

# MATH 400 Formula Sheet

## Linear First Order Equations

First Order Linear PDE:

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$$

## Simple Transport Equation

$$u_t + cu_x = 0, -\infty < x < \infty, t > 0$$

$$u(x, 0) = \phi(x), -\infty < x < \infty$$

Solution:

$$u(x, t) = \phi(x - ct)$$

## Linear Second Order Equations

Quasilinear PDE:

$$a_{11}(x, y, u, u_x, u_y)u_{xx} + 2a_{12}(x, y, u, u_x, u_y)u_{xy} + a_{22}(x, y, u, u_x, u_y)u_{yy} + a_{00}(x, y, u, u_x, u_y) = 0$$

Semilinear PDE:

$$a_{11}(x, y)u_{xx} + 2a_{12}(x, y)u_{xy} + a_{22}(x, y)u_{yy} + a_{00}(x, y, u, u_x, u_y) = 0$$

Linear PDE:

$$a_{11}(x, y)u_{xx} + 2a_{12}(x, y)u_{xy} + a_{22}(x, y)u_{yy} + a_1(x, y)u_x + a_2(x, y)u_y + a_0(x, y)u = f(x, y)$$

Discriminant for semilinear PDEs:

$$\mathcal{D}(x, y) = [a_{12}(x, y)]^2 - a_{11}(x, y)a_{22}(x, y)$$

Classification of semilinear PDEs:

$$\begin{cases} \mathcal{D}(x, y) > 0 & \text{Hyperbolic} \\ \mathcal{D}(x, y) < 0 & \text{Elliptic} \\ \mathcal{D}(x, y) = 0 & \text{Parabolic} \end{cases}$$

Change of variables:

$$\begin{cases} U_{\xi\eta} + b_{00}(\xi, \eta, U, U_{\xi}, U_{\eta}) = 0 & \text{Hyperbolic} \\ U_{\xi\xi} + U_{\eta\eta} + b_{00}(\xi, \eta, U, U_{\xi}, U_{\eta}) = 0 & \text{Elliptic} \\ U_{\xi\xi} + b_{00}(\xi, \eta, U, U_{\xi}, U_{\eta}) = 0 & \text{Parabolic} \end{cases}$$

## Wave Equation

General Solution:

$$u(x, t) = f(x + ct) + g(x - ct)$$

Initial Value Problem on the real line:

$$u_{tt} - c^2 u_{xx} = f(x, t), -\infty < x < \infty, t > 0$$

$$u(x, 0) = \phi(x), u_t(x, 0) = \psi(x) - \infty < x < \infty$$

d'Alembert's Formula:

$$u(x, t) = \frac{1}{2} [\phi(x - ct) + \phi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds$$

## Non-Homogenous Boundary Conditions

Shift the data ( $u = v + w$ ):

Dirichlet,  $u(0, t) = a(t), u(L, t) = b(t)$ :

$$w(x, t) = a(t) + \frac{x}{L} [b(t) - a(t)]$$

Neumann,  $u_x(0, t) = a(t), u_x(L, t) = b(t)$

$$w(x, t) = xa(t) + \frac{x^2}{2L} [b(t) - a(t)]$$

Mixed 1,  $u(0, t) = a(t), u_x(L, t) = b(t)$

$$w(x, t) = a(t) + xb(t)$$

Mixed 2,  $u_x(0, t) = a(t), u(L, t) = b(t)$

$$w(x, t) = (x - L)a(t) + b(t)$$

## Sturm-Liouville Theory

Consider the homogeneous linear second-order PDE:

$$r(x)u_t - (p(x)u_x)_x + q(x)u = 0$$

Separation of Variables:

$$-\frac{T'(t)}{T(t)} = -\frac{(p(x)X'(x))'}{r(x)X(x)} + \frac{q(x)}{r(x)} = \lambda$$

ODEs:

$$T' + \lambda T = 0$$

$$-(p(x)X')' + q(x)X = \lambda r(x)X$$

Associated Eigenvalue Problem:

$$-(p(x)X')' + q(x)X = \lambda r(x)X$$

General Boundary Conditions:

$$\mathcal{B}_1 X = \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0$$

$$\mathcal{B}_2 X = \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0$$

Boundary conditions are separated when:

$$\beta_1 = \delta_1 = 0 \quad \alpha_2 = \gamma_2 = 0$$

Lagrange's Identity. BCs are symmetric if for all functions  $f, g$  that satisfy the BCs:

$$[-p(f'g - fg')]_a^b = 0$$

Sturm-Liouville Problem ( $p(x) > 0, r(x) > 0$ ):

$$\begin{cases} \mathcal{L}X(x) = \lambda r(x)X, a < x < b \\ \mathcal{B}_1 X = 0, \mathcal{B}_2 X = 0 \end{cases}$$

Theorem 5.3.1:

If you have symmetric BCs, then any two eigenfunctions of a SL problem that correspond to distinct eigenvalues are orthogonal. If any function is expanded in a series of these eigenfunctions, the coefficients are determined.

Theorem 5.3.2:

Under Theorem 5.3.1, all the eigenvalues are real numbers and the eigenfunctions can be chosen to be real-valued.

Theorem 5.3.3:

Under 5.3.1, if  $q(x) \geq 0$  for all  $a \leq x \leq b$  and if for all real-valued functions  $f$  satisfying the BCs we satisfy

$$[p(x)f(x)f'(x)]_a^b \leq 0$$

then there can be no negative eigenvalues.

Theorem 5.4.1:

For any regular of periodic SL problem, there are an infinite amount of eigenvalues.

## Laplace's Equation

Polar Coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Spherical Coordinates:

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2 \sin \theta} [\sin \theta u_{\theta\theta}]_{\theta} + \frac{1}{r^2 \sin^2 \theta} u_{\phi\phi} = 0$$

Maximum Principle:

Let  $D$  be a connected bounded open set of  $\mathbb{R}^n$ ,  $n = 2, 3$ . If  $u(\mathbf{x})$  is harmonic in  $D$  and continuous on  $D \cup \partial D$  then  $u(\mathbf{x})$  attains its maximum and minimum values on  $\partial D$ .

## PDEs in 3 Dimensions

Elliptic PDE:

$$-\nabla \cdot [p(\mathbf{x})\nabla u] + q(\mathbf{x})u = f(\mathbf{x})$$

Parabolic PDE:

$$r(\mathbf{x})u_t - \nabla \cdot [p(\mathbf{x})\nabla u] + q(\mathbf{x})u = f(\mathbf{x}, t)$$

Hyperbolic PDE:

$$r(\mathbf{x})u_{tt} - \nabla \cdot [p(\mathbf{x})\nabla u] + q(\mathbf{x})u = f(\mathbf{x}, t)$$

Eigenvalue Problem:

$$-\nabla \cdot [p(\mathbf{x})\nabla X] + q(\mathbf{x})X = \lambda r(\mathbf{x})X$$

Green's First Identity:

Let  $D$  be an open connected domain in  $\mathbb{R}^3$  and let  $\partial D$  be its piecewise smooth boundary. Let  $f$  and  $g$  be smooth functions on  $D \cup \partial D$ . Then

$$\oint_{\partial D} f \frac{\partial g}{\partial n} ds = \iiint_D \nabla f \cdot \nabla g \, d\mathbf{x} + \iint_D f \nabla^2 g \, d\mathbf{x}$$

Bessel's Equation:

$$\rho^2 y'' + \rho y + (\rho^2 - n^2)y = 0$$

$$y(\rho) = c_1 J_n(\rho) + c_2 Y_n(\rho)$$

Associated Legendre Equation:

$$\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} P_l^m(x) \right] + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] P_l^m(x) = 0$$

Spherical Harmonics:

$$\nabla_{S^2}^2 Y_l^m = \frac{1}{\sin \theta} [\sin \theta Y]_{\theta} + \frac{1}{\sin^2 \theta} Y_{\phi\phi} = l(l+1) Y_l^m$$

$$Y_l^m(\theta, \phi) = P_l^{|m|}(\cos \theta) e^{im\phi}$$

$$l = 0, 1, 2, \dots \quad m = -l, -l+1, \dots, l-1, l$$

## Integral Transform Methods

Fourier Transform:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Inverse Fourier Transform:

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

Error Function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

$$\operatorname{erf}(0) = 0 \quad \lim_{x \rightarrow \infty} \operatorname{erf}(x) = 1$$

Laplace Transform:

Let  $f(t)$  be a piecewise continuous function on  $[0, \infty)$  and suppose  $\exists K, \gamma$  such that  $|f(t)| \leq K e^{\gamma t} \forall t > 0$ . Then

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Inverse Laplace Transform:

$$f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} F(s) e^{st} ds \quad \alpha > \gamma$$

## Solved 1D Eigenvalue Problems

### Dirichlet

$$\left. \begin{array}{l} -X'' = \lambda X \\ X(0) = 0, X(L) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3 \dots \\ X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \end{array} \right.$$

### Neumann

$$\left. \begin{array}{l} -X'' = \lambda X \\ X'(0) = 0, X'(L) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 0, 1, 2 \dots \\ X_n(x) = \cos\left(\frac{n\pi}{L}x\right) \end{array} \right.$$

### Periodic

$$\left. \begin{array}{l} -X'' = \lambda X \\ X(-\pi) = X(\pi) \\ X'(-\pi) = X'(\pi) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_n = n^2, n = 0, 1, 2 \dots \\ X_n(x) \in \{1, \cos(n\theta), \sin(n\theta)\} \end{array} \right.$$

### Mixed 1

$$\left. \begin{array}{l} -X'' = \lambda X \\ X(0) = 0, X'(L) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, n = 0, 1, 2 \dots \\ X_n(x) = \sin\left(\frac{(2n+1)\pi}{2L}x\right) \end{array} \right.$$

### Mixed 2

$$\left. \begin{array}{l} -X'' = \lambda X \\ X'(0) = 0, X(L) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, n = 0, 1, 2 \dots \\ X_n(x) = \cos\left(\frac{(2n+1)\pi}{2L}x\right) \end{array} \right.$$

## Solved PDE Problems

### Problem 1

$$\Delta u = 0 \quad x^2 + y^2 < a^2 \\ u = h(x, y) \quad \text{on} \quad x^2 + y^2 = a^2$$

Solution (Oct 21):

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{h(\phi)}{a^2 - 2ar \cos(\phi - \theta) + r^2} d\phi$$

### Problem 2

$$\Delta u = 0 \quad 0 < r < a, \quad 0 < \theta < \beta \\ u(r, 0) = 0, \quad u(r, \beta) = 0, \quad 0 < r < a \\ u_r(a, \theta) = h(\theta) \quad 0 < \theta < \beta$$

Solution (Oct 23):

$$u(r, \theta) = \sum_{n=1}^{\infty} C_n r^{n\pi/\beta} \sin\left(\frac{n\pi}{\beta}\theta\right) \\ C_n = \frac{2}{n\pi} a^{-n\pi/\beta+1} \int_0^{\beta} h(\phi) \sin\left(\frac{n\pi}{\beta}\theta\right) d\theta$$

### Problem 3

$$\Delta u = 0 \quad a < r < b, \quad 0 < \theta < 2\pi$$

General Solution (Oct 23):

$$u(r, \theta) = \frac{1}{2} (C_0 + D_0 \ln(r)) + \sum_{n=1}^{\infty} \left( C_n^{(1)} r^n + D_n^{(1)} r^{-n} \right) \cos(n\theta) + \left( C_n^{(2)} r^n + D_n^{(2)} r^{-n} \right) \sin(n\theta)$$

### Problem 4

$$S_t - DS_{xx} = 0 \quad -\infty < x < \infty, \quad t > 0 \\ S(x, 0) = \delta(x - y), \quad y \geq 0$$

Solution (Nov 16):

$$S(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-y)^2/(4Dt)}$$

### Problem 5

$$u_t - Du_{xx} = 0 \quad -\infty < x < \infty, \quad t > 0 \\ u(x, 0) = \varphi(x), \quad -\infty < x < \infty$$

Solution (Nov 18):

$$u(x, t) = \int_{-\infty}^{\infty} \varphi(y) S(x - y, t) dy$$

### Problem 6

$$u_t - Du_{xx} = 0 \quad 0 < x < \infty, \quad t > 0 \\ u(0, t) = 0, \quad t > 0 \\ u(x, 0) = \varphi(x), \quad -\infty < x < \infty$$

Solution (Nov 18):

$$u(x, t) = \int_0^{\infty} \phi(y) [S(x - y, t) - S(x + y, t)] dy$$

### Problem 7

$$\Delta u = 0 \quad -\infty < x < \infty, \quad y > 0 \\ u(x, 0) = h(x), \quad -\infty < x < \infty$$

Solution (Nov 20):

$$u(x, y) = \int_{-\infty}^{\infty} h(z) \frac{y}{\pi[(x-z)^2 + y^2]} dz$$

### Problem 8

$$u_t - Du_{xx} = 0 \quad 0 < x < \infty, \quad t > 0 \\ u(0, t) = h(t), \quad t > 0 \\ u(x, 0) = 0, \quad -\infty < x < \infty$$

Solution (Nov 23):

$$u(x, t) = \begin{cases} 0 & t < b \\ 1 - \operatorname{erf}\left(\frac{x}{\sqrt{4D(t-b)}}\right) & t > b \end{cases}$$

Updated December 11, 2020

<https://github.com/DonneyF/formula-sheets>