PHYS 408 Formula Sheet

Differential Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Wave Optics

Monochromatic Wave:
$$\mathcal{E} = \mathrm{Re}\{\mathbf{E}(\mathbf{r}) e^{-i\omega t}\}$$
Helmholtz Equation:
$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$
Plane wave solution to Helmholtz Equation:
$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$$
Spherical wave solution to Helmholtz Equation:
$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \frac{1}{r} e^{ikr} \qquad \mathbf{E}_0 = E_0 \hat{\theta}$$
Interference of multiple fields:
$$I = \sum_{k=1}^{N} I_k + v\epsilon\, \mathrm{Re}\{E_1 E_2^* + E_1 E_3^* + \cdots + E_{N-1} E_N^*\}$$
$$v\epsilon E_i E_j^* = 2\sqrt{I_i I_j} \exp[i(\theta_i - \theta_j)]$$
Fresnel (Paraxial) Approximation:
$$\sqrt{x^2 + y^2} \ll z$$
Paraxial Wave Equation:
$$\nabla_T^2 \mathbf{E} + i2k \frac{\partial \mathbf{E}}{\partial z} = \mathbf{0} \qquad \nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
Fresnel Approximation of a spherical wave:
$$\mathbf{E}(\mathbf{r}) \approx \frac{\mathbf{E}_0}{z} \exp\left[ik \frac{x^2 + y^2}{2z}\right] = \frac{\mathbf{E}_0}{z} \exp\left[ik \frac{\rho^2}{2z}\right]$$
Grating Equation:
$$\sin\theta_m = \sin\theta + m\lambda/d$$

Beam Optics

Gaussian beam with Rayleigh length z_0:
$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \frac{w_0}{w(z)} \exp\left[-\frac{\rho^2}{(w(z))^2}\right] \times$$
$$\exp\left[ik \frac{\rho^2}{2R(z)}\right] \exp\left[ikz - i\tan^{-1}\left(\frac{z}{z_0}\right)\right]$$
$$q(z) = z - iz_0$$
$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi} \frac{1}{(w(z))^2}$$
Beam waist:
$$w_0 = \sqrt{\frac{\lambda z_0}{\pi}}$$
Beam Radius:
$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$
Radius of curvature:
$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$
Gaussian beam intensity with center intensity I_0:
$$I(z, \rho) = I_0 \left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2\rho^2}{(w(z))^2}\right]$$
Gaussian beam total power:
$$P = \tfrac{I_0}{2} \pi w_0^2$$
Depth of focus (confocal parameter):
$$d = 2z_0 = 2\pi w_0^2/\lambda$$

Propagation of Paraxial Waves

ABCD Matrix of an optical system:
$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$
Free space propagation transfer matrix:
$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$
Refraction transfer matrix:
$$\begin{bmatrix} 1 & 0 \\ 0 & n_I/n_T \end{bmatrix}$$
Curved interface transfer matrix:
$$\begin{bmatrix} 1 & 0 \\ -\frac{n_T - n_I}{n_I R} & n_I/n_T \end{bmatrix}$$
Thin lens transfer matrix:
$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$
Gaussian Beam propagation in the paraxial regime:
$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

Fourier Optics

Fraunhofer (far field) limit:
$$x^2/\lambda, y^2/\lambda \ll z/\pi$$
Free space transfer function along $\mathbf{z}$:
$$\tilde{H}(k_x, k_y) = e^{ik_z z} \approx e^{ikz} e^{-i(k_x^2 + k_y^2))z/2k}$$
Free space propagation in paraxial limit:
$$E(x', y', z) \approx \frac{e^{ikz}}{i\lambda z} \exp\left[ik \frac{x'^2 + y'^2}{2z}\right] \times$$
$$\iint E(x, y, z) \exp\left[ik \frac{x^2 + y^2}{2z}\right] \exp\left[-\frac{ik}{z}(x'x + y'y)\right] dx\, dy$$
Free space propagation in Fraunhofer limit:
$$E(x', y', z) \approx \frac{e^{ikz}}{i\lambda z} \exp\left[ik \frac{x'^2 + y'^2}{2z}\right] \mathcal{F}\{E(x, y, z)\}$$
Thin lens thickness function:
$$\Delta(x, y) =$$
$$\Delta_0 - R_1\left(1 - \sqrt{1 - \frac{x^2 + y^2}{R_1^2}}\right) + R_2\left(1 - \sqrt{1 - \frac{x^2 + y^2}{R_2^2}}\right)$$
Thin lens thickness function in paraxial limit:
$$\Delta(x, y) \approx \Delta_0 - \tfrac{x^2 + y^2}{2}\left(\tfrac{1}{R_1} - \tfrac{1}{R_2}\right)$$
Thin lens phase function:
$$t_L(x, y) = \exp[ikn\Delta(x, y)] \exp[ik(\Delta_0 - \Delta(x, y))]$$
Thin lens phase function (ignore constant phase):
$$t_L(x, y) = \exp\left[-i\tfrac{k}{2f}(x^2 + y^2)\right]$$

Polarization Optics

E Parallel to Plane of Incidence

Field Magnitudes (Fresnel Equations):
$$\frac{E_R}{E_I} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{\alpha + \beta}\right)$$
Reflection and Transmission Coefficients:
$$R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2 \qquad T = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2$$
Brewster's Angle (no reflected wave, $\alpha = \beta$):
$$\tan\theta_B = n_2/n_1$$

E Perpendicular to Plane of Incidence

Field Magnitudes (Fresnel Equations):
$$\frac{E_R}{E_I} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right) \qquad \frac{E_T}{E_I} = \left(\frac{2}{1 + \alpha\beta}\right)$$
Reflection and Transmission Coefficients:
$$R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2 \qquad T = \alpha\beta\left(\frac{2}{1 + \alpha\beta}\right)^2$$

Basic Polarization States

Linear Polarization Vector:
$$\mathbf{E}_0 = E_{0x} e^{i\phi_x} \hat{\mathbf{x}} + E_{0y} e^{i\phi_y} \hat{\mathbf{y}} \qquad \phi = \phi_x - \phi_y \in \{0, \pi, -\pi\}$$
Linear Polarization Angle ψ relative to x-axis:
$$\tan\psi = E_{0x}/E_{0y}$$
Circular Polarization Vector ($+i$ is counter-clockwise):
$$\mathbf{E}_0 = E_0 \hat{\mathbf{x}} \pm iE_0 \hat{\mathbf{y}}$$
Elliptical Polarization Vector:
$$\mathbf{E}_0 = E_{0x} e^{i\phi_x} \hat{\mathbf{x}} + E_{0y} e^{i\phi_y} \hat{\mathbf{y}}$$
Elliptical Polarization Ellipticity:
$$\tan\chi = b/a$$

Jones Vectors

$$\mathbf{E}_0 = E_{0x}\hat{\mathbf{x}} + E_{0y}e^{i\phi}\hat{\mathbf{y}} = E_0 \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Linear Polarization:

$$\hat{\mathbf{e}}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{\mathbf{e}}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \hat{\mathbf{e}}_\alpha = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

Circular Polarization:

$$\hat{\mathbf{e}}_{\mathrm{CW}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \hat{\mathbf{e}}_{\mathrm{CCW}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Elliptical Polarization:

$$\hat{\mathbf{e}}_e = \frac{1}{\sqrt{2}}(\sin\chi + \cos\chi)e^{-i\psi}\hat{\mathbf{e}}_{\mathrm{CW}} + \frac{1}{\sqrt{2}}(\sin\chi - \cos\chi)e^{i\psi}\hat{\mathbf{e}}_{\mathrm{CCW}}$$

Jones Matrices

$$\mathbf{E_2} = T\mathbf{E_1}$$

Linear Polarizer in x-y basis:

$$T_p^{(x)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad T_p^{(y)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear Polarizer at angle α relative to x-axis:

$$T_p^{(\alpha)} = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$$

Arbitrary Phase Retarder in x-y basis:

$$T_\phi^{(x)} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

Phase retarder with fast axis at angle α relative to x-axis:

$$T_\phi^{(\alpha)} = e^{-\frac{i\phi}{2}} \begin{bmatrix} \cos^2\alpha + e^{i\phi}\sin^2\alpha & (1-e^{i\phi})\cos\alpha\sin\alpha \\ (1-e^{i\phi})\cos\alpha\sin\alpha & \sin^2\alpha + e^{i\phi}\cos^2\alpha \end{bmatrix}$$

Rotate x-y basis by an arbitrary angle:

$$T^{(\theta)} = R_\theta^{-1} T R_\theta \qquad R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Normal Modes:

$$T_{\mathrm{total}}\mathbf{E}_0 = \lambda_{1,2}\mathbf{E}_0$$

Stokes Vectors

$$\mathbf{E}_0 = E_{0x}\hat{\mathbf{x}} + E_{0y}e^{i\phi}\hat{\mathbf{y}}$$
$$S_0 = I_{\mathrm{tot}} = \langle E_{0x}^2 + E_{0y}^2 \rangle$$
$$S_1 = I_{\mathrm{pol}}\cos(2\psi)\cos(2\chi) = \langle E_{0x}^2 - E_{0y}^2 \rangle$$
$$S_2 = I_{\mathrm{pol}}\sin(2\psi)\cos(2\chi) = \langle 2E_{0x}E_{0y}\cos\phi \rangle$$
$$S_3 = I_{\mathrm{pol}}\sin(2\chi) = \langle 2E_{0x}E_{0y}\sin\phi \rangle$$

Polarization Degree:

$$\frac{I_{\mathrm{pol}}}{I_{\mathrm{tot}}} = \frac{1}{S_0}\sqrt{S_1^2 + S_2^2 + S_3^2}$$

Optical Cavities

ABCD matrix approach:

$$\begin{bmatrix} U_2^+ \\ U_2^- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U_1^+ \\ U_1^- \end{bmatrix} = M_{21} \begin{bmatrix} U_1^+ \\ U_1^- \end{bmatrix}$$

Two interfaces:

$$M_{21} = \frac{1}{2n_2} \begin{bmatrix} n_2 + n_1 & n_2 - n_1 \\ n_2 - n_1 & n_2 + n_1 \end{bmatrix}$$

Mirror:

$$M_m = \frac{i}{\sqrt{T}} \begin{bmatrix} 1 & -\sqrt{R} \\ \sqrt{R} & -1 \end{bmatrix} \qquad t = i\sqrt{T}, r = \sqrt{R}$$

Fabry-Perot Cavity (two mirrors a distance d apart):

$$M_{\mathrm{FP}} = \frac{1}{t_1^*} \begin{bmatrix} 1 & -r_1 \\ r_1 & -1 \end{bmatrix} \begin{bmatrix} e^{ik_0 d} & 0 \\ 0 & e^{-ik_0 d} \end{bmatrix} \frac{1}{t_2^*} \begin{bmatrix} 1 & -r_2 \\ r_2 & -1 \end{bmatrix}$$

$$t_{\mathrm{FP}} = \frac{t_1 t_2 e^{i\varphi}}{1 - r_1 r_2 e^{2i\varphi}} \qquad \varphi = k_0 d$$

$$T_{\mathrm{FP}} = [1 + (2\mathcal{F}/\pi)^2 \sin^2\varphi]^{-1}$$

Finesse:

$$\mathcal{F} = \frac{\pi\sqrt{R}}{1-R}$$

Free Spectral Range:

$$\nu_{\mathrm{FSR}} = c/2d$$

Spectral line width (FWHM):

$$\Delta\nu \approx \nu_{\mathrm{FSR}}/\mathcal{F}$$

Cavity Uncertainty Relation:

$$\Delta t \Delta f = \tau_\gamma \Delta\nu = 1/2\pi$$

Quality Factor:

$$Q = 2\pi \frac{\text{stored energy}}{\text{energy loss per oscillation cycle}} = q\mathcal{F} \quad q \in \mathbb{Z}$$

Resonator Stability:

$$0 \le (1 + d/R_1)(1 + d/R_2) \le 1$$

Resonant frequencies in a stable cavity:

$$\nu_{q,l,m} = \nu_{\mathrm{FSR}}\left[q + \tfrac{1}{\pi}(1 + l + m)\cos^{-1}(g_1 g_2)\right]$$

Gaussian beam parameters:

$$z_0^2 = \frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2g_1 g_2)^2} \qquad w_0^2 = \frac{d\lambda}{\pi} \frac{\sqrt{g_1 g_2 (1 - g_1 g_2)}}{g_1 + g_2 - 2g_1 g_2}$$

Lasers

Einstein coefficients in a two level system:

Spontaneous emission: Rate = AN_2

Absorption (stimulated): Rate = $B_{12}N_1\rho(\nu_{12})$

Stimulated emission: Rate = $B_{21}N_2\rho(\nu_{12})$

Spontaneous Emission ($d_{12} = \langle 1| e\hat{\mathbf{x}} |2\rangle$):

$$A = \frac{1}{\tau_{sp}} = \gamma_{sp} = \frac{1}{4\pi\varepsilon_0} \frac{4d_{12}^2\omega_0^3}{4\hbar c^3}$$

Einstein B coefficient (Blackbody radiation):

$$A/B = 8\pi h/\lambda^3$$

Gain Coefficient:

$$\tfrac{dI}{dz} = g(\omega)I$$

Absorption cross-section:

$$\sigma(\omega) = \tfrac{B\hbar\omega}{c\Delta\omega} = \tfrac{A\lambda^2}{4\Delta\omega}$$

Absorption & emission rate:

$$B\rho(\omega) = \tfrac{I\sigma}{\hbar\omega}$$

Saturation Intensity:

$$I_{\mathrm{sat}} = \hbar\omega/\sigma\tau$$

Pumping rate:

$$R_p = B\rho(\omega_p) = \tfrac{I_p\sigma(\omega_p)}{\hbar\omega_p}$$

Single-pass gain:

$$\ln\left(\frac{I_2}{I_1}\right) + \frac{I_2 - I_1}{I_{\mathrm{sat}}} = g_0 l_g$$

Four Level System Lasing:

$$N_2 = \frac{R_p\tau}{1 + R_p\tau + I(\omega_l)/I_{\mathrm{sat}}(\omega_l)} N_{\mathrm{tot}}$$

Gain medium in a cavity

Steady state, losses compensated:

$$I_2/I_1 \approx 1 + (A + T_{\mathrm{oc}})$$

Output intensity:

$$I_{\mathrm{out}} = I_{\mathrm{sat}}T_{\mathrm{oc}}\left[\tfrac{g_o l_g}{A + T_{\mathrm{oc}}} - 1\right]$$

Optimal output coupler:

$$T_{\mathrm{oc}} = \sqrt{g_o l_g A} - A$$

Density of cavity modes under gain bandwidth:

$$N_{\mathrm{cav}} = \tfrac{8\pi}{\lambda^3\omega}\Delta\omega$$

Cavity decay rate for gain medium loss a:

$$\gamma_c = ca$$

Threshold inversion:

$$N_t = \tfrac{\gamma_c}{\gamma}N_{\mathrm{cav}}$$

Normalized pumping rate:

$$p = \tfrac{P}{\gamma N_t}$$

Ultrafast Optics

Chirped pulse width:

$$\delta t \approx \tfrac{\phi''(\omega)}{\delta t_0}$$

Velocity:

$$v_\phi = \omega_0/k_0 \qquad v_g = 1/k_0'$$

Group Delay Dispersion:

$$\delta t(z) \approx k_0'' z/\delta t_0$$

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https://github.com/DonneyF/formula-sheets