Least Squares Curve Fit

Example

°F	°C
32	0
212	100

Linear relationship

$y = mx + b$ where °F $\Rightarrow x$
°C $\rightarrow y$

x to be set
y to be measured
Solve for m + b

The measurement y is corrupted by many possible errors we would like to minimize.

$$Y_i' = mx_i + b + \mathcal{E}_i$$

Solve for the errors and minimize

$$\mathcal{E}_i = Y_i' - mx_i - b$$

Take N readings, sum the errors

$$\sum_{i=1}^{N} \mathcal{E}_i = \sum_{i=1}^{N} (Y_i - mx_i - b)$$

Properties of sums

$$\sum_{i=1}^{N} 1 = 1+1+1+ \cdots 1 \rightarrow \text{Add } 1 \text{ } n \text{ times}$$
$$= N$$

$$\sum_{i=1}^{N} c = c+c+ \cdots c \rightarrow \text{Add } c \text{ } n \text{ times}$$
$$= cN$$

$$\sum_{i=1}^{N} i = 1+2+3+ \cdots N = n(N+1)/2$$

We need a positive function to minimize like the sum of squares

$$\sum_{i=1}^{N} \varepsilon_i^2 = \sum_{i=1}^{N} (y_i - mx_i - b)^2$$

To minimize, we take the derivative and set to 0. Then solve for m+b.

$$\partial \sum_{i=1}^{N} \varepsilon_i^2 \Big/ \partial m = \sum_{i=1}^{N} 2(y_i - mx_i - b)(-x_i) = 0$$

$$\sum_{i=1}^{N} y_i x_i - m \sum_{i=1}^{N} x_i^2 - b \sum_{i=1}^{N} x_i = 0$$

$$S_{xy} - m S_{x^2} - b S_x = 0$$

$$\partial \sum_{i=1}^{N} \varepsilon_i^2 / \partial b = \sum_{i=1}^{N} 2(y_i - m x_i - b)(-1)$$

$$= \sum_{i=1}^{N} y_i - m \sum_{i=1}^{N} x_i - b \sum_{i=1}^{N} 1$$

$$S_y - m S_x - bN = 0$$

We have 2 equations, 2 unknowns
m + b so solve

$$1)\quad S_{yx} - m S_{x^2} - b S_x$$
$$2)\quad S_y - m S_x - b N$$

solving for b from 2)

$$b = (S_y - m S_x)/N$$

plug b back into 1) and solve for
m

$$S_{yx} - m S_{x^2} - S_x (S_y - m S_x)/N = 0$$

$$(S_{yx} - S_y S_x / N) + m (S_x^2 / N - S_{x^2}) = 0$$

$$m = \frac{(S_y S_x - N S_{yx})}{(S_x^2 - N S_{x^2})}$$

Check with the original $N=2$ exact values

X	Y
°F	°C
32	0
212	100

$$S_x = 32 + 212 = 244$$
$$S_y = 0 + 100 = 100$$
$$S_x^2 = 244^2 = 59{,}536$$
$$S_{xy} = 0 \cdot 32 + 100 \cdot 212 = 21{,}200$$
$$S_{x^2} = 32 \cdot 32 + 212 \cdot 212 = 1024 + 44944 = 45{,}968$$

$$m = \frac{(S_y S_x - N S_{yx})}{(S_x^2 - N S_{x^2})} = \frac{(100 \cdot 244 - 2(21{,}200))}{(244^2 - 2(45968))}$$

$$= -18000 / -32400 = 180/324$$
$$= 5 \cdot \cancel{36} / 9 \cdot \cancel{36}$$
$$= 5/9$$

Then $b = (S_y - m S_x)/N$
$$= (100 - 5/9 \; 244)/2$$
$$= -5/9 \cdot 32$$

So $y = 5/9 \, x - 5/9 \cdot 32$

or $°C = 5/9(°F - 32)$

It Checks