MECH 466 Formula Sheet

Continuous Time Signals

Linearity:
$\mathcal{S}[\alpha x(t) + \beta y(t)] = \alpha \mathcal{S}[x(t)] + \beta \mathcal{S}[y(t)]$

Time Invariance:
If $\mathcal{S}[x(t)] = y(t)$ then $\mathcal{S}[x(t \pm \tau)] = y(t \pm \tau)$.

Laplace Transform

Initial Value Theorem:
$f(0+) \Leftrightarrow \lim_{s\to\infty} sF(s)$

Final Value Theorem:
$\lim_{t\to\infty} f(t) \Leftrightarrow \lim_{s\to 0} sF(s)$

Stability

- BIBO stability: any bounded input provides a bounded output
- Asymptotic stability: Initial conditions generates $y(t)$ converges to zero.

Characteristic Equation:
For $G(s) = N(s)/D(s)$, the characteristic equation is $D(s) = 0$.

Stability Condition in s-Domain:
All poles in open left hand plane $\Longleftrightarrow$ System is BIBO and asymptotically stable

Marginal Stability:

- $G(s)$ has no pole in the open RHP
- $G(s)$ has at least one simple pole on the imaginary axis
- $G(s)$ has no repeated pole on the imaginary axis
- BIBO stable except for sinusoidal inputs.
- For any non-zero initial condition, the output neither converges to zero nor diverge.

Polynomials:

- For 1st and 2nd order polynomials, all roots are in LHP $\Longleftrightarrow$ coefficients have the same sign
- For 3rd and higher orders, all roots are in LHP $\Longrightarrow$ coefficients have the same sign

Routh-Hurwitz Criterion

- The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.
- If zero row appears in the Routh array, roots are in imaginary axis or RHP.
- If zero row appears, replace the zero with the coefficients of derivative of auxiliary polynomial.
- Auxiliary polynomial: The polynomial above the zero row.

Steady State Error

Step Function:
$$r(t) = Ru(t) \implies e_{ss} = \frac{R}{1+K_p} = \frac{R}{1+L(0)}$$

Ramp Function:
$$r(t) = Rtu(t) \implies e_{ss} = \frac{R}{K_v} = \frac{R}{\lim_{s\to 0} sL(s)}$$

Parabolic Function:
$$r(t) = \frac{Rt}{2}u(t) \implies e_{ss} = \frac{R}{K_a} = \frac{R}{\lim_{s\to 0} s^2L(s)}$$

First Order System

$$G(s) = \frac{K}{Ts+1}$$

Step Response:
$y(t) = K(1 - e^{t/T})u(t)$

Time Constant:
$$T = \frac{1}{|\text{Real part of pole}|}$$

Response is slower the closer the pole is to the imaginary axis

Settling Time:
5%: $\approx 3T$ 2%: $\approx 4T$

Second Order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Underdamped Case ($0 < \zeta < 1$)

Step Response:
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \arccos\zeta)$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Poles:
$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

Settling Time:
5%: $\approx 3/\zeta\omega_n$ 2%: $\approx 4/\zeta\omega_n$

Peak Time:
$T_p = \pi/\omega_d$

Overshoot:
16%: $\zeta = 0.5$ 5%: $\zeta = \sqrt{2}/2$

$y_{\max} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Percent overshoot $= 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Root Locus Method

A graphical method to show how the poles move of a feedback system when K varies from 0 to ∞ in the form
$$H(S) = \frac{KL(s)}{1+KL(s)} \qquad L(s) = \frac{N(s)}{D(s)}$$

- RL includes all points on real axis to the left of an odd number of real poles/zeros
- RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.
- Number of asymptotes: $r = \deg(D(S)) - \deg(N(s))$
- Intersection of asymptotes: $\dfrac{\sum \text{poles} - \sum \text{zeroes}}{r}$
- Angle of asymptotes: $\dfrac{\pi(2k+1)}{r}$ $k = 0, 1, \ldots r-1$
- Breakaway points: Each root of $L'(s^\star) = 0$, where $K = -1/L(s^\star) > 0$
- Angle condition: s_0 is on the root locus $\Longleftrightarrow$ $\angle L(s_0) = 180°$

Lead & Lag Compensators

$$C(s) = K\frac{s+z}{s+p} \quad \text{or} \quad C(s) = K\frac{\frac{s}{z}+1}{\frac{s}{p}+1}$$

Lead ($z < p$)

- Moves intersection of asymptotes to the left
- Improves transient response (faster)

Lag $(z > p)$

- Moves intersection of asymptotes to the right
- Reduces steady state error

Phase-lag design:

1. Adjust DC gain of OL system by a constant gain K to satisfy low-frequency requirement.
2. On Bode plot, find the frequency ω_g where $\angle G(j\omega_g) = -180° + \phi_m + 5°$ for the required PM ϕ_m.
3. Select z and p: $z = 0.1\omega_g$, $p = z/\left|KG(j\omega_g)\right|$.

Phase-lead design:

1. Select z near uncompensated ω_g.
2. Select $p > z$ by trial and error.
3. Check PM and setting time.

Frequency Response

For a sinusoidal input $A\sin(\omega t)$ and system $G(s)$, the output is $A|G(j\omega)|\sin(\omega t + \angle G(j\omega))$

First Order System with corner frequency $1/T$:

$$G(j\omega) = \frac{K}{j\omega T + 1} \approx \begin{cases} K & 1 \gg \omega T \\ \frac{K}{j\omega T} & 1 \ll \omega T \end{cases}$$

Basic Functions

Constant Gain:
$|G(j\omega)| = K,\ \angle G(j\omega) = 0°$

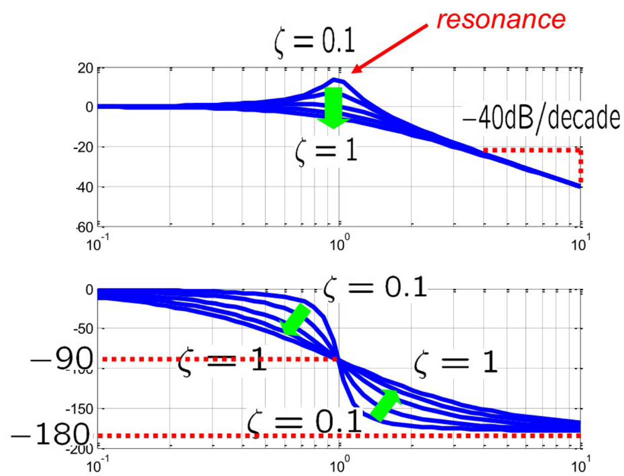
Differentiator:
$|G(j\omega)| = \omega,\ \angle G(j\omega) = \angle j\omega = 90°$

Integrator:
$|G(j\omega)| = \frac{1}{\omega},\ \angle G(j\omega) = \angle \frac{1}{j\omega} = -90°$

Double Integrator:
$|G(j\omega)| = \frac{1}{\omega^2},\ \angle G(j\omega) = \angle \frac{1}{(j\omega)^2} = -180°$

Time Delay:
$|G(j\omega)| = 1,\ \angle G(j\omega) = -\omega T$

Second Order System:



Resonant Frequency: $\omega_n\sqrt{1-2\zeta^2} \approx \omega_n$

Peak Gain: $\dfrac{1}{2\zeta\sqrt{1-\zeta^2}} \approx \dfrac{1}{2\zeta}$

At the resonant frequency, the gain is unity when $\zeta = 1/\sqrt{2}$.

Nyquist Stability

For a open-loop transfer function $L(s)$:

- CL system is stable $\iff Z := P + N = 0$
- Z: # of CL poles in open RHP.
- P: # of OL poles in open RHP.
- N: # of clockwise encirclement of -1
- $N = -1$ is a counter-clockwise encirclement.

Relative Stability

Gain crossover frequency ω_g:
$\left|L(j\omega_g)\right| = 1$

Phase crossover frequency ω_p
$\angle L(j\omega_p) = -180°$

Gain Margin:

$$\mathrm{GM} = 20\log_{10}\frac{1}{\left|L(j\omega_p)\right|}$$

Phase Margin:
$\mathrm{PM} = \angle L(j\omega_g) + 180°$

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https://github.com/DonneyF/formula-sheets