

ELEC 221 Formula Sheet

Continuous Time Signals

Even and Odd Components:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

A signal is periodic with fundamental period T_0 if

$$x(t + kT_0) = x(t) \quad \forall t \in (-\infty, \infty)$$

Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Power:

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

Causality:

- Causal: $x(t) = 0$ for $t < 0$.
- Anti-Causal: $x(t) = 0$ for $t \geq 0$.
- A-Causal or Non-Causal: Both of the above.

Continuous Time Systems

Dynamic systems have memory. Active systems can deliver energy to the outside world.

Linearity:

$$\mathcal{S}[\alpha x(t) + \beta y(t)] = \alpha \mathcal{S}[x(t)] + \beta \mathcal{S}[y(t)]$$

Time Invariance:

$$\text{If } \mathcal{S}[x(t)] = y(t) \text{ then } \mathcal{S}[x(t \pm \tau)] = y(t \pm \tau).$$

Zero-State Response:

Due to the input as the initial conditions are zero.

Zero-Input Response:

Due to the initial conditions as the input is zero.

Convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$$

Causality:

A continuous time system \mathcal{S} is causal if whenever $x(t) = 0$ and there are no initial conditions, $y(t) = 0$ and the output $y(t)$ does not depend on future inputs.

Bounded-Input Bounded-Output Stability:

If an input $x(t)$ bounded then the output of an BIBO system is also bounded.

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Laplace Transform

$$s = \sigma + j\omega$$

Eigenfunction Property:

$$\mathcal{S}[e^{s_0 t}] = H(s_0)e^{s_0 t}$$

One Sided Laplace Transform:

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

Region of Convergence:

- Finite support \rightarrow Entire s-plane.
- Causal function $\rightarrow \sigma > \max(\sigma_i), -\infty < \omega < \infty$.
- Anti-causal $\rightarrow \sigma < \min(\sigma_i), -\infty < \omega < \infty$.
- Non-causal $\rightarrow \mathcal{R} = \mathcal{R}_{\text{causal}} \cap \mathcal{R}_{\text{anti-causal}}$.

Initial Value Theorem:

$$f(0+) \Leftrightarrow \lim_{s \rightarrow \infty} sF(s)$$

Final Value Theorem:

$$\lim_{t \rightarrow \infty} f(t) \Leftrightarrow \lim_{s \rightarrow 0} sF(s)$$

Bounded-Input Bounded-Output Stability:

If the region of convergence contains the $j\omega$ -axis, then the system is BIBO stable.

Fourier Series

Fourier analysis in the steady state.

Eigenfunction Property:

$$\mathcal{S}[e^{j\omega_0 t}] = H(j\omega_0)e^{j\omega_0 t}$$

$$x(t) = \sum_k X_k e^{j\omega_k t} \implies y(t) = \sum_k X_k H(j\omega_k) e^{j\omega_k t} \\ = \sum_k X_k [H(j\omega_k)] e^{j(\omega_k t + \angle H(\omega_k))}$$

Fourier Series Coefficients (for any t_0):

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} dt$$

Parseval's Power Relation (for any t_0):

$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Symmetry of Line Spectra:

$$|X_k| = |X_{-k}|$$

$$\angle X_k = -\angle X_{-k}$$

Trigonometric Fourier Series:

$$x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \Theta_k)$$

$$x(t) = c_0 + 2 \sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)]$$

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\omega_0 t) dt \quad k = 0, 1, 2, \dots$$

$$d_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\omega_0 t) dt \quad k = 1, 2, 3, \dots$$

$$\Theta_k = -\arctan(d_k / c_k)$$

Fourier Coefficients from Laplace Transform:

If $x_1(t)$ is a single period of $x(t)$, then

$$X_k = \frac{1}{T_0} \mathcal{L}[x_1(t)]|_{s=jk\omega_0}$$

Response of LTI Systems to Periodic Signals:

If the input to an LTI system has Fourier Series

$x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \angle X_k)$, then the steady state response is $y(t) = X_0 |H(j\omega_0)| +$

$$2 \sum_{k=1}^{\infty} |X_k| |H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0))$$

Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier Transform from Laplace Transform (if $X(s)$ contains the $j\omega$ -axis):

$$\mathcal{F}[x(t)] = \mathcal{L}[x(t)]|_{s=j\omega} = X(s)|_{s=j\omega}$$

Fourier Transform of Periodic Signals:

$$\sum_k X_k e^{jk\omega_0 t} \xleftrightarrow{\mathcal{F}} \sum_k 2\pi \delta(\omega - k\omega_0)$$

Parseval's Energy Relation:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Symmetry of Spectral Representations:

$$|X(\omega)| = |X(-\omega)|$$

$$\text{Re}[X(\omega)] = \text{Re}[X(-\omega)]$$

$$\angle X(\omega) = -\angle X(-\omega)$$

$$\text{Im}[X(\omega)] = -\text{Im}[X(-\omega)]$$

Sampling Theory

$$x_s(t) = x(nT_s) = x(t)|_{t=nT_s} = x(t) \sum_n \delta(t - nT_s)$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Nyquist-Shannon Sampling Rate:

$$\omega_s = \frac{2\pi}{T_s} \geq 2\omega_{\max}$$

Aliasing occurs if $\omega_s < 2\omega_{\max}$.

Reconstruction $X(\omega) = X_s(\omega)H_{\text{lp}}(\omega)$:

$$H_{\text{lp}}(\omega) = \begin{cases} T_s & -\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

Signal Reconstruction from Sinc Interpolation:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin(\pi(t - nT_s)/T_s)}{\pi(t - nT_s)/T_s}$$

Discrete Time Signals

Define $x[n] = x(nT_0)$.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$$

Periodicity:

$$x[n + kN] = x[n] \quad \forall k \in \mathbb{Z}$$

When sampling an analog sinusoid of fundamental period T_0 , we obtain a periodic discrete sinusoid provided that m, N not divisible by each other:

$$T_s/T_0 = m/N$$

Aliasing occurs if:

$$T_s > T_0/2$$

Energy:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Power:

$$P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$$

Convolutional Sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

Bounded-Input Bounded-Output Stability:

$$\sum_k |h[n]|^2 < \infty$$

Z-Transform

$$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s) z^{-n}$$

Convergence:

$$|X(z)| = \sum_n |x[n]| r^{-n} < \infty$$

Initial Value Theorem:

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

Final Value Theorem:

$$\lim_{n \rightarrow \infty} x[n] = \lim_{z \rightarrow 1} (z - 1) X(z)$$

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<https://github.com/DonneyF/formula-sheets>

Fourier Transform Properties

	Time Domain	Frequency Domain
Expansion/Contraction	$x(\alpha t), \alpha \neq 0$	$\frac{1}{ \alpha } X\left(\frac{\omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Differentiation (t)	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Differentiation (ω)	$-jtx(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^t x(\tau) d\tau$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\omega} X(\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Modulation	$x(t) \cos(\omega_c t)$	$0.5[X(\omega - \omega_c) + X(\omega + \omega_c)]$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi} [X * Y](\omega)$

Z-Transform Properties

	Time Domain	Z-Domain
Time Shifting (causal)	$x[n - N], n \in \mathbb{Z}$	$z^{-N} X(z)$
Time Shifting (acausal)	$x[n - N]$	$z^{-N} X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \dots + x[-N]$
Time reversal	$x[-n]$	$X(z^{-1})$
Multiplication by n	$nx[n]$	$-z \frac{dX(z)}{dz}$
Multiplication by n^2	$n^2 x[n]$	$z^2 \frac{d^2 X(z)}{dz^2} + z \frac{dX(z)}{dz}$
Finite Difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z) - x[-1]$
Accumulation	$\sum_{k=0}^n x[k]$	$\frac{X(z)}{1 - z^{-1}}$

Fourier Transform Pairs

$f(t)$	$F(\omega)$
$\delta(t)$	1
$\delta(t - \tau)$	$e^{-j\omega\tau}$
$u(t)$	$1/j\omega + \pi\delta(\omega)$
$u(t)$	$-1/j\omega + \pi\delta(\omega)$
$\text{sign}(t)$	$2/j\omega$
α	$2\pi\alpha\delta(\omega)$
$e^{-a t }, a > 0$	$\frac{2a}{a^2 + \omega^2}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0)] + \delta(\omega + \omega_0)$
$\sin(\omega_0 t)$	$-j\pi[\delta(\omega - \omega_0)] - \delta(\omega + \omega_0)$
$\frac{\sin(\omega t)}{\pi t}$	$u(\omega + \omega_0) - u(\omega - \omega_0)$
$u(\omega + \tau) - u(\omega - \tau)$	$2\tau \frac{\sin(\omega\tau)}{\omega\tau}$

One-sided Z-Transforms

$f[n]$	$F(z)$
$\delta[n]$	1, Whole z-plane
$u[n]$	$\frac{1}{1 - z^{-1}}, z > 1$
$nu[n]$	$\frac{z^{-1}}{(1 - z^{-1})^2}, z > 1$
$n^2 u[n]$	$\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}, z > 1$
$\alpha^n u[n], \alpha < 1$	$\frac{1}{1 - \alpha z^{-1}}, z > \alpha $
$n\alpha^n u[n], \alpha < 1$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, z > \alpha $
$\cos(\omega_0 n) u[n]$	$\frac{1 - \cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}}, z > 1$
$\sin(\omega_0 n) u[n]$	$\frac{\sin(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}}, z > 1$