Quiz: 2D Affine Transforms [Solutions]

1. Find the 3×3 matrix homogeneous coordinates matrix for the affine transformation obtained by (1) scaling by 5 with respect to the origin, followed by (2) translation by $\langle -2, 7\rangle$. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   Let $\vec{v} = \langle -2, 7\rangle$. Then

$$\mathbf{A} = \mathbf{T}_{\vec{v}} \circ \mathbf{H}_{5,5} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 0 & -2 \\ 0 & 5 & 7 \\ 0 & 0 & 1 \end{bmatrix}}$$

   Note that since the translation is on the left of the composition $\mathbf{A} = \mathbf{T}_{\vec{v}} \circ \mathbf{H}_{5,5}$, we can immediately write the composition as a single matrix without performing any matrix multiplication.

2. Find the 3×3 homogeneous coordinates matrix for an affine transformation obtained by (1) translation by $\langle 4, 9\rangle$, followed by (2) rotation by 90° counterclockwise about the origin. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   Set $\vec{v} = \langle 4, 9\rangle$. First we compute the 3×3 matrix for rotation by $\theta = 90^\circ$:

$$\mathbf{R}_{\theta^\circ} = \begin{bmatrix} \cos\theta^\circ & -\sin\theta^\circ & 0 \\ \sin\theta^\circ & \cos\theta^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Next we compute:

$$\mathbf{B} = \mathbf{R}_{\theta^\circ} \circ \mathbf{T}_{\vec{v}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 & -9 \\ 1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}}$$

   In this case, since the translation occurs on the right of the composition $\mathbf{B} = \mathrm{R}_{\theta^\circ} \circ \mathbf{T}_{\vec{v}}$, we must perform the matrix multiplication.

3. Find the 3×3 homogeneous coordinates matrix for scaling by 3 with respect to the point $(6, -1)$. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   Set $\vec{v} = (0, 0) - (6, -1) = \langle -6, 1\rangle$. Then the desired transformation is

$$\mathbf{T}_{\vec{v}}^{-1} \circ \mathbf{H}_{3,3} \circ \mathbf{T}_{\vec{v}} = \left(\mathbf{T}_{\vec{v}}^{-1} \circ \mathbf{H}_{3,3}\right) \circ \mathbf{T}_{\vec{v}} = \begin{bmatrix} 3 & 0 & 6 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 0 & -12 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

   As a consistency check, note that the point $(6, -1)$ is invariant to the transformation:

$$\begin{bmatrix} 3 & 0 & -12 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}$$

   That is, the point $(6, -1)$ is unaffected by scaling with respect to the point $(6, -1)$.

4. Find the 3×3 homogeneous coordinates matrix for rotation by 270° about the point $(8, 3)$. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   Rotation by $\theta = 270^\circ$ is given by the 3×3 matrix:

$$\mathbf{R}_{\theta^\circ} = \begin{bmatrix} \cos\theta^\circ & -\sin\theta^\circ & 0 \\ \sin\theta^\circ & \cos\theta^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 270^\circ & -\sin 270^\circ & 0 \\ \sin 270^\circ & \cos 270^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Set $\vec{v} = (0, 0) - (8, 3) = \langle -8, -3 \rangle$. Then the desired transformation is

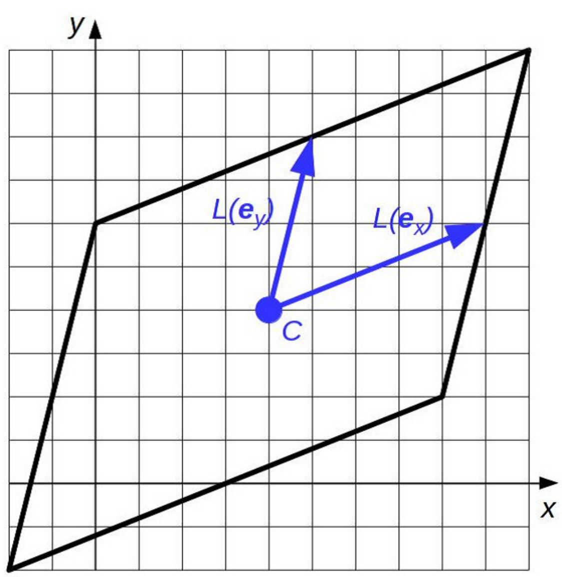
$$\mathbf{T}_{\vec{v}}^{-1} \circ \mathbf{R}_{\theta^\circ} \circ \mathbf{T}_{\vec{v}} = \left(\mathbf{T}_{\vec{v}}^{-1} \circ \mathbf{R}_{\theta^\circ}\right) \circ \mathbf{T}_{\vec{v}} = \begin{bmatrix} 0 & 1 & 8 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & 5 \\ -1 & 0 & 11 \\ 0 & 0 & 1 \end{bmatrix}}$$

   As a consistency check, note that the point $(8, 3)$ is invariant to the transformation:

$$\begin{bmatrix} 0 & 1 & 5 \\ -1 & 0 & 11 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix}$$

   That is, the rotation about point $(8, 3)$ necessarily leaves $(8, 3)$ unaffected by the rotation.

5. Find the 3×3 homogeneous coordinates matrix that sends the standard square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$, in this order, to the parallelogram with vertices $(10, 10)$, $(0, 6)$, $(-2, -2)$, $(8, 2)$, in this order. No computation is necessary! Some of the elements may evaluate to fractions and they must be written as rational expressions.

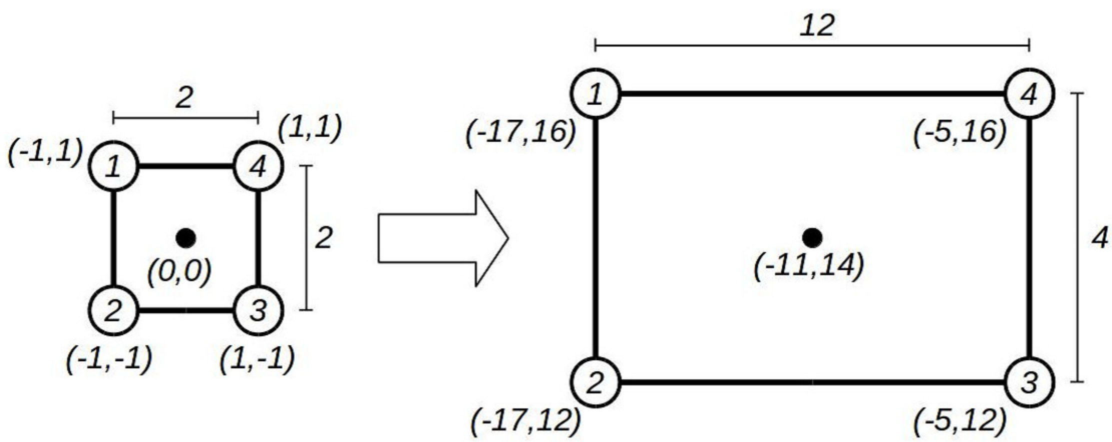


   From the figure above, we see that $\vec{e_x} = \langle 1, 0 \rangle$ maps to vector $\overrightarrow{L(\vec{e_x})} = \langle 5, 2 \rangle$; $\vec{e_y} = \langle 0, 1 \rangle$ maps to vector $\overrightarrow{L(\vec{e_y})} = \langle 1, 4 \rangle$, and origin $(0, 0)$ maps to point $P = (4, 4)$. Thus the necessary transformation is

$$\begin{bmatrix} \overrightarrow{L(\vec{e_x})} & \overrightarrow{L(\vec{e_y})} & P \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 1 & 4 \\ 2 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}}$$

6. Find the 3×3 homogeneous coordinates matrix for an affine transformation that maps the standard square with vertices $(-1, 1)$, $(-1, -1)$, $(1, -1)$, $(1, 1)$, in this order, to axis-aligned rectangle with vertices $(-17, 16)$, $(-17, 12)$, $(-5, 12)$, $(-5, 16)$, in this order. Some of the elements may evaluate to fractions and they must be written as rational expressions.

   The standard square is centered at the origin and has width and height of 2 units. We are to map this square into the rectangle whose center is $(-11, 14)$ and with width of 12 and height of 4 (why?). See the diagram below:



   Thus, the necessary transformation matrix $\mathbf{A}$ is obtained by (1) scaling along the $x-$axis by $\frac{12}{2} = 6$, and scaling along the $y-$axis by $\frac{4}{2} = 2$, followed by (2) translation by vector $\langle -11, 14 \rangle$:

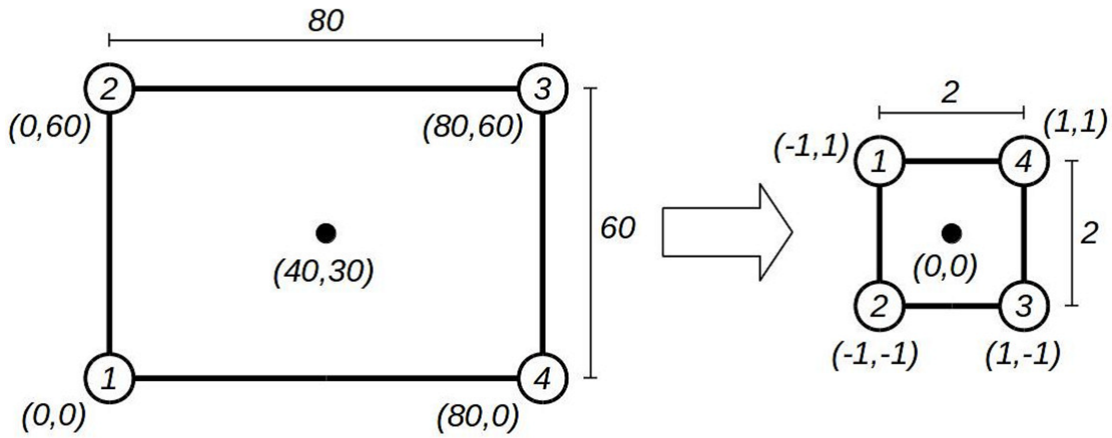
$$\mathbf{A} = \mathbf{T}_{\langle -11,14 \rangle} \circ \mathbf{H}_{6,2} = \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 14 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 0 & -11 \\ 0 & 2 & 14 \\ 0 & 0 & 1 \end{bmatrix}}$$

   We may check that A indeed maps the vertices of the standard square to the target vertices:

$$\begin{bmatrix} 6 & 0 & -11 \\ 0 & 2 & 14 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -17 & -17 & -5 & -5 \\ 16 & 12 & 12 & 16 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

7. Find the 3×3 homogeneous coordinates matrix for an affine transformation that maps the rectangle with vertices $(0, 0)$, $(0, 60)$, $(80, 60)$, $(80, 0)$, in this order, to the standard square with vertices $(-1, 1)$, $(-1, -1)$, $(1, -1)$, $(1, 1)$, in this order. Be careful of the vertex order! Some of the elements may evaluate to fractions and they must be written as rational expressions.

   We may obtain the necessary transformation matrix A by (1) translating the center of the rectangle $(40, 40)$ to origin $(0, 0)$ using displacement vector $\vec{v} = (0, 0) - (40, 30) = \langle -40, -30 \rangle$, followed by (2) scaling along the $x-$axis by $\frac{2}{80} = \frac{1}{40}$ and scaling along the $y-$axis by $\frac{-2}{60} = -\frac{1}{30}$ (we need to reflect along the $y-$axis in order to match up the vertices in the stated order).



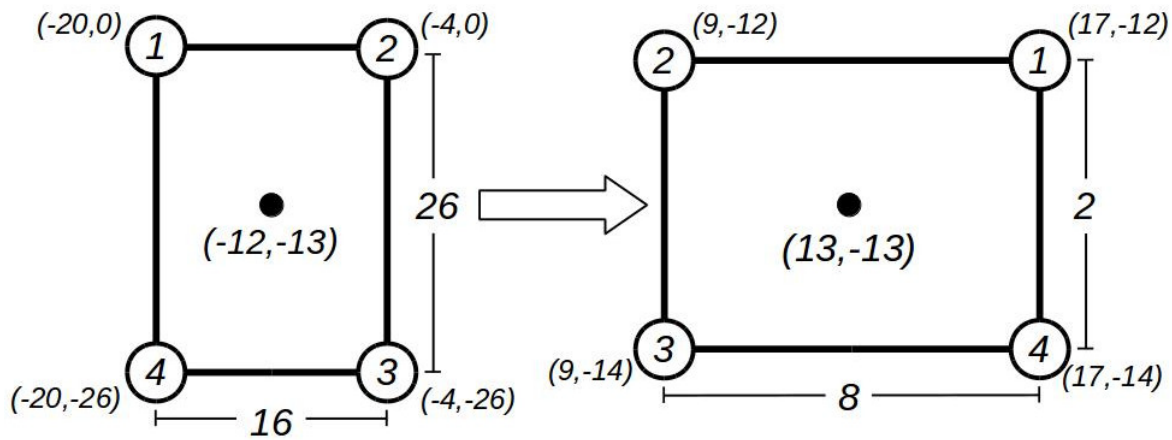
The required transformation matrix A is:

$$\mathbf{A} = \mathbf{H}_{\frac{1}{40},-\frac{1}{30}} \circ \mathbf{T}_{\langle -40,-30\rangle} = \begin{bmatrix} \frac{1}{40} & 0 & 0 \\ 0 & -\frac{1}{30} & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & -40 \\ 0 & 1 & -30 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{40} & 0 & -1 \\ 0 & -\frac{1}{30} & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

As an exercise, confirm that matrix A does indeed map the vertices of the rectangle to the target vertices.

8. Find the 3×3 homogeneous coordinates matrix for the affine transformation that maps the axis-aligned rectangle with vertices $(-20, 0)$, $(-4, 0)$, $(-4, -26)$, $(-20, -26)$, in this order, to the axis-aligned rectangle with respective vertices $(17, -12)$, $(9, -12)$, $(9, -14)$, $(17, -14)$, in this order. Be careful of the the vertex order! Some of the elements may evaluate to fractions and they must be written as rational expressions.

   The required mapping is obtained by (1) translating the center $(-12, -13)$ of the first rectangle to origin $(0, 0)$ using displacement vector $\vec{u} = (0, 0) - (-12, -13) = \langle 12, 13\rangle$ followed by (2) scaling along the $x-$axis by $-\frac{8}{16} = -\frac{1}{2}$ (the negative sign is needed to enforce the vertex order) and along the $y-$axis by $\frac{2}{26} = \frac{1}{13}$ followed by (3) translating from the origin $(0, 0)$ to the center $(13, -13)$ of the second rectangle using displacement vector $\vec{v} = (13, -13) - (0, 0) = \langle 13, -13\rangle$.



The required transformation matrix B is computed as:

$$\mathbf{B} = \mathbf{T}_{\vec{v}=\langle 13,-13\rangle} \circ \mathbf{H}_{-\frac{1}{2},\frac{1}{13}} \circ \mathbf{T}_{\vec{u}=\langle -12,-13\rangle} = \left(\mathbf{T}_{\vec{v}=\langle 13,-13\rangle} \circ \mathbf{H}_{-\frac{1}{2},\frac{1}{13}}\right) \circ \mathbf{T}_{\vec{u}=\langle -12,-13\rangle}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & 13 \\ 0 & \frac{1}{13} & -13 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 13 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \mathbf{B} = \boxed{\begin{bmatrix} -\frac{1}{2} & 0 & 7 \\ 0 & \frac{1}{13} & -12 \\ 0 & 0 & 1 \end{bmatrix}}$$