PHYS 350 Formula Sheet

Lagrangian Mechanics

Hamilton's Principle: $\mathcal{L}(\mathbf{q}, \mathbf{q}, t)$ minimizes

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \to S[t] = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)\, dt$$

Euler-Lagrangian Equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad i = 1, 2, \ldots, s$$

Conservation of Energy:

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

$$E = T + U = \sum_{i=1}^{s} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}$$

Conservation of Momentum:

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Systems with $s = 1$, $\frac{\partial L}{\partial t} = 0$:

- $\mathcal{L}(q, \dot{q}) = \dfrac{\alpha(q)\dot{q}^2}{2} - U(q)$
- $E \geq U(q)$
- $U(q_o) = E$ are turning points
- $\displaystyle\int_0^T dt = \int_0^Q \frac{1}{\sqrt{\frac{2}{\alpha(q)}(E - U(q))}} dq$

Two Body Problem

Generalized Coordinates:

$$\boldsymbol{r} = \mathbf{r_1} - \mathbf{r_2}$$

$$\mathbf{R}_{\text{CM}} = \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1 + m_2}$$

$$\dot{\mathbf{r_1}} = \frac{m_2}{m_1 + m_2}\dot{\boldsymbol{r}} + \mathbf{R}_{\text{CM}} \qquad \dot{\mathbf{r_2}} = -\frac{m_1}{m_1 + m_2}\dot{\boldsymbol{r}} + \mathbf{R}_{\text{CM}}$$

Lagrangian:

$$\mathcal{L} = \frac{\mu}{2}|\dot{\boldsymbol{r}}|^2 + \frac{M}{2}\left|\dot{\mathbf{R}}_{\text{CM}}\right|^2 - U(\boldsymbol{r})$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad M = m_1 + m_2$$

Reduction to independent problems:

$$\mathcal{L} = \mathcal{L}_{\text{CM}} + \mathcal{L}_{\text{rel}}$$

$$\mathcal{L}_{\text{CM}} = \frac{M}{2}\left|\dot{\mathbf{R}}_{\text{CM}}\right|^2$$

$$\mathcal{L}_{\text{rel}} = \frac{\mu}{2}|\dot{\boldsymbol{r}}|^2 - U(\boldsymbol{r})$$

Angular Momentum in Polar Coordinates:

$$\ell = \mu r^2 \dot{\phi}$$

Planetary Motion

$$U(r) = -\frac{G m_1 m_2}{r} = -\frac{\alpha}{r}$$

Eccentricity when $\ell \leq 0$:

$$e = \sqrt{1 + \frac{E_{\text{rel}}}{U_o}} \qquad U_o = |\min\{U_{\text{eff}}(r)\}|$$

$$\text{Trajectory} = \begin{cases} \text{Constant radius orbit} & e = 0 \\ \text{Ellipse} & 0 < e < 1 \\ \text{Parabola} & e = 1 \\ \text{Hyperbola} & e > 1 \end{cases}$$

Small Oscillations

Equilibrium Point:

$$\left.\frac{dU}{dq}\right|_{q=q_0} = 0$$

Stability Criterion of Equilibrium Point ($s = 1$):

$$\frac{d^2U}{dq^2} > 0$$

Stability Criterion of Equilibrium Point ($s \geq 2$):

$$\frac{\partial^2 U}{\partial q_i \partial q_j} = K_{ij} = K_{ji} \geq 0$$

Small Angle Approximation:

Taylor expand around equilibrium point of the Lagrangian q_0 and keep up to first term that contributes to the Lagrangian.

General Solution to $\ddot{x} = -\omega^2(x - x_0) + f(t)$:

Let $z(t) = \dot{x}(t) + i\omega x(t) \qquad x(t) = x_0 + \text{Im}\{z(t)\}/\omega$

$$z(t) = e^{i\omega t}\left[z(0) + \int_0^t e^{-i\omega\tau} f(\tau)\, d\tau\right]$$

Rigid Body Motion

Kinetic Energy:

$$T = \tfrac{M}{2}|\mathbf{V_0}|^2 + M r_{\text{CM}} \cdot (\mathbf{V_0} \times \mathbf{\Omega}) + \tfrac{1}{2}\mathbf{\Omega}\hat{I}_0\mathbf{\Omega}$$

where $\mathbf{V_o}$ is the velocity of the chosen reference point O, $r_{\text{CM}} = \mathbf{R}_{\text{CM}} - r_{\text{O}}$ is the position of the CM with respect to O, and $\mathbf{\Omega}$ is the angular velocity of the body.

Moment of Inertia:

$$I_{xx}^{(0)} = \int_{\mathcal{V}} \rho(\mathbf{r})(y^2 + z^2)\, dV \qquad I_{xy}^{(0)} = \int_{\mathcal{V}} \rho(\mathbf{r})xy\, dV$$

Parallel Axis Theorem:

$$I_{xx}^{(0)} = I_{xx}^{\text{CM}} + M(d_y^2 + d_z^2)$$

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https://github.com/DonneyF/formula-sheets