ELEC 433 Formula Sheet

Coding Approaches and Characteristics

Channel Capacity for Additive White Gaussian for bandwidth W and noise N_0:

$$C = W\log_2\left(1+\frac{P}{N_0W}\right)$$

BFSK bit error probability:

$$p = \frac{1}{2}e^{-E_b/2N_0}$$

Binary Linear Block Codes

(n, k, d) code:

- n - length of codeword
- k - number of message bits in codeword
- d - Code minimum distance

Number of codewords in a code C:

$$|C| = M = 2^k$$

Code rate:

$$R = \frac{\log_2(M)}{n} = \frac{k}{n}$$

Vector space dimensions:

$$\dim S + \dim S^{\perp} = \dim V$$

Def (*Binary Linear Block Codes*): A subset $C \subseteq V_n$ is a binary linear block code if:

- $\mathbf{u} + \mathbf{v} \in C \quad \forall \mathbf{u}, \mathbf{v} \in C$
- $a\mathbf{u} \in C \quad \forall \mathbf{u} \in C, a \in \{0, 1\}$

Hamming Weight:

$w(\mathbf{x})$ = number of non-zero elements in $\mathbf{x}$

Hamming Distance:

$d(\mathbf{x}, \mathbf{y})$ = number of places in which $\mathbf{x}$ and $\mathbf{y}$ differ

Hamming Distance for binary linear codes:

$$d(\mathbf{x}, \mathbf{y}) = w(\mathbf{x} + \mathbf{y})$$

Minimum Hamming Distance:

- $d(C) = \min\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}$
- A code C can detect up to v errors if $d(C) \geq v + 1$
- A code C can correct up to t errors if $d(C) \geq 2t + 1$

Singleton Bound:

$$d_{\min} \leq n - k + 1$$

Def (*Generator Matrix*): A $k \times n$ matrix whose rows for a basis for a linear (n, k) code of a subspace C is said to be a generator matrix for C.

Groups, Rings, and Fields

Def (*Group*) A group $(G, \cdot)$ is a set of objects G on which a binary operation $\cdot$ is defined: $a \cdot b \in G : \forall a, b \in G$. The operation must satisfy:

- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Identity: $\exists e \in G \mid \forall a \in G, a \cdot e = a$
- Inverse: $\forall a \in G, \exists$ unique $a^{-1} \in G \mid a \cdot a^{-1} = e$

Def (*Commutative Group*) A group is said to be commutative or abelian if it also satisfies: $\forall a, b \in G, a \cdot b = b \cdot a$

Def (*Ring*) A ring $(R, +, \cdot)$ is a set of objects R on which two binary operations (+ and $\cdot$) are defined. It has properties:

- $(R, +)$ is a commutative group under + with identity "0"
- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Distribution: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Def (*Commutative Ring*) A ring is said to be commutative if it also satisfies: $\forall a, b \in G, a \cdot b = b \cdot a$

Def (*Ring with Identity*) A ring is said to be a ring with identity if the operation $\cdot$ has an identity element "1"

Def (*Division Ring*) Let $(R, +, \cdot)$ be a ring, and $R^* = R - 0$. If the ring is a commutative ring with identity, and $(R^*, \cdot)$ is a group, then the ring is said to be a division ring.

Def (*Field*) A field $(F, +, \cdot)$ is a set of objects F for which two binary operations (+ and $\cdot$) are defined. F is said to be a field if and only if:

- $(F, +)$ is a commutative group under + with additive identity "0"
- $(F^*, \cdot)$ is a commutative group under $\cdot$ with multiplicative identity "1"
- Distribution: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Finite Integer Fields:

$S = \{0, 1, \ldots, p - 1\}$ form a finite field if p is prime.

Properties of Finite Fields:

- Order: A field of order q has cardinality $|F| = q$, denoted GF(q)
- Let $\beta \in \text{GF}(q), \beta \neq 0$. The order of β is the smallest positive integer m such that $\beta^m = 1$
- If t is the order of β, then $t \mid (q - 1)$
- In any finite field, there are on or more elements of order $q - 1$ called primitive elements.

Euler's Totient Function:

$\phi(t)$ = number of positive integers less than t that are relatively prime to t

Finite Fields and Euler's Totient Function:

- The number of elements in GF(q) of order t is $\phi(t)$
- In GF(q) there are exactly $\phi(q - 1)$ primitive elements
- If α is a primitive element, then $1, \alpha, \alpha^2, \ldots, \alpha^{q-2}$ must be non-zero elements of GF(q)

Def (*Primitive Polynomial*) If an irriducible polynomial $p(x)$ such that the smallest positive integer n for which $p(x)$ divides $x^n - 1$ is $n = p^m - 1$ for a prime p and positive integer m, the polynomial is said to be a primitive polynomial.

Encoding and Decoding

Codeword convention: Data appears unaltered at the start of the code word.

Thm (*Equivalence of Binary Linear Codes*) Two linear binary codes are called equivalent if one can be obtained from the other by permuting the positions of the code. Two $k \times n$ mbinary matrices generate equivalent linear (n, k, d) codes if one matrix can be obtained from the otehr by a sequence of row, column permutations and row addition.

Thm (*Systematic Codes*) Let G be a generator matrix of an (n, k) code. Then G can be transformed to the form $\left[I_k \mid P\right]$ where P is called the parity matrix.

Encoding of a message $\mathbf{m}$ with a code C:

$$c = \mathbf{m}G$$

Def (*Parity Check Matrix*). H satisfies $GH^T = 0$ and is a basis for the dual space. In systematic form $H = \left[P^T \mid I_{n-k}\right]$

Syndrome of a received word $\mathbf{r}$:

$$\mathbf{s} = \mathbf{r}H^T$$

Hamming Codes

Def (*Binary Hamming Code*) Let $m \in \mathbb{Z}$ and H be a $m \times (2^m - 1)$ matrix with columns which are the non-zero distinct words from a vector space V_m. The code having H as its parity-check matrix is a binary Hamming code of length $2^m - 1$

Hamming code parameters:

$$C : (2^m - 1, 2^m - 1 - m, 3) \quad C^\perp : (2^m - 1, m, 2^{m-1})$$

Decoding Hamming Codes where columns of H are arranged in order of increasing binary numbers:

1. Compute $S(\mathbf{r}) = \mathbf{r}H^T$
2. If $S(\mathbf{r}) = 0$, then $\mathbf{r}$ is a valid codeword
3. Else, $S(\mathbf{r})$ gives the binary position of the error

Hamming Bound:

$$\sum_{i=0}^{t} \binom{n}{i} \leq 2^{n-k}$$

Cyclic Codes

Def (*Cyclic Code*) A code C is cyclic if C is linear and a cyclic shift of any codeword is another codeword.

Properties of a (n, k) binary cyclic code C:

1. There exists a generator polynomial of minimal degree $n - k$
2. Every code polynomial in C can be expressed as $c(x) = m(x)g(x)$ where $m(x)$ has degree $< k - 1$
3. We can write $x^n - 1 = g(x)h(x)$ where $h(x)$ is the parity check polynomial.
4. If $g(x)$ is a primitive polynomial then C is also a Hamming code.

Generator Matrix:

$$G = \begin{bmatrix} g_0 & g_1 & \cdots & g_{n-k} & 0 & \cdots & 0 \\ 0 & g_0 & g_1 & \cdots & g_{n-k} & 0 & 0 \\ 0 & 0 & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & 0 & g_0 & g_1 & \cdots & g_{n-k} \end{bmatrix}$$

Parity Check Matrix:

$$h^*(x) = x^k h(x^{-1}) = h_k + h_{k-1}x + \ldots + h_0 x^k$$

$$H = \begin{bmatrix} h_k & \cdots & h_1 & h_0 & 0 & \cdots & 0 \\ 0 & h_k & \cdots & h_1 & h_0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & & \ddots & 0 \\ 0 & 0 & 0 & h_k & \cdots & h_1 & h_0 \end{bmatrix}$$

Systematic Generator Matrix:

1. For $i = n - k$ to $n - 1$, compute $x^i \mod g(x) = p_i(x)$
2. Rows of G are formed by $x^i + p_i(x)$.

Systematic Encoding:

$$c(x) = m(x)x^{n-k} + m(x)x^{n-k} \mod g(x)$$

Def (*Shortened Cyclic Code*) $(n, k) \to (n - u, k - u)$ by setting u most significant bits of codeword to zero. Typically not cyclic. A CRC code is a shortened cyclic code.

Minimal Polynomials

Def (*Minimal Polynomial*) Let $\alpha \in \mathrm{GF}(2^m)$. $p(x)$ is a minimal polynomial of α with respect to $\mathrm{GF}(2^m)$ if it is the smallest degree monic polynomial such that $p(\alpha) = 0$.

Properties of Minimal Polynomials:

- The degree of $p(x)$ is d, and $d \mid m$
- $f(\alpha) = 0 \implies p(x) \mid f(x)$
- $p(x)$ is irreducible in $\mathrm{GF}(2^m)$
- If α is primitive, $p(x)$ is a primitive polynomial.

Def (*Conjugacy Class*) Let $\alpha \in \mathrm{GF}(2^m)$. The conjugacy class of α is $\left\{\alpha, \alpha^2, \alpha^{2^2}, \cdots, \alpha^{2^{d-1}}\right\}$. If $p(\alpha) = 0$, any element of the conjugacy class is also a root.

Def (*Cyclotomic Coset*) The partition of powers of α by the conjugacy classes of a finite field is called the set of cyclotomic cosets.

BCH Codes

BCH codes are a generalization of cyclic Hamming codes. The generator polynomial $g(x)$ is a primitive polynomial. Codeword satisfies $g(\alpha) = 0 \implies c(\alpha) = 0$.

Thm (*BCH Bound*) Let C be a (n, k) 2-ary cyclic code with generator polynomial $g(x)$. Let $\alpha \in \mathrm{GF}(2^m)$ be an element of order n, $n \mid 2^m - 1$. If $g(x)$ is a minimal polynomial with roots $\alpha^b, \alpha^{b+1}, \cdots, \alpha^{b+\delta-2}$, then C has minimum distance at least δ. $g(x)$ is degree $n - k$ and is the product of the minimal polynomials of the roots

$$g(x) = \mathrm{LCM}\{m_b(x), m_{b+1}(x), \cdots, m_{b+\delta-2}(x)\}$$

Def (*Narrow-Sense BCH Code*) Narrow-Sense codes have parameter $b = 1$.

Def (*Binary Primitive BCH Codes*) For any m and $t < n/2$, there exists a binary primitive BCH code with parameters $n = 2^m - 1$, $d \geq 2t = 1$, $n - k \leq mt$, where d is the designed distance.

Construction of a t error correcting 2-ary BCH Code:

1. Find $\alpha \in \mathrm{GF}(2^m)$ where m is minimal.
2. Select $2t$ consecutive powers of α starting at α^b.
3. Find $g(x)$ as the LCM of the minimal polynomials for those powers of α.

Reed-Solomon Codes

Reed-Solomon codes are a subset of BCH codes and are non-binary. Properties:

- α is primitive.
- Generator polynomial $g(x) = (x - \alpha)(x - \alpha^2) \cdots (x - \alpha^{2t})$
- $n = 2^m - 1 \qquad n - k = 2t$

A RS code can correct up to s errors and r erasures if

$$2s + r < 2t$$

Convolutional Codes

Def (*Constraint Length*) The constraint length L is the length of longest input shift register with maximum number of memory elements plus one.

Coding Rate:

$$R = \frac{\text{symbols shifted in a cycle}}{\text{number of output symbols}}$$

LPDC Codes

Parity Check Matrix Representation

- Let W_r be the number of 1's in each row
- Let W_c be the number of 1's in each column
- A matrix is called low density if $W_c \ll n$ and $W_r \ll (n - k)$

Graph Representation

- Node are separated into variable nodes f_i and check nodes c_j
- An edge connects nodes f_i and c_j if $H_{ij} = 1$

Def (*Regular LPDC Code*) A LPDC code is said to be regular if W_c and $W_r = W_c(n/(n - l))$ are constant.

Updated April 26, 2022
https://github.com/DonneyF/formula-sheets