PHYS 408 Formula Sheet

Differential Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Integral Maxwell's Equations

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \qquad \oint \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \qquad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

Wave Optics

Monochromatic Wave:

$$\mathcal{E} = \text{Re}\left\{\mathbf{E}(\mathbf{r}) e^{-i\omega t}\right\}$$

Helmholtz Equation:

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$

Plane wave solution to Helmholtz Equation:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$$

Spherical wave solution to Helmholtz Equation:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \frac{1}{r} e^{ikr} \qquad \mathbf{E}_0 = E_0 \hat{\theta}$$

Interference of multiple fields:

$$I = \textstyle\sum_{k=1}^{N} I_k + v\epsilon\, \text{Re}\left\{E_1 E_2^* + E_1 E_3^* + \cdots + E_{N-1} E_N^*\right\}$$

$$v\epsilon E_i E_j^* = 2\sqrt{I_i I_j} \exp\left[i(\theta_i - \theta_j)\right]$$

Fresnel (Paraxial) Approximation:

$$\sqrt{x^2 + y^2} \ll z$$

Transverse Laplacian Operator:

$$\nabla_T^2 = \tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial y}$$

Paraxial Wave Equation:

$$\nabla_T^2 \mathbf{E} + i2k \frac{\partial \mathbf{E}}{\partial z} = \mathbf{0}$$

Fresnel Approximation of a spherical wave:

$$\mathbf{E}(\mathbf{r}) \approx \frac{\mathbf{E}_0}{z} \exp\left[ik\frac{x^2 + y^2}{2z}\right] = \frac{\mathbf{E}_0}{z} \exp\left[ik\frac{\rho^2}{2z}\right]$$

Beam Optics

Gaussian beam with Rayleigh length z_0:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \frac{w_0}{w(z)} \exp\left[-\frac{\rho^2}{(w(z))^2}\right] \times$$

$$\exp\left[ik\frac{\rho^2}{2R(z)}\right] \exp\left[ikz - i\tan^{-1}\left(\frac{z}{z_0}\right)\right]$$

$$q(z) = z - z_0$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi}\frac{1}{(w(z))^2}$$

Beam waist:

$$w_0 = \sqrt{\frac{\lambda z_0}{\pi}}$$

Beam Radius:

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$

Radius of curvature:

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

Gaussian beam intensity with center intensity I_0:

$$I(z, \rho) = I_0 \left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2\rho^2}{(w(z))^2}\right]$$

Gaussian beam total power:

$$P = \tfrac{I_0}{2} \pi w_0^2$$

Depth of focus (confocal parameter):

$$d = 2z_0 = \frac{2\pi w_0^2}{\lambda}$$

Propagation of Paraxial Waves

ABCD Matrix of an optical system:

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

Free space propagation transfer matrix:

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

Refraction transfer matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & n_I / n_T \end{bmatrix}$$

Curved interface transfer matrix:

$$\begin{bmatrix} 1 & 0 \\ -\frac{n_T - n_I}{n_I R} & n_I / n_T \end{bmatrix}$$

Thin lens transfer matrix:

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Gaussian Beam propagation in the paraxial regime:

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

Fourier Optics

Fraunhofer (far field) limit:

$$x^2/\lambda, y^2/\lambda \ll z/\pi$$

Free space transfer function along $\mathbf{z}$:

$$\tilde{H}(k_x, k_y) = e^{ik_z z} \approx e^{ikz} e^{-i(k_x^2 + k_y^2))z/2k}$$

Free space propagation in paraxial limit:

$$E(x', y', z) \approx \frac{e^{ikz}}{i\lambda z} \exp\left[ik\frac{x'^2 + y'^2}{2z}\right] \times$$

$$\iint E(x, y, z) \exp\left[ik\frac{x^2 + y^2}{2z}\right] \exp\left[-\frac{ik}{z}(x'x + y'y)\right] dx\, dy$$

Free space propagation in Fraunhofer limit:

$$E(x', y', z) \approx \frac{e^{ikz}}{i\lambda z} \exp\left[ik\frac{x'^2 + y'^2}{2z}\right] \mathcal{F}\{E(x, y, z)\}$$

Thin lens thickness function:

$$\Delta(x, y) =$$

$$\Delta_0 - R_1\left(1 - \sqrt{1 - \frac{x^2 + y^2}{R_1^2}}\right) + R_2\left(1 - \sqrt{1 - \frac{x^2 + y^2}{R_2^2}}\right)$$

Thin lens thickness function in paraxial limit:

$$\Delta(x, y) \approx \Delta_0 - \frac{x^2 + y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Thin lens phase function:

$$t_L(x, y) = \exp[ikn\Delta(x, y)] \exp[ik(\Delta_0 - \Delta(x, y))]$$

Thin lens phase function (ignore constant phase):

$$t_L(x, y) = \exp\left[-i\frac{k}{2f}(x^2 + y^2)\right]$$

Polarization Optics

Boundary Conditions

- $\varepsilon_1 \mathbf{E}_1^{\perp} = \varepsilon_2 \mathbf{E}_2^{\perp}$
- $\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}$
- $\mathbf{B}_1^{\perp} = \mathbf{B}_2^{\perp}$
- $\dfrac{\mathbf{B}_1^{\parallel}}{\mu_1} = \dfrac{\mathbf{B}_2^{\parallel}}{\mu_2}$

E Parallel to Plane of Incidence

$$\mathbf{k}_I = k_I(\cos\theta_I \hat{\mathbf{z}} + \sin\theta_I \hat{\mathbf{x}})$$

$$\mathbf{E}_I = E_I e^{i[k_I(\cos\theta_I z + \sin\theta_I x) - \omega t]}(\cos\theta_I \hat{\mathbf{x}} - \sin\theta_I \hat{\mathbf{z}})$$

$$\mathbf{k}_R = k_I(-\cos\theta_R \hat{\mathbf{z}} + \sin\theta_R \hat{\mathbf{x}})$$

$$\mathbf{E}_R = E_R e^{i[k_R(-\cos\theta_R z + \sin\theta_R x) - \omega t]}(\cos\theta_R \hat{\mathbf{x}} + \sin\theta_R \hat{\mathbf{z}})$$

$$\mathbf{k}_T = k_T(\cos\theta_T \hat{\mathbf{z}} + \sin\theta_T \hat{\mathbf{x}})$$

$$\mathbf{E}_T = E_T e^{i[k_T(\cos\theta_T z + \sin\theta_T x) - \omega t]}(\cos\theta_T \hat{\mathbf{x}} - \sin\theta_T \hat{\mathbf{z}})$$

Field Magnitudes (Fresnel Equations):

$$\frac{E_R}{E_I} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{\alpha + \beta}\right)$$

Reflection and Transmission Coefficients:

$$R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 \qquad T = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2$$

Brewster's Angle (no reflected wave, $\alpha = \beta$):

$$\tan\theta_B = n_2 / n_1$$

E Perpendicular to Plane of Incidence

$$\mathbf{k}_I = k_I(\cos\theta_I\hat{\mathbf{z}} + \sin\theta_I\hat{\mathbf{x}})$$
$$\mathbf{B}_I = B_I e^{i[k_I(\cos\theta_I z + \sin\theta_I x) - \omega t]}(-\cos\theta_I\hat{\mathbf{x}} + \sin\theta_I\hat{\mathbf{z}})$$
$$\mathbf{k}_R = k_I(-\cos\theta_R\hat{\mathbf{z}} + \sin\theta_R\hat{\mathbf{x}})$$
$$\mathbf{B}_R = B_R e^{i[k_R(-\cos\theta_R z + \sin\theta_R x) - \omega t]}(\cos\theta_R\hat{\mathbf{x}} + \sin\theta_R\hat{\mathbf{z}})$$
$$\mathbf{k}_T = k_T(\cos\theta_T\hat{\mathbf{z}} + \sin\theta_T\hat{\mathbf{x}})$$
$$\mathbf{B}_T = B_T e^{i[k_T(\cos\theta_T z + \sin\theta_T x) - \omega t]}(-\cos\theta_T\hat{\mathbf{x}} + \sin\theta_T\hat{\mathbf{z}})$$

Field Magnitudes (Fresnel Equations):

$$\frac{E_R}{E_I} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{1+\alpha\beta}\right)$$

Reflection and Transmission Coefficients:

$$R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 \qquad T = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

Basic Polarization States

Linear Polarization Vector:

$$\mathbf{E}_0 = E_{0x}e^{i\phi_x}\hat{\mathbf{x}} + E_{0y}e^{i\phi_y}\hat{\mathbf{y}}$$
$$\phi = \phi_x - \phi_y \in \{0, \pi, -\pi\}$$

Linear Polarization Angle ψ relative to x-axis:

$$\tan\psi = E_{0x}/E_{0y}$$

Circular Polarization Vector ($+i$ is counter-clockwise):

$$\mathbf{E}_0 = E_0\hat{\mathbf{x}} \pm iE_0\hat{\mathbf{y}}$$

Elliptical Polarization Vector:

$$\mathbf{E}_0 = E_{0x}e^{i\phi_x}\hat{\mathbf{x}} + E_{0y}e^{i\phi_y}\hat{\mathbf{y}}$$

Elliptical Polarization Ellipticity:

$$\tan\chi = b/a$$

Jones Vectors

$$\mathbf{E}_0 = E_{0x}\hat{\mathbf{x}} + E_{0y}e^{i\phi}\hat{\mathbf{y}} = E_0\begin{bmatrix}E_x\\E_y\end{bmatrix}$$
$$|E_x|^2 + |E_y|^2 = 1$$

Linear Polarization:

$$\hat{\mathbf{e}}_x = \begin{bmatrix}1\\0\end{bmatrix} \qquad \hat{\mathbf{e}}_y = \begin{bmatrix}0\\1\end{bmatrix} \qquad \hat{\mathbf{e}}_\alpha = \begin{bmatrix}\cos\alpha\\\sin\alpha\end{bmatrix}$$

Circular Polarization:

$$\hat{\mathbf{e}}_{\mathrm{CW}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix} \qquad \hat{\mathbf{e}}_{\mathrm{CCW}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$$

Elliptical Polarization:

$$\hat{\mathbf{e}}_e = \frac{1}{\sqrt{2}}(\sin\chi + \cos\chi)e^{-i\psi}\hat{\mathbf{e}}_{\mathrm{CW}} + \frac{1}{\sqrt{2}}(\sin\chi - \cos\chi)e^{i\psi}\hat{\mathbf{e}}_{\mathrm{CCW}}$$

Jones Matrices

$$\mathbf{E_2} = T\mathbf{E_1}$$

Linear Polarizer in x-y basis:

$$T_p^{(x)} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \qquad T_p^{(y)} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

Linear Polarizer at angle α relative to x-axis:

$$T_p^{(\alpha)} = \begin{bmatrix}\cos^2\alpha & \cos\alpha\sin\alpha\\\cos\alpha\sin\alpha & \sin^2\alpha\end{bmatrix}$$

Arbitrary Phase Retarder in x-y basis:

$$T_\phi^{(x)} = \begin{bmatrix}1 & 0\\0 & e^{i\phi}\end{bmatrix}$$

Phase retarder with fast axis at angle α relative to x-axis:

$$T_\phi^{(\alpha)} = e^{-\frac{i\phi}{2}}\begin{bmatrix}\cos^2\alpha + e^{i\phi}sin^2\alpha & (1-e^{i\phi})\cos\alpha\sin\alpha\\(1-e^{i\phi})\cos\alpha\sin\alpha & \sin^2\alpha + e^{i\phi}\cos^2\alpha\end{bmatrix}$$

Rotate x-y basis by an arbitrary angle:

$$T^{(\theta)} = R_\theta^{-1}TR_\theta \qquad R_\theta = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$$

Normal Modes:

$$T_{\mathrm{total}}\mathbf{E}_0 = \lambda_{1,2}\mathbf{E}_0$$

Stokes Vectors

$$\mathbf{E}_0 = E_{0x}\hat{\mathbf{x}} + E_{0y}e^{i\phi}\hat{\mathbf{y}}$$
$$S_0 = I_{\mathrm{tot}} = \langle E_{0x}^2 + E_{0y}^2\rangle$$
$$S_1 = I_{\mathrm{pol}}\cos(2\psi)\cos(2\chi) = \langle E_{0x}^2 - E_{0y}^2\rangle$$
$$S_2 = I_{\mathrm{pol}}\sin(2\psi)\cos(2\chi) = \langle 2E_{0x}E_{0y}\cos\phi\rangle$$
$$S_3 = I_{\mathrm{pol}}\sin(2\chi) = \langle 2E_{0x}E_{0y}\sin\phi\rangle$$

Polarization Degree:

$$\frac{I_{\mathrm{pol}}}{I_{\mathrm{tot}}} = \frac{1}{S_0}\sqrt{S_1^2 + S_2^2 + S_3^2}$$

Vector Derivatives

Cartesian

$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}} \qquad d\tau = dx\,dy\,dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Spherical

$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl:

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} +$$
$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Cylindrical

$d\tau = s\, ds\, d\phi\, dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Fundamental Theorems

Fundamental Theorem of Line Integrals:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence Theorem:

$$\int (\nabla\cdot\mathbf{A})\, d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$$

Stoke's Theorem:

$$\int (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$$

Vector Identities

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}$$

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Spherical Coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right)$$
$$\phi = \tan^{-1}(y/x)$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Cylindrical Coordinates

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$s = \sqrt{x^2+y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

$$\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Trig Identities

$$2\cos\theta\cos\phi = \cos(\theta-\phi) + \cos(\theta+\phi)$$
$$2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi)$$
$$2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi)$$
$$2\cos\theta\sin\phi = \sin(\theta+\phi) - \sin(\theta-\phi)$$

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https://github.com/DonneyF/formula-sheets