of the causal sinusoid $x[n] = \cos(\omega_0 n)u[n]$ is

$$X(z) = \frac{1 - (\cos\omega_0)z^{-1}}{1 - 2(\cos\omega_0)z^{-1} + z^{-2}}, \quad \text{ROC: } |z| > 1 \tag{4.166}$$

and has a pair of complex conjugate poles on the unit circle at $z = e^{\pm j\omega_0}$. The DTFT exist for $\omega \neq \pm\omega_0 + 2\pi k$.

The DTFT of sequences with poles on the unit circle can be formally defined for all values of ω by allowing Dirac impulse functions at the frequencies of the poles; however, this is not necessary for the needs of this text.

Learning summary

- The time domain and frequency domain representations of signals contain the same information in a different form. However, some signal characteristics and properties are better reflected in the frequency domain.
- The representation of a signal in the frequency domain (spectrum) consists of the amplitudes, frequencies, and phases of all sinusoidal components required to "build" the signal.
- The form of the formulas required to find the spectrum of a signal or synthesize a signal from its spectrum depends on whether:
  - the time variable is continuous or discrete;
  - the signal is periodic or nonperiodic.

  Therefore, there are four types of signal and related Fourier transform and series representations which are summarized in Figure 4.33.
- All Fourier representations share a set of properties that show how different characteristics of signals and how different operations upon signals are reflected in their spectra. The exact mathematical descriptions of these properties are different for each representation; however, the underlying concept is the same.

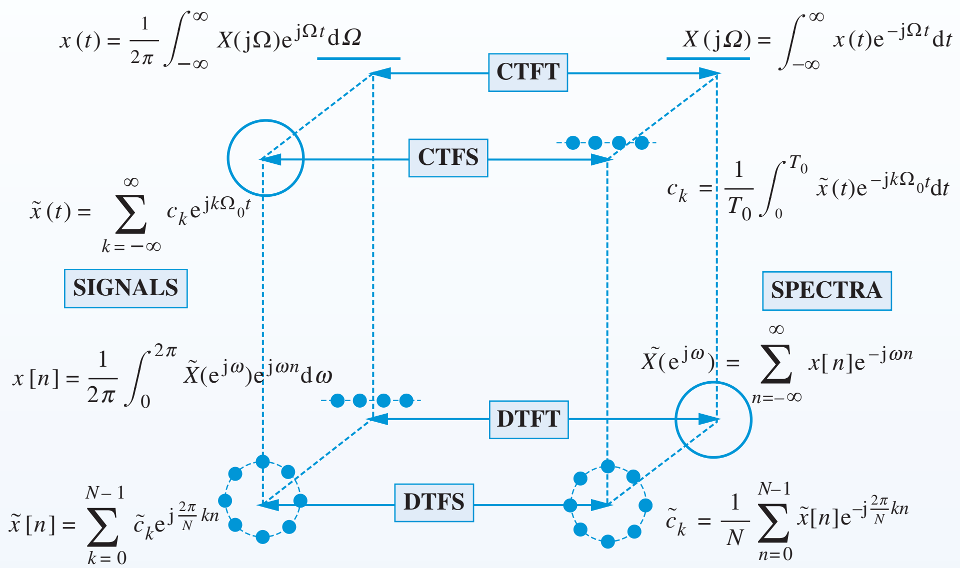


Figure 4.33 Summary of four Fourier representations.

TERMS AND CONCEPTS

Amplitude spectrum A graph of the Fourier series coefficients or transform as a function of frequency when these quantities are real-valued.

Analog frequency Represents a number of occurrences of a repeating event per unit time. For sinusoidal signals, the linear frequency, F, is measured in cycles per second (or Hz) while the angular (or radian) frequency, $\Omega = 2\pi F$, is measured in radians per second.

Autocorrelation sequence A sequence defined by $r_x[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[n-\ell]$ that measures a degree of similarity between samples of a real-valued sequence $x[n]$ at a lag ℓ.

Continuous-Time Fourier Series (CTFS) Expresses a continuous-time periodic signal $x(t)$ as a sum of scaled complex exponentials (or sinusoids) at harmonics kF_0 of the fundamental frequency F_0 of the signal. The scaling factors are called Fourier series coefficients c_k.

Continuous-Time Fourier Transform (CTFT) Expresses a continuous-time aperiodic signal $x(t)$ as an integral of scaled complex exponentials (or sinusoids) of all frequencies. The scaling factor is denoted by $X(j2\pi F)$.

Correlation coefficient A sequence denoted by $\rho_{xy}[\ell]$ which is a normalized correlation between samples of two real-valued sequences $x[n]$ and $y[n]$ at a lag ℓ and measures similarity between the two.

Correlation sequence A sequence defined by $r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[n-\ell]$ that measures similarity between samples of two real-valued sequences $x[n]$ and $y[n]$ at a lag ℓ.

Dirichlet conditions Requirements on the signals that are needed to determine Fourier series or transform of continuous- or discrete-time signals.

Dirichlet's function A periodic sinc function denoted by $D_L(x)$ and defined as $\frac{\sin(xL/2)}{L\sin(x/2)}$. Its value is 1 at $x = 0$.

Discrete-Time Fourier Series (DTFS) Expresses a discrete-time periodic signal $x[n]$ as a finite sum of scaled complex exponentials (or sinusoids) at harmonics k/N

of the fundamental frequency $1/N$ of the signal. The scaling factors are called Fourier series coefficients, c_k, which themselves form a periodic sequence.

Discrete-Time Fourier Transform (DTFT) Expresses a discrete-time aperiodic signal $x(t)$ as an integral of scaled complex exponentials (or sinusoids) of all frequencies. The scaling factor is denoted by $X(\omega)$.

Energy density spectrum A graph of $|X(j2\pi F)|^2$ or $|X(j\Omega)|^2$ as a function of frequency. It is a continuous spectrum.

Fundamental frequency Defined for periodic signals, it is the reciprocal of fundamental period. For continuous-time periodic signals it is denoted by $F_0 = 1/T_0$ in cycles per second, while for discrete-time signals it is denoted by $f_0 = 1/N$ in cycles per sample.

Fundamental harmonic The complex exponential (or sinusoid) associated with the fundamental period in set of harmonically-related complex exponentials.

Fundamental period Defined for periodic signals, it is the smallest period with respect to which a periodic signal repeats itself. For continuous-time periodic signals the fundamental period is T_0 in seconds, while for discrete-time signals the fundamental period is N in samples.

Harmonic frequencies or Harmonics Frequencies that are integer multiples of the fundamental frequency.

Harmonically-related complex exponentials A set of complex exponential signals with frequencies that are integer multiples of the fundamental frequency.

Magnitude spectrum A graph of the magnitude of the Fourier series coefficients or transform as a function of frequency.

Normalized frequency Defined for discrete-time sinusoids, it represents a number of occurrences of a repeating event per sample. For sinusoidal signals, the linear normalized frequency, f, is measured in cycles per sample while the normalized angular (or radian) frequency, $\omega = 2\pi f$, is measured in radians per sample.

Orthogonality property Defined for harmonically-related complex exponentials. For continuous-time complex exponentials it is given by

$$\frac{1}{T_0}\int_{T_0} e^{jk\Omega_0 t} e^{-jm\Omega_0 t}\,dt = \delta[k-m],$$

and for discrete-time complex exponentials it is given by

$$\frac{1}{N}\sum_{n=<N>} e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}mn} = \delta[k-m].$$

Phase spectrum A graph of the phase of the Fourier series coefficients or transform as a function of frequency.

Power spectrum A graph of $|c_k|^2$ as a function of harmonic frequency. It is a line spectrum.

Sinc function Denoted by $\mathrm{sinc}(x)$ and defined as $\frac{\sin(\pi x)}{\pi x}$. Its value is 1 at $x = 0$.

Learning summary

- The response of a stable LTI system to an everlasting complex exponential sequence is a complex exponential sequence with the same frequency; only the amplitude and phase are changed by the system. More specifically,

$$x[n] = A\mathrm{e}^{\mathrm{j}(\omega n+\phi)} \xrightarrow{\mathcal{H}} y[n] = A|H(\mathrm{e}^{\mathrm{j}\omega})|\mathrm{e}^{\mathrm{j}[\omega n+\phi+\angle H(\mathrm{e}^{\mathrm{j}\omega})]},$$

  where $H(\mathrm{e}^{\mathrm{j}\omega})$ is the frequency response function of the system. The complex exponential sequences are said to be eigenfunctions of LTI systems.

- The response of an LTI system to a periodic input sequence $x[n]$ is a periodic sequence $y[n]$ with the same fundamental period N, that is,

$$c_k^{(y)} = H\left(\frac{2\pi}{N}k\right)c_k^{(x)}, \quad -\infty < k < \infty$$

  where $c_k^{(x)}$ and $c_k^{(y)}$ are the DTFS coefficients of $x[n]$ and $y[n]$, respectively.

- The response of an LTI system to an aperiodic input signal $x(t)$ with Fourier transform $X(\mathrm{e}^{\mathrm{j}\omega})$ is a signal $y(t)$ with Fourier transform given by

$$Y(\mathrm{e}^{\mathrm{j}\omega}) = H(\mathrm{e}^{\mathrm{j}\omega})X(\mathrm{e}^{\mathrm{j}\omega}),$$

  which corresponds to point-by-point weighting of the input frequency components by the frequency response function.

- A system with distortionless response is defined by $y[n] = Gx[n-n_\mathrm{d}]$, where $G > 0$ and n_d are constants. The frequency response function of distortionless systems has constant magnitude, $|H(\mathrm{e}^{\mathrm{j}\omega})| = G$, and linear phase $\angle H(\mathrm{e}^{\mathrm{j}\omega}) = -\omega n_\mathrm{d}$; deviations from these conditions result in magnitude and phase distortions, respectively.

- The shape of the magnitude and phase frequency responses is determined by the locations of the poles and zeros with respect to the unit circle. Poles (zeros) close to the unit circle amplify (attenuate) input frequency components corresponding to the angle of these poles (zeros).

- A system with constant magnitude response $|H(\mathrm{e}^{\mathrm{j}\omega})| = G$ is called allpass. A causal and stable LTI system which has a causal and stable inverse is known as a minimum-phase system. Systems with rational system functions are minimum phase if all poles and zeros are inside the unit circle.

- The magnitude and phase responses of an arbitrary LTI system are independent. However, for minimum-phase systems the magnitude (phase) response uniquely specifies the phase (magnitude) response to within a scale factor. Every nonminimum-phase system can be expressed as the cascade connection of a minimum-phase system and an allpass system.

TERMS AND CONCEPTS

Allpass system Systems that have constant magnitude (>0) at all frequencies which are obtained by placing a complex reciprocal zero for each pole inside the unit circle.

Comb filter These are filters with multiple passbands and stopbands and are obtained by placing several poles near the unit circle.

Continuous-phase function A phase function that varies continuously without any jumps of 2π due to periodicity, denoted by $\Psi(\omega)$. Also known as an unwrapped-phase function.

Delay distortion A distortion in the shape of the response if the phase response is not a linear function of ω, defined by $\tau_{\text{pd}}(\omega) = -\angle H(\omega)/\omega$. Also known as the phase distortion.

Discrete-time oscillator A marginally stable system that has poles on the unit circle. Useful for generating sinusoidal carrier signals.

Discrete-time resonator A system that has a large magnitude response (that is, it resonates) in the vicinity of a pole location. It is essentially a bandpass filter.

Distortionless system A system whose input $x[n]$ and output $y[n]$ have the same shape, that is, $y[n] = Gx[n - n_{\text{d}}]$ or $H(\omega) = Ge^{-j\omega n_{\text{d}}}$.

Eigenfunctions of LTI systems The complex exponential, $e^{j\omega n}$, signals are the eigenfunctions since they are not distorted as they travel from the input to the output of an LTI system.

Energy or power gain The logarithm of $|H(\omega)|^2$, measured in decibels, is called the energy or power gain. Termed as attenuation if the value is negative.

Frequency response function It is the response of a stable LTI system to the complex exponential signal. It is denoted by $H(\omega)$ and is an eigenvalue of an LTI system.

Group delay Defined as the negative of the slope of the phase response, $\tau_{\text{gd}}(\omega) = -\mathrm{d}\Psi(\omega)/\mathrm{d}\omega$. Useful in checking the linearity of the phase response.

Ideal frequency-selective filters Has a distortionless response over one or more frequency bands and zero response elsewhere. Major categories are: lowpass (LPF), highpass (HPF), bandpass (BPF), and bandstop (BSF) filter.

Invertible system If we can determine the input $x[n]$ uniquely for each output $y[n]$ the system is invertible.

Linear FM (LFM) A sinusoidal signal with a frequency that grows linearly with time.

Magnitude distortion A system introduces magnitude distortion if $|H(\omega)| \neq$ a constant.

Magnitude response The magnitude, $|H(\omega)|$, of the frequency response function is called the magnitude response. It is also known as the gain of the system.

Maximum-phase system An anticausal and stable system with an anticausal and stable inverse is called a maximum-phase system. It has all poles and zeros outside the unit circle and imparts the maximum phase or group delay to the input signal.

Minimum-phase system A causal and stable system with a causal and stable inverse is called a minimum-phase system. It has all poles and zeros inside the unit circle and imparts the minimum phase or group delay to the input signal.

Mixed-phase system It is a system that is neither minimum phase nor maximum phase and has all poles inside the unit circle but zeros can be inside or outside the unit circle.

Notch filter These are filters with perfect null at certain frequencies and are obtained by placing zeros at those frequencies.

Phase distortion A distortion in the shape of the response if the phase response is not a linear function of ω, defined by $\tau_{\text{pd}}(\omega) = -\angle H(\omega)/\omega$. Also known as the delay distortion.

Phase response The angle, $\angle H(\omega)$, of the the frequency response function is called the phase response.

Practical or nonideal filters Approximation of ideal filters that are stable and realizable.

Principal-phase function A piecewise function with jumps of 2π due to periodicity, denoted by $\angle H(\omega)$, and is a result of the MATLAB `angle` function. Also known as a wrapped-phase function.

Steady-state response A response of a stable LTI system that continues or persists as $n \to \infty$. It is either a constant or sinusoidal in nature.

System gain The magnitude, $|H(\omega)|$, of the the frequency response function is called the system gain. Also known as the magnitude response of the system.

Transient response A response of an LTI system that decays to zero as $n \to \infty$.

Unwrapped-phase function A phase function that varies continuously without any jumps of 2π due to periodicity, denoted by $\Psi(\omega)$. Also known as a continuous-phase function.

Wrapped-phase function A piecewise function with jumps of 2π due to periodicity, denoted by $\angle H(\omega)$, and is a result of the MATLAB `angle` function. Also known as a principal-phase function.

Zero-state response A response of an LTI system due to an applied input when no initial conditions are present.

MATLAB functions and scripts

Name	Description	Page
`abs`	Computes the magnitude of frequency response	226
`angle`	Computes the principal value of phase response	208, 226
`contphase`	Computes the continuous phase from group delay	228
`fft`	Computes equidistant values of the DTFT	226
`freqs`	Computes continuous-time frequency response	267
`freqz`	Computes the frequency response function	226
`freqz0`*	Computes the frequency response function	227
`grpdelay`	Computes the group-delay response	228
`grpdelay0`*	Computes the group-delay response	228
`fvtool`	Filter analysis and visualization tool	229
`phasez`	Computes phase response in radians	229
`phasedelay`	Computes phase delay in “samples”	229
`polystab`	Converts polynomial to minimum phase	289
`splane`	Plots poles and zeros of a rational $H(s)$	267

*Part of the MATLAB toolbox accompanying the book.

FURTHER READING

- A detailed treatment of continuous-time and discrete-time Fourier series and transforms at the same level as in this book is given in Oppenheim *et al.* (1997) and Lathi (2005).
- The standard references for Fourier transforms from an electrical engineering perspective are Bracewell (2000) and Papoulis (1962).
- A mathematical treatment of Fourier series and transforms is given in Walker (1988) and Kammler (2000).

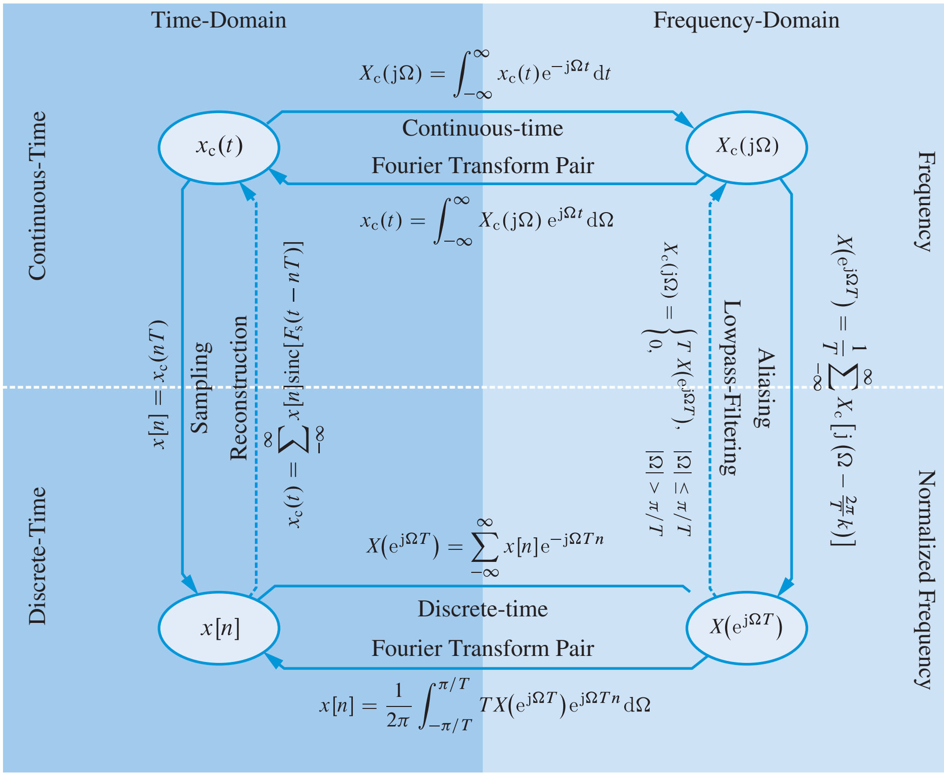


Figure 6.9 Relationships between the spectra of a continuous-time signal $x_c(t)$ and the discrete-time signal $x[n] = x_c(nT)$ obtained by periodic sampling. The dashed paths hold for bandlimited signals sampled at a rate $F_s > 2F_H$.

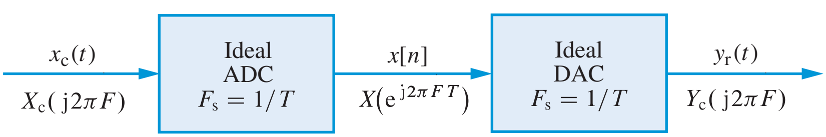


Figure 6.10 A talk-through system. Ideally, the reconstructed signal $y_r(t)$ should be identical to the input signal $x_c(t)$.

The phenomenon of aliasing has a clear meaning in the time-domain. Two continuous-time sinusoids of different frequencies appear at the *same* frequency when sampled. Since we cannot distinguish them based on their samples alone, they assume the same identity ("alias") and they produce the same continuous-time sinusoidal signal.

In this section we shall provide more insight into the causes and effects of aliasing using the system shown in Figure 6.10. This "talk-through" system is often used in practice to verify the correct operation and limitations of A/D and D/A converters before their actual application.

Learning summary

- Any time a continuous-time signal $x_c(t)$ is uniformly sampled with sampling period $T = 1/F_s$, the spectrum of $x[n] = x_c(nT)$ is obtained by scaling the spectrum of $x_c(t)$ by $1/T$ and putting copies at all integer multiples of F_s

$$X\left(e^{j2\pi FT}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c[j2\pi(F - kF_s)].$$

- A bandlimited signal with $X_c(j2\pi F) = 0$ for $|F| > F_H$ can be exactly reconstructed from the sequence of samples $x_c(nT)$, where $F_s = 1/T \geq 2F_H$, using the ideal bandlimited interpolation formula

$$x_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}.$$

  The highest frequency F_H present in $x_c(t)$ is called the Nyquist frequency. The minimum sampling rate, $2F_H$, required to avoid overlap of the repeated copies of $X_c(j2\pi F)$ (aliasing distortion) is known as the Nyquist rate.
- The sampling theorem makes possible the discrete-time processing of continuous-time signals. This is accomplished by sampling a continuous-time signal, applying discrete-time processing algorithms to the sequence of samples, and reconstructing a continuous-time signal from the resulting sequence.
- In practice, the value of each sample is represented numerically using a finite number of bits. This process, known as quantization, destroys information in the same way that adding noise destroys precision. For most signals of practical interest, the signal-to-quantization error ratio increases 6 dB for each additional bit used in the representation.
- A bandpass signal $x_c(t)$, with spectrum $X_c(j2\pi F) = 0$ outside the range $0 < F_L \leq |F| \leq F_H < \infty$, can be reconstructed from its samples without aliasing using a sampling rate in the range $2B \leq F_s \leq 4B$, where $B = F_H - F_L$, instead of the Nyquist rate $2F_H$. The minimum sampling rate of $2B$ will be adequate under the condition that F_H/B is an integer.
- Perfect reconstruction of a bandlimited image $s_c(x, y)$, from a set of samples $s_c(m\Delta x, n\Delta y)$ without aliasing is possible if both the horizontal and vertical sampling frequencies satisfy the sampling theorem.

TERMS AND CONCEPTS

Aliasing distortion A signal distortion caused by overlapping spectra of the signal samples in which frequencies higher than folding frequency are aliased into lower frequencies. Also known as aliasing.

Apparent frequency The lowest frequency of a sinusoid that has exactly the same samples as the input sinusoid and is denoted by F_a.

Arbitrary-band positioning A condition in a bandlimited bandpass signal whose highest bandwidth is not an integer multiple of its bandwidth.

Bandlimited bandpass signal A signal whose spectrum has a finite bandwidth around a center frequency that is much larger than its bandwidth.

Bandlimited lowpass signal A baseband signal whose spectrum is zero above a finite maximum frequency, called bandwidth.

Effective continuous-time filter A continuous-time system realized through the A/D converter – digital filter – D/A filter operation, that is, through a discrete-time processing.

Folding frequency The highest signal frequency that is retained in an input signal after sampling and is equal to half the sampling frequency, or $F_s/2$. All frequencies above $F_s/2$ are aliased into frequencies below $F_s/2$.

Guard band A band of frequencies created when the sampling frequency is greater than the Nyquist rate. It contains no signal spectra.

Ideal digital-to-analog converter (DAC) An idealized operation that reconstructs a continuous-time signal from its samples.

Ideal sampling An idealized operation that periodically picks values of a continuous-time signal resulting in a discrete-time signal. Also called ideal analog-to-digital conversion (ADC) or uniform sampling.

Ideal bandlimited interpolation An idealized reconstruction of a bandlimited signal from its samples using an ideal lowpass filter or using a sinc interpolating function.

Impulse-invariance transformation A procedure of converting a continuous-time filter into an equivalent discrete-time filter so that the shape of the impulse response is preserved.

Integer-band positioning A condition in a bandlimited bandpass signal whose highest bandwidth is an exact integer multiple of its bandwidth.

Interpolation An operation that fills-in values between signal samples according to a predetermined function.

Lowpass antialiasing filter A continuous-time lowpass filter that prevents aliasing by removing frequencies above the folding frequency prior to sampling.

Moiré pattern A visual sampling effect in image sampling created by frequencies close to folding frequency and results in beat-like modulation pattern.

Nyquist frequency The highest frequency in a continuous-time signal. Also called the bandwidth of the signal.

Nyquist rate The minimum sampling rate that avoids aliasing in a bandlimited signal and is equal to twice the Nyquist frequency.

Practical DAC An implementable system that converts samples into a continuous-time signal by implementing a sample-and-hold circuit followed by a carefully designed lowpass post-filter.

Quantization noise An unavoidable error created by the quantization operation in a practical ADC. It is measured via signal-to-quantization noise ratio (SQNR) in dB.

Quantization A process of approximating a continuous range of signal values by a relatively small but finite number of discrete values. Results in an error called quantization noise.

Sample-and-hold circuit A relatively simple circuit in an ADC or DAC that is designed to hold applied input value steady for one sampling interval while the converter performs some operation.

Sampling ADC A practical analog-to-digital converter that has a built-in sample-and-hold circuit.

Sampling frequency A measure of number of samples in one second, expressed in samples per second.

Sampling rate A measure of number of samples in one second, expressed in Hz.

Sampling theorem A fundamental result that states that if a signal contains no frequencies above the highest (or Nyquist) frequency F_H, then it can be completely determined by its samples spaced at-most $1/(2F_\mathrm{H})$ seconds apart.

Talk-through system A simple discrete-time system consisting of an ADC followed by a DAC and used for verifying correct operation of sampling and reconstruction or limitations of A/D or D/A converters.

MATLAB functions and scripts

Name	Description	Page
`audiorecorder`	Records sound as an object using an audio input device	323
`audioplayer`	Creates a player object for use with the `play` function	327
`ceil`	Quantizes a number to the nearest integer towards ∞	321
`fix`	Quantizes a number to the nearest integer towards 0	321
`floor`	Quantizes a number to the nearest integer towards $-\infty$	321
`play`	Plays a player object through an audio output device	327
`round`	Quantizes a number to the nearest integer	321
`sinc`	Computes the $\sin(\pi x)/(\pi x)$ interpolating function	309
`sound`	Plays sampled signal as a sound through speakers	327
`wavrecord`	Records sound through mic or input-line (PC only)	323
`wavplay`	Plays sampled signal as a sound through speakers (PC only)	327

FURTHER READING

1. A detailed treatment of sampling theory, at the same level as in this book, is given in Oppenheim and Schafer (2010) and Proakis and Manolakis (2007). A clear and concise discussion of the sampling theorem, including the original derivation, is given in Shannon (1949).
2. The practical aspects of A/D and D/A conversion are discussed in Hoeschele (1994) and Kester (2005). Williston (2009) provides an introduction to all practical aspects of DSP, including A/D and D/A conversion.
3. Bandpass sampling, which is used extensively in radar and communications, is discussed in Linden (1959), Vaughan *et al.* (1991), and Coulson (1995). A tutorial introduction is given in Proakis and Manolakis (2007).
4. Two-dimensional sampling is discussed in the standard image processing references by Gonzalez and Woods (2008) and Pratt (2007). The implications of sampling in computer graphics are discussed in Foley *et al.* (1995).

7.1 Computational Fourier analysis

The basic premise of Fourier analysis is that any signal can be expressed as a linear superposition, that is, a sum or integral of sinusoidal signals. The exact mathematical form of the representation depends on whether the signal is continuous-time or discrete-time and whether it is periodic or aperiodic (see Table 7.1). Note that in this chapter we use the tilde ($\tilde{}$) to emphasize that a sequence or function is periodic. Indeed, $\tilde{x}[n+N] = \tilde{x}[n]$, $\tilde{x}_c(t+T_0) = \tilde{x}_c(t)$, $\tilde{c}_{k+N} = \tilde{c}_k$, and $\tilde{X}(e^{j\Omega + j2\pi}) = \tilde{X}(e^{j\Omega})$.

Careful inspection of Table 7.1 reveals that the equations for the DTFS involve computation of a finite number of coefficients or samples using finite sums of products. Therefore, they can be *exactly* evaluated by numerical computation. All other series or transforms can be computed only *approximately* because they involve infinite summations, integrals, and computation of signals or spectra at a continuous range of values. To illustrate these issues we discuss how we compute the CTFT, DTFT, and CTFS in practice.

Computing the CTFT A simple numerical approximation of $X_c(j\Omega)$ can be obtained by first sampling $x_c(t)$ and then replacing the Fourier integral by a sum

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t} dt \approx \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega nT} (T) \triangleq \hat{X}_c(j\Omega). \tag{7.1}$$

Table 7.1 Summary of direct and inverse Fourier transforms and the key computational operations required for their evaluation. The presence of an infinite sum or integral prevents exact numerical computation of the corresponding transform

	Direct transform (spectral analysis)	Inverse transform (signal reconstruction)	Exact computation
DTFS	$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$ finite summation	$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$ finite summation	yes
DTFT	$\tilde{X}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ infinite summation	$x[n] = \frac{1}{2\pi}\int_0^{2\pi} \tilde{X}(e^{j\Omega}) e^{j\Omega} d\omega$ integration	no
CTFS	$c_k = \frac{1}{T_0}\int_0^{T_0} \tilde{x}_c(t) e^{-jk\Omega_0 t} dt$ integration	$\tilde{x}_c(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$ infinite summation	no
CTFT	$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t} dt$ integration	$x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega) e^{j\Omega t} d\Omega$ integration	no

finite duration sequence. Second, the DFT provides samples of the DTFT of the sequence at a set of equally spaced frequencies. This sampling process results in the inherent periodicity of the DFT. Understanding the underlying periodicity of DFT is absolutely critical for the correct application of DFT and meaningful interpretation of the results obtained.

To understand the relationship between the CTFT and the DFT, we consider the illustration in Figure 7.8. Suppose that we are given a continuous-time signal $x_c(t)$ with Fourier transform $X_c(j\Omega)$. Application of discrete-time signal processing starts by uniformly sampling $x_c(t)$ at $t = nT$. This results in a discrete-time signal $x[n] = x_c(nT)$ with DTFT specified by

$$\tilde{X}(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega T n} = \frac{1}{T}\sum_{m=-\infty}^{\infty} X_c\left(j\Omega - j\frac{2\pi}{T}m\right). \tag{7.73}$$

Since $\omega = \Omega T$, the N-point DFT $X[k]$ is obtained by sampling the DTFT $\tilde{X}(e^{j\omega})$ at $\omega = 2\pi k/N$ or $\tilde{X}(e^{j\Omega T})$ at $\Omega = 2\pi k/NT$ for $0 \le k \le N-1$. The result is

$$X[k] = \frac{1}{T}\sum_{m=-\infty}^{\infty} X_c\left(j\frac{2\pi k}{NT} - j\frac{2\pi}{T}m\right). \quad k = 0, 1, \ldots, N-1 \tag{7.74}$$

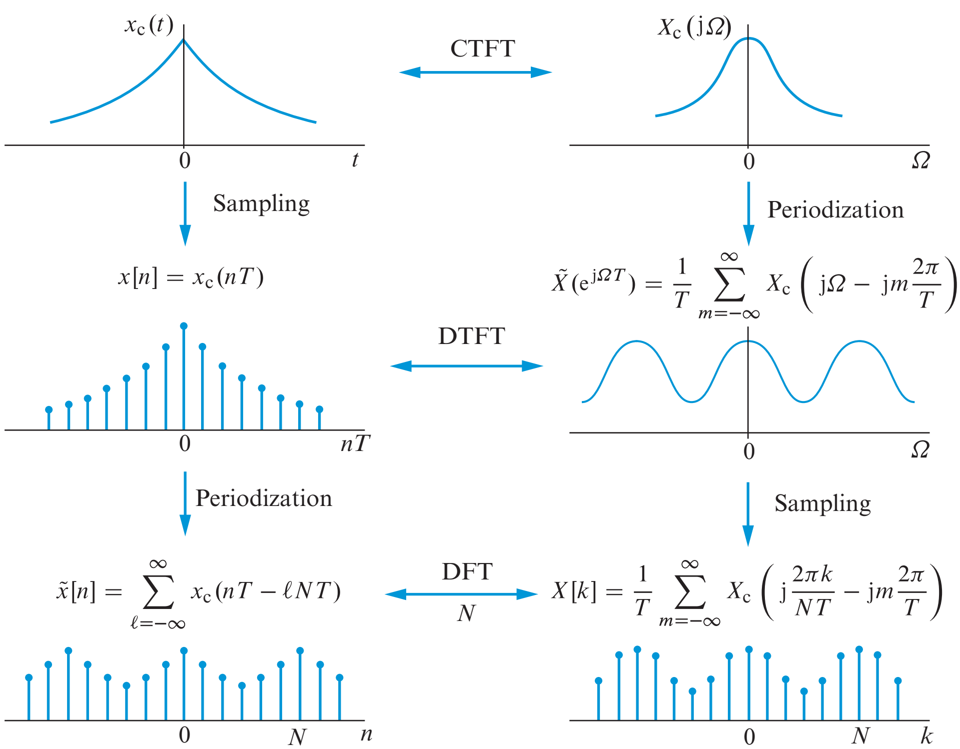


Figure 7.8 Operations and steps required to establish the relationship between CTFT, DTFT, and DFT. We note that sampling in one domain is equivalent to periodization in the other domain. Periodic replication may cause frequency-domain or time-domain aliasing.

Table 7.3 Symmetry properties of the DFT

N-point Sequence	N-point DFT
Complex signals	
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$x_R[n]$	$X^{cce}[k] = \frac{1}{2}(X[k] + X^*[\langle -k\rangle_N])$
$jx_I[n]$	$X^{cco}[k] = \frac{1}{2}(X[k] - X^*[\langle -k\rangle_N])$
$x^{cce}[n] = \frac{1}{2}(x[n] + x^*[\langle -n\rangle_N])$	$X_R[k]$
$x^{cco}[n] = \frac{1}{2}(x[n] - x^*[\langle -n\rangle_N])$	$jX_I[k]$
Real signals	
{Any real $x[n]$	$\begin{cases} X[k] = \tilde{X}^*[\langle -k\rangle_N] \\ X_R[k] = X_R[\langle -k\rangle_N] \\ X_I[k] = -X_I[\langle -k\rangle_N] \\ \lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert \\ \angle X[k] = -\angle X[\langle -k\rangle_N] \end{cases}$

with DFT $X[k]$, then using Table 7.3, we can show that (see Tutorial Problem 12)

$$X_1[k] = X^{cce}[k] \quad \text{and} \quad jX_2[k] = X^{cco}[k]. \tag{7.117}$$

Thus, using one DFT computation followed by a conjugate-symmetry decomposition gives two required DFTs.

7.4.4 Circular shift of a sequence

Consider a sequence $x[n]$ with discrete-time Fourier transform $X(e^{j\omega})$. As we observed in Section 4.5.3, $e^{-j\omega m}X(e^{j\omega})$ is the Fourier transform of the time-shifted sequence $x[n-m]$. It is then natural to ask what happens to an N-point sequence $x[n]$ if we multiply its N-point DFT $X[k]$ by $W_N^{mk} = e^{-j(2\pi m/N)k}$. The result is a sequence $z[n]$ obtained by the inverse DFT of $Z[k] = W_N^{mk}X[k]$. Hence, we have

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \tag{7.118}$$

$$z[n] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{kn}X[k]W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-k(n-m)}. \tag{7.119}$$

A superficial comparison of (7.118) and (7.119) yields $z[n] = x[n-m]$. However, because $x[n]$ is unavailable outside the interval $0 \le n \le N-1$, we cannot obtain $z[n]$ by a simple

Table 7.4 Operational properties of the DFT

	Property	N-point sequence	N-point DFT
		$x[n], h[n], v[n]$	$X[k], H[k], V[k]$
		$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1[k] + a_2X_2[k]$
2.	Time shifting	$x[\langle n-m\rangle_N]$	$W_N^{km}X[k]$
3.	Frequency shifting	$W_N^{-mn}x[n]$	$X[\langle k-m\rangle_N]$
4.	Modulation	$x[n]\cos(2\pi/N)k_0n$	$\frac{1}{2}X[\langle k+k_0\rangle_N] + \frac{1}{2}X[\langle k-k_0\rangle_N]$
5.	Folding	$x[\langle -n\rangle_N]$	$X^*[k]$
6.	Conjugation	$x^*[n]$	$X^*[\langle -k\rangle_N]$
7.	Duality	$X[n]$	$Nx[\langle -k\rangle]_N]$
8.	Convolution	$h[n] \, (N) \, x[n]$	$H[k]X[k]$
9.	Correlation	$x[n] \, (N) \, y[\langle -n\rangle_N]$	$X[k]Y^*[k]$
10.	Windowing	$v[n]x[n]$	$\frac{1}{N}V[k] \, (N) \, X[k]$
11.	Parseval's theorem	$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[n]$	
12.	Parseval's relation	$\sum_{n=0}^{N-1} \|x[n]\|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \|X[k]\|^2$	

7.4.8 Summary of properties of the DFT

Table 7.4 summarizes the operational properties of the DFT; the symmetry properties were summarized in Table 7.3. We emphasize that the presence of the modulo-N indexing operator ensures that all sequences and their DFTs are specified in the range from 0 to $N-1$. The fact that the DFT and inverse DFT formulas differ only in a factor of $1/N$ and in the sign of the exponent of W_N, results in a strong duality between the two transforms. To establish this duality, we first replace n by $-n$ in the inverse DFT formula to obtain

$$Nx[-n] = \sum_{k=0}^{N-1} X[k]W_N^{kn}. \tag{7.145}$$

Interchanging the roles of n and k in (7.145) yields

$$Nx[-k] = \sum_{n=0}^{N-1} X[n]W_N^{nk}. \tag{7.146}$$

Learning summary

- The Discrete Fourier Transform (DFT) is a finite orthogonal transform which provides a unique representation of N consecutive samples $x[n]$, $0 \leq n \leq N-1$ of a sequence through a set of N DFT coefficients $X[k]$, $0 \leq k \leq N-1$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} x[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n]e^{j2\pi kn/N}.$$

  The DFT does not provide any information about the unavailable samples of the sequence, that is, the samples *not* used in the computation. The interpretation or physical meaning of the DFT coefficients depends upon the assumptions we make about the unavailable samples of the sequence.

- If we create a periodic sequence $\tilde{x}[n]$ by repeating the N consecutive samples, the DFT coefficients and the DTFS coefficients $\tilde{c}_k$ of $\tilde{x}[n]$ are related by $X[k] = N\tilde{c}_k$. Thus, the DFT "treats" the finite segment as one period of a periodic sequence. If $x[n] = x[n+N_0]$ and $N \neq N_0$, the period "seen" by the DFT differs from the actual period of the analyzed sequence.
- If $X(e^{j\Omega})$ is the DTFT of the entire sequence and $X_N(e^{j\Omega})$ the DTFT of the finite segment $x[n]$, $0 \leq n \leq N-1$, we have:
  - The DFT provides samples of $X_N(e^{j\Omega})$ at equally spaced points on the unit circle, that is, $X[k] = X_N(^{j2\pi k/N})$. If $x[n]$ has length $L \leq N$, then $X(e^{j\Omega}) = X_N(e^{j\Omega})$.
  - If $\tilde{X}[k] \triangleq X(^{j2\pi k/N})$, $0 \leq k \leq N-1$, the inverse DFT yields an aliased version of $x[n]$, that is, $\tilde{x}[n] = \sum_\ell x[n-\ell N]$. If $x[n]$ has length $L \leq N$, we have $X(e^{j\Omega}) = X_N(e^{j\Omega})$ and $x[n] = \tilde{x}[n]$.
- The multiplication of two N-point DFTs is equivalent to the circular convolution of the corresponding N-point sequences. Since circular convolution is related to linear convolution, we can use the DFT to compute the output of an FIR filter to an indefinitely long input sequence.
- The DFT is widely used in practical applications to determine the frequency content of continuous-time signals (spectral analysis). The basic steps are: (a) sampling the continuous-time signal, (b) multiplication with a finite-length window (Hann or Hamming) to reduce leakage, (c) computing the DFT of the windowed segment, with zero-padding, to obtain an oversampled estimate of the spectrum. The frequency resolution, which is about $8\pi/L$ rads, is determined by the length L of the window.
- The value of the DFT stems from its relation to the DTFT, its relation to convolution and correlation operations, and the existence of very efficient algorithms for its computation. These algorithms are collectively known as Fast Fourier Transform (FFT) algorithms. The FFT is not a new transform; it is simply an efficient algorithm for computing the DFT.

TERMS AND CONCEPTS

CTFS Expresses a continuous-time periodic signal $\tilde{x}_c(t)$ as a sum of scaled complex exponentials (or sinusoids) at harmonics kF_0 of the fundamental frequency F_0 of the signal. The scaling factors are called Fourier series coefficients c_k.

CTFT Expresses a continuous-time aperiodic signal $x(t)$ as an integral of scaled complex exponentials (or sinusoids) of all frequencies. The scaling factor is denoted by $X_c(j\Omega)$.

Circular addressing A "wrap around" operation on integers after they reach integer multiples of N. It produces integers between 0 and $N-1$. Also called modulo-N operation. Denoted by $\langle n\rangle_N$.

Circular buffer Memory storage in which data are stored in a circular fashion and accessed by modulo-N addressing.

Circular convolution A convolution between two N-point sequences using circular shift resulting in another N-point sequence; it is denoted and given by

$$x_1[n] \textcircled{N} x_2[n] = \sum_{k=0}^{N-1} x_1[k]x_2[\langle n-k\rangle_N].$$

Circular folding A time- or frequency-reversal operation implemented according to the modulo-N circular addressing on a finite-length sequence. The sample at 0 remains at its position while the remaining samples are arranged in reverse order. Denoted by $\langle -n\rangle_N$.

Circular shift A shifting operation on a finite-length sequence implemented using the modulo-N circular addressing. Denoted by $\langle n-m\rangle_N$.

Circular-even symmetry A kind of even symmetry created when an even sequence is wrapped around a circle and then recovered by unwrapping and laying the axis flat.

Circular-odd symmetry A kind of odd symmetry created when an odd sequence is wrapped around a circle and then recovered by unwrapping and laying the axis flat.

DFS The periodic extension of the DFT $X[k]$ for all k and denoted by $\tilde{X}$.

DFT matrix An $N \times N$ matrix formed using Nth roots of unity and denoted by $\boldsymbol{W}_N$.

DFT A transform-like operation on a finite-length N-point sequence $x[n]$ resulting in a finite-length N-point sequence $X[k]$ given by

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}.$$

DTFS Expresses a discrete-time periodic signal $\tilde{x}[n]$ as a finite sum of scaled complex exponentials (or sinusoids) at harmonics k/N of the fundamental frequency $1/N$ of the signal. The scaling factors are called Fourier series coefficients $\tilde{c}_k$, and they themselves form a periodic sequence.

DTFT Expresses a discrete-time aperiodic signal $x(t)$ as an integral of scaled complex exponentials (or sinusoids) of all frequencies. The scaling factor is denoted by $\tilde{X}(\omega)$.

Data window A finite-length function (or sequence) used to truncate an infinite-length signal (or sequence) into a finite-length one by way of multiplication.

Gaussian pulse A Gaussian shaped signal that satisfies the uncertainty principle with equality, that is, the time-bandwidth product is the smallest for the Gaussian pulse.

IDFS An inverse of the DFS and a periodic extension of the IDFT $x[n]$ denoted by $\tilde{x}[n]$.

IDFT An inverse of DFT resulting in a finite-length N-point given by

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j2\pi nk/N}.$$

Inherent periodicity An intrinsic periodicity imposed by the DFT or IDFT operations on a finite-length sequence over the entire axis.

Modulo-N operation A "wrap around" operation on integers after they reach integer multiples of N. It produces integers between 0 and $N-1$. Also called circular addressing operation. Denoted by $\langle n\rangle_N$.

Overlap–add method A block convolution approach for convolving a very long input sequence with a finite-length M-point impulse response. The successive output blocks are overlapped by $(M-1)$ samples and the contributions from overlapping blocks are added to assemble the correct result.

Overlap–save method A block convolution approach for convolving a very long input sequence with a finite-length M-point impulse response. The successive input blocks are overlapped by $(M-1)$ samples and the last $(M-1)$ samples saved from the previous output block are used to assemble the correct result.

Periodic extension or Periodization Creation of a periodic sequence from an N-point sequence as "seen" by the DFT. It is the result of the IDFT operation.

Resolvability Ability of the time-windowing operation to separate two closely spaced sharp peaks in the spectra. It is related to the length of the time-window.

Short-time DFT A DFT computed over short but overlapping data segments to illustrate time-dependent distribution of spectral power.

Spectral leakage A time-windowing effect that transfers (or leaks) power from one band to another caused by the nonzero sidelobes of the window spectra.

Spectral spreading or smearing A blurring, introduced in the spectral shape by the time-windowing operation, which affects the ability to resolve sharp peaks in the spectrum. This is due to the finite nonzero width of the mainlobe of the window spectra.

Spectrogram A time-frequency plot used to illustrate time-varying spectral distribution of power. It is computed using short-time DFT.

Time-domain aliasing When the DTFT is sampled at N equally-spaced frequencies followed by an N-point IDFT, the resulting periodic sequence is obtained by adding overlapping shifted replicas of the original sequence creating aliasing.

Uncertainty principle A principle similar to one in quantum mechanics. It states that the duration, σ_t, and bandwidth, σ_Ω, of any signal cannot be arbitrarily small simultaneously, that is, $\sigma_t\sigma_\Omega \geq 1/2$.

Zero-padding Appending zeros to a sequence prior to taking its DFT which results in a densely sampled DTFT spectrum and is a practical approach for DTFT reconstruction.

Learning summary

- Direct computation of the N-point DFT using the defining formula

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \quad k = 0, 1, \ldots, N-1$$

requires $\mathrm{O}(N^2)$ complex operations. It is assumed that the twiddle factors $W_N^{kn} = \mathrm{e}^{-2\pi kn/N}$ are obtained from a look-up table.
- Fast Fourier Transform (FFT) algorithms reduce the computational complexity from $\mathrm{O}(N^2)$ to $\mathrm{O}(N \log_2 N)$ operations. This immense reduction in complexity is obtained by a divide-and-conquer strategy which exploits the periodicity and symmetry properties of the twiddle factors.
- The decimation-in-time radix-2 FFT algorithm, which requires $N = 2^\nu$, is widely used because it is easy to derive, simple to program, and extremely fast. More complicated FFTs can reduce the number of operations, compared to the radix-2 FFT, only by a small percentage.
- For many years the time for the computation of FFT algorithms was dominated by multiplications and additions. The performance of FFTs on current computers depends upon the structure of the algorithm, the compiler, and the architecture of the machine.
- For applications which do *not* require all N-DFT coefficients, we can use algorithms based on linear filtering operations; these include Goertzel's algorithm and the chirp transform algorithm.

TERMS AND CONCEPTS

Bit-reversed order The sequential arrangement of data values in which the order is determined by reversing the bit arrangement from its natural order, that is, the bit arrangement $b_B \cdots b_1 b_0$ in natural order is reversed to $b_0 b_1 \cdots b_B$.

Butterfly computation A set of computations that combines the results of smaller DFTs into a larger DFT, or vice versa (that is, breaking a larger DFT up into smaller DFTs). Also known as a merging formula.

Chirp z-transform (CZT) A modified version of the CTA that can be used to compute z-transform over a spiral contour in the z-plane.

Chirp signal A sinusoidal signal with a frequency that grows linearly with time.

Chirp transform algorithm (CTA) A DFT algorithm that uses convolution with a chirp signal as the computation strategy, which is useful in performing high-density narrowband spectral analysis.

Computational cost or complexity The number of arithmetic operations needed to compute one DFT coefficient (per sample cost) or all coefficients (algorithm cost). It is defined using several metrics: real/complex additions, real/complex multiplications, etc. It is stated using the $\mathrm{O}(\cdot)$ notation.

Decimation-in-frequency (DIF) FFT A class of divide-and-conquer FFT algorithms that decompose the DFT sequence $X[k]$ into smaller sequences for DFT computations.

Decimation-in-time (DIT) FFT A class of divide-and-conquer FFT algorithms that decompose the signal sequence $x[n]$ into smaller sequences for DFT computations.

DFT matrix An $N \times N$ matrix formed using Nth roots of unity and denoted by $\boldsymbol{W}_N$.

Direct DFT algorithm A direct implementation of the DFT algorithm to compute coefficients at a computational cost proportional to N^2.

Discrete Fourier Transform (DFT) A transform-like operation on a finite-length N-point sequence $x[n]$ resulting in a finite-length N-point sequence $X[k]$ given by

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}.$$

Divide-and-conquer approach A computation scheme in which an exponential cost of computation is reduced significantly by dividing a long sequence into smaller sub-sequences, performing the computations on sub-sequences, and then combining results into the larger computation.

Fast Fourier Transform (FFT) The Cooley–Tukey class of efficient DFT computation algorithms designed to obtain all DFT coefficients as a block, with computational cost proportional to $N\log_2 N$.

FFTW algorithm Acronym for the Fastest Fourier Transform in the West algorithm which is a software library that optimizes FFT implementations for the best performance given a target computer architecture.

Goertzel's algorithm A recursive algorithm that is efficient for the computation of some DFT coefficients and that uses only the periodicity property of the twiddle factors.

Horner's rule An efficient polynomial computation technique in which exponential evaluations are replaced by nested multiplications.

In-place algorithm An algorithm that uses the same memory locations to store both the input and output sequences.

Merging formula An operation that combines two shorter-length DFTs into a longer-length DFT. Also known as a FFT butterfly.

Mixed radix FFT algorithm A divide-and-conquer algorithm in which length N can be factored as $N = R_1 R_2 \dots R_M$ for splitting and merging.

Natural order The sequential arrangement of data values according to the order in which they are obtained in nature, that is, in order from $1, 2, \dots,$ and so on.

Prime factor algorithm (PFA) A family of more efficient algorithms for length N that can be factored into relatively prime factors in which efficiency is achieved by eliminating all twiddle factors.

Quick Fourier transform (QFT) An algorithm to compute DFT that systematically uses basic symmetries in the DFT definition to achieve efficiency through real arithmetic.

Radix-2 FFT algorithm The basic Cooley–Tukey algorithm that uses length N as a power of 2 for splitting sequences into two sub-sequences of equal length at each stage of the algorithm.

Radix-R FFT algorithm A divide-and-conquer algorithm that uses length N as a power of R for splitting sequences into R sub-sequences of equal length at each stage of the algorithm.

Reverse carry algorithm A recursive and efficient algorithm to determine the bit-reversed order that propagates a carry from right-to-left (or a reverse carry).

Shuffling operation A reordering of sequences performed prior to merging (DIT-FFT) or after merging (DIF-FFT) operations. For radix-2 FFT, this shuffling results in bit-reversed ordering.

Sliding DFT (SDFT) A recursive algorithm that computes shorter fixed-length DFTs at each n for a very-long length signal to obtain a time-frequency plot or spectrogram that captures time-varying spectral properties of the signal.

Split-radix FFT algorithm (SR-FFT) A specialized algorithm that for $N = 2^\nu$ uses radix-2 DIF-FFT for computation of even-indexed coefficients and radix-4 DIF-FFT for computation of odd-indexed coefficients resulting in an overall faster algorithm.

Spectrogram A 2D time–frequency plot that captures time-varying spectral properties of a signal.

Twiddle factor The root-of-unity, $W_N = e^{-j\frac{2\pi}{N}}$, complex multiplicative constants in the butterfly operations of the Cooley–Tukey FFT algorithm.

Winograd Fourier transform algorithm (WFTA) An FFT algorithm that minimizes the number of multiplications at the expense of an increased number of additions.

Zoom FFT algorithm (ZFA) An efficient algorithm that computes DFT over a narrow band of frequencies along the unit circle without using the convolution approach used in CTA.

MATLAB functions and scripts

Name	Description	Page
`dftdirect`*	Direct computation of the DFT	436
`fftrecur`*	Recursive computation using divide & conquer	439
`bitrev`*	Bit-reversal algorithm based on Gold and Rader (1969)	446
`fftditr2`*	Decimation-in-time radix-2 FFT algorithm	449
`gafft`*	Goertzel's algorithm	461
`fft`	Fast algorithm for computation of the 1D-DFT	458
`fft2`	Fast algorithm for computation of the 2D-DFT	429
`fftshift`	Moves the zero-frequency component to the center	458
`cta`*	Chirp transform algorithm	464
`czt`*	Chirp z-transform	465
`ifft`	Fast algorithm for computation of the inverse 1D-DFT	458
`ifft2`	Fast algorithm for computation of the inverse 2D-DFT	429
`ifftshift`	Moves time-origin component to the center	458
`zfa`*	Zoom FFT algorithm	466

*Part of the MATLAB toolbox accompanying the book.

FURTHER READING

1. A detailed treatment of FFT algorithms, at the same level as in this book, is given in Oppenheim and Schafer (2010), Proakis and Manolakis (2007), Mitra (2006), and Brigham (1988).
2. The classical introduction to the FFT is given by Cochran *et al.* (1967) in a paper entitled "What is the Fast Fourier Transform?" The history of the FFT is discussed by Cooley *et al.* (1967), Cooley (1992), and Heideman *et al.* (1984).
3. Van Loan (2000) provides a unified derivation and organization of most known FFT algorithms using a matrix factorization approach. A detailed algebraic treatment of FFT algorithms is given in Blahut (2010). Duhamel and Vetterli (1990) present a tutorial review of various FFT algorithms.
4. The two-dimensional FFT and its applications in image processing are discussed in Gonzalez and Woods (2008) and Pratt (2007). Applications of three-dimensional FFT to video processing can be found in Woods (2006).