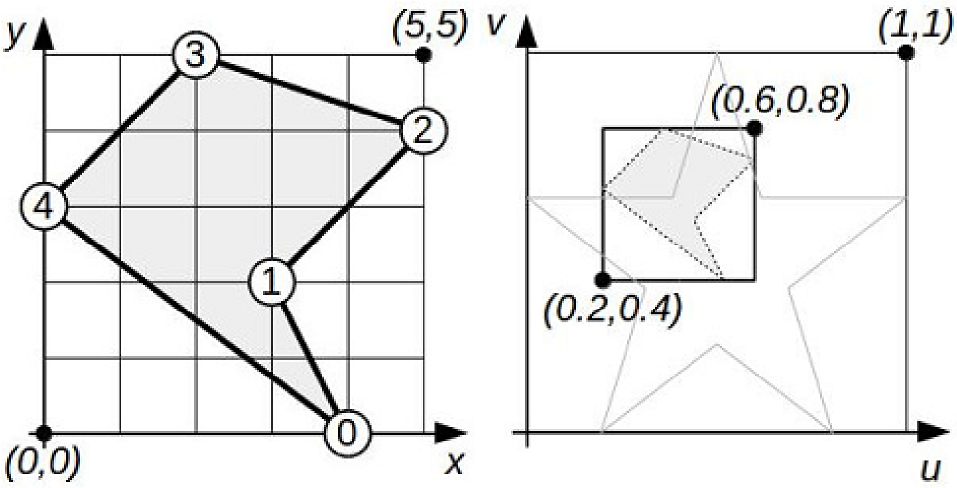
CSD2100 Worksheet: Texture Mapping [SOLUTIONS]

$[1]$ We are to texture map the model on the left so that the vertices of the model map to the points in texture space as indicated on the right.



1. Find the 3×3 model-to-texture transformation matrix that performs the desired mapping.

   From the picture, the source square box on the left having width of 5 units with bottom-left corner at $(0, 0)$ must be mapped to a destination square box on the right having width of 0.4 units with bottom-left corner at $(0.2, 0.4)$. To implement this mapping, the source box is first scaled by uniform scaling factor $s_{\hat{i}} = s_{\hat{j}} = \frac{0.4}{5}$ to have the same size as the destination box. Next, the scaled box is displaced by $\vec{t} = (0.2, 0.4) - (0, 0) = \langle 0.2, 0.4 \rangle$ to position it at the same location as the destination box. The 3×3 manifestation of this matrix is:

$$\mathrm{T}_{\vec{t}} \circ \mathrm{H}_{(s_{\hat{i}}, s_{\hat{j}})} = \begin{bmatrix} \vec{u} & \vec{v} & O + \vec{t} \end{bmatrix} = \begin{bmatrix} s_{\hat{i}} & 0 & t_{\hat{i}} \\ 0 & s_{\hat{j}} & t_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0.08 & 0 & 0.20 \\ 0 & 0.08 & 0.40 \\ 0 & 0 & 1 \end{bmatrix}}$$

2. What are the $2-$tuple texture coordinates that should be assigned to model vertex labeled 0?

   Vertex V_0 has position coordinates $(4, 0)$. Applying the above model-to-texture transformation:

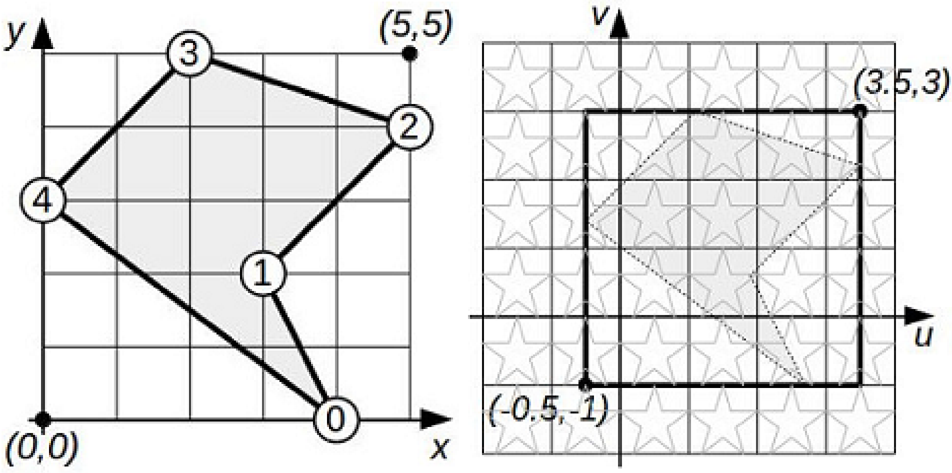
$$\begin{bmatrix} 0.08 & 0 & 0.2 \\ 0 & 0.08 & 0.4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.52 \\ 0.4 \\ 1 \end{bmatrix} \implies (s, t) = \boxed{(0.52, 0.40)}$$

3. What are the $2-$tuple texture coordinates that should be assigned to model vertex labeled 1?

   Vertex V_1 has position coordinates $(3, 2)$. Applying the above model-to-texture transformation:

$$\begin{bmatrix} 0.08 & 0 & 0.2 \\ 0 & 0.08 & 0.4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.44 \\ 0.56 \\ 1 \end{bmatrix} \implies (s, t) = \boxed{(0.44, 0.56)}$$

$[2]$ **We are now to map the vertices of the same model to the points in texture space as indicated in the following picture. Note that the texture will be tiled onto the model.**



1. Find the 3×3 model-to-texture transformation that performs the desired mapping.

   Even though the texture is tiled or repeated on the model, the model-to-texture transformation matrix is computed exactly as in Problem $[1]$. From the picture, the source square box on the left having width of 5 units with bottom-left corner at $(0, 0)$ must be mapped to a destination square box on the right having width of 4 units with bottom-left corner at $(-0.5, -1)$. To implement this mapping, the source box is first scaled by uniform scaling factor $s_{\hat{i}} = s_{\hat{j}} = \frac{4}{5}$ to have the same size as the destination box. Next, the scaled box is displaced by vector $\vec{t} = (-0.5, -1) - (0, 0) = \langle -0.5, -1 \rangle$ to position it at the same location as the destination box. The 3×3 manifestation of this matrix is:

$$\mathrm{T}_{\vec{t}} \circ \mathrm{H}_{(s_{\hat{i}}, s_{\hat{j}})} = \begin{bmatrix} \vec{u} & \vec{v} & O + \vec{t} \end{bmatrix} = \begin{bmatrix} s_{\hat{i}} & 0 & t_{\hat{i}} \\ 0 & s_{\hat{j}} & t_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0.80 & 0 & -0.50 \\ 0 & 0.80 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

2. What are the $2-$tuple texture coordinates that should be assigned to model vertex labeled 3?

   Vertex V_3 has position coordinates $(2, 5)$. Applying the above model-to-texture transformation:

$$\begin{bmatrix} 0.8 & 0 & -0.5 \\ 0 & 0.8 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.10 \\ 3 \\ 1 \end{bmatrix} \implies (s, t) = \boxed{(1.10, 3)}$$

   Since the texture is tiled, the texture coordinates are not clamped to the range $[0, 1]$. Instead, the triangle rasterizer will interpolate the (unclamped) texture coordinates across the triangle surface. At each fragment, the fragment shader will sample the texture image by first clamping texture coordinates (s, t) to texel coordinates (S, T).

3. What are the $2-$tuple texture coordinates that should be assigned to model vertex labeled 1?

   Vertex V_1 has position coordinates $(3, 2)$. Applying the above model-to-texture transformation:

$$\begin{bmatrix} 0.8 & 0 & -0.5 \\ 0 & 0.8 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.9 \\ 0.6 \\ 1 \end{bmatrix} \implies (s, t) = \boxed{(1.90, 0.60)}$$

[3] An image to be used as a texture map has width 50 pixels and height 50 pixels.

1. Find the 3×3 homogeneous coordinates matrix that maps texel coordinates to texture coordinates (that is, the unit square $0 \leq s \leq 1, 0 \leq t \leq 1$).

   The source texture map is a rectangle of size 50×50 with its bottom-left corner at $(0, 0)$. The destination texture space is a rectangle of size 1×1 with its bottom-left corner at $(0, 0)$. To map the destination rectangle to the source rectangle, the source rectangle is scaled by uniform scaling factor $s_{\hat{i}} = s_{\hat{j}} = \frac{1}{50}$:

$$\mathrm{H}_{(s_{\hat{i}}, s_{\hat{j}})} = [\vec{u} \quad \vec{v} \quad O] = \begin{bmatrix} s_{\hat{i}} & 0 & 0 \\ 0 & s_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{1}{50} & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \boxed{\begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

2. What are the $2-$tuple texture coordinates corresponding to texel coordinates $(28, 20)$?

   Applying the above 3×3 texel-to-texture matrix on texel $(28, 20)$:

$$\begin{bmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{1}{50} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 28 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{14}{25} \\ \frac{2}{5} \\ 1 \end{bmatrix}$$

$$\implies (s, t) = \boxed{\left(\frac{14}{25}, \frac{2}{5}\right)} = \boxed{(0.56, 0.40)}$$

3. Find the 3×3 homogeneous coordinate matrix that maps texture coordinates to texel coordinates.

   The texture-to-texel matrix is just the inverse of the previously computed 3×3 texel-to-texture matrix:

$$\begin{bmatrix} s_{\hat{i}} & 0 & 0 \\ 0 & s_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s_{\hat{i}}} & 0 & 0 \\ 0 & \frac{1}{s_{\hat{j}}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

4. Find the real-valued $2-$tuple texel coordinates (S_r, T_r) corresponding to texture coordinates $(s, t) = (0.487, 0.794)$.

$$\begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.487 \\ 0.794 \\ 1 \end{bmatrix} = \begin{bmatrix} 24.35 \\ 39.7 \\ 1 \end{bmatrix} \implies (S_r, T_r) = \boxed{(24.35, 39.70)}$$

5. Use point sampling to determine the integral-valued coordinates of the texel in the texture image for the real-valued texel computed in the previous problem.

$$(\lfloor S_r \rfloor, \lfloor T_r \rfloor) = (\lfloor 24.35 \rfloor, \lfloor 39.70 \rfloor) = (S, T) = \boxed{(24, 39)}$$

6. Assuming texture wrapping mode, compute the (integer) coordinates of the point sampled texel in the texture image corresponding to texture coordinates $(s, t) = (8.652, 5.896)$.

   Assuming texture wrapping, the sampled texture coordinates are:

$$wrap(s, t) = (s - \lfloor s \rfloor, t - \lfloor t \rfloor - t) = (8.652 - \lfloor 8.652 \rfloor, 5.896 - \lfloor 5.896 \rfloor) = (0.652, 0.896)$$
$$\implies (s', t') = (0.652, 0.896)$$

The real-valued texel coordinates (S_r, T_r) corresponding to sampled texture coordinates $(s', t') = (0.652, 0.896)$ are:

$$\begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.652 \\ 0.896 \\ 1 \end{bmatrix} = \begin{bmatrix} 32.6 \\ 44.8 \\ 1 \end{bmatrix} \implies (S_r, T_r) = (32.6, 44.8)$$

The point sampled texel closest to real-valued texel coordinates $(S_r, T_r) = (32.6, 44.8)$ are:

$$(\lfloor S_r \rfloor, \lfloor T_r \rfloor) = (\lfloor 32.6 \rfloor, \lfloor 44.8 \rfloor) = (S, T) = \boxed{(32, 44)}$$

$[4]$ **We are to render a figure using a texture. In model space, the figure has axis-aligned bounding box $\mathcal{B}$ that is centered at the origin with width 3, and height 2. The texture image has a width and height of 201 texels and 542 texels, respectively. The figure should be texture mapped in such a way that bounding box $\mathcal{B}$ maps to rectangle $\mathcal{R}$ in texel space, where (1) texel $(S, T) = (93, 175)$ is $\mathcal{R}$'s center, (2) $\mathcal{R}$'s width is 72 texels, and (3) $\mathcal{B}$ and $\mathcal{R}$ have the same aspect ratios.**

1. What is the aspect ratio of $\mathcal{B}$?

$$\begin{aligned} aspect\ ratio &= \frac{width}{height} = \frac{3}{2} \\ \implies aspect\ ratio &= \boxed{1.50} \end{aligned}$$

2. What is the height of $\mathcal{R}$?

$$\begin{aligned} height\ of\ \mathcal{R} &= \frac{\mathcal{R}'s\ width}{aspect\ ratio} = \frac{72}{1.5} \\ \implies height\ of\ \mathcal{R} &= \boxed{48} \end{aligned}$$

3. Find the 3×3 model-to-texel transform matrix that maps $\mathcal{B}$ to $\mathcal{R}$.

   The source AABB $\mathcal{B}$ having width 3, height 2, center $(0, 0)$ must be mapped to AABB $\mathcal{R}$ having width 72 units, height 48 units, and center $(93, 175)$. To implement this mapping, the AABB is first scaled by scaling factors $s_{\hat{i}} = \frac{72}{3} = 24$ and $s_{\hat{j}} = \frac{48}{3} = 24$ followed by a displacement $\vec{t} = (93, 175)$ from the origin. The 3×3 manifestation of this matrix is:

$$\mathrm{M}_1 = \mathrm{T}_{\vec{t}} \circ \mathrm{H}_{(s_{\hat{i}}, s_{\hat{j}})} = [\vec{u} \quad \vec{v} \quad O + \vec{t}] = \begin{bmatrix} s_{\hat{i}} & 0 & t_{\hat{i}} \\ 0 & s_{\hat{j}} & t_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 24 & 0 & 93 \\ 0 & 24 & 175 \\ 0 & 0 & 1 \end{bmatrix}}$$

4. Find the 3×3 texel-to-texture transform matrix that maps the texture image to texture coordinates. Write fractions as rational numbers.

   The source rectangular texture image with width 201 texels, height of 542 texels, and center $(0, 0)$ at bottom-left corner must be mapped to destination texture space with width 1 unit, height 1 unit, and center $(0, 0)$ at bottom-left corner. This mapping is implemented by scaling the source texture image by scaling factors $s_{\hat{i}} = \frac{1}{201}$ and $s_{\hat{j}} = \frac{1}{542}$. The 3×3 manifestation of this matrix is:

$$\mathrm{M}_2 = \mathrm{H}_{(s_{\hat{i}}, s_{\hat{j}})} = [\vec{u} \quad \vec{v} \quad O] = \begin{bmatrix} s_{\hat{i}} & 0 & 0 \\ 0 & s_{\hat{j}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{201} & 0 & 0 \\ 0 & \frac{1}{542} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

5. Find the 3×3 model-to-texture transform matrix that maps $\mathcal{B}$ to texture space.

$$\mathrm{M}_2 \circ \mathrm{M}_1 = \begin{bmatrix} \frac{1}{201} & 0 & 0 \\ 0 & \frac{1}{542} & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 24 & 0 & 93 \\ 0 & 24 & 175 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{24}{201} & 0 & \frac{93}{201} \\ 0 & \frac{24}{542} & \frac{175}{542} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \mathrm{M}_2 \circ \mathrm{M}_1 \approx \boxed{\begin{bmatrix} 0.1194 & 0 & 0.4627 \\ 0 & 0.0443 & 0.3229 \\ 0 & 0 & 1 \end{bmatrix}}$$

6. Compute $2-$tuple texture coordinates for $\mathcal{B}$'s lower left-hand corner.

   Since $\mathcal{B}$ has width 3, height 2, and center $(0,0)$, the lower left-hand corner is $(-1.5,-1)$. Applying the 3×3 model-to-texture transform matrix from question (5), we get

$$\begin{bmatrix} 0.1194 & 0 & 0.4627 \\ 0 & 0.0443 & 0.3229 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.5 \\ -1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.2836 \\ 0.2786 \\ 1 \end{bmatrix}$$

$$\boxed{\implies \text{lower left-hand corner} = (0.2836, 0.2786)}$$

7. Compute $2-$tuple texture coordinates for $\mathcal{B}$'s upper right-hand corner.

   Since $\mathcal{B}$ has width 3, height 2, and center $(0,0)$, the upper right-hand corner is $(1.5,1)$. Applying the 3×3 model-to-texture transform matrix from question (5), we get

$$\begin{bmatrix} 0.1194 & 0 & 0.4627 \\ 0 & 0.0443 & 0.3229 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.6418 \\ 0.3672 \\ 1 \end{bmatrix}$$

$$\boxed{\implies \text{upper right-hand corner} = (0.6418, 0.3672)}$$

8. The texture coordinate $(s,t) = (0.545, 0.328)$ is inside of the image of $\mathcal{B}$ in texture space. Compute integer-valued $2-$tuple coordinates of the point sampled texel in the texture image corresponding to texture coordinates (s,t).

   The 3×3 texture-to-texel transform matrix is

$$\mathrm{H}_{\langle 201,542 \rangle} = \begin{bmatrix} 201 & 0 & 0 \\ 0 & 542 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   We compute the corresponding texel coordinates:

$$\begin{bmatrix} 201 & 0 & 0 \\ 0 & 542 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.545 \\ 0.328 \\ 0 \end{bmatrix} = \begin{bmatrix} 109.545 \\ 177.776 \\ 0 \end{bmatrix}$$

$$\implies (S_r, T_r) = (109.545, 177.776)$$

   Thus, the nearest texel will have coordinates

$$(\lfloor S_r \rfloor, \lfloor T_r \rfloor) = (\lfloor 109.545 \rfloor, \lfloor 177.776 \rfloor)$$

$$\implies (S,T) = \boxed{(109, 177)}$$

9. Suppose each texel in the texture image is an RGB triplet requiring $24-$bits of storage. Also suppose the size of each row of texels is rounded up to a multiple of 4 bytes by padding with multiple padded rows stored consecutively. What is the offset into the texel data for the texel from previous question (8)?

First, we need to compute the stride (or width) in bytes of each row in the image:

$$\begin{aligned} (\text{minimum bytes}) &= 201 \cdot 3 = 603 & \frac{603}{4} &= 150\frac{3}{4} \\ (\text{pad}) &= 4 - 3 = 1 & \implies R &= (\text{minimum bytes}) + (\text{pad}) = 603 + 1 = 604 \end{aligned}$$

So, the offset for texel $(S, T) = (109, 177)$ is

$$(R)(T) + (3)(S) = (604)(177) + (3)(109) = \boxed{107235 \text{ bytes}}$$