# ELEC 221 Formula Sheet

## Continuous Time Signals

Even and Odd Components:
$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$
$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

A signal is periodic with fundamental period $T_0$ if
$x(t + kT_0) = x(t) \quad \forall t \in (-\infty, \infty)$

Energy:
$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$

Power:
$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

Causality:

- Causal: $x(t) = 0$ for $t < 0$.
- Anti-Causal: $x(t) = 0$ for $t \geq 0$.
- A-Causal or Non-Causal: Both of the above.

## Continuous Time Systems

Dynamic systems have memory. Active systems can deliver energy to the outside world.

Linearity:
$\mathcal{S}[\alpha x(t) + \beta y(t)] = \alpha\mathcal{S}[x(t)] + \beta\mathcal{S}[y(t)]$

Time Invariance:
If $\mathcal{S}[x(t)] = y(t)$ then $\mathcal{S}[x(t \pm \tau)] = y(t \pm \tau)$.

Zero-State Response:
Due to the input as the initial conditions are zero.

Zero-Input Response:
Due to the initial conditions as the input is zero.

Convolution:
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Causality:
A continuous time system $\mathcal{S}$ is causal if whenever $x(t) = 0$ and there are no initial conditions, $y(t) = 0$ and the output $y(t)$ does not depend on future inputs.

Bounded-Input Bounded-Output Stability:
If an input $x(t)$ bounded then the output of an BIBO system is also bounded.
$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$

## Laplace Transform

$s = \sigma + j\omega$

Eigenfunction Property:
$\mathcal{S}[e^{s_0 t}] = H(s_0)e^{s_0 t}$

One Sided Laplace Transform:
$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0^-}^{\infty} f(t)e^{-st}\, dt$$

| Signal Support | ROC |
|---|---|
| Finite support | Entire s-plane. |
| Causal function | $\sigma > \max(\sigma_i)$, $-\infty < \omega < \infty$ |
| Anti-causal | $\sigma < \min(\sigma_i)$, $-\infty < \omega < \infty$ |
| Non-causal | $\mathcal{R} = \mathcal{R}_{\text{causal}} \cap \mathcal{R}_{\text{anti-causal}}$ |

Initial Value Theorem:
$f(0+) \Leftrightarrow \lim_{s\to\infty} sF(s)$

Final Value Theorem:
$\lim_{t\to\infty} f(t) \Leftrightarrow \lim_{s\to 0} sF(s)$

Bounded-Input Bounded-Output Stability:
If the region of convergence contains the $j\omega$-axis, then the system is BIBO stable.

## Fourier Series

Fourier analysis in the steady state.

Eigenfunction Property:
$\mathcal{S}[e^{j\omega_0 t}] = H(j\omega_0)e^{j\omega_0 t}$
$x(t) = \sum_k X_k e^{j\omega_k t} \implies y(t) = \sum_k X_k H(j\omega_k) e^{j\omega_k t}$
$= \sum_k X_k |H(j\omega_k)| e^{j(\omega_k t + \angle H(\omega_k))}$

Fourier Series Coefficients (for any $t_0$):
$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)e^{-jk\omega_0 t}\, dt$$

Parseval's Power Relation (for any $t_0$):
$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Symmetry of Line Spectra:
$|X_k| = |X_{-k}|$
$\angle X_k = -\angle X_{-k}$

Trigonometric Fourier Series:
$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \Theta_k)$
$x(t) = c_0 + 2\sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)]$
$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\cos(k\omega_0 t)\, dt \quad k = 0,1,2\ldots$
$d_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\sin(k\omega_0 t)\, dt \quad k = 1,2,3\ldots$
$\Theta_k = -\arctan(d_k/c_k)$

Fourier Coefficients from Laplace Transform:
If $x_1(t)$ is a single period of $x(t)$, then
$X_k = \frac{1}{T_0}\mathcal{L}[x_1(t)]\big|_{s=jk\omega_0}$

Response of LTI Systems to Periodic Signals:
If the input to an LTI system has Fourier Series $x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \angle X_k)$, then the steady state response is $y(t) = X_0|H(j0)| +$
$$2\sum_{k=1}^{\infty} |X_k||H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0))$$

## Fourier Transform

$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$

Fourier Transform from Laplace Transform (if $X(s)$ contains the $j\omega$-axis):
$\mathcal{F}[x(t)] = \mathcal{L}[x(t)]\big|_{s=j\omega} = X(s)\big|_{s=j\omega}$

Fourier Transform of Periodic Signals:
$$\sum_k X_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} \sum_k 2\pi X_k \delta(\omega - k\omega_0)$$

Parseval's Energy Relation:
$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Symmetry of Spectral Representations:
$|X(\omega)| = |X(-\omega)|$
$\text{Re}[X(\omega)] = \text{Re}[X(-\omega)]$
$\angle X(\omega) = -\angle X(-\omega)$
$\text{Im}[X(\omega)] = -\text{Im}[X(-\omega)]$

## Sampling Theory

$x_s(t) = x(nT_s) = x(t)|_{t=nT_s} = x(t)\sum_n \delta(t - nT_s)$
$$X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Nyquist-Shannon Sampling Rate:
$\omega_s = \frac{2\pi}{T_s} \geq 2\omega_{\max}$
Aliasing occurs if $\omega_s < 2\omega_{\max}$.

Reconstruction $X(\omega) = X_s(\omega)H_{\text{lp}}(\omega)$:
$$H_{\text{lp}}(\omega) = \begin{cases} T_s & -\frac{\omega_s}{2} \leq \omega \leq \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

(Inclusive bounds. If represented by a unit step function the bounds are inclusive for filters (Refer to lecture notes))

Signal Reconstruction from Sinc Interpolation:
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin(\pi(t - nT_s))/T_s}{\pi(t-nT_s)/T_s}$$

## Discrete Time Signals

Define $x[n] = x(nT_0)$.

$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Periodicity:

$x[n+kN] = x[n] \qquad \forall k \in \mathbb{Z}$

When sampling an analog sinusoid of fundamental period $T_0$, we obtain a periodic discrete sinusoid provided that $m$, $N$ not divisible by each-other:

$T_s/T_0 = m/N$

Aliasing occurs if:

$T_s > T_0/2$

Energy:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Power:

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Convolutional Sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Bounded-Input Bounded-Output Stability:

$\sum_k |h[n]| < \infty$

Solution to Autoregressive Discrete System:

$y[n] = ay[n-1] + bx[n],\ n \geq 0$

$y[n] = \sum_{k=0}^{n} ba^k x[n-k],\ n \geq 0$

## Z-Transform

$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n}$

Convergence:

$|X(z)| = \sum_n |x[n]|r^{-n} < \infty$

Initial Value Theorem:

$x[0] = \lim_{z\to\infty} X(z)$

Final Value Theorem:

$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

BIBO Stability:

If the ROC contains radius $z=1$, then the system is BIBO stable.

Updated December 18, 2018
https://github.com/DonneyF/formula-sheets

### Partial Fraction Decomposition

| Fraction | Partial Fraction | Solution |
|---|---|---|
| $\frac{px+q}{(x-a)(x-b)}$ | $\frac{A}{x-a}+\frac{B}{x-b}$ | $A=\frac{pa+q}{a-b}$ $\quad B=\frac{pb+q}{b-a}$ |
| $\frac{px+q}{(x-a)^2}$ | $\frac{A}{x-a}+\frac{B}{(x-a)^2}$ | $A=p$ $\quad B=pa+q$ |
| $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$ | $\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$ | $A=\frac{pa^2+qa+r}{(a-b)(a-c)}$ $\quad B=\frac{pb^2+qb+r}{(b-a)(b-c)}$ $\quad C=\frac{pc^2+qc+r}{(c-a)(c-b)}$ |
| $\frac{px^2+qx+r}{(x-a)^2(x-b)}$ | $\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$ | |
| $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ | $\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$ | |

### Two-Sided Z-Transforms

| $f[n]$ | $F(z)$ | ROC |
|---|---|---|
| $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $|z|<1$ |
| $-\alpha^n u[-n-1]$ | $\frac{1}{1-\alpha z^{-1}}$ | $|z|<|\alpha|$ |
| $-n\alpha^n u[-n-1]$ | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $|z|<|\alpha|$ |
| $\alpha^{|n|}, |\alpha|<1$ | $\frac{1}{1-\alpha z^{-1}}-\frac{1}{1-\alpha^{-1}z^{-1}}$ | $|\alpha|<z<\left|\frac{1}{\alpha}\right|$ |

### Interconnection of LTI Systems

| Connection | Time | Laplace/Z |
|---|---|---|
| Series | $[h_1 * h_2](t)$ | $H_1(s)H_2(s)$ |
| Parallel | $h_1(t)+h_2(t)$ | $H_1(s)+H_2(s)$ |

### Misc. Identities

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \xrightarrow{n\to\infty \text{ and } x<1} \frac{1}{1-x}$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$