

# PHYS 304 Formula Sheet

## The Wave Function

Time dependent Schrodinger Equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

Standard Deviation:

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

Momentum:

$$\langle p \rangle = m \frac{d \langle x \rangle}{dt} = \int \Psi^* \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi dx$$

## Infinite Square Well

Time Independent Schrodinger Equation:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi$$

Eigenstate Expansion:

$$\Psi(x, t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} = \sum_{n=1}^{\infty} c_n \Psi_n(x, t)$$

Energy In Infinite Square Well:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

Stationary States:

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

Determining Coefficients:

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x, 0) dx$$

Expectation Value of Energy:

$$\langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

## Harmonic Oscillator

$$k = \omega^2 m$$

Ladder Operators:

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}} (\mp i p + m\omega x)$$

$$a_+ \psi_n = \sqrt{n+1} \psi_{n+1}$$

$$a_- \psi_n = \sqrt{n} \psi_{n-1}$$

Operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-)$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}} (a_+ - a_-)$$

$$x^2 = \frac{\hbar}{2m\omega} [(a_+)^2 + (a_+ a_-) + (a_- a_+) + (a_-)^2]$$

Commutation:

$$[x, p] = i\hbar$$

$$[a_-, a_+] = 1$$

Hamiltonian:

$$H = \hbar\omega \left( a_+ a_- + \frac{1}{2} \right)$$

$$a_+ a_- + a_- a_+ = 2 \left( \frac{H}{\hbar\omega} \right)$$

States:

$$\psi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}$$

$$\psi_n = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0$$

Energy:

$$E_n = \left( n + \frac{1}{2} \right) \hbar\omega$$

## Generalized Statistical Interpretation

Momentum Expansion:

$$\Phi(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x, t) dx$$

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \Phi(p, t) dp$$

## Angular Momentum

$$L_x = yp_z - zp_y \quad L_y = zp_x - xp_z \quad L_z = xp_y - yp_x$$

Commutators:

$$[L_x, L_y] = i\hbar L_z \quad [L_y, L_z] = i\hbar L_x \quad [L_z, L_x] = i\hbar L_y$$

$$[L_z, x] = i\hbar y \quad [L_z, y] = -i\hbar x \quad [L_z, z] = 0$$

$$[L_z, p_x] = i\hbar p_y \quad [L_z, p_y] = -i\hbar p_x \quad [L_z, p_z] = 0$$

Square of Angular Momentum:

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2 \quad [L^2, \mathbf{L}] = 0$$

Ladder Operators:

$$L_{\pm} = L_x \pm iL_y \quad [L_z, L_{\pm}] = \pm i\hbar L_{\pm}$$

$$L^2 = L_{\pm} L_{\mp} + L_z^2 \mp \hbar L_z$$

$$L_x = \frac{L_+ + L_-}{2} \quad L_y = \frac{L_+ - L_-}{2i}$$

$$L_+ |l, l\rangle = 0 \quad L_- |l, -l\rangle = 0$$

Eigenvalues:

$$L^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle$$

$$L_z |l, m\rangle = \hbar m |l, m\rangle$$

$$L_{\pm} |l, m\rangle = A_l^m |l, m \pm 1\rangle$$

$$A_l^m = \hbar \sqrt{l(l+1) - m(m \pm 1)}$$

Expectation Value when  $L_x |\psi\rangle = L_y |\psi\rangle$ :

$$\langle \psi | L_x^2 | \psi \rangle = \langle \psi | L_y^2 | \psi \rangle = \frac{1}{2} \langle \psi | L^2 - L_z^2 | \psi \rangle$$

Spherical Coordinate Representation:

$$L_x = \frac{\hbar}{i} \left( -\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi} \right)$$

$$L_y = \frac{\hbar}{i} \left( +\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi} \right)$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial\theta}$$

$$L_{\pm} = \pm \hbar e^{\pm i\theta} \left( \frac{\partial}{\partial\theta} \pm i \cot\theta \frac{\partial}{\partial\phi} \right)$$

$$L_+ L_- = -\hbar^2 \left( \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \cot^2\theta \frac{\partial^2}{\partial\phi^2} + i \frac{\partial}{\partial\phi} \right)$$

$$L^2 = -\hbar^2 \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

## Spin

Commutators:

$$[S_x, S_y] = i\hbar S_z \quad [S_y, S_z] = i\hbar S_x \quad [S_z, S_x] = i\hbar S_y$$

Eigenvalues:

$$S^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle$$

$$S_z |s, m\rangle = \hbar m |s, m\rangle$$

$$S_{\pm} |s, m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)} |s, m \pm 1\rangle$$

Square of Angular Momentum:

$$S^2 \equiv S_x^2 + S_y^2 + S_z^2 \quad [S^2, \mathbf{S}] = 0$$

## Spin 1/2

Spinors:

$$\chi = \begin{bmatrix} a \\ b \end{bmatrix}, |a|^2 + |b|^2 = 1 \quad \chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Spin Operators:

$$\mathbf{S}_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{S}_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \mathbf{S}_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{S}^2 = \frac{3}{4} \hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{S}_+ = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{S}_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

## The Hydrogen Atom

$$n = j_{\max} + l + 1$$

Wave Function:  $\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi)$

Radial Equation:

$$R_{nl}(r) = \frac{1}{r} \rho^{l+1} e^{-\rho} v(\rho) \quad \rho = \frac{r}{an}$$

$$v(\rho) = \sum_0^{j_{\max}} c_j \rho^j \quad c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)} c_j$$

Energy:

$$E_n = \frac{E_1}{n^2} \quad E_1 = -13.6 \text{ eV}$$

## Addition of Angular Momenta

Possible values of total spin  $s$ :

$$s = (s_1 + s_2), (s_1 + s_2 - 1), \dots, |s_1 - s_2|$$

Operators:

$$S^2 = (S^{(1)})^2 + (S^{(2)})^2 + 2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$$

$$2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} = S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)} + S_z^{(1)} S_z^{(2)}$$

$$[S^2, \mathbf{S}^{(1)}] = 2i\hbar(\mathbf{S}^{(1)} \times \mathbf{S}^{(2)})$$

Combined state with total spin  $s$ , and  $z$ -component  $m$ :

$$|s, m\rangle = \sum_{m_1+m_2=m} C_{m_1 m_2 m}^{s_1 s_2 s} |s_1, m_1\rangle |s_2, m_2\rangle$$

$$|s_1, m_1\rangle |s_2, m_2\rangle = \sum_s C_{m_1 m_2 m}^{s_1 s_2 s} |s, m\rangle$$

## Trig Identities

$$2 \cos \theta \cos \phi = \cos(\theta - \phi) + \cos(\theta + \phi)$$

$$2 \sin \theta \sin \phi = \cos(\theta - \phi) - \cos(\theta + \phi)$$

$$2 \sin \theta \cos \phi = \sin(\theta + \phi) + \sin(\theta - \phi)$$

$$2 \cos \theta \sin \phi = \sin(\theta + \phi) - \sin(\theta - \phi)$$

## Integral Identities

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{1}{4a} \sin 2ax + C$$

$$\int \sin^3 ax \, dx = \frac{\cos 3ax}{12a} - \frac{3 \cos ax}{4a} + C$$

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{1}{4a} \sin 2ax + C$$

$$\int \cos^3 ax \, dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a} + C$$

$$\int \cos^n(ax) \sin(ax) \, dx = -\frac{\cos^{n+1}(ax)}{a(n+1)} + C$$

$$\int \sin^n(ax) \cos(ax) \, dx = \frac{\sin^{n+1}(ax)}{a(n+1)} + C$$

$$\int_0^\pi \cos^{2n+1}(ax) \sin(ax) \, dx = 0 \quad n = 0, 1, 2, \dots$$

$$\int_0^\pi \cos^{2n}(x) \sin(x) \, dx = \frac{2}{2n+1} \quad n = 0, 1, 2, \dots$$

$$\int_0^\infty x^n e^{-x/a} \, dx = n! a^{n+1}$$

$$\int_0^\infty x^{2n} e^{-x^2/a^2} \, dx = \sqrt{\pi} \frac{(2n)!}{n!} \left(\frac{a}{2}\right)^{2n+1}$$

$$\int_0^\infty x^{2n+1} e^{-x^2/a^2} \, dx = \frac{n!}{2} a^{2n+2}$$

## Spherical Harmonics

$l$	$m$	$Y_l^m(\theta, \phi)$
0	0	$Y_0^0 = \sqrt{\frac{1}{4\pi}}$
1	0	$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$
	$\pm 1$	$Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$
2	0	$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$
	$\pm 1$	$Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$
	$\pm 2$	$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$
3	0	$Y_3^0 = \sqrt{\frac{7}{16\pi}} (5 \cos^3 \theta - 3 \cos \theta)$
	$\pm 1$	$Y_3^{\pm 1} = \mp \sqrt{\frac{21}{64\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{\pm i\phi}$
	$\pm 2$	$Y_3^{\pm 2} = \sqrt{\frac{105}{32\pi}} \sin^2 \theta \cos \theta e^{\pm 2i\phi}$
	$\pm 3$	$Y_3^{\pm 3} = \mp \sqrt{\frac{35}{64\pi}} \sin^3 \theta e^{\pm 3i\phi}$

## Hydrogen Radial Wave Functions

$$R_{10} = 2a^{-3/2} e^{-r/a}$$

$$R_{20} = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a}\right) e^{-r/2a}$$

$$R_{21} = \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}$$

$$R_{30} = \frac{2}{\sqrt{27}} a^{-3/2} \left(1 - \frac{2}{3} \frac{r}{a} + \frac{2}{27} \left(\frac{r}{a}\right)^2\right) e^{-r/3a}$$

$$R_{31} = \frac{8}{27\sqrt{6}} a^{-3/2} \left(1 - \frac{1}{6} \frac{r}{a}\right) \left(\frac{r}{a}\right) e^{-r/3a}$$

$$R_{32} = \frac{4}{81\sqrt{30}} a^{-3/2} \left(\frac{r}{a}\right)^2 e^{-r/3a}$$

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<https://github.com/DonneyF/formula-sheets>