MATH 217 Cheat Sheet

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Vectors & Geometry of Space

Distance between two points:
$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$
Equation of a sphere:
$R^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2$
Unit vector:
$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$
Dot Product:
$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta)$
Scalar projection of $\mathbf{a}$ on to $\mathbf{b}$:
$\text{comp}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}| \cos\theta = |\mathbf{a}| \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \, |\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$
Vector projection of $\mathbf{a}$ on to $\mathbf{b}$:
$\text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{a_1}\hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \frac{\mathbf{b}}{|\mathbf{b}|}$
Orthogonal projection of $\mathbf{a}$ on to $\mathbf{b}$:
$\mathbf{v} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}$
Cross product:
$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta)$

Lines & Planes

Vector equation of a line:
$\mathbf{r}(t) = \mathbf{r_0} + \mathbf{v}(t)$
Vector equation of a plane:
$\mathbf{r} - \mathbf{r_0} \cdot \mathbf{n} = \mathbf{0}$
Scalar equation of a plane, where a, b, c are components of the normal vector:
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
Distance from the point $P = (x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$:
$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$

Quadric Surfaces

Equation of an ellipsoid:
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Equation of an elliptic paraboloid:
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$
Equation of a hyperbolic paraboloid:
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$
Equation of a elliptic hyperboloid of one sheet:
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Equation of an elliptic hyperboloid of two sheets:
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

Vector Functions

Arc length of a vector function:
$\int_a^b |\mathbf{r}'(t)| \, dt$
Length of the curve $y = f(x)$:
$\int_a^b \sqrt{1 + (f'(x))^2} \, dx$
If $\mathbf{r}'(t)$ is differentiable at $t = a$ and $\mathbf{r}'(a) \neq \mathbf{r}(0)$, the tangent line to the curve given by $\mathbf{r}'(t)$ is the line through $\mathbf{r}'(a)$ in the direction of $\mathbf{r}'(a)$

Partial Derivatives

$f(x, y)$ is continuous at (a, b) if
$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$
Suppose that $z = f(x, y)$, f is differentiable, $x = g(t)$, and $y = h(t)$. Assuming that the relevant derivatives exist,
$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.
If $f(x, y)$ is defined on a domain D that contains the point (a, b). If $\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial z^2}{\partial x \partial y}$ are continuous on D, then
$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$
Equation of a tangent plane:
$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$
The differential of $f(x, y)$is:
$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$
Suppose that $f(x, y)$ is a function and $x = g(s, t)$ and $y = h(s, t)$:
$\frac{\partial f}{\partial s} = f_x g_s + f_y h_s \qquad \frac{\partial f}{\partial t} = f_x g_t + f_y h_t$.
The slope of a surface given by $z = f(x, y)$ in the direction of a vector $\mathbf{u}$ is called the directional derivative of f, written $D_u f$
$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta$
The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{v})$ is $|\nabla f(\mathbf{v})|$and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{v})$.

The gradient vector $\nabla f(a, b, c)$ is orthogonal to the level surface S through (a, b, c)

Discriminant of $f(x, y)$:

$$D(x, y) = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

If $f_{xx} > 0$ or $f_{yy} > 0$ and $D(a, b) > 0$, then $f(a, b)$ is a local minimum.
If $f_{xx} < 0$or $f_{yy} < 0$ and $D(a, b) > 0$, then $f(a, b)$ is a local maximum.
If $D(a, b) < 0$, then $f(a, b)$ is a saddle point.
If $D(a, b) = 0$, no information is given.

Optimization

The extreme values of $f(x, y)$ can only occur at:

- Interior critical points, where both partials exist.
- Boundary points of the domain of the function.

To maximize or minimize $f(x, y)$ subject to the constraint $g(x, y) = C$, we solve:

- $\nabla f(x, y) = \lambda \nabla g(x, y)$
- $g(x, y) = C$

Multiple Integrals

Fubini's Theorem: If $f(x, y)$ is continuous on the domain D:
$\iint_D f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$
Area of the domain D:
$A = \iint_D 1dA$
Average value of a $f(x, y)$ over domain D:
$f_{avg} = \frac{1}{A} \iint_D f(x, y)dA$
Mass of a lamina D with density $\rho(x, y)$:
$m = \iint_D \rho(x, y)dA$
Moment of a mass about the x-axis:
$M_x = \iint_D x\rho(x, y)dA$
Center of mass of a lamina:
$\bar{x} = M_x/m \qquad \bar{y} = M_y/m \qquad \bar{z} = M_z/m$
Surface area of $f(x, y)$:
$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \sqrt{f_x^2 + f_y^2 + 1} \, dy \, dx$.

Cylindrical Coordinates

- $x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$
- $dA = r \, dr \, d\theta \qquad dV = r \, dr \, dz \, d\theta$
- $x^2 + y^2 = r^2$

Spherical Coordinates

- $x = \rho \sin\phi \cos\theta$
- $y = \rho \sin\phi \sin\theta$
- $z = \rho \cos\phi$
- $x^2 + y^2 + z^2 = \rho^2$
- $x^2 + y^2 = \rho^2 \sin^2\phi$
- $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Change of Variables

Suppose that $f(x, y)$ is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S:

$\iint_R F(x,y)\,dV = \iint_S F(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

3D case: $\iiint_R F(x,y,z)\,dV =$

$\iiint_S F(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$

Jacobian:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}$$

Vector Caclulus

A vector field $\mathbf{F}$ is conservative if there is a scalar function f such that:

$\mathbf{F} = \nabla f$

Other properties:

curl $\mathbf{F} = \nabla \times \mathbf{F}$

div $\mathbf{F} = \nabla \cdot \mathbf{F}$

$\nabla \times (\nabla f) = \mathbf{0}$

If $\nabla \cdot \mathbf{F} \neq 0$, then $\mathbf{F}$ cannot be a curl of another vector field.

Line Integrals

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$$

where $\mathbf{r}(t)$ is the parametrization of C.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(\mathbf{t})) \cdot \mathbf{r}'(t)dt = \int_a^b \mathbf{F} \cdot \mathbf{T} ds$$

where $\mathbf{T}$ is the unit tangential vector $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Fundamental Theorem of Line Integrals:

$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Independence of Path & Conservativeness

- Let $\mathbf{F}$ be a continuous vector field on the domain D. We have independence of path in D if:
  $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$
- If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
- If Clairaut's Theorem fails, then $\mathbf{F}$ is not conservative.
- If D is an open (boundary points are not on the domain) and connected region and $\mathbf{F}$ is a continuous vector field of D, then if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is conservative.
- Green's Theorem evaluates to 0 for any conservative vector field.
- $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = 0$
- If div(curl $\mathbf{F}$) $= 0$, then $\mathbf{F}$ is conservative.

Green's Theorem

Let C be a simple piecewise smooth curve that bounds a region D in the plane. If $P(x, y)$ and $Q(x, y)$ have continuous partials in an open region containing D, then

$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

If $\mathbf{F}$ is a vector field with third component 0 defined on a domain D enclosed by boundary C then

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA.$

Similarly, if C is defined by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$

$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D \nabla \cdot \mathbf{F}\,dA$

Parametric Surfaces

Plane through $\mathbf{r_0}$ parallel to the non-parallel vectors $\mathbf{v}_1$, $\mathbf{v}_2$:

$\mathbf{r}(s,t) = \mathbf{r_0} + s\mathbf{v}_1 + t\mathbf{v}_2$

Graph of a function $z = f(x, y)$:

$\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle$

Cylinder about the x-axis:

$\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$

A cone given by $z = a\sqrt{x^2 + y^2}$:

$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, ar\rangle$

An ellipsoid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\mathbf{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, b\sin\phi\sin\theta, c\cos\theta\rangle$

A smooth parametric surface given by the equation $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ in a domain D and S is covered once throughout the parameter domain D, then the area of S is:

$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$

Surface area of a graph of a function:

$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$

Surface Integrals

$\iint_S f(x,y,z)\,d\mathbf{S} = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$

Graph of a function $z = g(x, y)$:

$\iint_S f(x,y,z)\,d\mathbf{S} =$

$\iint_D f(x,y,g(x,y))\sqrt{1 + (g_x)^2 + (g_y)^2}\,dA$

If we have a surface S given by a graph $z = g(x, y)$, we write:

$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = \langle P, Q, R\rangle \cdot \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right\rangle$ so

$\iint_S \mathbf{F} \cdot\ d\mathbf{S} = \iint_D (-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R)\,dA$

Vector field:

$\iint_S f(x,y,z)\,d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,dA$

Stoke's Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F})\,d\mathbf{S} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dA$

Divergence Theorem

Let E be a solid region in $\mathbb{R}^3$ with piecewise smooth boundary surface S (given the outward orientation). Let $\mathbf{F}$ be a vector field with continuous partial derivatives on a region containing E. Then

$\iint_D \mathbf{F} \cdot\ d\mathbf{S} = \iint_D \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_E \nabla \cdot \mathbf{F}\,dV.$

The flux of $\mathbf{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E.

For cases where it is too difficult to parameterize a surface, the surface is not closed, and we cannot use Stoke's Theorem, we can close the surface and apply the Divergence Theorem. (Later subtract the contribution from the closed surface)

Trigonometric Identities

- $\sin^2 x = \frac{1 - \cos 2x}{2}$
- $\cos^2 x = \frac{1 + \cos 2x}{2}$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$