Introduction to Computer Graphics

2D Affine Geometry

Prasanna Ghali

2D Coordinate System

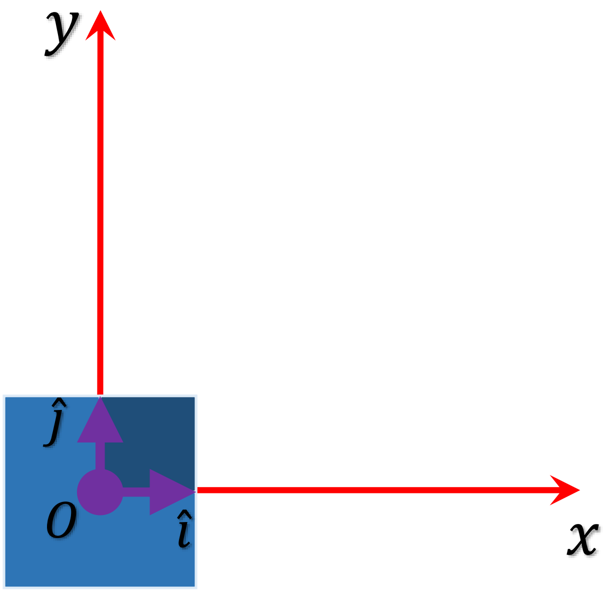
$$O = (0, 0)$$

$$\hat{\imath} = (1, 0)$$

$$\hat{\jmath} = (0, 1)$$

$$||\hat{\imath}|| = ||\hat{\jmath}|| = 1$$

$$\hat{\imath} \cdot \hat{\jmath} = 0$$



Homogeneous Coordinates (1/3)

- Points
  - Standard representation $P = (P_1, P_2) = \left(P_x, P_y\right) = \left(P_{\hat{\imath}}, P_{\hat{\jmath}}\right)$
  - Column vector representation

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} P_{\hat{\imath}} \\ P_{\hat{\jmath}} \end{bmatrix}$$

  - Homogeneous coordinate representation

$$P = \begin{bmatrix} P_1 \\ P_2 \\ 1 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} = \begin{bmatrix} P_{\hat{\imath}} \\ P_{\hat{\jmath}} \\ 1 \end{bmatrix}$$

Homogeneous Coordinates (2/3)

- Vectors
  - Standard representation $\vec{v} = \langle v_1, v_2 \rangle = \langle v_x, v_y \rangle = \langle v_{\hat{\imath}}, v_{\hat{\jmath}} \rangle$
  - Column vector representation

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_{\hat{\imath}} \\ v_{\hat{\jmath}} \end{bmatrix}$$

  - Homogeneous coordinate representation

$$P = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} v_{\hat{\imath}} \\ v_{\hat{\jmath}} \\ 0 \end{bmatrix}$$

Homogeneous Coordinates (3/3)

- Combining points and vectors
  - We may add, subtract, and multiply by scalars
  - Result is geometrically meaningful only when
    - The w- component is 1 (a point)
    - The w-component is 0 (a vector)
- Example

$$P = (2, 1), Q = (-4, 5), \vec{v} = \langle 3, 7 \rangle$$

$$3P - 2Q - \vec{v} = 3\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -14 \\ 1 \end{bmatrix}$$

Affine Transformations (1/2)

- Compositional notation: $\mathrm{A} = \mathrm{T}_{\vec{t}} \circ \mathrm{L}$
  - L is linear part
  - $\vec{t}$ is translation part
- Standard representation

$$\mathrm{A}(P) = \mathrm{L}(P) + \vec{t}$$

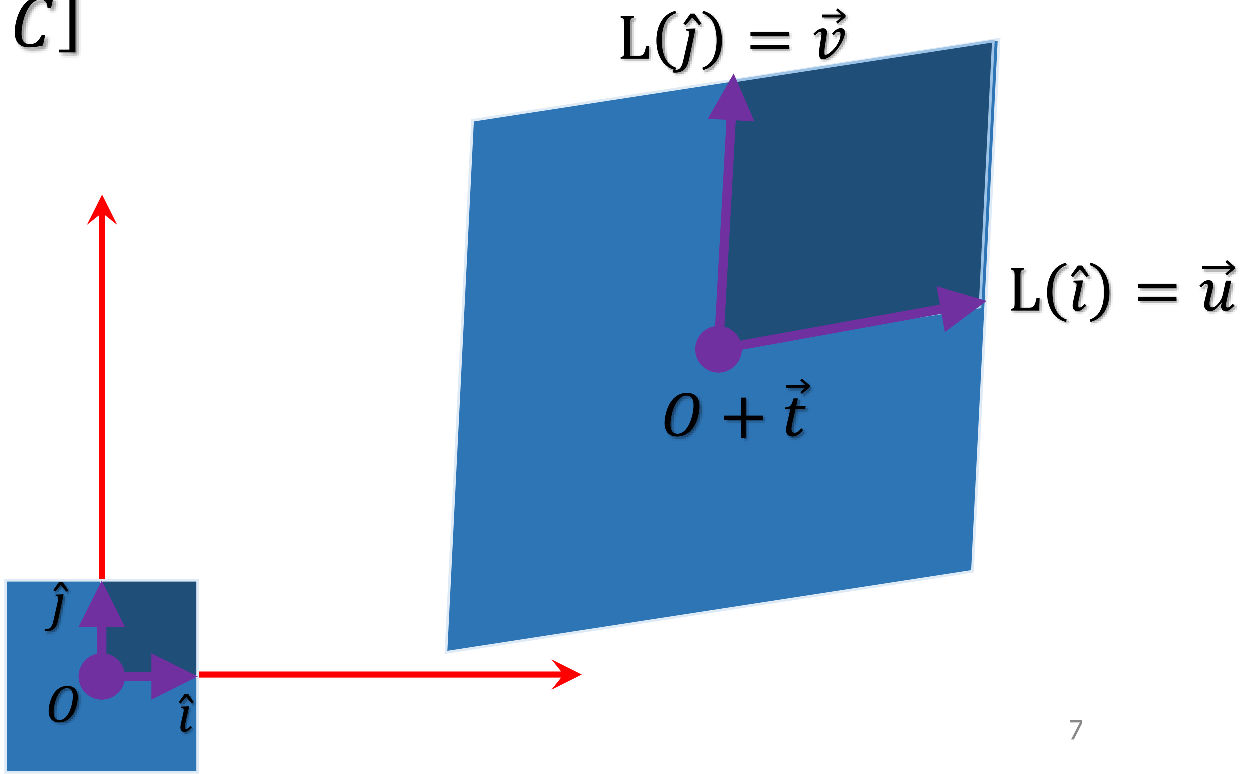
$$\mathrm{L} = \begin{bmatrix} \mathrm{L}_{11} & \mathrm{L}_{12} \\ \mathrm{L}_{21} & \mathrm{L}_{22} \end{bmatrix}, \vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

Affine Transformations (2/2)

- Homogeneous coordinate representation: $A = \begin{bmatrix} L_{11} & L_{12} & t_1 \\ L_{21} & L_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} L(\hat{\imath}) & L(\hat{\jmath}) & O + \vec{t} \end{bmatrix} = \begin{bmatrix} \vec{u} & \vec{v} & C \end{bmatrix}$$

$$A = \begin{bmatrix} u_{\hat{\imath}} & v_{\hat{\imath}} & C_{\hat{\imath}} \\ u_{\hat{\jmath}} & v_{\hat{\jmath}} & C_{\hat{\jmath}} \\ 0 & 0 & 1 \end{bmatrix}$$

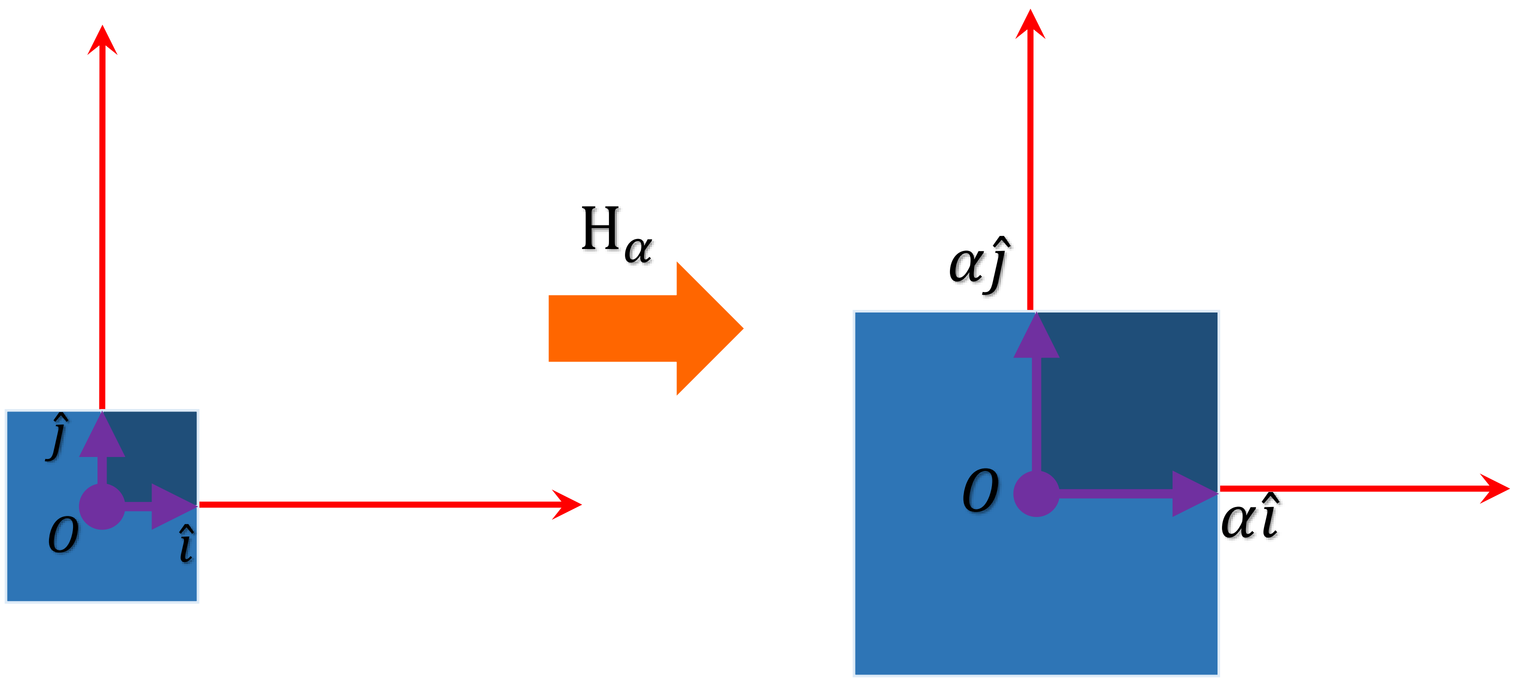


Basic Transformations

- Scaling
- Rotation
- Translation

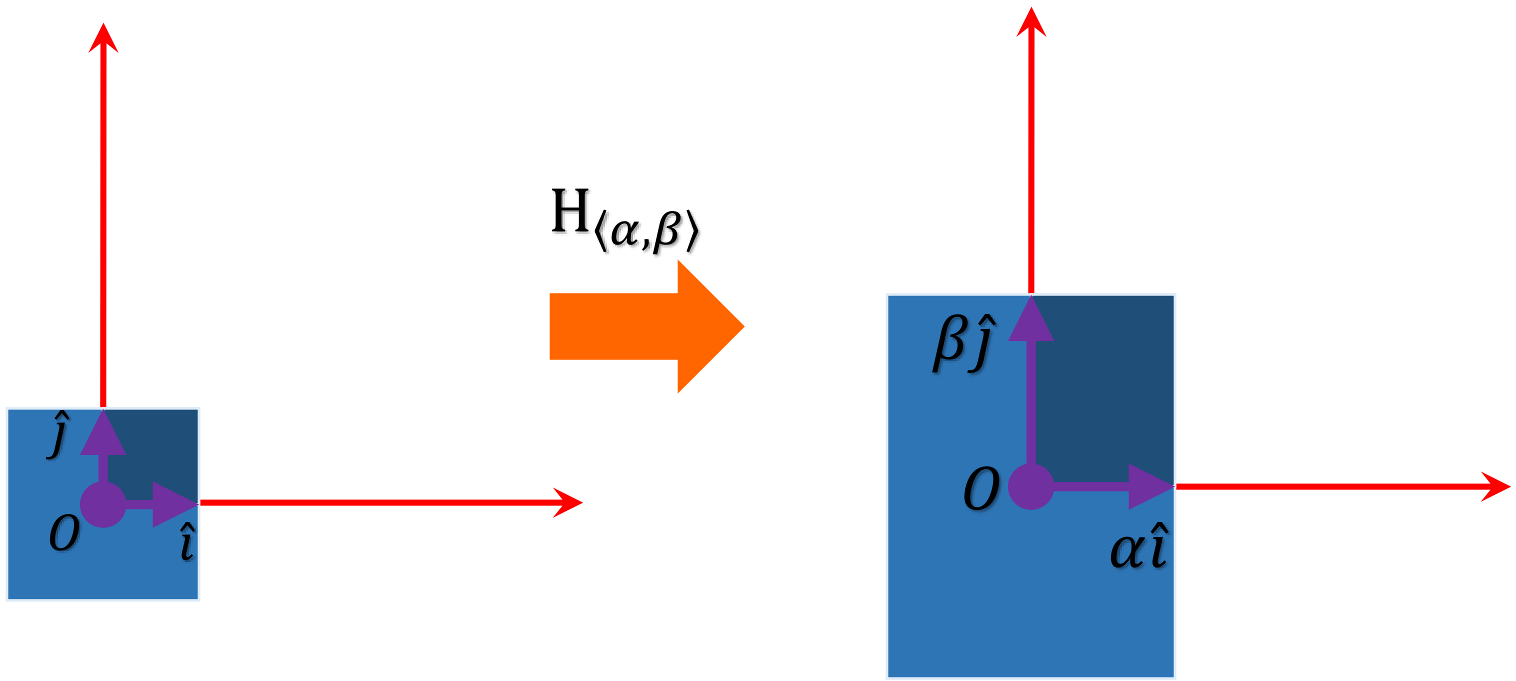
Scaling Transformations (1/2)

- Uniform $\mathrm{H}_\alpha = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Scaling Transformations (2/2)

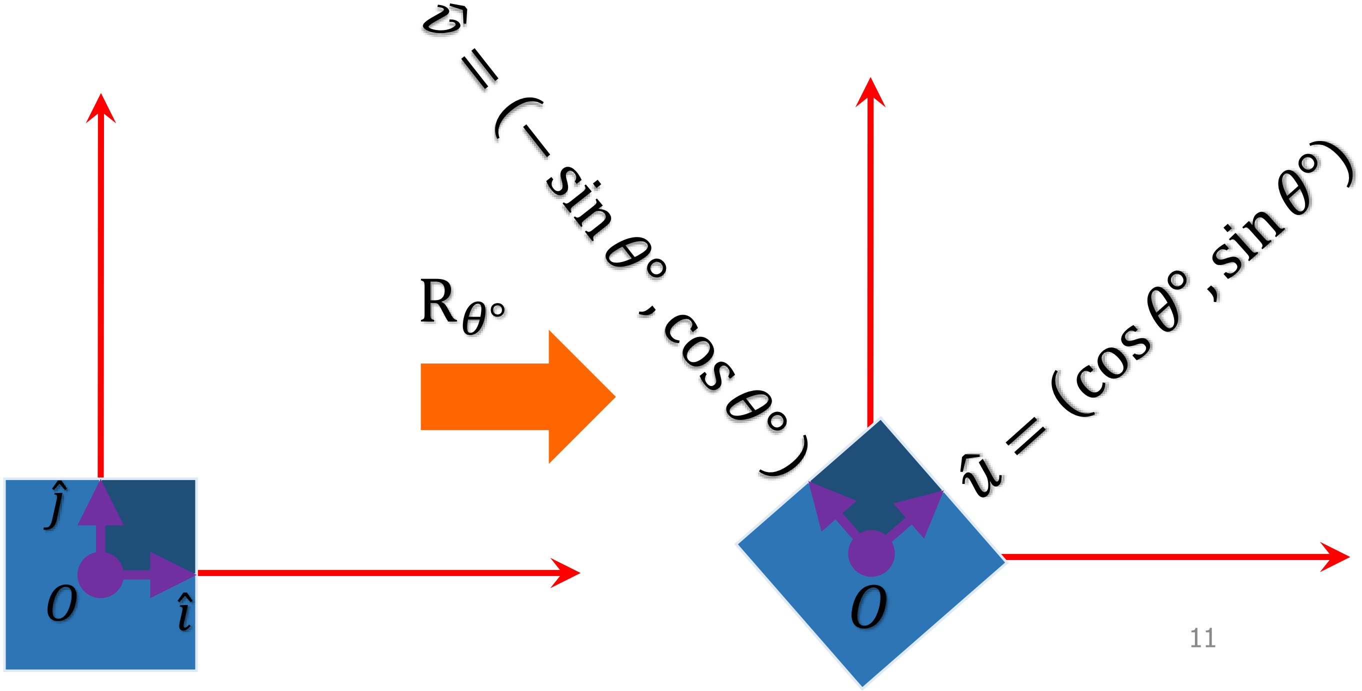
- Non-uniform $\mathrm{H}_{\langle\alpha,\beta\rangle} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Rotation Transformations

- Counterclockwise rotation about z axis by angle $\theta°$

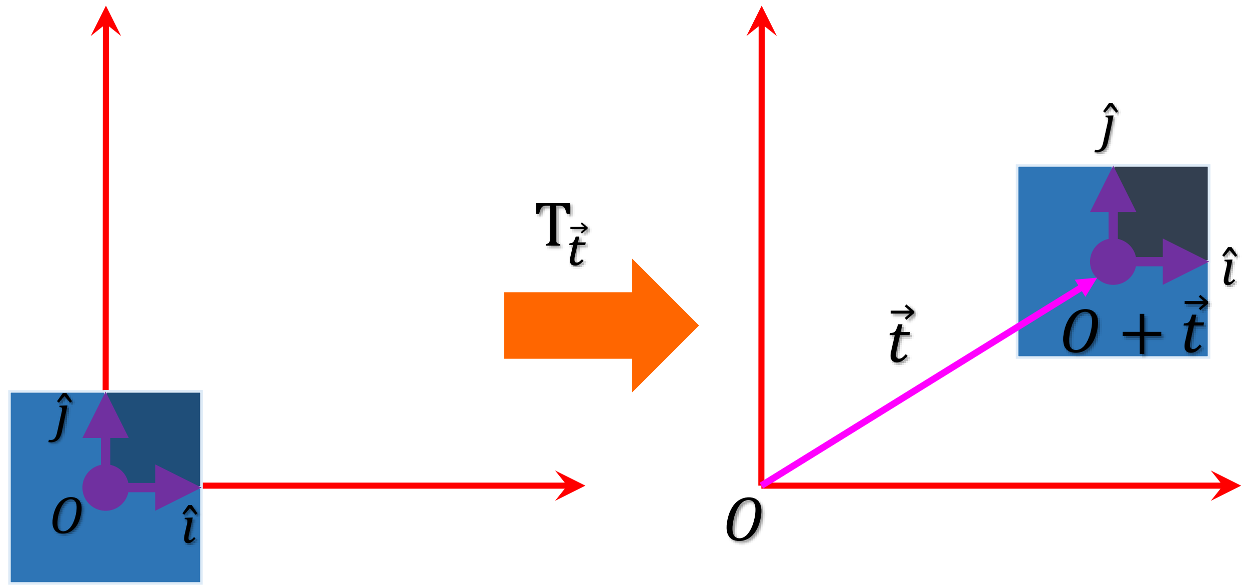
$$\mathrm{R}_{\theta°} = \begin{bmatrix} \cos\theta° & -\sin\theta° & 0 \\ \sin\theta° & \cos\theta° & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Translation Transformations

- Move or displace

$$\mathrm{T}_{\vec{t}} = \begin{bmatrix} 1 & 0 & t_i \\ 0 & 1 & t_j \\ 0 & 0 & 1 \end{bmatrix}$$

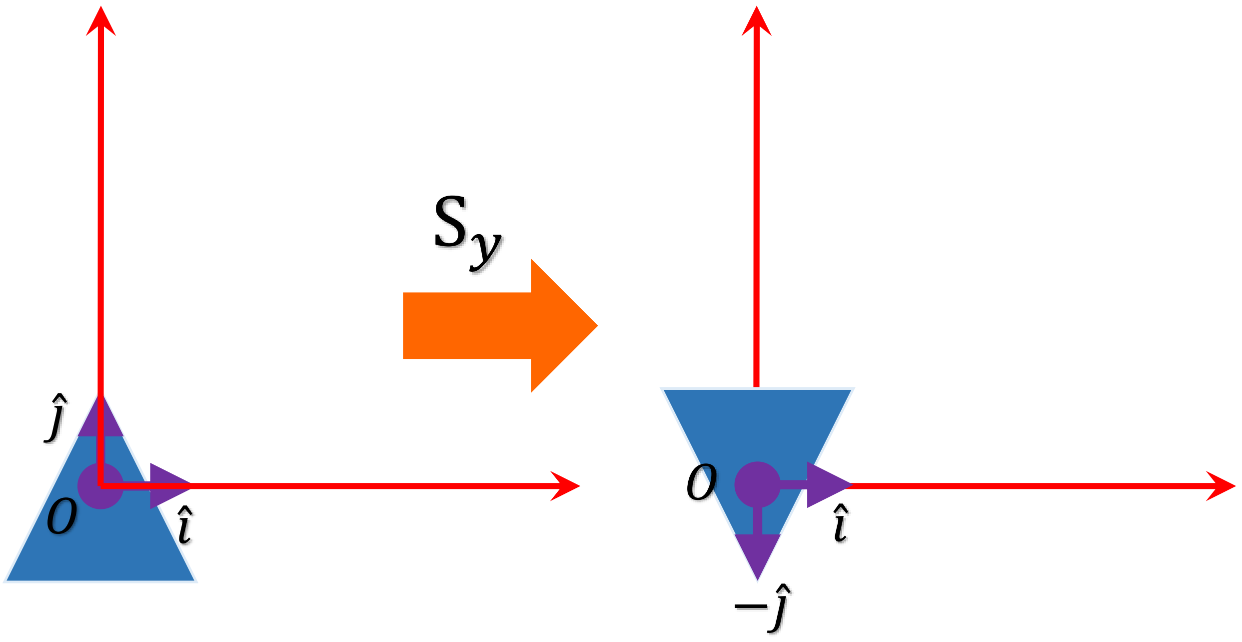


Reflection Transformations

- Mirroring
- Not a rotation by 180° about z axis!!!

$$S_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

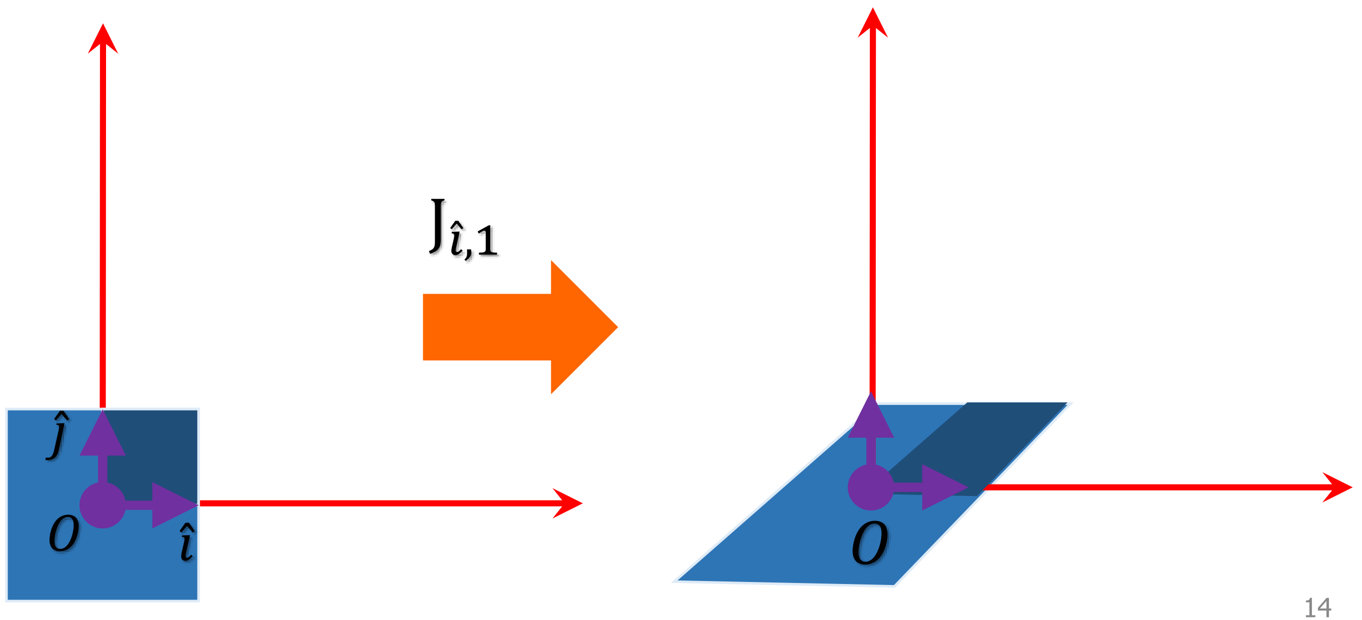


Shear Transformations

- Rectangle to parallelogram,
- For example, standard fonts become *italic*

$$\mathrm{J}_{\hat{\imath},\alpha} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{J}_{\hat{\jmath},\beta} = \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composing Transformations

- Transform A followed by transform B followed by C is
  - $\mathrm{C} \circ \mathrm{B} \circ \mathrm{A}$
- Order of composition is *right to left!!!*
- Applied to a point P: $(\mathrm{C} \circ \mathrm{B} \circ \mathrm{A})(P) = \mathrm{C}\Big(\mathrm{B}\big(\mathrm{A}(P)\big)\Big)$

Composing Transformations: Example

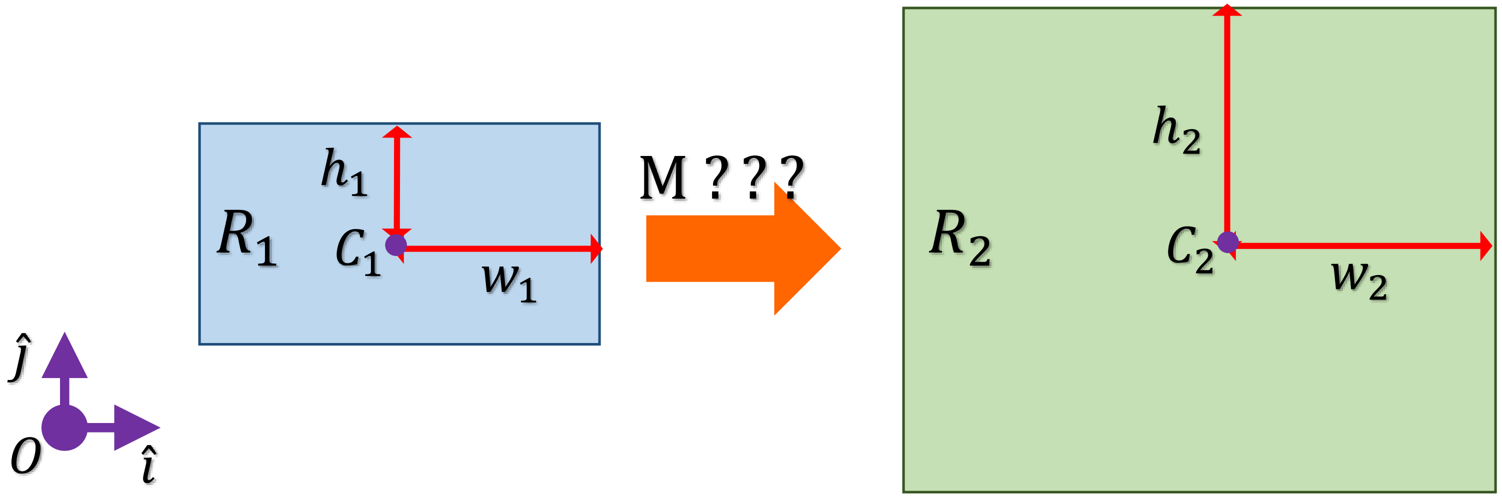
- Translation by $\vec{m} = \langle 3, 1 \rangle$, followed by
- Uniform scaling by 4, followed by
- Translation by $\vec{n} = \langle -2, 5 \rangle$

$$\mathrm{T}_{\vec{n}} \circ \mathrm{H}_4 \circ \mathrm{T}_{\vec{m}} = \mathrm{T}_{\vec{n}} \circ (\mathrm{H}_4 \circ \mathrm{T}_{\vec{m}}) = (\mathrm{T}_{\vec{n}} \circ \mathrm{H}_4) \circ \mathrm{T}_{\vec{m}}$$

$$= \begin{bmatrix} 4 & 0 & -2 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 10 \\ 0 & 4 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

Mapping Axis-Aligned Rectangles (1/2)

- Input: axis-aligned rectangles R_1 and R_2
- Wanted: transformation matrix that maps R_1 to R_2



Mapping Axis-Aligned Rectangles (2/2)

- Strategy
  - Translate so that R_1's center is origin
  - Scale so that R_1's geometry matches R_2
  - Translate origin to R_2's center

Axis-Aligned Rectangles: Example

- R_1: width 2, height 3, center $(9, -5)$
- R_2: width 6, height 6, center $(4, 7)$

$$\mathrm{T}_{\langle 4,7\rangle} \circ \mathrm{H}_{\langle 6/2,6/3\rangle} \circ \mathrm{T}_{\langle -9,5\rangle} = \left(\mathrm{T}_{\langle 4,7\rangle} \circ \mathrm{H}_{\langle 3,2\rangle}\right) \circ \mathrm{T}_{\langle -9,5\rangle}$$

$$= \begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -23 \\ 0 & 2 & 17 \\ 0 & 0 & 1 \end{bmatrix}$$

Action With Respect to a Point

- *Linear* transformation L acts with respect to origin
  - Rotation will rotate a thing about origin
- Corresponding *affine* transformation that has L acting with respect to point P is $\mathrm{T}_P \circ \mathrm{L} \circ \mathrm{T}_{-P}$

Action With Respect to a Point: Example

- Find action of $\mathrm{L} = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}$ wrt point $(4,5)$

$$\mathrm{T}_P \circ \mathrm{L} \circ \mathrm{T}_{-P} = (\mathrm{T}_P \circ \mathrm{L}) \circ \mathrm{T}_{-P}$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 3 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 7 \\ 3 & 0 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$