PHYS 401 Formula Sheet

Differential Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Integral Maxwell's Equations

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \qquad \oint \mathbf{B} \cdot d\mathbf{a} = 0$$
$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \qquad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

Electromagnetic Waves

Wave Speed:
$$v = \frac{1}{\sqrt{\mu\varepsilon}} = c/n$$
Poynting Vector:
$$\mathbf{S} = \frac{1}{\mu}\mathbf{E} \times \mathbf{B}$$
Momentum Density:
$$\mathbf{P} = \frac{1}{c^2}\mathbf{S}$$
Energy per unit volume:
$$u = \frac{1}{2}\left(\varepsilon|\mathbf{E}|^2 + \frac{|\mathbf{B}|^2}{\mu}\right)$$
Intensity:
$$I = \langle \mathbf{S} \cdot \hat{\mathbf{n}} \rangle$$
Potentials:
$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{B}$$
Gauge Transformation:
$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda \qquad V' = V - \frac{\partial \Lambda}{\partial t}$$
Lorentz Gauge:
$$\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0$$
Lorentz Transformation:
$$\mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu \mathbf{J}$$
$$\mu\varepsilon \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\varepsilon}$$

Plane Waves

$$\mathbf{E} = \mathbf{E}_0 e^{i(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}} - \omega t)} \qquad \mathbf{B} = \mathbf{B}_0 e^{i(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}} - \omega t)}$$
Phase and Group Velocity:
$$v_p = \frac{\omega}{k} \qquad v_g = \frac{d\omega}{dk}$$

Magnetic Field from Electric Field:
$$\mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega}$$
Intensity:
$$I = \frac{1}{2} v \varepsilon E_0^2$$
Law of Refraction:
$$\theta_I = \theta_R$$
Snell's Law:
$$n_I \sin\theta_I = n_T \sin\theta_T$$
Complex Index of Refraction:
$$n \approx \sqrt{\varepsilon_r} = n_R + i n_I$$
Reflection and Transmission Coefficients:
$$R = \frac{I_R}{I_I} = \frac{I_R^{\text{beam}} \cos\theta_R}{I_I^{\text{beam}} \cos\theta_I} = \frac{I_R^{\text{beam}}}{I_I^{\text{beam}}}$$
$$T = \frac{I_T}{I_I} = \frac{I_T^{\text{beam}} \cos\theta_T}{I_I^{\text{beam}} \cos\theta_I}$$
Misc. Constants:
$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Boundary Conditions

- $\varepsilon_1 \mathbf{E}_1^{\perp} = \varepsilon_2 \mathbf{E}_2^{\perp}$
- $\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}$
- $\mathbf{B}_1^{\perp} = \mathbf{B}_2^{\perp}$
- $\dfrac{\mathbf{B}_1^{\parallel}}{\mu_1} = \dfrac{\mathbf{B}_2^{\parallel}}{\mu_2}$

E Parallel to Plane of Incidence

$$\mathbf{k}_I = k_I(\cos\theta_I \hat{\mathbf{z}} + \sin\theta_I \hat{\mathbf{x}})$$
$$\mathbf{E}_I = E_I e^{i[k_I(\cos\theta_I z + \sin\theta_I x) - \omega t]}(\cos\theta_I \hat{\mathbf{x}} - \sin\theta_I \hat{\mathbf{z}})$$
$$\mathbf{k}_R = k_I(-\cos\theta_R \hat{\mathbf{z}} + \sin\theta_R \hat{\mathbf{x}})$$
$$\mathbf{E}_R = E_R e^{i[k_R(-\cos\theta_R z + \sin\theta_R x) - \omega t]}(\cos\theta_R \hat{\mathbf{x}} + \sin\theta_R \hat{\mathbf{z}})$$
$$\mathbf{k}_T = k_T(\cos\theta_T \hat{\mathbf{z}} + \sin\theta_T \hat{\mathbf{x}})$$
$$\mathbf{E}_T = E_T e^{i[k_T(\cos\theta_T z + \sin\theta_T x) - \omega t]}(\cos\theta_T \hat{\mathbf{x}} - \sin\theta_T \hat{\mathbf{z}})$$
Field Magnitudes (Fresnel Equations):
$$\frac{E_R}{E_I} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{\alpha + \beta}\right)$$
Reflection and Transmission Coefficients:
$$R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 \qquad T = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2$$
Brewster's Angle (no reflected wave, $\alpha = \beta$):
$$\tan\theta_B = n_2/n_1$$

E Perpendicular to Plane of Incidence

$$\mathbf{k}_I = k_I(\cos\theta_I \hat{\mathbf{z}} + \sin\theta_I \hat{\mathbf{x}})$$
$$\mathbf{B}_I = B_I e^{i[k_I(\cos\theta_I z + \sin\theta_I x) - \omega t]}(-\cos\theta_I \hat{\mathbf{x}} + \sin\theta_I \hat{\mathbf{z}})$$
$$\mathbf{k}_R = k_I(-\cos\theta_R \hat{\mathbf{z}} + \sin\theta_R \hat{\mathbf{x}})$$
$$\mathbf{B}_R = B_R e^{i[k_R(-\cos\theta_R z + \sin\theta_R x) - \omega t]}(\cos\theta_R \hat{\mathbf{x}} + \sin\theta_R \hat{\mathbf{z}})$$
$$\mathbf{k}_T = k_T(\cos\theta_T \hat{\mathbf{z}} + \sin\theta_T \hat{\mathbf{x}})$$
$$\mathbf{B}_T = B_T e^{i[k_T(\cos\theta_T z + \sin\theta_T x) - \omega t]}(-\cos\theta_T \hat{\mathbf{x}} + \sin\theta_T \hat{\mathbf{z}})$$
Field Magnitudes (Fresnel Equations):
$$\frac{E_R}{E_I} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{1 + \alpha\beta}\right)$$
Reflection and Transmission Coefficients:
$$R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2 \qquad T = \alpha\beta\left(\frac{2}{1 + \alpha\beta}\right)^2$$

Electromagnetic Waves in Ohmic Conductors

(Follow are applicable only to normal incidence waves with **E** perpendicular to plane of incidence)

Wave Number:
$$\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \qquad \tilde{k} = k + i/\delta = (n_R + i n_I)\omega/c$$
$$k = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}$$
$$\frac{1}{\delta} = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$$
Good and Poor Conductors:
$$\delta_{\text{good}} \approx \sqrt{\frac{2}{\sigma\mu\omega}} \qquad \delta_{\text{poor}} \approx \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$
Field Magnitudes (Fresnel Equations):
$$\frac{E_R}{E_I} = \left(\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}}\right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{1 + \tilde{\beta}}\right) \qquad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$
Reflection and Transmission Coefficients:
$$R = \left|\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}}\right|^2 \qquad T = 1 - \left|\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}}\right|^2$$

Polarization of Hydrogen

Complex permittivity for N/2 molecules of H_2.
$$\varepsilon = \varepsilon_0\left(1 + \frac{Nq^2/m\varepsilon_0}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)$$
For optical materials (small complex index of refraction):
$$\text{Re}\{n\} = 1 + \frac{Nq^2}{2m\varepsilon_0}\left(\frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2 - \gamma^2\omega^2}\right)$$
$$\text{Im}\{n\} = \frac{Nq^2}{2m\varepsilon_0}\frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$
Cauchy's Formula:
$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right)$$

Dilute Plasmas

Conductivity:

$$\sigma_{\text{plasma}} \approx \frac{iN_e q^2/m}{\omega}$$

Wave Number:

$$\tilde{k}^2 = \mu_0\varepsilon_0\omega^2 - \frac{\mu_0 N_e q^2}{m} = \frac{\omega^2 - \omega_p^2}{c^2}$$

Plasma Frequency:

$$\omega_p = \sqrt{\frac{c^2\mu_0 N_e q^2}{m}}$$

Phase and Group Velocities:

$$v_p = \frac{c\omega}{\sqrt{\omega^2 - \omega_p^2}} \qquad v_g = \frac{c\sqrt{\omega^2 - \omega_p^2}}{\omega}$$

Index of Refraction:

$$n = \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2}$$

Critical Angle (angle for which $\theta_T = \pi/2$):

$$\sin\theta_C = \frac{n_T}{n_I} = \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2}$$

Rectangular Wave-Guide

Dispersion Relation:

$$v^2 k^2 = \omega^2 - \omega_{mn}^2$$

Frequencies:

$$\omega_{mn} = v\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

z-components:

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{i(kz-\omega t)}$$

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{i(kz-\omega t)}$$

x-components:

$$E_x = \frac{iv^2}{\omega_{mn}^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$B_x = \frac{iv^2}{\omega_{mn}^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{v^2}\frac{\partial E_z}{\partial y}\right)$$

y-components:

$$E_y = \frac{iv^2}{\omega_{mn}^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_y = \frac{iv^2}{\omega_{mn}^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{v^2}\frac{\partial E_z}{\partial x}\right)$$

Intensity:

$$\langle \mathbf{S}\cdot\hat{\mathbf{z}}\rangle = \langle \tfrac{1}{\mu}\left(E_x B_y - E_y B_x\right)\rangle$$

Transmission Lines

Fields of a coaxial-cable:

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon s}e^{i(kz-\omega t)}\hat{\mathbf{s}} \qquad \mathbf{B} = \frac{\lambda}{2\pi\varepsilon s v}e^{i(kz-\omega t)}\hat{\phi}$$

Inductance and Capacitance per unit length:

$$C\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial z} \qquad \frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} - RI$$

Impedance of a perfectly conductive transmission line:

$$Z = \sqrt{L/C}$$

Conservation of Charge:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{J} \qquad \frac{d\lambda}{dt} + \frac{dI}{dz} = 0$$

Vector Derivatives

Cartesian

$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$ $\qquad d\tau = dx\,dy\,dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Spherical

$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} +$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Cylindrical

$d\tau = s\, ds\, d\phi\, dz$

Gradient:

$$\nabla f = \frac{\partial f}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \mathbf{v} =$$

$$\left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Fundamental Theorems

Fundamental Theorem of Line Integrals:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence Theorem:

$$\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$$

Stoke's Theorem:

$$\int (\nabla \times \mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$$

Vector Identities

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}$$

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Spherical Coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$$
$$\phi = \tan^{-1}(y/x)$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

Cylindrical Coordinates

$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$s = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

$$\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

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https://github.com/DonneyF/formula-sheets