MATH 400 Formula Sheet

Linear First Order Equations

First Order Linear PDE:
$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$

Simple Transport Equation

$$u_t + cu_x = 0\,, -\infty < x < \infty, t > 0$$
$$u(x,0) = \phi(x)\,, -\infty < x < \infty$$

Solution:
$u(x,t) = \phi(x - ct)$

Linear Second Order Equations

Quasilinear PDE:
$a_{11}(x,y,u,u_x,u_y)u_{xx} + 2a_{12}(x,y,u,u_x,u_y)u_{xy} + a_{22}(x,y,u,u_x,u_y)u_{yy} + a_{00}(x,y,u,u_x,u_y) = 0$

Semilinear PDE:
$a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} + a_{00}(x,y,u,u_x,u_y) = 0$

Linear PDE:
$a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} + a_1(x,y)u_x + a_2(x,y)u_y + a_0(x,y)u = f(x,y)$

Discriminant for semilinear PDEs:
$\mathcal{D}(x,y) = [a_{12}(x,y)]^2 - a_{11}(x,y)a_{22}(x,y)$

Classification of semilinear PDEs:

$$\begin{cases} \mathcal{D}(x,y) > 0 & \text{Hyperbolic} \\ \mathcal{D}(x,y) < 0 & \text{Elliptic} \\ \mathcal{D}(x,y) = 0 & \text{Parabolic} \end{cases}$$

Change of variables:

$$\begin{cases} U_{\xi\eta} + b_{00}(\xi,\eta,U,U_\xi,U_\eta) = 0 & \text{Hyperbolic} \\ U_{\xi\xi} + U_{\eta\eta} + b_{00}(\xi,\eta,U,U_\xi,U_\eta) = 0 & \text{Elliptic} \\ U_{\xi\xi} + b_{00}(\xi,\eta,U,U_\xi,U_\eta) = 0 & \text{Parabolic} \end{cases}$$

Wave Equation

General Solution:
$u(x,t) = f(x+ct) + g(x-ct)$

Initial Value Problem on the real line:

$$\begin{cases} u_{tt} - c^2u_{xx} = f(x,t)\,, -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x)\,, u_t(x,0) = \psi(x) - \infty < x < \infty \end{cases}$$

d'Alembert's Formula:
$$u(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

Non-Homogenous Boundary Conditions

Shift the data ($u = v + w$):

Dirichlet, $u(0,t) = a(t)$, $u(L,t) = b(t)$:
$w(x,t) = a(t) + \frac{x}{L}[b(t) - a(t)]$

Neumann, $u_x(0,t) = a(t)$, $u_x(L,t) = b(t)$
$w(x,t) = xa(t) + \frac{x^2}{2L}[b(t) - a(t)]$

Mixed 1, $u(0,t) = a(t)$, $u_x(L,t) = b(t)$
$w(x,t) = a(t) + xb(t)$

Mixed 2, $u_x(0,t) = a(t)$, $u(L,t) = b(t)$
$w(x,t) = (x-L)a(t) + b(t)$

Sturm-Liouville Theory

Consider the homogeneous linear second-order PDE:
$r(x)u_t - (p(x)u_x)_x + q(x)u = 0$

Separation of Variables:
$-\frac{T'(t)}{T(t)} = -\frac{(p(x)X'(x))'}{r(x)X(x)} + \frac{q(x)}{r(x)} = \lambda$

ODEs:

$$\begin{cases} T' + \lambda T = 0 \\ -(p(x)X')' + q(x)X = \lambda r(x)X \end{cases}$$

Associated Eigenvalue Problem:
$-(p(x)X')' + q(x)X = \lambda r(x)X$

General Boundary Conditions:
$\mathcal{B}_1X = \alpha_1X(a) + \beta_1X(b) + \gamma_1X'(a) + \delta_1X'(b) = 0$
$\mathcal{B}_2X = \alpha_2X(a) + \beta_2X(b) + \gamma_2X'(a) + \delta_2X'(b) = 0$

Boundary conditions are separated when:
$\beta_1 = \delta_1 = 0 \qquad \alpha_2 = \gamma_2 = 0$

Lagrange's Identity. BCs are symmetric if for all functions f, g that satisfy the BCs:
$[-p\,(f'g - fg')]_a^b = 0$

Sturm-Liouville Problem ($p(x) > 0, r(x) > 0$):

$$\begin{cases} \mathcal{L}X(x) = \lambda r(x)X,\ a < x < b \\ \mathcal{B}_1X = 0,\ \mathcal{B}_2X = 0 \end{cases}$$

Theorem 5.3.1:

If you have symmetric BCs, then any to eigenfunctions of a SL problem that correspond to distinct eigenvalues are orthogonal. If any function is expanded in a series of these eigenfunctions, the coefficients are determined.

Theorem 5.3.2:

Under Theorem 5.3.1, all the eigenvalues are real numbers and the eigenfunctions can be chosen to be real-valued.

Theorem 5.3.3:

Under 5.3.1, if $q(x) \geq 0$ for all $a \leq x \leq b$ and if for all real-valued functions f satisfying the BCs we satisfy

$$[p(x)f(x)f'(x)]_a^b \leq 0$$

then there can be no negative eigenvalues.

Theorem 5.4.1:

For any regular of periodic SL problem, there are an infinite amount of eigenvalues.

Laplace's Equation

Polar Coordinates:
$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

Spherical Coordinates:
$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin\theta}[\sin\theta\, u_\theta]_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} = 0$

Maximum Principle:

Let D be a connected bounded open set of $\mathbb{R}^n$, $n = 2, 3$. If $u(\mathbf{x})$ is harmonic in D and continuous on $D \cup \partial D$ then $u(\mathbf{x})$ attains its maximum and minimum values on ∂D.

PDEs in 3 Dimensions

Elliptic PDE:

$-\nabla \cdot [p(\mathbf{x})\nabla u] + q(\mathbf{x})u = f(\mathbf{x})$

Parabolic PDE:

$r(\mathbf{x})u_t - \nabla \cdot [p(\mathbf{x})\nabla u] + q(\mathbf{x})u = f(\mathbf{x}, t)$

Hyperbolic PDE:

$r(\mathbf{x})u_{tt} - \nabla \cdot [p(\mathbf{x})\nabla u] + q(\mathbf{x})u = f(\mathbf{x}, t)$

Eigenvalue Problem:

$-\nabla \cdot [p(\mathbf{x})\nabla X] + q(\mathbf{x})X = \lambda r(\mathbf{x})X$

Green's First Identity:

Let D be an open connected domain in $\mathbb{R}^3$ and let ∂D be its piecewise smooth boundary Let f and g be smooth functions on $D \cup \partial D$. Then

$$\oint_{\partial D} f \frac{\partial g}{\partial n} \, ds = \iint_D \nabla F \cdot \nabla g \, d\mathbf{x} + \iint_D f \nabla^2 g \, d\mathbf{x}$$

Bessel's Equation:

$\rho^2 y'' + \rho y + (\rho^2 - n^2)y = 0$

$y(\rho) = c_1 J_n(\rho) + c_2 Y_n(\rho)$

Associated Legendre Equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_l^m(x)\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P_l^m(x) = 0$$

Spherical Harmonics:

$$\nabla_{S^2}^2 Y_l^m = \frac{1}{\sin\theta}[\sin\theta\, Y]_\theta + \frac{1}{\sin^2\theta} Y_{\phi\phi} = l(l+1)Y_l^m$$

$Y_l^m(\theta, \phi) = P_l^{|m|}(\cos\theta)e^{im\phi}$

$l = 0, 1, 2 \ldots \quad m = -l, -l+1, \ldots, l-1, l$

Integral Transform Methods

Fourier Transform:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} \, dx$$

Inverse Fourier Transform:

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ikx} \, dx$$

Error Function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2} \, dp$$

$\operatorname{erf}(0) = 0 \quad \lim_{x\to\infty} \operatorname{erf}(x) = 1$

Laplace Transform:

Let $f(t)$ be a piecewise continuous function on $[0, \infty)$ and suppose $\exists K, \gamma$ such that $|f(t)| \leq Ke^{\gamma t}\ \forall t > 0$. Then

$$F(s) = \int_0^{\infty} f(t)e^{-st} \, dt$$

Inverse Laplace Transform:

$$f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} F(s)e^{st} \, ds \quad \alpha > \gamma$$

Solved 1D Eigenvalue Problems

Dirichlet

$$\left.\begin{aligned}-X'' &= \lambda X\\ X(0)=0, X(L)&=0\end{aligned}\right\} \implies \begin{cases}\lambda_n = \frac{n\pi}{L},\ n = 1,2,3\dots\\ X_n(x) = \sin\left(\frac{n\pi}{L}x\right)\end{cases}$$

Neumann

$$\left.\begin{aligned}-X'' &= \lambda X\\ X'(0)=0, X'(L)&=0\end{aligned}\right\} \implies \begin{cases}\lambda_n = \frac{n\pi}{L},\ n = 0,1,2\dots\\ X_n(x) = \cos\left(\frac{n\pi}{L}x\right)\end{cases}$$

Periodic

$$\left.\begin{aligned}-X'' &= \lambda X\\ X(-\pi) &= X(\pi)\\ X'(-\pi) &= X'(\pi)\end{aligned}\right\} \implies \begin{cases}\lambda_n = n^2,\ n = 0,1,2\dots\\ X_n(x) \in \{1, \cos(n\theta), \sin(n\theta)\}\end{cases}$$

Mixed 1

$$\left.\begin{aligned}-X'' &= \lambda X\\ X(0)=0, X'(L)&=0\end{aligned}\right\} \implies \begin{cases}\lambda_n = \frac{(2n+1)\pi}{2L},\ n = 0,1,2\dots\\ X_n(x) = \sin\left(\frac{(2n+1)\pi}{2L}x\right)\end{cases}$$

Mixed 2

$$\left.\begin{aligned}-X'' &= \lambda X\\ X'(0)=0, X(L)&=0\end{aligned}\right\} \implies \begin{cases}\lambda_n = \frac{(2n+1)\pi}{2L},\ n = 0,1,2\dots\\ X_n(x) = \cos\left(\frac{(2n+1)\pi}{2L}x\right)\end{cases}$$

Solved PDE Problems

Problem 1

$$\Delta u = 0 \quad x^2+y^2 < a^2$$

$$u = h(x,y) \quad \text{on} \quad x^2+y^2 = a^2$$

Solution (Oct 21):

$$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{h(\phi)}{a^2-2ar\cos(\phi-\theta)+r^2}\,d\phi$$

Problem 2

$$\Delta u = 0 \quad 0<r<a, \quad 0<\theta<\beta$$

$$u(r,0)=0, \quad u(r,\beta)=0, \quad 0<r<a$$

$$u_r(a,\theta) = h(\theta) \quad 0<\theta<\beta$$

Solution (Oct 23):

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n r^{n\pi/\beta}\sin\left(\frac{n\pi}{\beta}\theta\right)$$

$$C_n = \frac{2}{n\pi}a^{-n\pi/\beta+1}\int_0^{\beta}h(\phi)\sin\left(\frac{n\pi}{\beta}\theta\right)$$

Problem 3

$$\Delta u = 0 \quad a<r<b, \quad 0<\theta<2\pi$$

General Solution (Oct 23):

$$u(r,\theta) = \frac{1}{2}\left(C_0 + D_0\ln(r)\right) + \sum_{n=1}^{\infty}\left(C_n^{(1)}r^n + D_n^{(1)}r^{-n}\right)\cos(n\theta) + \left(C_n^{(2)}r^n + D_n^{(2)}r^{-n}\right)\sin(n\theta)$$

Problem 4

$$S_t - DS_{xx} = 0 \quad -\infty<x<\infty, \quad t>0$$

$$S(x,0) = \delta(x-y), \quad y \ge 0$$

Solution (Nov 16):

$$S(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-(x-y)^2/(4Dt)}$$

Problem 5

$$u_t - Du_{xx} = 0 \quad -\infty<x<\infty, \quad t>0$$

$$u(x,0) = \varphi(x), \quad -\infty<x<\infty$$

Solution (Nov 18):

$$u(x,t) = \int_{-\infty}^{\infty}\varphi(y)S(x-y,t)\,dy$$

Problem 6

$$u_t - Du_{xx} = 0 \quad 0<x<\infty, \quad t>0$$

$$u(0,t) = 0, \quad t>0$$

$$u(x,0) = \varphi(x), \quad -\infty<x<\infty$$

Solution (Nov 18):

$$u(x,t) = \int_0^{\infty}\phi(y)[S(x-y,t) - S(x+y,t)]\,dy$$

Problem 7

$$\Delta u = 0 \quad -\infty<x<\infty, \quad y>0$$

$$u(x,0) = h(x), \quad -\infty<x<\infty$$

Solution (Nov 20):

$$u(x,y) = \int_{-\infty}^{\infty}h(z)\frac{y}{\pi[(x-z)^2-y^2]}\,dz$$

Problem 8

$$u_t - Du_{xx} = 0 \quad 0<x<\infty, \quad t>0$$

$$u(0,t) = h(t), \quad t>0$$

$$u(x,0) = 0, \quad -\infty<x<\infty$$

Solution (Nov 23):

$$u(x,t) = \begin{cases}0 & t<b\\ 1-\operatorname{erf}\left(\frac{x}{\sqrt{4D(t-b)}}\right) & t>b\end{cases}$$

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https://github.com/DonneyF/formula-sheets