PHYS 301 Formula Sheet

Differential Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Integral Maxwell's Equations

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \oint \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \qquad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

Electrostatics

The Electric Field

Coulomb's Law:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}} \hat{\boldsymbol{\mathscr{r}}}$$

The Electric Field:

$$\mathbf{F} = Q\mathbf{E} \qquad \mathbf{E} = -\nabla V$$

Electric Field due to discrete point charges:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\boldsymbol{\mathscr{r}}}_i$$

Electric Field due to a continuous charge distribution:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, dq$$

Electric Potential

$$V(b) - V(a) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Poisson's Equation:

$$\nabla^2 V = \rho/\epsilon_0$$

Potential due to a localized charge distribution:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, dV$$

Work and Energy in Electrostatics

Energy stored in a point charge distribution:

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i)$$

Energy of a continuous charge distribution:

$$W = \frac{1}{2} \int \rho V \, d\tau$$

Total energy of a continuous charge distribution:

$$W = \frac{\epsilon_0}{2} \int E^2 \, d\tau \quad \text{(all space)}$$

Conductors

$\mathbf{E} = \mathbf{0}$ inside a conductor.

Electric Field immediately outside a conductor:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

Surface charge:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Capacitors:

$$Q = CV$$

$$W = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QV$$

Potentials

Laplace's Equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Separation of Variables

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ a/2 & \text{if } n = m \end{cases}$$

Legendre Polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = (3x^2 - 1)/2$
- $P_3(x) = (5x^3 - 3x)/2$

Solution to Laplace in spherical (ϕ independent):

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Solution to Laplace in cylindrical (z independent):

$$V(s, \phi) = A_0 \ln(s) + B_0 + \sum_{n=1}^{\infty} (A_n s^{-n} + B_n s^{-n})(C_n \cos(n\phi) + D_n \sin(n\phi))$$

Multipole Expansion

Potential at large distances (α is between $\mathbf{r}$ and $\mathbf{r}'$):

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha) \rho(\mathbf{r}')\, d\tau$$

Dipole Moment:

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i = \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau$$

Electric Dipole Potential:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

Electric Fields in Matter

Bound Charges:

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_b = -\nabla \cdot \mathbf{P}$$

The Electric Displacement:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \nabla \cdot \mathbf{D} = \rho_f \qquad \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f,\text{enc}}$$

Linear Dielectrics

Polarization:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

Electric Displacement:

$$\mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon \mathbf{E}$$

Energy in a Dielectric System:

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E}\, d\tau$$

Boundary Conditions in Electrostatics

- $\mathbf{D}^{\perp}_{\text{above}} - \mathbf{D}^{\perp}_{\text{below}} = \sigma_f$
- $\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$
- $\mathbf{E}^{\perp}_{\text{above}} - \mathbf{E}^{\perp}_{\text{below}} = \sigma/\epsilon_0$
- $\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \mathbf{0}$
- $V_{\text{above}} = V_{\text{below}}$
- $\epsilon_{\text{above}} \mathbf{E}^{\perp}_{\text{above}} - \epsilon_{\text{below}} \mathbf{E}^{\perp}_{\text{below}} = \sigma_f$
- $\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = \sigma_f$

Magnetostatics

Lorentz Force Law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Currents:

$$\mathbf{I} = \lambda \mathbf{v} \qquad \mathbf{K} = \sigma \mathbf{v} \qquad \mathbf{J} = \rho \mathbf{v}$$

Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

Magnetic Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{where} \quad \nabla \cdot \mathbf{A} = 0$$

Vector Potential Poisson's Equation:
$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Vector Potential when $\mathbf{A} \to \mathbf{0}$ at infinity:
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau$$

Multipole Expansion of a current loop:
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha)\, d\mathbf{l}'$$
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

Magnetic Dipole Moment for a vector area $\mathbf{a}$:
$$\mathbf{m} = I \int d\mathbf{A} = I\mathbf{a}$$

Magnetic Fields in Matter

Bound Currents:
$$\mathbf{J}_B = \nabla \times \mathbf{M} \qquad \mathbf{K}_B = \mathbf{M} \times \mathbf{n}$$

Auxillary Field:
$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad \nabla \times \mathbf{H} = \mathbf{J}_f \qquad \oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{free}}$$

Linear Media

Magnetization in linear media:
$$\mathbf{M} = \chi_m \mathbf{H}$$

Auxillary Field:
$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}$$

Volume bound current:
$$\mathbf{J}_B = \chi_m \mathbf{J}_f$$

Boundary Conditions in Magnetostatics

- $\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0 \mathbf{K}$
- $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$
- $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$
- $\dfrac{\partial A_{\text{above}}}{\partial n} - \dfrac{\partial A_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$
- $\mathbf{H}^{\perp}_{\text{above}} - \mathbf{H}^{\perp}_{\text{below}} = -(\mathbf{M}^{\perp}_{\text{above}} - \mathbf{M}^{\perp}_{\text{below}})$
- $\mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}$

Vector Derivatives

Cartesian

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}} \qquad d\tau = dx\,dy\,dz$$

Gradient:
$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:
$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:
$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Spherical

$$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Gradient:
$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence:
$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl:
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} +$$
$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Cylindrical

$$d\tau = s\, ds\, d\phi\, dz$$

Gradient:
$$\nabla f = \frac{\partial f}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:
$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:
$$\nabla \times \mathbf{v} =$$
$$\left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Fundamental Theorems

Fundamental Theorem of Line Integrals:
$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence Theorem:
$$\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$$

Stoke's Theorem:
$$\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$$

Vector Identities

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$
$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}$$
$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Spherical Coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$$
$$\phi = \tan^{-1}(y/x)$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Cylindrical Coordinates

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$s = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

$$\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Updated January 14, 2021
https://github.com/DonneyF/formula-sheets