

ELEC 221 Formula Sheet

Continuous Time Signals

Even and Odd Components:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

A signal is periodic with fundamental period T_0 if

$$x(t + kT_0) = x(t) \quad \forall t \in (-\infty, \infty)$$

Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Power:

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

Causality:

- Causal: $x(t) = 0$ for $t < 0$.
- Anti-Causal: $x(t) = 0$ for $t \geq 0$.
- A-Causal or Non-Causal: Both of the above.

Continuous Time Systems

Dynamic systems have memory. Active systems can deliver energy to the outside world.

Linearity:

$$\mathcal{S}[\alpha x(t) + \beta y(t)] = \alpha \mathcal{S}[x(t)] + \beta \mathcal{S}[y(t)]$$

Time Invariance:

$$\text{If } \mathcal{S}[x(t)] = y(t) \text{ then } \mathcal{S}[x(t \pm \tau)] = y(t \pm \tau).$$

Zero-State Response:

Due to the input as the initial conditions are zero.

Zero-Input Response:

Due to the initial conditions as the input is zero.

Convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$$

Causality:

A continuous time system \mathcal{S} is causal if whenever $x(t) = 0$ and there are no initial conditions, $y(t) = 0$ and the output $y(t)$ does not depend on future inputs.

Bounded-Input Bounded-Output Stability:

If an input $x(t)$ bounded then the output of an BIBO system is also bounded.

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Laplace Transform

$$s = \sigma + j\omega$$

Eigenfunction Property:

$$\mathcal{S}[e^{s_0 t}] = H(s_0)e^{s_0 t}$$

One Sided Laplace Transform:

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0^-}^{\infty} f(t)e^{-st} dt$$

| Signal Support | ROC |
|-----------------|---|
| Finite support | Entire s-plane. |
| Causal function | $\sigma > \max(\sigma_i), -\infty < \omega < \infty$ |
| Anti-causal | $\sigma < \min(\sigma_i), -\infty < \omega < \infty$ |
| Non-causal | $\mathcal{R} = \mathcal{R}_{\text{causal}} \cap \mathcal{R}_{\text{anti-causal}}$ |

Initial Value Theorem:

$$f(0+) \Leftrightarrow \lim_{s \rightarrow \infty} sF(s)$$

Final Value Theorem:

$$\lim_{t \rightarrow \infty} f(t) \Leftrightarrow \lim_{s \rightarrow 0} sF(s)$$

Bounded-Input Bounded-Output Stability:

If the region of convergence contains the $j\omega$ -axis, then the system is BIBO stable.

Fourier Series

Fourier analysis in the steady state.

Eigenfunction Property:

$$\mathcal{S}[e^{j\omega_0 t}] = H(j\omega_0)e^{j\omega_0 t}$$

$$x(t) = \sum_k X_k e^{j\omega_k t} \implies y(t) = \sum_k X_k H(j\omega_k) e^{j\omega_k t} \\ = \sum_k X_k |H(j\omega_k)| e^{j(\omega_k t + \angle H(\omega_k))}$$

Fourier Series Coefficients (for any t_0):

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} dt$$

Parseval's Power Relation (for any t_0):

$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Symmetry of Line Spectra:

$$|X_k| = |X_{-k}|$$

$$\angle X_k = -\angle X_{-k}$$

Trigonometric Fourier Series:

$$x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \Theta_k)$$

$$x(t) = c_0 + 2 \sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)]$$

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\omega_0 t) dt \quad k = 0, 1, 2, \dots$$

$$d_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\omega_0 t) dt \quad k = 1, 2, 3, \dots$$

$$\Theta_k = -\arctan(d_k/c_k)$$

Fourier Coefficients from Laplace Transform:

If $x_1(t)$ is a single period of $x(t)$, then

$$X_k = \frac{1}{T_0} \mathcal{L}[x_1(t)]|_{s=jk\omega_0}$$

Response of LTI Systems to Periodic Signals:

If the input to an LTI system has Fourier Series

$x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \angle X_k)$, then the steady state response is $y(t) = X_0 |H(j\omega)| +$

$$2 \sum_{k=1}^{\infty} |X_k| |H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0))$$

Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier Transform from Laplace Transform (if $X(s)$ contains the $j\omega$ -axis):

$$\mathcal{F}[x(t)] = \mathcal{L}[x(t)]|_{s=j\omega} = X(s)|_{s=j\omega}$$

Fourier Transform of Periodic Signals:

$$\sum_k X_k e^{jk\omega_0 t} \xLeftrightarrow{\mathcal{F}} \sum_k 2\pi \delta(\omega - k\omega_0)$$

Parseval's Energy Relation:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Symmetry of Spectral Representations:

$$|X(\omega)| = |X(-\omega)|$$

$$\text{Re}[X(\omega)] = \text{Re}[X(-\omega)]$$

$$\angle X(\omega) = -\angle X(-\omega)$$

$$\text{Im}[X(\omega)] = -\text{Im}[X(-\omega)]$$

Sampling Theory

$$x_s(t) = x(nT_s) = x(t)|_{t=nT_s} = x(t) \sum_n \delta(t - nT_s)$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Nyquist-Shannon Sampling Rate:

$$\omega_s = \frac{2\pi}{T_s} \geq 2\omega_{\max}$$

Aliasing occurs if $\omega_s < 2\omega_{\max}$.

Reconstruction $X(\omega) = X_s(\omega)H_{\text{lp}}(\omega)$:

$$H_{\text{lp}}(\omega) = \begin{cases} T_s & -\frac{\omega_s}{2} \leq \omega \leq \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

(Inclusive bounds. If represented by a unit step function the bounds are inclusive for filters (Refer to lecture notes))

Signal Reconstruction from Sinc Interpolation:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin(\pi(t - nT_s)/T_s)}{\pi(t - nT_s)/T_s}$$

Discrete Time Signals

Define $x[n] = x(nT_0)$.
 $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k]$
Periodicity:
 $x[n + kN] = x[n] \quad \forall k \in \mathbb{Z}$
When sampling an analog sinusoid of fundamental period T_0 , we obtain a periodic discrete sinusoid provided that m, N not divisible by each-other:
 $T_s/T_0 = m/N$
Aliasing occurs if:
 $T_s > T_0/2$
Energy:
 $E = \sum_{n=-\infty}^{\infty} |x[n]|^2$
Power:

$$P_x = \lim_{N \rightarrow \infty} \frac{1}{2N + 1} \sum_{n=-N}^N |x[n]|^2$$

Convolutional Sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

Bounded-Input Bounded-Output Stability:
 $\sum_k |h[n]|^2 < \infty$
Solution to Autoregressive Discrete System:
 $y[n] = ay[n - 1] + bx[n], n \geq 0$
 $y[n] = \sum_{k=0}^n ba^k x[n - k], n \geq 0$

Z-Transform

$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n}$
Convergence:
 $|X(z)| = \sum_n |x[n]|r^{-n} < \infty$
Initial Value Theorem:
 $x[0] = \lim_{z \rightarrow \infty} X(z)$
Final Value Theorem:
 $\lim_{n \rightarrow \infty} x[n] = \lim_{z \rightarrow 1} (z - 1)X(z)$
BIBO Stability:

If the ROC contains radius $z = 1$, then the system is BIBO stable.

Partial Fraction Decomposition

| Fraction | Partial Fraction | Solution |
|---|---|--|
| $\frac{px + q}{(x - a)(x - b)}$ | $\frac{A}{x - a} + \frac{B}{x - b}$ | $A = \frac{pa + q}{a - b} \quad B = \frac{pb + q}{b - a}$ |
| $\frac{px + q}{(x - a)^2}$ | $\frac{A}{x - a} + \frac{B}{(x - a)^2}$ | $A = p \quad B = pa + q$ |
| $\frac{px^2 + qx + r}{(x - a)(x - b)(x - c)}$ | $\frac{A}{x - a} + \frac{B}{x - b} + \frac{C}{x - c}$ | $A = \frac{pa^2 + qa + r}{(a - b)(a - c)} \quad B = \frac{pb^2 + qb + r}{(b - a)(b - c)} \quad C = \frac{pc^2 + qc + r}{(c - a)(c - b)}$ |
| $\frac{px^2 + qx + r}{(x - a)^2(x - b)}$ | $\frac{A}{x - a} + \frac{B}{(x - a)^2} + \frac{C}{x - b}$ | |
| $\frac{px^2 + qx + r}{(x - a)(x^2 + bx + c)}$ | $\frac{A}{x - a} + \frac{Bx + C}{x^2 + bx + c}$ | |

Two-Sided Z-Transforms

| $f[n]$ | $F(z)$ | ROC |
|------------------------------|---|--|
| $-u[-n - 1]$ | $\frac{1}{1 - z^{-1}}$ | $ z < 1$ |
| $-\alpha^n u[-n - 1]$ | $\frac{1}{1 - \alpha z^{-1}}$ | $ z < \alpha $ |
| $-n\alpha^n u[-n - 1]$ | $\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$ | $ z < \alpha $ |
| $\alpha^{ n }, \alpha < 1$ | $\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \alpha^{-1}z^{-1}}$ | $ \alpha < z < \left \frac{1}{\alpha}\right $ |

Interconnection of LTI Systems

| Connection | Time | Laplace/Z |
|------------|-------------------|-------------------|
| Series | $[h_1 * h_2](t)$ | $H_1(s)H_2(s)$ |
| Parallel | $h_1(t) + h_2(t)$ | $H_1(s) + H_2(s)$ |

Misc. Identities

$$\sum_{k=0}^n x^k = \frac{1 - x^{n+1}}{1 - x} \xrightarrow{n \rightarrow \infty \text{ and } x < 1} \frac{1}{1 - x}$$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$