ELEC 221 Formula Sheet

Continuous Time Signals

Even and Odd Components:
$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$
$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$
A signal is periodic with fundamental period T_0 if
$x(t + kT_0) = x(t) \quad \forall t \in (-\infty, \infty)$
Energy:
$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$
Power:
$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$
Causality:

- Causal: $x(t) = 0$ for $t < 0$.
- Anti-Causal: $x(t) = 0$ for $t \geq 0$.
- A-Causal or Non-Causal: Both of the above.

Continuous Time Systems

Dynamic systems have memory. Active systems can deliver energy to the outside world.
Linearity:
$\mathcal{S}[\alpha x(t) + \beta y(t)] = \alpha\mathcal{S}[x(t)] + \beta\mathcal{S}[y(t)]$
Time Invariance:
If $\mathcal{S}[x(t)] = y(t)$ then $\mathcal{S}[x(t \pm \tau)] = y(t \pm \tau)$.
Zero-State Response:
Due to the input as the initial conditions are zero.
Zero-Input Response:
Due to the initial conditions as the input is zero.
Convolution Integral:
$$[x * y](t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\, d\tau$$
Total Response from Impulse Response:
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$
Causality:
A continuous time system $\mathcal{S}$ is causal if whenever $x(t) = 0$ and there are no initial conditions, $y(t) = 0$ and the output $y(t)$ does not depend on future inputs.

Bounded-Input Bounded-Output Stability:
If an input $x(t)$ bounded then the output of an BIBO system is also bounded.
$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Laplace Transform

$s = \sigma + j\omega$
Eigenfunction Property:
$\mathcal{S}[e^{s_0 t}] = H(s_0)e^{s_0 t}$
One Sided Laplace Transform:
$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0^-}^{\infty} f(t)e^{-st}\, dt$$
Region of Convergence:

- Finite support $\rightarrow$ Entire s-plane.
- Causal function $\rightarrow \sigma > \max(\sigma_i)$, $-\infty < \omega < \infty$.
- Anti-causal $\rightarrow \sigma < \min(\sigma_i)$, $-\infty < \omega < \infty$.
- Non-causal $\rightarrow \mathcal{R} = \mathcal{R}_{\text{causal}} \cap \mathcal{R}_{\text{anti-causal}}$.

Initial Value Theorem:
$f(0+) \Leftrightarrow \lim_{s\to\infty} sF(s)$
Final Value Theorem:
$\lim_{t\to\infty} f(t) \Leftrightarrow \lim_{s\to 0} sF(s)$
Bounded-Input Bounded-Output Stability:
If the region of convergence contains the $j\omega$-axis, then the system is BIBO stable.

Fourier Series

Fourier analysis in the steady state.
Eigenfunction Property:
$\mathcal{S}[e^{j\omega_0 t}] = H(j\omega_0)e^{j\omega_0 t}$
$$x(t) = \sum_k X_k e^{j\omega_k t} \implies y(t) = \sum_k X_k H(j\omega_k)e^{j\omega_k t} = \sum_k X_k |H(j\omega_k)| e^{j(\omega_k t + \angle H(\omega_k))}$$
Fourier Series Coefficients (for any t_0):
$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\omega_0 t}\, dt$$
Parseval's Power Relation (for any t_0):
$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$
Symmetry of Line Spectra:
$|X_k| = |X_{-k}|$
$\angle X_k = -\angle X_{-k}$

Trigonometric Fourier Series:
$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \Theta_k)$
$x(t) = c_0 + 2\sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)]$
$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\omega_0 t)\, dt \qquad k = 0,1,2\ldots$$
$$d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\omega_0 t)\, dt \qquad k = 1,2,3\ldots$$
$\Theta_k = -\arctan(d_k/c_k)$

Fourier Coefficients from Laplace Transform:
If $x_1(t)$ is a single period of $x(t)$, then
$$X_k = \frac{1}{T_0}\mathcal{L}[x_1(t)]\Big|_{s=jk\omega_0}$$
Response of LTI Systems to Periodic Signals:
If the input to an LTI system has Fourier Series $x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \angle X_k)$, then the steady state response is $y(t) = X_0|H(j0)| + 2\sum_{k=1}^{\infty} |X_k||H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0))$

Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$
Fourier Transform from Laplace Transform (if $X(s)$ contains the $j\omega$-axis):
$\mathcal{F}[x(t)] = \mathcal{L}[x(t)]\Big|_{s=j\omega} = X(s)\Big|_{s=j\omega}$
Duality:
$X(t) \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi x(-\omega)$
Fourier Transform of Periodic Signals:
$$\sum_k X_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} \sum_k 2\pi_k \delta(\omega - k\omega_0)$$
Parseval's Energy Relation:
$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}|X(\omega)|^2\, d\omega$$
Symmetry of Spectral Representations:
$|X(\omega)| = |X(-\omega)|$
$\mathrm{Re}[X(\omega)] = \mathrm{Re}[X(-\omega)]$
$\angle X(\omega) = -\angle X(-\omega)$
$\mathrm{Im}[X(\omega)] = -\mathrm{Im}[X(-\omega)]$

Updated December 1, 2018
https://github.com/DonneyF/formula-sheets