CSD2100 Worksheet: 2D View Transforms

1. Let A be the affine transformation matrix obtained by (1) translation by $\langle 9,-7\rangle$ followed by (2) uniform scaling by 4 with respect to the origin, followed by (3) translation by $\langle 4,8\rangle$. Find the 3×3 homogeneous coordinates matrix for A^{-1} without computing matrix A. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   Matrix A can be symbolically expressed as:

$$\mathbf{A} = \mathbf{T}_{\langle 4,8\rangle} \circ \mathbf{H}_{4,4} \circ \mathbf{T}_{\langle 9,-7\rangle}$$

   Therefore, the inverse of A is:

$$\begin{aligned}
\mathbf{A}^{-1} &= \left(\mathbf{T}_{\langle 4,8\rangle} \circ \mathbf{H}_{4,4} \circ \mathbf{T}_{\langle 9,-7\rangle}\right)^{-1} = \mathbf{T}^{-1}_{\langle 9,-7\rangle} \circ \mathbf{H}^{-1}_{4,4} \circ \mathbf{T}^{-1}_{\langle 4,8\rangle} \\
&= \mathbf{T}_{\langle -9,7\rangle} \circ \mathbf{H}_{\frac{1}{4},\frac{1}{4}} \circ \mathbf{T}_{\langle -4,-8\rangle} = \left(\mathbf{T}_{\langle -9,7\rangle} \circ \mathbf{H}_{\frac{1}{4},\frac{1}{4}}\right) \circ \mathbf{T}_{\langle -4,-8\rangle} \\
&= \begin{bmatrix} \frac{1}{4} & 0 & -9 \\ 0 & \frac{1}{4} & 7 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \\
\implies \mathrm{A}^{-1} &= \boxed{\begin{bmatrix} \frac{1}{4} & 0 & -10 \\ 0 & \frac{1}{4} & 5 \\ 0 & 0 & 1 \end{bmatrix}}
\end{aligned}$$

2. Use the formula for the inverse of a 2×2 matrix to find the inverse of matrix $\begin{bmatrix} 8 & -5 & -2 \\ -4 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix}$. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   First observe that the inverse of the linear (scaling and rotational) part is

$$\begin{bmatrix} 8 & -5 \\ -4 & 3 \end{bmatrix}^{-1} = \frac{1}{(8)(3)-(-5)(-4)} \begin{bmatrix} 3 & 5 \\ 4 & 8 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 5 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{5}{4} \\ 1 & 2 \end{bmatrix}$$

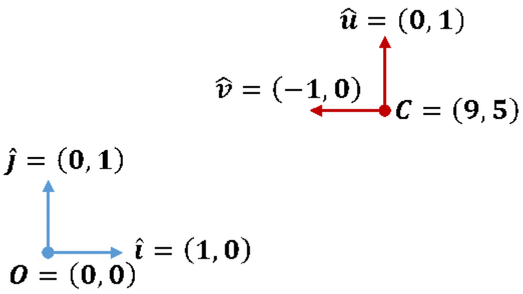
   Expanding the formula for the inverse of 2×2 matrix to a 3×3 matrix, we've:

$$\begin{aligned}
\begin{bmatrix} 8 & -5 & -2 \\ -4 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix}^{-1} &= \left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 8 & -5 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1} \\
&= \begin{bmatrix} 8 & -5 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \circ \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \frac{3}{4} & \frac{5}{4} & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\
\implies \begin{bmatrix} 8 & -5 & -2 \\ -4 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix}^{-1} &= \boxed{\begin{bmatrix} \frac{3}{4} & \frac{5}{4} & -6 \\ 1 & 2 & -10 \\ 0 & 0 & 1 \end{bmatrix}}
\end{aligned}$$

3. Consider a camera located at position $(9, 5)$ in a game world defined in the canonical world coordinate system. The camera's orientation is specified by a right vector $\langle 0, 1\rangle$ and an up vector $\langle -1, 0\rangle$. The camera's window to the world has width 20 units and height 30 units with the window's center coincident with the camera's position. Note that all positions and vectors are defined in the world coordinate system. Some of the elements may evaluate to fractions and they must be written as rational expressions.

    1. Find the 3×3 homogeneous coordinates matrix for transforming points defined in the camera coordinate system to the world coordinate system.

        The camera coordinate system is defined relative to the canonical world coordinate system with its origin located at $C = (9, 5)$, its right axis in direction $\hat{u} = \langle 0, 1\rangle$, and up axis in direction $\hat{v} = \langle -1, 0\rangle$. The following picture illustrates the (child) camera coordinate system in the (parent) world coordinate system:

        The matrix M for transforming a point defined in the camera coordinate system into the world coordinate system can be written by stacking from left to right the orientations of the right axis, the up axis, and the origin of the camera coordinate system:

        $$\mathbf{M} = [\hat{u} \quad \hat{v} \quad C] = \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & C_{\hat{i}} \\ u_{\hat{j}} & u_{\hat{j}} & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 & 9 \\ 1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}}$$

        Confirm that matrix M does transform points defined in camera coordinate system into the world coordinate system by transforming the following points: the point $(0, 0)$ which represents the origin of the camera coordinate system and must have world coordinates $(9, 5)$; the point $(1, 0)$ which represents a point on the camera's right axis and must have world coordinates $(9, 6)$; the point $(0, 1)$ which represents a point on the camera's up axis and must have world coordinates $(8, 5)$:

        $$\begin{bmatrix} 0 & -1 & 9 \\ 1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 8 \\ 5 & 6 & 5 \\ 1 & 1 & 1 \end{bmatrix}$$

    2. Find the 3×3 homogeneous coordinates matrix for transforming points defined in world coordinate system to the camera coordinate system.

        We must compute M^{-1} where M is the matrix computed in the previous question. Since $\mathrm{M} = [\hat{u} \quad \hat{v} \quad C]$, we can write M^{-1} as:

        $$\mathbf{M}^{-1} = [\hat{u} \quad \hat{v} \quad C]^{-1} = \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & C_{\hat{i}} \\ u_{\hat{j}} & u_{\hat{j}} & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \left(\begin{bmatrix} 1 & 0 & C_{\hat{i}} \\ 0 & 1 & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & 0 \\ u_{\hat{j}} & u_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$

        $$\implies \mathbf{M}^{-1} = \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & 0 \\ u_{\hat{j}} & u_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \circ \begin{bmatrix} 1 & 0 & C_{\hat{i}} \\ 0 & 1 & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Since $\begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & 0 \\ u_{\hat{j}} & v_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a rotation matrix, its inverse is its transpose:

$$\begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & 0 \\ u_{\hat{j}} & u_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & 0 \\ u_{\hat{j}} & u_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} u_{\hat{i}} & u_{\hat{j}} & 0 \\ v_{\hat{j}} & u_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\begin{bmatrix} 1 & 0 & C_{\hat{i}} \\ 0 & 1 & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix}$ is a translation matrix, its inverse can be written as:

$$\begin{bmatrix} 1 & 0 & C_{\hat{i}} \\ 0 & 1 & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -C_{\hat{i}} \\ 0 & 1 & -C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we can write M^{-1} as:

$$\mathbf{M}^{-1} = \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & 0 \\ u_{\hat{j}} & u_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \circ \begin{bmatrix} 1 & 0 & C_{\hat{i}} \\ 0 & 1 & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} u_{\hat{i}} & u_{\hat{j}} & 0 \\ v_{\hat{j}} & u_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -C_{\hat{i}} \\ 0 & 1 & -C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \mathbf{M}^{-1} = \begin{bmatrix} u_{\hat{i}} & u_{\hat{j}} & -\hat{u} \cdot C \\ v_{\hat{j}} & u_{\hat{j}} & -\hat{v} \cdot C \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 9 \\ 0 & 0 & 1 \end{bmatrix}}$$

3. Find the 3×3 homogeneous coordinates matrix for transforming camera coordinates to the NDC defined by the standard square, where $x^n \in [-1, 1]$ and $y^n \in [-1, 1]$.

   We're to map the *window to the world* in camera coordinate system to the standard square. In camera coordinate system, the window-to-the-world rectangle is the axis aligned rectangle with width $w = 20$, height $h = 30$, and centered at the origin. Since the standard square is centered at the origin and has width and length of 2, the required matrix is just $\mathrm{H}_{\frac{2}{20}, \frac{2}{30}}$:

$$\boxed{\begin{bmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{15} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

4. Consider a camera located at position $(7, 1)$ in a game world defined in the canonical world coordinate system. The camera's orientation is specified by a right vector $\langle \frac{4}{5}, -\frac{3}{5} \rangle$ and an up vector $\langle \frac{3}{5}, \frac{4}{5} \rangle$. The camera's window to the world has width 60 units and height 30 units with the window's center coincident with the camera's position. Note that all positions and vectors are defined in the world coordinate system. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   1. Compute the 3×3 homogeneous coordinates matrix for transforming points defined in the camera coordinate system to the world coordinate system.

$$\mathbf{M} = [\hat{u} \quad \hat{v} \quad C] = \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & C_{\hat{i}} \\ u_{\hat{j}} & u_{\hat{j}} & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 7 \\ -\frac{3}{5} & \frac{4}{5} & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

2. Find the 3×3 homogeneous coordinates matrix for transforming points defined in world coordinate system to the camera coordinate system.

$$\mathbf{M}^{-1} = \begin{bmatrix} u_{\hat{i}} & u_{\hat{j}} & -\hat{u} \cdot C \\ v_{\hat{j}} & u_{\hat{j}} & -\hat{v} \cdot C \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & -5 \\ \frac{3}{5} & \frac{4}{5} & -5 \\ 0 & 0 & 1 \end{bmatrix}}$$

3. Find the 3×3 homogeneous coordinates matrix for transforming camera coordinates to the NDC defined by the unit square, where $x^n \in [0,1]$ and $y^n \in [0,1]$.

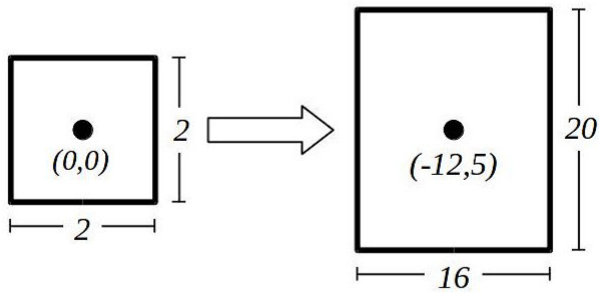
   Here we're to find the transformation that maps the *window-to-the-world* rectangle in camera coordinate system to the square centered at $(\frac{1}{2}, \frac{1}{2})$ with sides of length 1. Since the *window-to-the-world* rectangle in the camera coordinate system is centered at the origin and has width $w = 60$ and height $h = 30$, the desired transformation matrix is:

$$\mathbf{T}_{\langle \frac{1}{2}, \frac{1}{2} \rangle} \circ \mathbf{H}_{\frac{1}{10}, \frac{1}{15}} = \boxed{\begin{bmatrix} \frac{1}{60} & 0 & \frac{1}{2} \\ 0 & \frac{1}{30} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}}$$

5. We're to display a figure using OpenGL. In model coordinate system, the figure is the standard square (that is, the 2×2 square centered at the origin). In world coordinate system, the figure is the axis-aligned rectangle with width 16, height 20, and center at position $(-12, 5)$. The figure is viewed using a camera with right vector $\langle 0, 1 \rangle$, up vector $\langle -1, 0 \rangle$ and a window to the world having center $(40, -15)$, width 10, and height 8. To display the figure, we're to use the standard OpenGL NDC. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   1. Find the 3×3 model-to-world transformation matrix for the figure.

      The following picture illustrates the required transformation:



      The desired transformation matrix is:

$$\mathbf{T}_{\langle -12, 5 \rangle} \circ \mathbf{H}_{\frac{16}{2}, \frac{20}{2}} = \boxed{\begin{bmatrix} 8 & 0 & -12 \\ 0 & 10 & 5 \\ 0 & 0 & 1 \end{bmatrix}}$$

   2. Find the 3×3 world-to-camera transformation matrix for the figure.

      First, we find the camera-to-world transformation matrix:

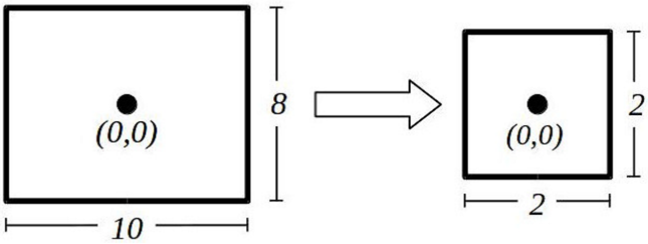
$$\mathbf{M} = [\hat{u} \quad \hat{v} \quad C] = \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & C_{\hat{i}} \\ u_{\hat{j}} & u_{\hat{j}} & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 40 \\ 1 & 0 & -15 \\ 0 & 0 & 1 \end{bmatrix}$$

   The world-to-camera transformation is:

$$\mathbf{M}^{-1} = \begin{bmatrix} u_{\hat{i}} & u_{\hat{j}} & -\hat{u}\cdot C \\ v_{\hat{j}} & u_{\hat{j}} & -\hat{v}\cdot C \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & 15 \\ -1 & 0 & 40 \\ 0 & 0 & 1 \end{bmatrix}}$$

3. Find the 3×3 camera-to-NDC transformation matrix for the figure.

   The mapping of the camera's *window-to-the-world* to the standard OpenGL NDC is shown by the following picture:



   The camera-to-NDC transformation matrix is just $\mathbf{H}_{\frac{2}{10},\frac{2}{8}}$:

$$\boxed{\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

4. Find the cumulative 3×3 transformation matrix that maps the figure from model coordinate system to NDC.

   The cumulative 3×3 transformation matrix that maps the figure from model coordinate system to NDC is obtained by concatenating the model-to-world, world-to-camera, and camera-to-NDC matrices, in that order:

$$\mathbf{H}_{\frac{1}{5},\frac{1}{4}} \circ \mathbf{M}^{-1} \circ \left(\mathbf{T}_{\langle -12,5\rangle} \circ \mathbf{H}_{\frac{16}{2},\frac{20}{2}}\right) = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 15 \\ -1 & 0 & 40 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 8 & 0 & -12 \\ 0 & 10 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \mathbf{N} = \boxed{\begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 13 \\ 0 & 0 & 1 \end{bmatrix}}$$