

PHYS 301 Formula Sheet

Differential Maxwell's Equations

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

Integral Maxwell's Equations

$$\begin{aligned}\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} & \oint \mathbf{B} \cdot d\mathbf{a} &= 0 \\ \oint \mathbf{E} \cdot d\mathbf{l} &= -\frac{d\Phi_B}{dt} & \oint \mathbf{B} \cdot d\mathbf{l} &= \mu_0 I_{\text{enc}} + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}\end{aligned}$$

Electrostatics

The Electric Field

Coulomb's Law:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{r}}$$

The Electric Field:

$$\mathbf{F} = Q\mathbf{E} \quad \mathbf{E} = -\nabla V$$

Electric Field due to discrete point charges:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i$$

Electric Field due to a continuous charge distribution:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbf{r}}}{r^2} dq$$

Electric Potential

$$V(b) - V(a) = - \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Poisson's Equation:

$$\nabla^2 V = \rho/\epsilon_0$$

Potential due to a localized charge distribution:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} dV$$

Work and Energy in Electrostatics

Energy stored in a point charge distribution:

$$W = \frac{1}{2} \sum_{i=1}^n q_i V(\mathbf{r}_i)$$

Energy of a continuous charge distribution:

$$W = \frac{1}{2} \int \rho V d\tau$$

Total energy of a continuous charge distribution:

$$W = \frac{\epsilon_0}{2} \int E^2 d\tau \quad (\text{all space})$$

Conductors

$\mathbf{E} = \mathbf{0}$ inside a conductor.

Electric Field immediately outside a conductor:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

Surface charge:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Capacitors:

$$Q = CV$$

$$W = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} QV$$

Potentials

Laplace's Equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Separation of Variables

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ a/2 & \text{if } n = m \end{cases}$$

Legendre Polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = (3x^2 - 1)/2$
- $P_3(x) = (5x^3 - 3x)/2$

Solution to Laplace in spherical (ϕ independent):

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Solution to Laplace in cylindrical (z independent):

$$\begin{aligned}V(s, \phi) &= A_0 \ln(s) + B_0 + \\ &\sum_{n=1}^{\infty} (A_n s^{-n} + B_n s^{-n})(C_n \cos(n\phi) + D_n \sin(n\phi))\end{aligned}$$

Multipole Expansion

Potential at large distances (α is between \mathbf{r} and \mathbf{r}'):

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos \alpha) \rho(\mathbf{r}') d\tau$$

Dipole Moment:

$$\mathbf{p} = \sum_{i=1}^n q_i \mathbf{r}'_i = \int \mathbf{r}' \rho(\mathbf{r}') d\tau$$

Electric Dipole Potential:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

Electric Fields in Matter

Bound Charges:

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \quad \rho_b = -\nabla \cdot \mathbf{P}$$

The Electric Displacement:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad \nabla \cdot \mathbf{D} = \rho_f \quad \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f,\text{enc}}$$

Linear Dielectrics

Polarization:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

Electric Displacement:

$$\mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon \mathbf{E}$$

Energy in a Dielectric System:

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau$$

Boundary Conditions in Electrostatics

- $\mathbf{D}_{\text{above}}^{\perp} - \mathbf{D}_{\text{below}}^{\perp} = \sigma_f$
- $\mathbf{D}_{\text{above}}^{\parallel} - \mathbf{D}_{\text{below}}^{\parallel} = \mathbf{P}_{\text{above}}^{\parallel} - \mathbf{P}_{\text{below}}^{\parallel}$
- $\mathbf{E}_{\text{above}}^{\perp} - \mathbf{E}_{\text{below}}^{\perp} = \sigma/\epsilon_0$
- $\mathbf{E}_{\text{above}}^{\parallel} - \mathbf{E}_{\text{below}}^{\parallel} = 0$
- $V_{\text{above}} = V_{\text{below}}$
- $\epsilon_{\text{above}} \mathbf{E}_{\text{above}}^{\perp} - \epsilon_{\text{below}} \mathbf{E}_{\text{below}}^{\perp} = \sigma_f$
- $\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = \sigma_f$

Magnetostatics

Lorentz Force Law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Currents:

$$\mathbf{I} = \lambda \mathbf{v} \quad \mathbf{K} = \sigma \mathbf{v} \quad \mathbf{J} = \rho \mathbf{v}$$

Biot-Savart Law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{r}}}{r^2} dl' = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2}$$

Magnetic Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{where} \quad \nabla \cdot \mathbf{A} = 0$$

Vector Potential Poisson's Equation:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Vector Potential when $\mathbf{A} \rightarrow \mathbf{0}$ at infinity:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau$$

Multipole Expansion of a current loop:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos \alpha) d\mathbf{l}'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

Magnetic Dipole Moment for a vector area \mathbf{a} :

$$\mathbf{m} = I \int d\mathbf{A} = I \mathbf{a}$$

Magnetic Fields in Matter

Bound Currents:

$$\mathbf{J}_B = \nabla \times \mathbf{M} \quad \mathbf{K}_B = \mathbf{M} \times \mathbf{n}$$

Auxillary Field:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad \nabla \times \mathbf{H} = \mathbf{J}_f \quad \oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{free}}$$

Linear Media

Magnetization in linear media:

$$\mathbf{M} = \chi_m \mathbf{H}$$

Auxillary Field:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}$$

Volume bound current:

$$\mathbf{J}_B = \chi_m \mathbf{J}_f$$

Boundary Conditions in Magnetostatics

- $\mathbf{B}_{\text{above}}^{\parallel} - \mathbf{B}_{\text{below}}^{\parallel} = \mu_0 \mathbf{K}$
- $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}})$
- $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$
- $\frac{\partial A_{\text{above}}}{\partial n} - \frac{\partial A_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$
- $\mathbf{H}_{\text{above}}^{\perp} - \mathbf{H}_{\text{below}}^{\perp} = -(\mathbf{M}_{\text{above}}^{\perp} - \mathbf{M}_{\text{below}}^{\perp})$
- $\mathbf{H}_{\text{above}}^{\parallel} - \mathbf{H}_{\text{below}}^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$

Vector Derivatives

Cartesian

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}} \quad d\tau = dx dy dz$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

Spherical

$$d\tau = r^2 \sin \theta dr d\theta d\phi$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_{\theta}) + \frac{1}{r \sin \theta} \frac{\partial v_{\phi}}{\partial \phi}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_{\phi}) - \frac{\partial v_{\theta}}{\partial \phi} \right] \hat{\mathbf{r}} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_{\phi}) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_{\theta}) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$$

Cylindrical

$$d\tau = s ds d\phi dz$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_{\phi}}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_{\phi}}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_{\phi}) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

Fundamental Theorems

Fundamental Theorem of Line Integrals:

$$\int_a^b (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence Theorem:

$$\int (\nabla \cdot \mathbf{A}) d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$$

Stoke's Theorem:

$$\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$$

Vector Identities

$$\nabla \cdot \left(\frac{\hat{\mathbf{z}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r})$$

$$\nabla \left(\frac{1}{r} \right) = -\frac{\hat{\mathbf{z}}}{r}$$

$$\delta(kx) = \frac{1}{|k|} \delta(x)$$

Spherical Coordinates

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \left(\sqrt{x^2 + y^2} / z \right)$$

$$\phi = \tan^{-1}(y/x)$$

$$\hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$$

Cylindrical Coordinates

$$x = s \cos \phi$$

$$y = s \sin \phi$$

$$z = z$$

$$\hat{\mathbf{x}} = \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$s = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}(y/x)$$

$$z = z$$

$$\hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Updated December 11, 2018

<https://github.com/DonneyF/formula-sheets>