PHYS 401 Formula Sheet

Differential Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Integral Maxwell's Equations

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \qquad \oint \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \qquad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

Electromagnetic Waves

Wave Speed:

$$v = \frac{1}{\sqrt{\mu \varepsilon}} = c/n$$

Poynting Vector:

$$\mathbf{S} = \frac{1}{\mu} \mathbf{E} \times \mathbf{B}$$

Momentum Density:

$$\mathbf{P} = \frac{1}{c^2} \mathbf{S}$$

Energy per unit volume:

$$u = \frac{1}{2} \left(\varepsilon |\mathbf{E}|^2 + \frac{|\mathbf{B}|^2}{\mu} \right)$$

Intensity:

$$I = \langle \mathbf{S} \cdot \hat{\mathbf{n}} \rangle$$

Potentials:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{B}$$

Gauge Transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla \Lambda \qquad V' = V - \frac{\partial \Lambda}{\partial t}$$

Lorentz Gauge:

$$\nabla \cdot \mathbf{A} + \mu \varepsilon \frac{\partial V}{\partial t} = 0$$

Lorentz Transformation:

$$\mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu \mathbf{J}$$

$$\mu \varepsilon \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\varepsilon}$$

Plane Waves

$$\mathbf{E} = \mathbf{E}_0 e^{i(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}} - \omega t)} \qquad \mathbf{B} = \mathbf{B}_0 e^{i(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}} - \omega t)}$$

Phase and Group Velocity:

$$v_p = \frac{\omega}{k} \qquad v_g = \frac{d\omega}{dk}$$

Magnetic Field from Electric Field:

$$\mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega}$$

Intensity:

$$I = \frac{1}{2} v \varepsilon E_0^2 = \frac{1}{2} \frac{E_0^2}{\mu v}$$

Law of Refraction:

$$\theta_I = \theta_R$$

Snell's Law:

$$n_I \sin \theta_I = n_T \sin \theta_T$$

Complex Index of Refraction:

$$n \approx \sqrt{\varepsilon_r} = n_R + i n_I$$

Reflection and Transmission Coefficients:

$$R = \frac{I_R}{I_I} = \frac{I_R^{\text{beam}} \cos \theta_R}{I_I^{\text{beam}} \cos \theta_I} = \frac{I_R^{\text{beam}}}{I_I^{\text{beam}}}$$

$$T = \frac{I_T}{I_I} = \frac{I_T^{\text{beam}} \cos \theta_T}{I_I^{\text{beam}} \cos \theta_I}$$

Misc. Constants:

$$\alpha = \frac{\cos \theta_T}{\cos \theta_I} \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Boundary Conditions

- $\varepsilon_1 \mathbf{E}_1^{\perp} = \varepsilon_2 \mathbf{E}_2^{\perp}$
- $\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}$
- $\mathbf{B}_1^{\perp} = \mathbf{B}_2^{\perp}$
- $\dfrac{\mathbf{B}_1^{\parallel}}{\mu_1} = \dfrac{\mathbf{B}_2^{\parallel}}{\mu_2}$

E Parallel to Plane of Incidence

$$\mathbf{k}_I = k_I (\cos \theta_I \hat{\mathbf{z}} + \sin \theta_I \hat{\mathbf{x}})$$

$$\mathbf{E}_I = E_I e^{i[k_I (\cos \theta_I z + \sin \theta_I x) - \omega t]} (\cos \theta_I \hat{\mathbf{x}} - \sin \theta_I \hat{\mathbf{z}})$$

$$\mathbf{k}_R = k_I (-\cos \theta_R \hat{\mathbf{z}} + \sin \theta_R \hat{\mathbf{x}})$$

$$\mathbf{E}_R = E_R e^{i[k_R (-\cos \theta_R z + \sin \theta_R x) - \omega t]} (\cos \theta_R \hat{\mathbf{x}} + \sin \theta_R \hat{\mathbf{z}})$$

$$\mathbf{k}_T = k_T (\cos \theta_T \hat{\mathbf{z}} + \sin \theta_T \hat{\mathbf{x}})$$

$$\mathbf{E}_T = E_T e^{i[k_T (\cos \theta_T z + \sin \theta_T x) - \omega t]} (\cos \theta_T \hat{\mathbf{x}} - \sin \theta_T \hat{\mathbf{z}})$$

Field Magnitudes (Fresnel Equations):

$$\frac{E_R}{E_I} = \left(\frac{\alpha - \beta}{\alpha + \beta} \right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{\alpha + \beta} \right)$$

Reflection and Transmission Coefficients:

$$R = \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2 \qquad T = \alpha \beta \left(\frac{2}{\alpha + \beta} \right)^2$$

Brewster's Angle (no reflected wave, $\alpha = \beta$):

$$\tan \theta_B = n_2 / n_1$$

E Perpendicular to Plane of Incidence

$$\mathbf{k}_I = k_I (\cos \theta_I \hat{\mathbf{z}} + \sin \theta_I \hat{\mathbf{x}})$$

$$\mathbf{B}_I = B_I e^{i[k_I (\cos \theta_I z + \sin \theta_I x) - \omega t]} (-\cos \theta_I \hat{\mathbf{x}} + \sin \theta_I \hat{\mathbf{z}})$$

$$\mathbf{k}_R = k_I (-\cos \theta_R \hat{\mathbf{z}} + \sin \theta_R \hat{\mathbf{x}})$$

$$\mathbf{B}_R = B_R e^{i[k_R (-\cos \theta_R z + \sin \theta_R x) - \omega t]} (\cos \theta_R \hat{\mathbf{x}} + \sin \theta_R \hat{\mathbf{z}})$$

$$\mathbf{k}_T = k_T (\cos \theta_T \hat{\mathbf{z}} + \sin \theta_T \hat{\mathbf{x}})$$

$$\mathbf{B}_T = B_T e^{i[k_T (\cos \theta_T z + \sin \theta_T x) - \omega t]} (-\cos \theta_T \hat{\mathbf{x}} + \sin \theta_T \hat{\mathbf{z}})$$

Field Magnitudes (Fresnel Equations):

$$\frac{E_R}{E_I} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta} \right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{1 + \alpha\beta} \right)$$

Reflection and Transmission Coefficients:

$$R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta} \right)^2 \qquad T = \alpha \beta \left(\frac{2}{1 + \alpha\beta} \right)^2$$

Electromagnetic Waves in Ohmic Conductors

(Follow are applicable only to normal incidence waves with **E** perpendicular to plane of incidence)

Wave Number:

$$\tilde{k}^2 = \mu \varepsilon \omega^2 + i \mu \sigma \omega \qquad \tilde{k} = k + i/\delta = (n_R + i n_I)\omega/c$$

$$k = \omega \sqrt{\frac{\mu \varepsilon}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon \omega} \right)^2} + 1 \right]^{1/2}$$

$$\frac{1}{\delta} = \omega \sqrt{\frac{\mu \varepsilon}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon \omega} \right)^2} - 1 \right]^{1/2}$$

Good ($\sigma \gg \varepsilon \omega$ or $k \approx 1/\delta$) and poor conductors:

$$\delta_{\text{good}} \approx \sqrt{\frac{2}{\sigma \mu \omega}} \qquad \delta_{\text{poor}} \approx \frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu}}$$

Field Magnitudes (Fresnel Equations):

$$\frac{E_R}{E_I} = \left(\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{1 + \tilde{\beta}} \right) \qquad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2$$

Reflection and Transmission Coefficients:

$$R = \left| \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right|^2 \qquad T = 1 - R = \left| \frac{2}{1 + \tilde{\beta}} \right|^2 \frac{v_1}{\omega} |\tilde{k}| \cos(\angle \tilde{k})$$

Polarization of Hydrogen

Complex permittivity for N/2 molecules of H_2.

$$\varepsilon = \varepsilon_0 \left(1 + \frac{N q^2 / m \varepsilon_0}{\omega_0^2 - \omega^2 - i \gamma \omega} \right)$$

For optical materials (small complex index of refraction):

$$\text{Re}\{n\} = 1 + \frac{N q^2}{2 m \varepsilon_0} \left(\frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2 - \gamma^2 \omega^2} \right)$$

$$\text{Im}\{n\} = \frac{N q^2}{2 m \varepsilon_0} \frac{\gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

Cauchy's Formula:

$$n = 1 + A \left(1 + \frac{B}{\lambda^2} \right)$$

Dilute Plasmas

Conductivity:

$$\sigma_{\text{plasma}} \approx \frac{iN_e q^2/m}{\omega}$$

Wave Number:

$$\tilde{k}^2 = \mu_0\varepsilon_0\omega^2 - \frac{\mu_0 N_e q^2}{m} = \frac{\omega^2 - \omega_p^2}{c^2}$$

Plasma Frequency:

$$\omega_p = \sqrt{\frac{c^2\mu_0 N_e q^2}{m}}$$

Phase and Group Velocities:

$$v_p = \frac{c\omega}{\sqrt{\omega^2 - \omega_p^2}} \qquad v_g = \frac{c\sqrt{\omega^2 - \omega_p^2}}{\omega}$$

Index of Refraction:

$$n = \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2}$$

Critical Angle (angle for which $\theta_T = \pi/2$):

$$\sin\theta_C = \frac{n_T}{n_I} = \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2}$$

Rectangular Wave-Guide

Dispersion Relation:

$$v^2 k^2 = \omega^2 - \omega_{mn}^2$$

Frequencies:

$$\omega_{mn} = v\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

z-components:

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{i(kz-\omega t)}$$

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{i(kz-\omega t)}$$

x-components:

$$E_x = \frac{iv^2}{\omega_{mn}^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$B_x = \frac{iv^2}{\omega_{mn}^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{v^2}\frac{\partial E_z}{\partial y}\right)$$

y-components:

$$E_y = \frac{iv^2}{\omega_{mn}^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_y = \frac{iv^2}{\omega_{mn}^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{v^2}\frac{\partial E_z}{\partial x}\right)$$

Intensity:

$$\langle \mathbf{S}\cdot\hat{\mathbf{z}}\rangle = \langle \tfrac{1}{\mu}\left(E_x B_y - E_y B_x\right)\rangle$$

Transmission Lines

Fields of a coaxial-cable:

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon s}e^{i(kz-\omega t)}\hat{\mathbf{s}} \qquad \mathbf{B} = \frac{\lambda}{2\pi\varepsilon s v}e^{i(kz-\omega t)}\hat{\phi}$$

Inductance and Capacitance per unit length:

$$C\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial z} \qquad \frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} - RI$$

Impedance of a perfectly conductive transmission line:

$$Z = \sqrt{L/C}$$

Conservation of Charge:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J} \qquad \frac{d\lambda}{dt} + \frac{dI}{dz} = 0$$

Electromagnetic Radiation

Retarded Potentials

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau'$$

Dipole Radiation

Assumptions:

$$d \ll r \qquad d \ll c/\omega \qquad r \gg c/\omega$$

Oscillating electric dipole ($\mathbf{p} = p_0\cos(\omega t)\hat{\mathbf{z}}$):

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin(\omega t_r)$$

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin(\omega t_r)\hat{\mathbf{z}}$$

Radiation from an arbitrary source with approximations $r' \ll r$, $r' \ll c/|\ddot{\rho}/\dot{\rho}|$ and dropping $1/r^2$ terms:

$$\mathbf{E}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi r}[(\mathbf{r}\cdot\ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}[\mathbf{r}\times(\mathbf{r}\times\ddot{\mathbf{p}})]$$

$$\mathbf{B}(\mathbf{r},t) \approx -\frac{\mu_0}{4\pi r c}[\mathbf{r}\times\ddot{\mathbf{p}}]$$

$$\mathbf{S}(\mathbf{r},t) \approx \frac{\mu_0}{c}\left(\frac{1}{4\pi r}\right)^2[|\ddot{\mathbf{p}}|^2 - (\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}})^2]\hat{\mathbf{r}}$$

Point Charges

Liénard-Wiechert Potential and Fields ($\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$):

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}(\mathbf{r},t)$$

Larmor Formula:

$$P = \frac{\mu_0 q a^2}{6\pi c}$$

Special Relativity

4-Vectors:

$$x = (ct, \mathbf{x})$$

$$\eta = \frac{dx}{d\tau} = \left(\gamma c, \frac{d\mathbf{x}}{d\tau}\right)$$

$$p = m\eta = (E/c, \mathbf{p})$$

$$K = (\omega/c, \mathbf{k})$$

Invariance of Length-Squared:

$$a_\mu b^\mu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

Proper Time:

$$d\tau = \sqrt{1 - u^2/c^2}\,dt = \gamma^{-1}\,dt$$

Space-time Lorentz Transform with inertial frame, $\mathbf{v} = v\hat{\mathbf{x}}$:

$$c\bar{t} = \gamma\left(ct - \frac{v}{c}x\right) \qquad \bar{x} = \gamma(x - vt)$$

$$\bar{y} = y \qquad \bar{z} = z$$

Energy-momentum Lorentz Transform with inertial frame, $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\frac{\bar{E}}{c} = \gamma\left(\frac{E}{c} - \frac{v}{c}p_x\right) \qquad \bar{p}_x = \gamma\left(p_x - \frac{v}{c}\frac{E}{c}\right)$$

$$\bar{p}_y = p_y \qquad \bar{p}_z = p_z$$

Relativistic Energy:

$$E = \gamma m c^2$$

Relativistic Energy and Momentum:

$$E^2 = p^2c^2 + m^2c^4$$

Relativistic Electrodynamics

Field transform for an inertial frame moving $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\bar{E}_x = E_x \qquad \bar{B}_x = B_x$$

$$\bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$\bar{E}_z = \gamma(E_z + vB_y) \qquad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

Vector Derivatives

Cartesian

$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}} \qquad d\tau = dx\,dy\,dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Spherical

$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Cylindrical

$d\tau = s\, ds\, d\phi\, dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Fundamental Theorems

Fundamental Theorem of Line Integrals:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence Theorem:

$$\int (\nabla\cdot\mathbf{A})\, d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$$

Stoke's Theorem:

$$\int (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$$

Vector Identities

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}$$

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Spherical Coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)$$
$$\phi = \tan^{-1}(y/x)$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Cylindrical Coordinates

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$s = \sqrt{x^2+y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

$$\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

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https://github.com/DonneyF/formula-sheets