MATH 305 Formula Sheet

Complex Numbers

Operators:

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$$

Conjugation:

$$\overline{z_1 z_1} = \overline{z_1} \cdot \overline{z_2} \qquad \overline{z_1/z_1} = \overline{z_1}/\overline{z_2}$$

Modulus:

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}} \qquad |z| = |\overline{z}|$$

$$|z_1 z_1| = |z_1||z_2|$$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

Argument Function:

$$\arg(z) = \arg(|z|e^{i\varphi}) = \{\varphi + 2\pi k \mid k \in \mathbb{Z}\}$$

$$\mathrm{Arg}(z) = \{\arg(z) + 2\pi k \mid k \in \mathbb{Z} \wedge -\pi < \mathrm{Arg}(z) \le \pi\}$$

De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Roots of Unity:

For $z = re^{i\theta}$, the n-th root of z is

$$z^{1/n} = r^{1/n}\exp\left(i\frac{\theta + 2\pi k}{n}\right) \qquad k = 0, 1, 2, \ldots, n-1$$

$$1^{1/n} = e^{i2\pi k/m}$$

Complex Functions

Def (*Continuity*): Suppose $f(z)$ is defined on a domain D and $z_0 \in D$. Then $f(z)$ is continuous at z_0 if:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Def (*Differentiability*): Let f be defined in a neighborhood of z_0. Then f is differentiable at z_0 if the following limit exists:

$$\frac{df}{dz}(z_0) \equiv f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Def (*Analyticity*): A complex valued function $f(z)$ is said to be analytic on an open domain D if it has a derivative at every point in D.

Cauchy-Riemann Equations:

For $f(z) = U(x, y) + iV(x, y)$, the CR equations are

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Theorem 1. Let $f(z) = U(x, y) + iV(x, y)$ be defined in some open set G containing the point z_0. If the first partial derivatives of U and V exist in G, are continuous at z_0, and satisfy the Cauchy-Riemann equations at z_0, then f is differentiable at z_0.

Theorem 2. If $\dfrac{\partial f}{\partial \overline{z}} = 0$, then f is differentiable.

Def (*Harmonic*): A function $u(x, y)$ is harmonic if $\Delta u = 0$.

Theorem 3. If $f(z) = U(x, y) + iV(x, y)$ is analytic, then U and V are harmonic functions.

Harmonic Conjugate:

Let $f(z) = U + Vi$ be an analytic function. Then V is a harmonic conjugate of U.

Elementary Functions

$$e^z = e^x(\cos y + i\sin y)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\log z = \ln|z| + i\arg(z)$$

$$\sin^{-1} z = -i\log\left(iz \pm \sqrt{1 - z^2}\right)$$

$$\cos^{-1} z = -i\log\left(z + \sqrt{z^2 - 1}\right)$$

$$\tan^{-1} z = \frac{i}{2}\log\left(\frac{1 - iz}{1 + iz}\right)$$

Theorem 4. $\mathrm{Arg}(z)$ is harmonic in $\mathbb{C} - \{x < 0 \mid x \in \mathbb{R}\}$.

Def (*Branch*): A function $F(z)$ is a branch of a multiple-valued function $f(z)$ if $F(z)$ is single-valued, continuous in some domain, and $F(z) \in f(z)$. **Def** (*Branch Cut*): Discontinuous points of an argument function.

Complex Integration

Fundamental Theorem of Calculus:

$$\int_a^b f(z)\,dz = F(b) - F(a)$$

Contour Integral:

$$\int_\Gamma f(z) = \int_a^b f(r(t))r'(t)\,dt$$

Theorem 5. If f is continuous on the contour Γ and if $|f(z)| < M$ for all z on Γ, then $\left|\int_\Gamma f(z)\,dz\right| \le M\ell(\Gamma)$, where $\ell(\Gamma)$ is the length of Γ.

Path Independence:

If $f(z)$ is continuous in an domain D and has an anti-derivative $F(z)$ throughout D, then with initial point z_I and terminal point z_T, for any $\Gamma \in D$, $\int_\Gamma f(z)\,dz = F(z_T) - F(z_I)$

Deformation Invariance:

Let f be analytic in a domain D containing the loops Γ_0 and Γ_1. If the loops can be deformed continuously to one-another, then $\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz$

Cauchy's Theorem:

If f is analytic in a simply connected domain D and Γ is any closed contour, then $\int_\Gamma f(z)\,dz = 0$.

Cauchy's Integral Formula:

Let Γ be a simple closed positively oriented contour. If f is analytic in the domain enclosed by Γ, and z_0 is any point inside Γ, then

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz$$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\sigma)}{(\sigma - z)^{n+1}}\,d\sigma$$

Complex Analysis

Theorem 6. A bounded entire function must be constant.

Maximum Modulus Principle:

If f is analytic in a domain D and $|f(z)|$ achieves its maximum value at a point z_0 in D, then f is constant in D.

Theorem 7. A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.

Theorem 8. Suppose at each point in some closed domain enclosed by Γ f is analytic or is a pole. Then $N_0(f) - N_p(f) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f'(z)}{f(z)}\,dz$

Argument Principle:

$$\int_\Gamma \frac{f'(z)}{f(z)}\,dz = i(\text{Total change of } \arg(f(z)) \text{ along } \Gamma)$$

Nyquist Stability Criterion:

Let Γ_+ be the countour from $(0, \infty)$ to $(0, 0)$. For a polynomial function $p(z)$ of order n, the number of zeroes in the right-half plane is given by:

$$N_0(p(z)) = \frac{1}{2\pi}\left(n\pi + 2\Delta_{\Gamma_+}(\arg(p(z)))\right)$$

Rouche's Theorem:

Suppose f and h are analytic functions on a domain enclosed by Γ and that $|h(z)| < |f(z)|\, \forall z \in \Gamma$. Then

$N_0(f) = N_0(f + h)$.

Laurent Series:

Assume f is analytic in some annulus $r < |z - z_0| < R$. Then we can write $\sum_{j=-\infty}^{\infty} a_j(z - z_0)^j$.

Singularities:

Let f have an isolated singularity at z_0, and let f have a Laurent series expansion in $r < |z - z_0| < R$. Then

- If $a_j = 0$ for all $j < 0$, we say z_0 is a removable singularity.
- If $a_{-m} \neq 0$ for some positive integer m, but $a_j = 0$ for all $j < -m$, we say that z_0 is a pole of order m for f.
- If $a_j \neq 0$ for all $j < 0$, then we say z_0 is an essential singularity of f.

Residue Theory

Def (*Residue*): Suppose f has a Laurent series expansion around a point z_0. Then $\text{Res}(f, z_0) = a_{-1}$.

Theorem 9. Suppose $f(z) = P(z)/Q(z)$ and $Q(z)$ has a simple zero at z_0. Then $\text{Res}(f, z_0) = \dfrac{P(z_0)}{Q'(z_0)}$.

Theorem 10. If f has a pole of order m at z_0, then $\text{Res}(f, z_0) = \lim_{z \to z_0} \dfrac{1}{(m-1)!} \dfrac{d^{m-1}}{dz^{m-1}} [f(z)(z - z_0)^m]$

Cauchy's Residue Theorem:

If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at the points $z_1, z_2, \ldots, z_n$, then $\int_\Gamma f(z)\, dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j)$

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https://github.com/DonneyF/formula-sheets