Derivation of relevant expressions for non-ideal single-ended Colpitts oscilllator

These notes derive the steady-state oscillation frequency and oscillation condition for a single-ended common-drain Colpitts oscillator. The derivations focus on finding the characteristic equation (CE) and then expressing the relevant terms.

These notes are organized in the following order:

1. Derivation when considering inductor equivalent series resistance (ESR) only, by using determinant approach

2. Derivation when considering inductor ESR only, by using KCL approach

3. Derivation when considering inductor ESR only, by using loop gain approach

4. Derivation when considering inductor equivalent parallel capacitance (EPC) only, by using determinant approach

5. Derivation when considering inductor EPC only, by using KCL approach

6. Derivation when considering inductor EPC only, by using loop gain approach

7. Derivation when considering inductor ESR and EPC, by using loop gain approach

Derivation of same expressions were executed using multiple approaches in order to minimize errors.

Relevant variables:

L - LC resonant circuit inductance

C_1/C_2 - LC resonant circuit capacitances

R - Total circuit resistance

r_o - MOSFET small signal output resistance

C_T - LC resonant circuit equivalent capacitance

g_m - Transistor transconductance

ω_0 - steady-state oscillation frequency

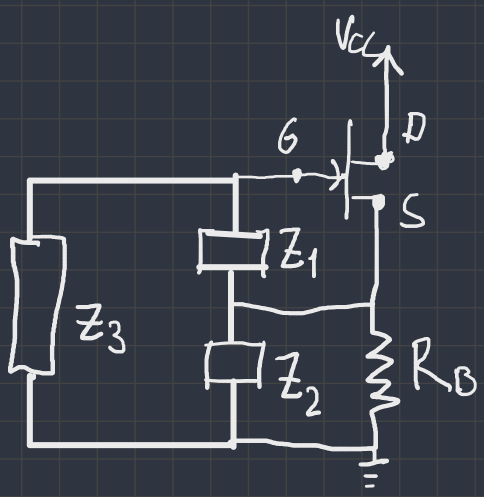
R_S - LC inductor equivalent series parasitic resistance

C_p - LC inductor equivalent parallel parasitic capacitance

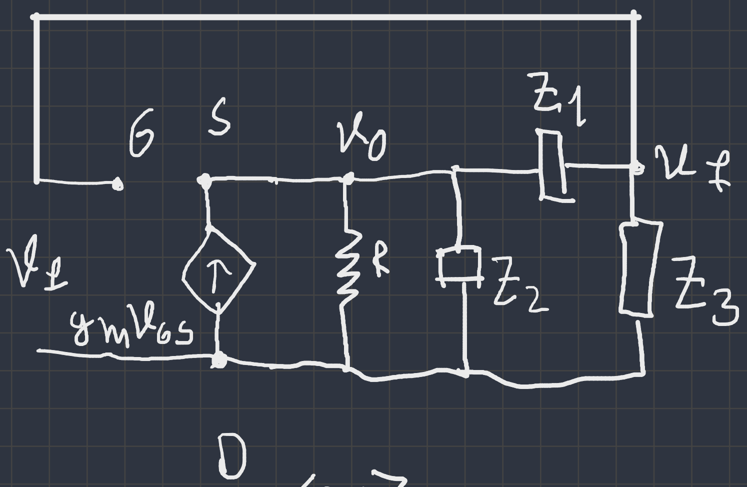
$Re(j\omega)$ - real part

$Im(j\omega)$ - imaginary part

Considered circuit



1. Derivation when considering inductor ESR only by using determinant approach



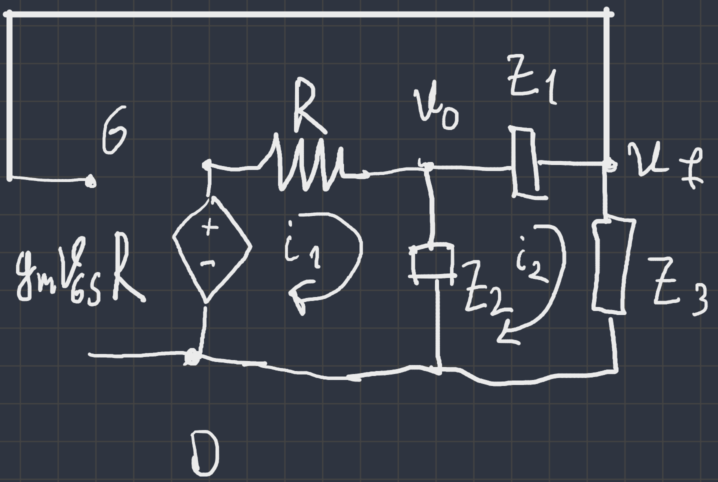
$$Z_1 = \frac{1}{sC_1}$$

$$Z_2 = \frac{1}{sC_2}$$

$$Z_3 = Ls + R_s$$

$$R = R_B \| r_0$$

$\Leftrightarrow$



$$\begin{cases} \left(R + \frac{1}{sC_2}\right) i_1 - \frac{1}{sC_2} i_2 = g_m v_{GS} R \\ -\frac{1}{sC_2} i_1 + \left(Ls + R_s + \frac{1}{sC_1} + \frac{1}{sC_2}\right) i_2 = 0 \end{cases}$$

$$v_{GS} = v_f - v_0 = Z_3 i_2 - Z_2 (i_1 - i_2) =$$
$$= \left(sL + R_s + \frac{1}{sC_2}\right) i_2 - \frac{1}{sC_2} i_1$$

$$\begin{cases} \left(R + \frac{1}{sC_2}\right) i_1 - \frac{1}{sC_2} i_2 = g_m R \left(sL + R_s + \frac{1}{sC_2}\right) i_2 - g_m R \frac{1}{sC_2} i_1 \\ -\frac{1}{sC_2} i_1 + \left(sL + R_s + \frac{1}{sC_2} + \frac{1}{sC_1}\right) i_2 = 0 \end{cases}$$

$$\begin{cases} \left(R + \frac{1}{sC_2} + g_m R \frac{1}{sC_2}\right) i_1 - \left(s g_m R L + g_m R R_s + \frac{1}{sC_2}(g_m R + 1)\right) i_2 = 0 \\ -\frac{1}{sC_2} i_1 + \left(sL + R_s + \frac{1}{sC_1} + \frac{1}{sC_2}\right) i_2 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} R + \frac{1}{sC_2} + g_m R \frac{1}{sC_2} & -\left(s g_m R L + g_m R R_s + \frac{1}{sC_2}(g_m R + 1)\right) \\ -\frac{1}{sC_2} & sL + R_s + \frac{1}{sC_2} + \frac{1}{sC_2} \end{vmatrix}$$

$$= \frac{C_1 C_2 L R s^3 + (C_1 L + C_1 C_2 R R_s) s^2 + (C_1 R + C_2 R + C_1 R_s) s + R g_m + 1}{C_1 C_2 s^2}$$

For oscillations to be obtained, the system determinant needs to equal to zero. $\Rightarrow$ $\Delta = 0$ $\Rightarrow$

Characteristic equation (CE):

(1)

$$\boxed{LC_1C_2R_s s^3 + (LC_1 + C_1C_2RR_s)s^2 + (C_1R + C_2R + C_1R_s)s + Rg_m + 1 = 0}$$

From (1), substituting $s = j\omega$ $\Rightarrow$

$$Re(j\omega) + Im(j\omega) = 0$$

$$Re(j\omega) = Rg_m - C_1L\omega^2 - C_1C_2RR_s\omega^2 + 1 = 0$$

$$Im(j\omega) = \omega(-C_1C_2LR\omega^2 + R(C_1 + C_2) + C_1R_s) = 0$$

From $Im(j\omega) = 0$ $\Rightarrow$

$$\omega_0 = \sqrt{\frac{R(C_1+C_2) + C_1R_s}{C_1C_2LR}} = \sqrt{\frac{1}{\frac{C_1C_2}{C_1+C_2}L} + \frac{R_s}{C_2LR}} =$$

$$\boxed{\omega_0 = \sqrt{\frac{1}{C_T L} + \frac{R_s}{C_2LR}}} \quad (2)$$

From $\mathrm{Re}(y\omega)=0$ and (2)

$$R g_m - \frac{C_1}{C_2} - \frac{C_1 R_s^2 + C_1 R R_s + C_2 R R_s}{L} - \frac{C_1 R_s}{C_2 R} = 0$$

$$R g_m = \frac{C_1}{C_2} + \frac{R R_s (C_1 + C_2) + C_1 R_s^2}{L} + \frac{C_1}{C_2} \cdot \frac{R_s}{R}$$

$$R g_m = \frac{C_1}{C_2} + R R_s \frac{C_1 + C_2}{L} + \frac{C_1 C_2 R R_s^2 + L C_1 R_s}{C_2 L R}$$

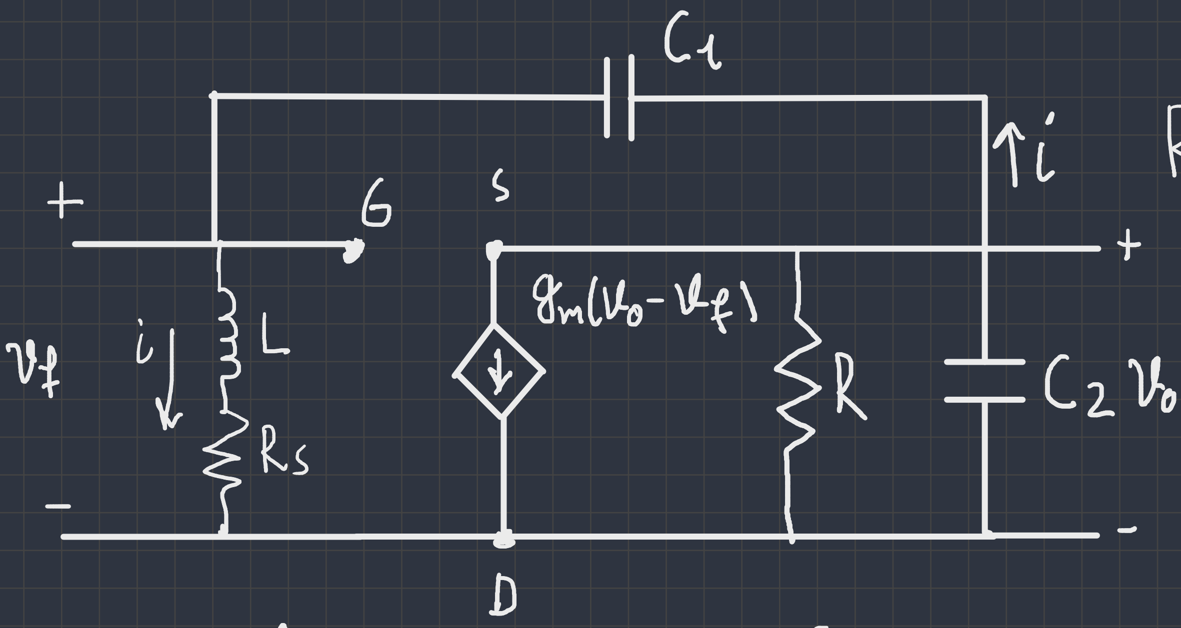
(3)

$$\boxed{R g_m = \frac{C_1}{C_2} + R R_s \frac{C_1 + C_2}{L} + \frac{R_s}{C_2 L R}(C_1 C_2 R R_s + L C_1)}$$

As long as Rs ≪ R we can make some approximations, such as:

$$\boxed{W_0 \approx \sqrt{\frac{1}{C_T L}} \qquad R g_m = \frac{C_1}{C_2} + R R_s \frac{C_1 + C_2}{L}}$$

2. Derivation when considering inductor ESR only by using KCL approach



$R = R_B \| r_o$

$$i = \frac{v_o}{\left(sL + R_s + \frac{1}{sC_1}\right)} = v_o \frac{sC_1}{s^2LC_1 + sR_sC_1 + 1} \quad (1)$$

$$i = \frac{v_f}{sL + R_s} \quad (2)$$

$(1) = (2) \Rightarrow$

$$v_o = v_f \frac{s^2LC_1 + sR_sC_1 + 1}{s^2LC_1 + sC_1R_s}$$

From KCL at S terminal:

$$\frac{sC_1}{s^2LC_1 + sR_sC_1 + 1} v_o + \frac{sC_2R + 1}{R} v_o + g_m(v_o - v_f) = 0$$

$$\Rightarrow \left(\frac{sC_1}{s^2LC_1 + sR_sC_1 + 1} + \frac{sC_2R + 1}{R} + g_m\right) v_o - g_m v_f = 0 \quad (4)$$

Substituting (3) in (4):

$$\left(\frac{sC_1}{s^2LC_1+sR_sC_1+1}+\frac{sC_2R+1}{R}+g_m\right)\left(\frac{s^2LC_1+sR_sC_1+1}{s^2LC_1+sC_1R_s}\right)V_f-g_mV_f=0$$

Assuming oscillation have begun: $V_f \neq 0$

$$\left[\frac{sRC_1+(s^2LC_1+sR_sC_1+1)[(sC_2R+1)+g_mR]}{R\cancel{(s^2LC_1+sR_sC_1+1)}}\right]\cdot\frac{\cancel{s^2LC_1+sR_sC_1+1}}{s^2LC_1+sC_1R_s}-g_mV_f=0\Big/\frac{1}{V_f}$$

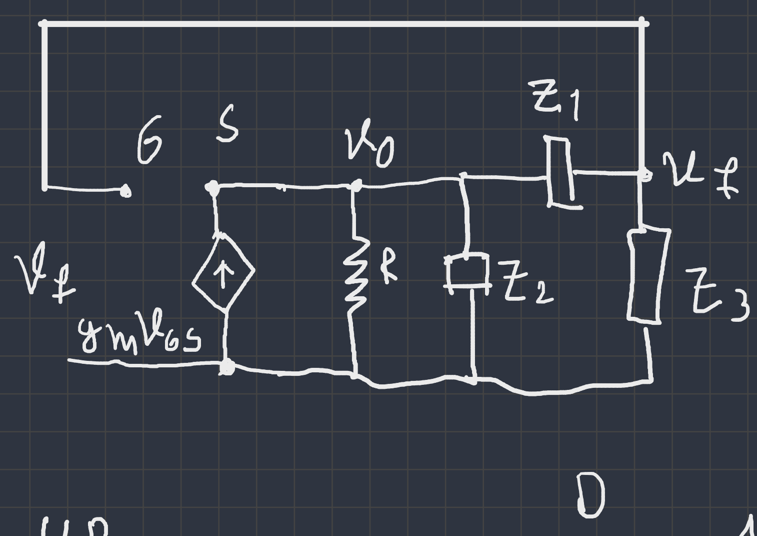
$$\frac{sRC_1+s^3LC_1C_2R+s^2LC_1+s^2g_mLC_1R+s^2C_1C_2RR_s+sR_sC_1+sg_mC_1RR_s+sC_2R+1+g_mR}{s^2LC_1R+sC_1R_sR}-g_m=0$$

Characteristic equation (CE):

$$\boxed{LC_1C_2R\cdot s^3+(LC_1+C_1C_2RR_s)s^2+(C_1R+C_2R+C_1R_s)s+g_mR+1=0}$$

Considering that the CE is the same as previously, it can be concluded that this approach will lead to the same results as previously.

3. Derivation when considering inductor ESR only by using loop gain approach



$R = r_o \| R_B$

$Z_1 = \frac{1}{sC_1}$ $Z_2 = \frac{1}{sC_3}$ $Z_3 = sL + R_S$

t - tank

$$Z_t = \frac{Z_2(Z_1 + Z_3)}{Z_1 + Z_2 + Z_3} = \frac{1}{sC_1} \| \left(\frac{1}{sC_3} + sL + R_s\right)$$

$Z_L = R \| Z_t$

$$A_v = \frac{V_o}{V_f} = \frac{g_m V_{GS} Z_L}{V_{GS} + V_S} = \frac{g_m V_{GS} Z_L}{V_{GS} + V_o} =$$

$$= \frac{g_m V_{GS} Z_L}{V_{GS} + g_m V_{GS} Z_L} = \frac{g_m Z_L}{1 + g_m Z_L}$$

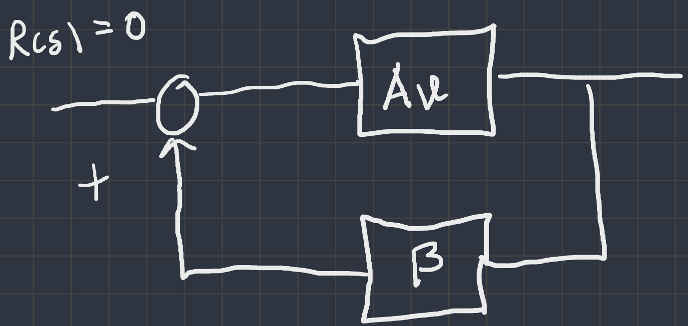
$$\beta = \frac{V_f}{V_o} = \frac{V_o Z_3}{V_o(Z_1 + Z_3)}$$

$$\beta = \frac{(sL+R_s)\,sC_1}{s^2LC_1 + sC_1R_s + 1} = \frac{s^2LC_1 + sC_1R_s}{s^2LC_1 + sC_1R_s + 1}$$

$$A_v = \frac{Rg_m\,(LC_1s^2 + C_1R_s + 1)}{LC_1C_2Rs^3 + (LC_1 + C_1C_2RR_s + LC_1Rg_m)s^2 + (C_1R + C_2R + C_1R_s + C_1RR_sg_m)s + Rg_m + 1}$$

The closed loop gain of the oscillator is:

$$A_f = \frac{A_v}{1 - A_v \cdot \beta}$$



According to the Barkhausen criterion for steady-state oscillation:

$$A_v \cdot \beta = 1 \Rightarrow$$

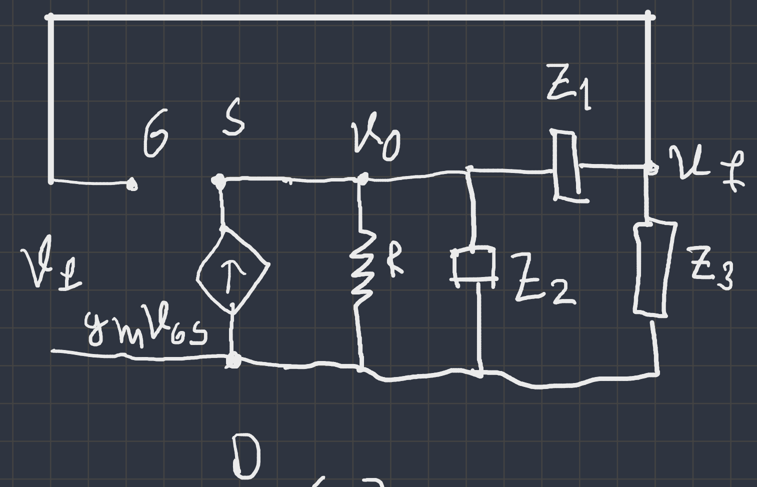
$$\frac{Rg_m\,(s^2LC_1 + sC_1R_s)}{LC_1C_2Rs^3 + (LC_1 + C_1C_2RR_s + LC_1Rg_m)s^2 + (C_1R + C_2R + C_1R_s + C_1RR_sg_m)s + Rg_m + 1} = 1$$

Characteristic equation (CE):

$$\boxed{LC_1C_2R \cdot s^3 + (LC_1 + C_1C_2RR_s)s^2 + (C_1R + C_2R + C_1R_s)s + g_mR + 1 = 0}$$

Considering that the CE is the same as previously, it can be concluded that this approach will lead to the same results as previously.

4. Derivation when considering inductor equivalent parallel capacitance (EPC) only by using determinant approach



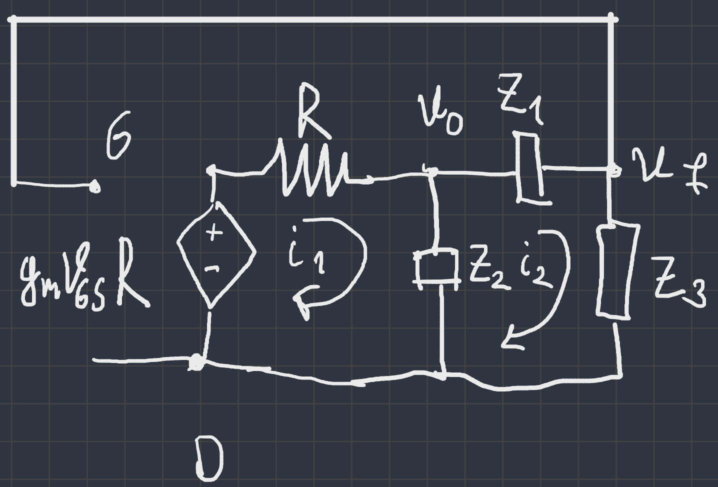
$$Z_1 = \frac{1}{sC_1}$$

$$Z_2 = \frac{1}{sC_2}$$

$$Z_3 = \frac{Ls}{LC_p s^2 + 1}$$

$$R = R_B \| r_o$$

$\Leftrightarrow$



$$\begin{cases} \left(R + \frac{1}{sC_2}\right) i_1 - \frac{1}{sC_2} i_2 = g_m v_{GS} R \\ -\frac{1}{sC_2} i_1 + \left(\frac{Ls}{LC_p s^2 + 1} + \frac{1}{sC_1} + \frac{1}{sC_2}\right) i_2 = 0 \end{cases}$$

$$v_{GS} = v_f - v_o = Z_3 i_2 - Z_2 (i_1 - i_2) =$$

$$= \left(\frac{Ls}{LC_p s^2 + 1} + \frac{1}{sC_2}\right) i_2 - \frac{1}{sC_2} i_1$$

$$\begin{cases} \left(R + \frac{1}{sC_2}\right) i_1 - \frac{1}{sC_2} i_2 = g_m R \left(\frac{Ls}{LC_p s^2 + 1} + \frac{1}{sC_2}\right) i_2 - g_m R \frac{1}{sC_2} i_1 \\ -\frac{1}{sC_2} i_1 + \left(\frac{Ls}{LC_p s^2 + 1} + \frac{1}{sC_2} + \frac{1}{sC_1}\right) i_2 = 0 \end{cases}$$

$$\begin{cases} \left(R + \frac{1}{sC_2} + g_m R \frac{1}{sC_2}\right) i_1 - \left(g_m R \frac{Ls}{LC_p s^2 + 1} + \frac{1}{sC_2}(g_m R + 1)\right) i_2 = 0 \\ -\frac{1}{sC_2} i_1 + \left(\frac{Ls}{LC_p s^2 + 1} + \frac{1}{sC_1} + \frac{1}{sC_2}\right) i_2 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} R + \frac{1}{sC_2} + g_m R \frac{1}{sC_2} & -\left(g_m R \frac{Ls}{LC_p s^2 + 1} + \frac{1}{sC_2}(g_m R + 1)\right) \\ -\frac{1}{sC_2} & \frac{Ls}{LC_p s^2 + 1} + \frac{1}{sC_2} + \frac{1}{sC_2} \end{vmatrix}$$

$$= \frac{a_3 s^3 + a_2 s^2 + a_1 s + R g_m + 1}{C_1 C_2 C_p L s^4 + C_1 C_2 s^2} = 0$$

$$a_3 = C_1 C_2 L R + C_1 C_p L R + C_2 C_p L R$$

$$a_2 = C_1 L + C_p L + C_p L R g_m$$

$$a_1 = R(C_1 + C_2)$$

Characteristic equation (CE):

$$\Rightarrow \boxed{a_3 s^3 + a_2 s^2 + a_1 s + R g_m + 1 = 0} \quad (1)$$

From (1), substituting $s = j\omega \Rightarrow$

$$Re(j\omega) + Im(j\omega) = 0$$

$$\Rightarrow Re(j\omega) = R g_m + C_1 L \omega^2 - C_p L \omega^2 - C_p L R g_m \omega^2 + 1 = 0$$

$$Im(j\omega) = C_1 R \omega + C_2 R \omega - C_1 C_2 L R \omega^3 - C_1 C_p L R \omega^3 - C_2 C_p L R \omega^3 = 0$$

From $Im(j\omega) = 0 \Rightarrow$

$$\omega_0 = \sqrt{\frac{C_1 + C_2}{L(C_1 C_2 + C_1 C_p + C_2 C_p)}} = \frac{1}{\sqrt{L\left(\frac{C_1 C_2}{C_1 + C_2} + C_p\right)}} =$$

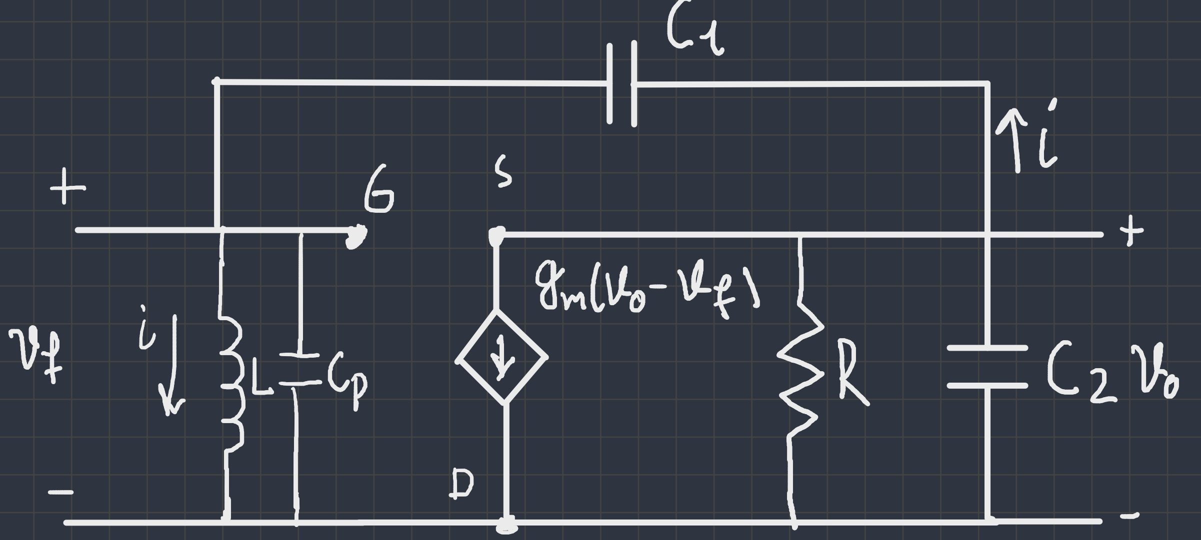
$$\boxed{\omega_0 = \frac{1}{\sqrt{L(C_T + C_p)}}} \quad (2)$$

From $Re(j\omega) = 0$ and (2) $\Rightarrow$

$$\frac{-(C_1(C_1 - C_2 R g_m)}{C_p(C_1 + C_2) + C_1 C_2} = 0$$

$$\Rightarrow \boxed{R g_m = \frac{C_1}{C_2}} \quad (3)$$

5. Derivation when considering inductor EPC only by using KCL approach



$R = R_B \| r_o$

$$X_3 = \frac{Ls \cdot \frac{1}{C_p s}}{Ls + \frac{1}{C_p s}} = \frac{Ls}{LC_p s^2 + 1}$$

$$i = \frac{v_o}{\left(X_3 + \frac{1}{sC_1}\right)} = \frac{LC_1C_p s^3 + C_1 s}{C_1 L s^2 + C_p L s^2 + 1} \quad (1)$$

$$i = \frac{v_f}{X_3} = v_f \frac{LC_p s^2 + 1}{Ls} \quad (2)$$

$(1) = (2) \Rightarrow$

$$v_o = \frac{L(C_1 + C_p)s^2 + 1}{LC_1 s^2} \quad (3)$$

From KCL at S terminal:

$$\frac{LC_1C_p s^3 + C_1 s}{L(C_1 + C_p)s^2 + 1} v_o + \frac{sC_2 R + 1}{R} v_o + g_m(v_o - v_f) = 0$$

$$\Rightarrow \left(\frac{LC_1C_p s^3 + C_1 s}{L(C_1 + C_p)s^2 + 1} + \frac{sC_2 R + 1}{R} + g_m \right) v_o - g_m v_f = 0 \quad (4)$$

Substituting (3) in (4):

$$\left(\frac{LC_1C_ps^3+C_1s}{L(C_1+C_p)s^2+1}+\frac{sC_2R+1}{R}+g_m\right)\frac{L(C_1+C_p)s^2+1}{LC_1s^2}v_f-g_mv_f=0$$

Assuming oscillation have begun: $v_f \neq 0$

$$\frac{R(LC_1C_ps^3+C_1s)+(sC_2R+1+g_mR)(L(C_1+C_p)s^2+1)}{R\cancel{(L(C_1+C_p)s^2+1)}}\frac{\cancel{L(C_1+C_p)s^2+1}}{LC_1s^2}-g_mv_f=0\Big/\frac{1}{v_f}$$

$$\frac{a_3s^3+a_2s^2+a_1s+Rg_m+1}{LC_1Rs^2}=0$$

$$a_3=C_1C_2LR+C_1C_pLR+C_2C_pLR$$

$$a_2=C_1L+C_pL+C_pLRg_m$$

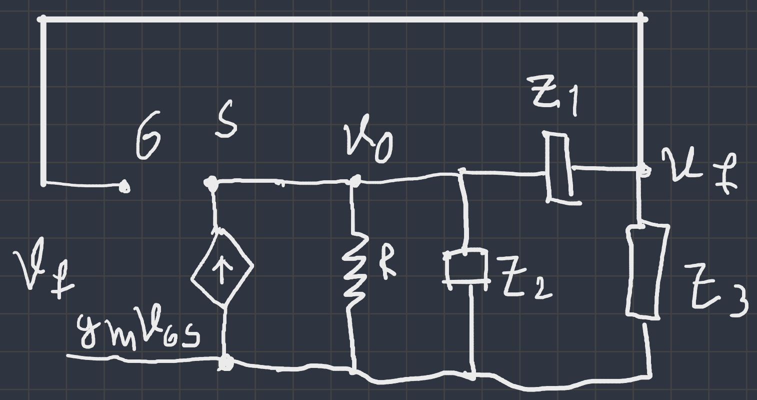
$$a_1=R(C_1+C_2)$$

Characteristic equation (CE):

$$\boxed{a_3s^3+a_2s^2+a_1s+Rg_m+1=0}$$

Considering that the CE is the same as previously, it can be concluded that this approach will lead to the same results as previously.

6. Derivation when considering inductor EPC only by using loop gain approach



$R = r_0 \| R_D$

$$Z_1 = \frac{1}{sC_1} \quad Z_2 = \frac{1}{sC_3} \quad Z_3 = sL \| \frac{1}{sC_p}$$

t - tank

$$Z_t = \frac{Z_2(Z_1 + Z_3)}{Z_1 + Z_2 + Z_3} = \frac{1}{sC_1} \| \left(\frac{1}{sC_3} + sL \| \frac{1}{sC_p}\right)$$

$$Z_L = R \| Z_t$$

$$A_v = \frac{V_0}{V_f} = \frac{g_m V_{GS} Z_L}{V_{GS} + V_S} = \frac{g_m V_{GS} Z_L}{V_{GS} + V_0} =$$

$$= \frac{g_m V_{GS} Z_L}{V_{GS} + g_m V_{GS} Z_L} = \frac{g_m Z_L}{1 + g_m Z_L}$$

$$\beta = \frac{V_f}{V_0} = \frac{V_0 Z_3}{V_0(Z_1 + Z_3)} = \frac{Z_3}{Z_1 + Z_3}$$

$$\beta = \frac{LC_1 s^2}{L(C_1+C_p)s^2 + 1}$$

$$A_v = \frac{LRg_m(C_1+C_p)s^2 + Rg_m}{a_3 s^3 + (a_2 + LC_1Rg_m)s^2 + a_1 s + Rg_m + 1}$$

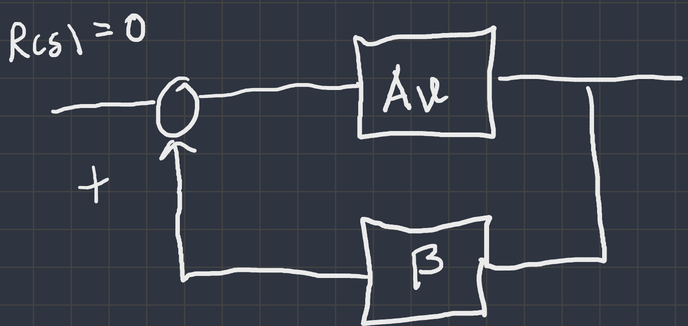
$$a_3 = C_1C_2LR + C_1C_pLR + C_2C_pLR$$

$$a_2 = C_1L + C_pL + C_pLRg_m$$

$$a_1 = R(C_1+C_2)$$

The closed loop gain of the oscillator is:

$$A_f = \frac{A_v}{1 - A_v \cdot \beta}$$



According to the Barkhausen criterion for steady-state oscillation:

$$A_v \cdot \beta = 1 \Rightarrow$$

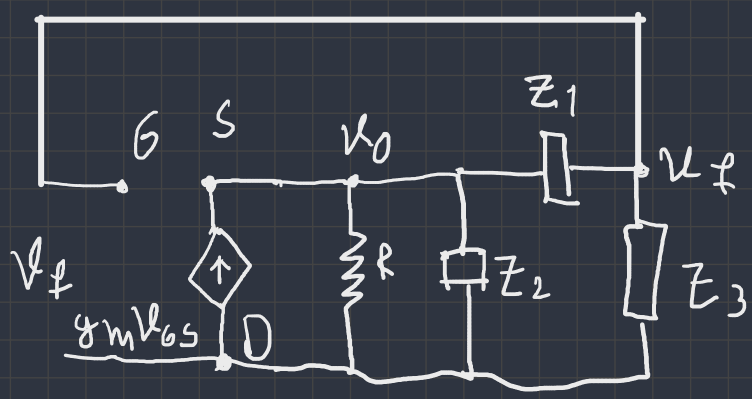
$$\frac{LC_1Rg_m s^2}{a_3 s^3 + (a_2 + C_1 L R g_m) s^2 + a_1 s + R g_m + 1} = 1$$

$\Rightarrow$ Characteristic equation (CE):

$$\boxed{a_3 s^3 + a_2 s^2 + a_1 s + R g_m + 1 = 0}$$

Considering that the CE is the same as previously, it can be concluded that this approach will lead to the same results.

7. Derivation when considering inductor ESR and EPC by using loop gain approach



$$R = r_o \,||\, R_D \quad Z_1 = \frac{1}{sC_1} \quad Z_2 = \frac{1}{sC_3} \quad Z_3 = (sL + R_s) \,||\, \frac{1}{sC_p}$$

t - tank

$$Z_t = \frac{Z_2 (Z_1 + Z_3)}{Z_1 + Z_2 + Z_3} = \frac{1}{sC_1} \,||\, \left(\frac{1}{sC_3} + (sL + R_s) \,||\, \frac{1}{sC_p} \right)$$

$$Z_L = R \,||\, Z_t$$

$$A_v = \frac{V_o}{V_f} = \frac{g_m V_{GS} Z_L}{V_{GS} + V_S} = \frac{g_m V_{GS} Z_L}{V_{GS} + V_o} =$$

$$= \frac{g_m V_{GS} Z_L}{V_{GS} + g_m V_{GS} Z_L} = \frac{g_m Z_L}{1 + g_m Z_L}$$

$$\beta = \frac{V_f}{V_o} = \frac{V_o Z_3}{V_o (Z_1 + Z_3)} = \frac{Z_3}{Z_1 + Z_3}$$

$$\beta = \frac{L C_1 s^2 + C_1 R_s \cdot s}{L(C_1+C_p)s^2 + R_s(C_1+C_p)s + 1}$$

$$A_v = \frac{R g_m \left[L(C_1+C_p)s^2 + R_s(C_1+C_p)s + 1 \right]}{a_3 s^3 + (a_2 + L C_1 R g_m)s^2 + (a_1 + C_1 R_s R g_m)s + R g_m + 1}$$

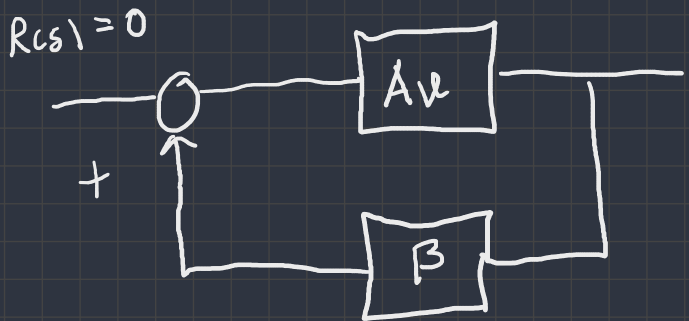
$$a_3 = L C_1 C_2 R + L C_1 C_p R + L C_2 C_p R$$

$$a_2 = L(C_1+C_p) + R R_s (C_1 C_2 + C_1 C_p + C_2 C_p) + L C_p R g_m$$

$$a_1 = R(C_1+C_2) + R_s(C_1+C_p) + C_p R_s R g_m$$

The closed loop gain of the oscillator is:

$$A_f = \frac{A_v}{1 - A_v \cdot \beta}$$



According to the Barkhausen criterion for steady-state oscillation:

$$A_v \cdot \beta = 1 \Rightarrow$$

$$\frac{LC_1Rg_m s^2 + C_1R_sRg_m s}{a_3s^3 + (a_2 + LC_1Rg_m)s^2 + (a_1 + C_1R_sRg_m)s + Rg_m + 1} = 1$$

$\Rightarrow$ Characteristic equation (CE):

$$\boxed{a_3s^3 + a_2s^2 + a_1s + Rg_m + 1 = 0} \qquad (1)$$

From (1), substituting $s = j\omega \Rightarrow$

$$Re(j\omega) + Im(j\omega) = 0$$

$$Re(j\omega) = -\left[L(C_1+C_p) + RR_s(C_1C_2 + C_1C_p + C_2C_p) + LC_pRg_m\right]\omega^2 + Rg_m + 1$$

$$Im(j\omega) = -\left[LR(C_1C_2 + C_1C_p + C_2C_p)\right]\omega^3 + \left[R(C_1+C_2) + R_s(C_1+C_p) + C_pRR_sg_m\right]\omega$$

From $Im(j\omega) = 0 \Rightarrow$

$$\omega_0 = \sqrt{\frac{R(C_1+C_2) + C_1R_s + C_pR_s + C_pRR_sg_m}{LR(C_1C_2 + C_p(C_1+C_2))}}$$

$$\omega_0 = \sqrt{\frac{(C_1+C_2)}{L\cdot[C_1C_2 + C_p(C_1+C_2)]} + \frac{R_s}{R}\cdot\frac{C_1+C_p}{C_1+C_2}\cdot\frac{1}{L(C_T+C_p)} + \frac{C_pR_sg_m}{C_1+C_2}\,\frac{1}{L(C_T+C_p)}}$$

$$\omega_0 = \sqrt{\frac{1}{L(C_T+C_p)} + \frac{R_s}{R}\,\frac{C_1+C_p}{C_1+C_2}\,\frac{1}{L(C_T+C_p)} + \frac{C_p}{C_1+C_2}\cdot R_s g_m \frac{1}{L(C_T+C_p)}}$$

As long as Rs ≪ R, Cp ≪ C1 and Cp ≪ C1 we can make some approximations, such as:

$$\boxed{\omega_0 = \sqrt{\frac{1}{L(C_T+C_p)}}} \quad (2) \qquad C_T = \frac{C_1 C_2}{C_1+C_2}$$

From $\mathrm{Re}\{\omega\}=0$ and (2):

$$LC_1^2 + LC_1C_2 + C_1C_2^2RR_s + C_1^2C_2RR_s + C_1^2C_pRR_s + C_2^2C_pRR_s + 2C_1C_2C_pRR_s = LC_1C_2(Rg_m+1)$$

$$\frac{C_1}{C_2} \cancel{+1} + \frac{C_2}{L}RR_s + \frac{C_1}{L}RR_s + \frac{C_1}{LC_2}C_pRR_s + \frac{C_2}{LC_1}C_pRR_s + \frac{2C_pRR_s}{L} = Rg_m \cancel{+1}$$

$$Rg_m = \frac{C_1}{C_2} + RR_s\frac{C_1+C_2}{L} + \frac{C_pRR_s}{L}\left(\frac{C_1}{C_2} + \frac{C_2}{C_1} + 2\right)$$

$$Rg_m = \frac{C_1}{C_2} + RR_s\frac{C_1+C_2}{L} + \frac{C_pRR_s}{L}\cdot\frac{C_1^2+C_2^2+2C_1C_2}{C_1C_2}$$

$$Rg_m = \frac{C_1}{C_2} + RR_s\frac{C_1+C_2}{L} + \frac{C_pRR_s}{L}\,\frac{(C_1+C_2)^2}{C_1C_2}$$

$$Rg_m = \frac{C_1}{C_2} + RR_s\frac{C_1+C_2}{L} + RR_s\frac{C_p}{C_T}\,\frac{C_1+C_2}{L}$$

$$\boxed{Rg_m = \frac{C_1}{C_2} + RR_s\left(1+\frac{C_p}{C_T}\right)\frac{C_1+C_2}{L}} \quad (3)$$