PHYS 408 Formula Sheet

Differential Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Integral Maxwell's Equations

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \qquad \oint \mathbf{B} \cdot d\mathbf{a} = 0$$
$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \qquad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

Wave Optics

Monochromatic Wave:
$$\mathcal{E} = \text{Re}\left\{\mathbf{E}(\mathbf{r}) e^{-i\omega t}\right\}$$
Helmholtz Equation:
$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$
Plane wave solution to Helmholtz Equation:
$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$$
Spherical wave solution to Helmholtz Equation:
$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \frac{1}{r} e^{ikr} \qquad \mathbf{E}_0 = E_0 \hat{\theta}$$
Interference of multiple fields:
$$I = \sum_{k=1}^{N} I_k + v\epsilon \,\text{Re}\left\{E_1 E_2^* + E_1 E_3^* + \cdots + E_{N-1} E_N^*\right\}$$
$$v\epsilon E_i E_j^* = 2\sqrt{I_i I_j} \exp\left[i(\theta_i - \theta_j)\right]$$
Fresnel (Paraxial) Approximation:
$$\sqrt{x^2 + y^2} \ll z$$
Transverse Laplacian Operator:
$$\nabla_T^2 = \tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial y}$$
Paraxial Wave Equation:
$$\nabla_T^2 \mathbf{E} + i2k \frac{\partial \mathbf{E}}{\partial z} = \mathbf{0}$$
Fresnel Approximation of a spherical wave:
$$\mathbf{E}(\mathbf{r}) \approx \frac{\mathbf{E}_0}{z} \exp\left[ik \frac{x^2 + y^2}{2z}\right] = \frac{\mathbf{E}_0}{z} \exp\left[ik \frac{\rho^2}{2z}\right]$$

Beam Optics

Gaussian beam with Rayleigh length z_0:
$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \frac{w_0}{w(z)} \exp\left[-\frac{\rho^2}{(w(z))^2}\right] \times$$
$$\exp\left[ik \frac{\rho^2}{2R(z)}\right] \exp\left[ikz - i\tan^{-1}\left(\frac{z}{z_0}\right)\right]$$
$$q(z) = z - z_0$$
$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi} \frac{1}{(w(z))^2}$$
Beam waist:
$$w_0 = \sqrt{\frac{\lambda z_0}{\pi}}$$
Beam Radius:
$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$
Radius of curvature:
$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$
Gaussian beam intensity with center intensity I_0:
$$I(z, \rho) = I_0 \left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2\rho^2}{(w(z))^2}\right]$$
Gaussian beam total power:
$$P = \tfrac{I_0}{2} \pi w_0^2$$
Depth of focus (confocal parameter):
$$d = 2z_0 = \frac{2\pi w_0^2}{\lambda}$$

Propagation of Paraxial Waves

ABCD Matrix of an optical system:
$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$
Free space propagation transfer matrix:
$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$
Refraction transfer matrix:
$$\begin{bmatrix} 1 & 0 \\ 0 & n_I/n_T \end{bmatrix}$$
Curved interface transfer matrix:
$$\begin{bmatrix} 1 & 0 \\ -\frac{n_T - n_I}{n_I R} & n_I/n_T \end{bmatrix}$$
Thin lens transfer matrix:
$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$
Gaussian Beam propagation in the paraxial regime:
$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

Fourier Optics

Fraunhofer (far field) limit:
$$x^2/\lambda, y^2/\lambda \ll z/\pi$$
Free space transfer function along $\mathbf{z}$:
$$\tilde{H}(k_x, k_y) = e^{ik_z z} \approx e^{ikz} e^{-i(k_x^2 + k_y^2))z/2k}$$
Free space propagation in paraxial limit:
$$E(x', y', z) \approx \frac{e^{ikz}}{i\lambda z} \exp\left[ik \frac{x'^2 + y'^2}{2z}\right] \times$$
$$\iint E(x, y, z) \exp\left[ik \frac{x^2 + y^2}{2z}\right] \exp\left[-\frac{ik}{z}(x'x + y'y)\right] \,dx\,dy$$
Free space propagation in Fraunhofer limit:
$$E(x', y', z) \approx \frac{e^{ikz}}{i\lambda z} \exp\left[ik \frac{x'^2 + y'^2}{2z}\right] \mathcal{F}\{E(x, y, z)\}$$
Thin lens thickness function:
$$\Delta(x, y) =$$
$$\Delta_0 - R_1 \left(1 - \sqrt{1 - \frac{x^2 + y^2}{R_1^2}}\right) + R_2 \left(1 - \sqrt{1 - \frac{x^2 + y^2}{R_2^2}}\right)$$
Thin lens thickness function in paraxial limit:
$$\Delta(x, y) \approx \Delta_0 - \frac{x^2 + y^2}{2} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$
Thin lens phase function:
$$t_L(x, y) = \exp[ikn\Delta(x, y)] \exp[ik(\Delta_0 - \Delta(x, y))]$$
Thin lens phase function (ignore constant phase)
$$t_L(x, y) = \exp\left[-i\frac{k}{2f}(x^2 + y^2)\right]$$

Vector Derivatives

Cartesian

$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}} \qquad d\tau = dx\,dy\,dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Spherical

$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Cylindrical

$d\tau = s\, ds\, d\phi\, dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Fundamental Theorems

Fundamental Theorem of Line Integrals:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence Theorem:

$$\int (\nabla\cdot\mathbf{A})\, d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$$

Stoke's Theorem:

$$\int (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$$

Vector Identities

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}$$

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Spherical Coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$$
$$\phi = \tan^{-1}(y/x)$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Cylindrical Coordinates

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$s = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

$$\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Trig Identities

$$2\cos\theta\cos\phi = \cos(\theta-\phi) + \cos(\theta+\phi)$$
$$2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi)$$
$$2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi)$$
$$2\cos\theta\sin\phi = \sin(\theta+\phi) - \sin(\theta-\phi)$$

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https://github.com/DonneyF/formula-sheets