ELEC 221 Formula Sheet

Continuous Time Signals

Even and Odd Components:
$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$
$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$
A signal is periodic with fundamental period T_0 if
$x(t + kT_0) = x(t) \quad \forall t \in (-\infty, \infty)$
Energy:
$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$
Power:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

Causality:

- Causal: $x(t) = 0$ for $t < 0$.
- Anti-Causal: $x(t) = 0$ for $t \geq 0$.
- A-Causal or Non-Causal: Both of the above.

Continuous Time Systems

Dynamic systems have memory. Active systems can deliver energy to the outside world.
Linearity:
$\mathcal{S}[\alpha x(t) + \beta y(t)] = \alpha \mathcal{S}[x(t)] + \beta \mathcal{S}[y(t)]$
Time Invariance:
If $\mathcal{S}[x(t)] = y(t)$ then $\mathcal{S}[x(t \pm \tau)] = y(t \pm \tau)$.
Zero-State Response:
Due to the input as the initial conditions are zero.
Zero-Input Response:
Due to the initial conditions as the input is zero.
Convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Causality:
A continuous time system $\mathcal{S}$ is causal if whenever $x(t) = 0$ and there are no initial conditions, $y(t) = 0$ and the output $y(t)$ does not depend on future inputs.

Bounded-Input Bounded-Output Stability:
If an input $x(t)$ bounded then the output of an BIBO system is also bounded.
$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$

Laplace Transform

$s = \sigma + j\omega$
Eigenfunction Property:
$\mathcal{S}[e^{s_0 t}] = H(s_0) e^{s_0 t}$
One Sided Laplace Transform:

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0^-}^{\infty} f(t) e^{-st} \, dt$$

Region of Convergence:

- Finite support $\rightarrow$ Entire s-plane.
- Causal function $\rightarrow \sigma > \max(\sigma_i), -\infty < \omega < \infty$.
- Anti-causal $\rightarrow \sigma < \min(\sigma_i), -\infty < \omega < \infty$.
- Non-causal $\rightarrow \mathcal{R} = \mathcal{R}_{\text{causal}} \cap \mathcal{R}_{\text{anti-causal}}$.

Initial Value Theorem:
$f(0+) \Leftrightarrow \lim_{s\to\infty} sF(s)$
Final Value Theorem:
$\lim_{t\to\infty} f(t) \Leftrightarrow \lim_{s\to 0} sF(s)$
Bounded-Input Bounded-Output Stability:
If the region of convergence contains the $j\omega$-axis, then the system is BIBO stable.

Fourier Series

Fourier analysis in the steady state.
Eigenfunction Property:
$\mathcal{S}[e^{j\omega_0 t}] = H(j\omega_0) e^{j\omega_0 t}$
$x(t) = \sum_k X_k e^{j\omega_k t} \implies y(t) = \sum_k X_k H(j\omega_k) e^{j\omega_k t}$
$= \sum_k X_k |H(j\omega_k)| e^{j(\omega_k t + \angle H(\omega_k))}$
Fourier Series Coefficients (for any t_0):

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} \, dt$$

Parseval's Power Relation (for any t_0):

$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Symmetry of Line Spectra:
$|X_k| = |X_{-k}|$
$\angle X_k = -\angle X_{-k}$
Trigonometric Fourier Series:
$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \Theta_k)$
$x(t) = c_0 + 2\sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)]$
$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\omega_0 t)\, dt \quad k = 0,1,2\ldots$
$d_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\omega_0 t)\, dt \quad k = 1,2,3\ldots$
$\Theta_k = -\arctan(d_k / c_k)$

Fourier Coefficients from Laplace Transform:
If $x_1(t)$ is a single period of $x(t)$, then
$X_k = \frac{1}{T_0} \mathcal{L}[x_1(t)]\big|_{s=jk\omega_0}$
Response of LTI Systems to Periodic Signals:
If the input to an LTI system has Fourier Series $x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \angle X_k)$, then the steady state response is $y(t) = X_0 |H(j0)| +$

$$2\sum_{k=1}^{\infty} |X_k||H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0))$$

Fourier Transform

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$
$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$
Fourier Transform from Laplace Transform (if $X(s)$ contains the $j\omega$-axis):
$\mathcal{F}[x(t)] = \mathcal{L}[x(t)]\big|_{s=j\omega} = X(s)\big|_{s=j\omega}$
Fourier Transform of Periodic Signals:

$$\sum_k X_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} \sum_k 2\pi_k \delta(\omega - k\omega_0)$$

Parseval's Energy Relation:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} |X(\omega)|^2 \, d\omega$$

Symmetry of Spectral Representations:
$|X(\omega)| = |X(-\omega)|$
$\mathrm{Re}[X(\omega)] = \mathrm{Re}[X(-\omega)]$
$\angle X(\omega) = -\angle X(-\omega)$
$\mathrm{Im}[X(\omega)] = -\mathrm{Im}[X(-\omega)]$

Sampling Theory

$x_s(t) = x(nT_s) = x(t)|_{t=nT_s} = x(t)\sum_n \delta(t - nT_s)$

$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Nyquist-Shannon Sampling Rate:
$\omega_s = \frac{2\pi}{T_s} \geq 2\omega_{\max}$
Aliasing occurs if $\omega_s < 2\omega_{\max}$.
Reconstruction $X(\omega) = X_s(\omega) H_{\text{lp}}(\omega)$:

$$H_{\text{lp}}(\omega) = \begin{cases} T_s & -\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

Signal Reconstruction from Sinc Interpolation:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin(\pi(t - nT_s))/T_s}{\pi(t - nT_s)/T_s}$$

Discrete Time Signals

Define $x[n] = x(nT_0)$.
$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Periodicity:
$x[n+kN] = x[n] \qquad \forall k \in \mathbb{Z}$

When sampling an analog sinusoid of fundamental period T_0, we obtain a periodic discrete sinusoid provided that m, N not divisible by eachother:
$T_s/T_0 = m/N$

Aliasing occurs if:
$T_s > T_0/2$

Energy:
$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Power:
$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Convolutional Sum:
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Bounded-Input Bounded-Output Stability:
$\sum_k |h[n]|^2 < \infty$

Solution to Autoregressive Discrete System:
$y[n] = ay[n-1] + bx[n], n \geq 0$
$y[n] = \sum_{k=0}^{n} ba^k x[n-k], n \geq 0$

Z-Transform

$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n}$

Convergence:
$|X(z)| = \sum_n |x[n]|r^{-n} < \infty$

Initial Value Theorem:
$x[0] = \lim_{z\to\infty} X(z)$

Final Value Theorem:
$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

Updated December 11, 2018
https://github.com/DonneyF/formula-sheets

Fourier Transform Properties

	Time Domain	Frequency Domain
Expansion/Contraction	$x(\alpha t)$, $\alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X(\frac{\omega}{\alpha})$
Reflection	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Differentiation (t)	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Differentiation (ω)	$-jtx(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
Time shifting	$x(t-\alpha)$	$e^{-j\alpha\omega}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Modulation	$x(t)\cos(\omega_c t)$	$0.5[X(\omega-\omega_c) + X(\omega+\omega_c)]$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$

Z-Transform Properties

	Time Domain	Z-Domain
Time Shifting (causal)	$x[n-N]$, $n \in \mathbb{Z}$	$z^{-N}X(z)$
Time Shifting (acausal)	$x[n-N]$	$z^{-N}X(z) + x[-1]z^{-N+1}$ $+x[-2]z^{-N+2} + \cdots x[-N]$
Time reversal	$x[-n]$	$X(z^{-1})$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$
Multiplication by n^2	$n^2x[n]$	$z^2\frac{d^2X(z)}{dz^2} + z\frac{dX(z)}{dz}$
Finite Difference	$x[n] - x[n-1]$	$(1-z^{-1})X(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{X(z)}{1-z^{-1}}$

Misc. Identities

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \xrightarrow{n\to\infty \text{ and } x<1} \frac{1}{1-x}$$
$\cos\theta = 1/2(e^{j\theta} + e^{-j\theta})$
$\sin\theta = 1/2j(e^{j\theta} - e^{-j\theta})$

Fourier Transform Pairs

$f(t)$	$F(\omega)$
$\delta(t)$	1
$\delta(t-\tau)$	$e^{-j\omega\tau}$
$u(t)$	$1/j\omega + \pi\delta(\omega)$
$u(t)$	$-1/j\omega + \pi\delta(\omega)$
$\text{sign}(t)$	$2/j\omega$
α	$2\pi\alpha\delta(\omega)$
$e^{-a\lvert t\rvert}$, $a > 0$	$\frac{2a}{a^2+\omega^2}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)] + \delta(\omega+\omega_0)]$
$\sin(\omega_0 t)$	$-j\pi[\delta(\omega-\omega_0)] - \delta(\omega+\omega_0)]$
$\frac{\sin(\omega t)}{\pi t}$	$u(\omega+\omega_0) - u(\omega-\omega_0)$
$u(\omega+\tau) - u(\omega-\tau)$	$2\tau\frac{\sin(\omega\tau)}{\omega\tau}$
$\text{rect}(t/T)$	$T\,\text{sinc}(T\omega/2)$

One-sided Z-Transforms

$f[n]$	$F(z)$
$\delta[n]$	1, Whole z-plane
$u[n]$	$\frac{1}{1-z^{-1}}$, $\lvert z\rvert > 1$
$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$, $\lvert z\rvert > 1$
$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$, $\lvert z\rvert > 1$
$\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha z^{-1}}$, $\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $\lvert z\rvert > \lvert\alpha\rvert$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$