PHYS 304 Formula Sheet

The Wave Function

Time dependent Schrodinger Equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

Standard Deviation:

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2}$$

Momentum:

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \int \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\,dx$$

Infinite Square Well

Time Independent Schrodinger Equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$$

Eigenstate Expansion:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} = \sum_{n=1}^{\infty} c_n\Psi_n(x,t)$$

Energy In Infinite Square Well:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Stationary States:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

Determining Coefficients:

$$c_n = \sqrt{\frac{2}{a}}\int_0^a \sin\left(\frac{n\pi}{a}x\right)\Psi(x,0)\,dx$$

Expectation Value of Energy:

$$\langle H\rangle = \textstyle\sum_{n=1}^{\infty} |c_n|^2 E_n$$

Harmonic Oscillator

$$k = \omega^2 m$$

Ladder Operators:

$$a_\pm = \tfrac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$
$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$$
$$a_-\psi_n = \sqrt{n}\,\psi_{n-1}$$

Operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$$
$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$
$$x^2 = \frac{\hbar}{2m\omega}\left[(a_+)^2 + (a_+a_-) + (a_-a_+) + (a_-)^2\right]$$

Commutation:

$$[x,p] = i\hbar$$
$$[a_-,a_+] = 1$$
$$[AB,C] = A[B,C] + [A,C]B$$

Hamiltonian:

$$H = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)$$
$$a_+a_- + a_-a_+ = 2\left(\frac{H}{\hbar\omega}\right)$$

States:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$
$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0$$

Energy:

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$

Generalized Statistical Interpretation

Momentum Expansion:

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi(x,t)\,dx$$
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{ipx/\hbar}\,\Phi(p,t)\,dp$$

Angular Momentum

$$L_x = yp_z - zp_y \qquad L_y = zp_x - xp_z \qquad L_z = xp_y - yp_x$$

Commutators:

$$[L_x,L_y] = i\hbar L_z \qquad [L_y,L_z] = i\hbar L_x \qquad [L_z,L_x] = i\hbar L_y$$
$$[L_z,x] = i\hbar y \qquad [L_z,y] = -i\hbar x \qquad [L_z,z] = 0$$
$$[L_z,p_x] = i\hbar p_y \qquad [L_z,p_y] = -i\hbar p_x \qquad [L_z,p_z] = 0$$

Square of Angular Momentum:

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2 \qquad [L^2,\mathbf{L}] = 0$$

Ladder Operators:

$$L_\pm = L_x \pm iL_y \qquad [L_z,L_\pm] = \pm i\hbar L_\pm$$
$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$
$$L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2i}$$
$$L_+|l,l\rangle = 0 \qquad L_-|l,-l\rangle = 0$$

Eigenvalues:

$$L^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle$$
$$L_z|l,m\rangle = \hbar m|l,m\rangle$$
$$L_\pm|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\,|l,m\pm1\rangle$$

Expectation Value when $L_x|\psi\rangle = L_y|\psi\rangle$:

$$\langle\psi|L_x^2|\psi\rangle = \langle\psi|L_y^2|\psi\rangle = \frac{1}{2}\langle\psi|L^2 - L_z^2|\psi\rangle$$

Spherical Coordinate Representation:

$$L_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$
$$L_y = \frac{\hbar}{i}\left(+\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$
$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\theta}$$
$$L_\pm = \pm\hbar e^{\pm i\theta}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$
$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)$$
$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Spin

Commutators:

$$[S_x,S_y] = i\hbar S_z \qquad [S_y,S_z] = i\hbar S_x \qquad [S_z,S_x] = i\hbar S_y$$

Eigenvalues:

$$S^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$$
$$S_z|s,m\rangle = \hbar m|s,m\rangle$$
$$S_\pm|s,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\,|s,m\pm1\rangle$$

Square of Angular Momentum:

$$S^2 \equiv S_x^2 + S_y^2 + S_z^2 \qquad [S^2,\mathbf{S}] = 0$$

Spin 1/2

Spinors & Eigenspinors:

$$\chi = \begin{bmatrix}a\\b\end{bmatrix},\ |a|^2 + |b|^2 = 1 \qquad \chi_+^z = \begin{bmatrix}1\\0\end{bmatrix} \qquad \chi_-^z = \begin{bmatrix}0\\1\end{bmatrix}$$
$$\chi_+^x = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad \chi_-^x = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$
$$\chi_+^y = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} \qquad \chi_-^y = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

Spin Operators:

$$\mathbf{S}_x = \frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix} \qquad \mathbf{S}_y = \frac{\hbar}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix} \qquad \mathbf{S}_z = \frac{\hbar}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$
$$\mathbf{S}^2 = \frac{3}{4}\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix} \qquad \mathbf{S}_+ = \hbar\begin{bmatrix}0&1\\0&0\end{bmatrix} \qquad \mathbf{S}_- = \hbar\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

Spin along $\hat{\mathbf{r}}$:

$$\mathbf{S}_r = \mathbf{S}\cdot\hat{\mathbf{r}} = \frac{\hbar}{2}\begin{bmatrix}\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & -\cos\theta\end{bmatrix}$$
$$\chi_+^r = \begin{bmatrix}\cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\end{bmatrix} \qquad \chi_-^r = \begin{bmatrix}e^{-i\phi}\sin(\theta/2)\\ -\cos(\theta/2)\end{bmatrix}$$

Lamor Precession:

$$\chi(t) = \begin{bmatrix}\cos(\alpha/2)e^{i\gamma B_0t/2}\\ \sin(\alpha/2)e^{-i\gamma B_0t/2}\end{bmatrix}$$

The Hydrogen Atom

$n = j_{\max} + l + 1$

Wave Function: $\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi)$

Radial Equation:

$$R_{nl}(r) = \frac{1}{r}\rho^{l+1}e^{-\rho}v(\rho) \qquad \rho = \frac{r}{an}$$

$$v(\rho) = \sum_0^{j_{\max}} c_j\rho^j \qquad c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)}c_j$$

Energy:

$$E_n = \frac{E_1}{n^2} \qquad E_1 = -13.6 \text{ eV}$$

Addition of Angular Momenta

Possible values of total spin s:

$$s = (s_1+s_2), (s_1+s_2-1), \ldots, |s_1-s_2|$$

Operators:

$$S^2 = (S^{(1)})^2 + (S^{(2)})^2 + 2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}$$

$$2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} = S_+^{(1)}S_-^{(2)} + S_-^{(1)}S_+^{(2)} + S_z^{(1)}S_z^{(2)}$$

$$\left[S^2, \mathbf{S}^{(1)}\right] = 2i\hbar(\mathbf{S}^{(1)}\times\mathbf{S}^{(2)})$$

Combined state with total spin s, and z-component m:

$$|s,m\rangle = \sum_{m_1+m_2=m} C_{m_1m_2m}^{s_1s_2s}\,|s_1,m_1\rangle\,|s_2,m_2\rangle$$

$$|s_1,m_1\rangle\,|s_2,m_2\rangle = \sum_s C_{m_1m_2m}^{s_1s_2s}\,|s,m\rangle$$

Trig Identities

$$2\cos\theta\cos\phi = \cos(\theta-\phi) + \cos(\theta+\phi)$$
$$2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi)$$
$$2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi)$$
$$2\cos\theta\sin\phi = \sin(\theta+\phi) - \sin(\theta-\phi)$$

Integral Identities

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax + C$$

$$\int \sin^3 ax\,dx = \frac{\cos 3ax}{12a} - \frac{3\cos ax}{4a} + C$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax + C$$

$$\int \cos^3 ax\,dx = \frac{\sin ax}{a} - \frac{\sin^3 ax}{3a} + C$$

$$\int \cos^n(ax)\sin(ax)\,dx = -\frac{\cos^{n+1}(ax)}{a(n+1)} + C$$

$$\int \sin^n(ax)\cos(ax)\,dx = \frac{\sin^{n+1}(ax)}{a(n+1)} + C$$

$$\int_0^\pi \cos^{2n+1}(ax)\sin(ax)\,dx = 0 \qquad n = 0,1,2\ldots$$

$$\int_0^\pi \cos^{2n}(x)\sin(x)\,dx = \frac{2}{2n+1} \qquad n = 0,1,2\ldots$$

$$\int_0^\pi \sin^n(ax)\cos(ax)\,dx = 0 \qquad n = 0,1,2\ldots$$

$$\int_0^\infty x^n e^{-x/a}\,dx = n!a^{n+1}$$

$$\int_0^\infty x^{2n}e^{-x^2/a^2}\,dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$$

$$\int_0^\infty x^{2n+1}e^{-x^2/a^2}\,dx = \frac{n!}{2}a^{2n+2}$$

Spherical Harmonics

l	m	$Y_l^m(\theta,\phi)$
0	0	$Y_0^0 = \sqrt{\frac{1}{4\pi}}$
1	0	$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$
	±1	$Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$
2	0	$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$
	±1	$Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}$
	±2	$Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi}$
3	0	$Y_3^0 = \sqrt{\frac{7}{16\pi}}(5\cos^3\theta - 3\cos\theta)$
	±1	$Y_3^{\pm1} = \mp\sqrt{\frac{21}{64\pi}}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$
	±2	$Y_3^{\pm2}\sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta e^{\pm2i\phi}$
	±3	$Y_3^{\pm3} = \mp\sqrt{\frac{35}{64\pi}}\sin^3\theta e^{\pm3i\phi}$

Hydrogen Radial Wave Functions

$$R_{10} = 2a^{-3/2}e^{-r/a}$$

$$R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-r/2a}$$

$$R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\frac{r}{a}e^{-r/2a}$$

$$R_{30} = \frac{2}{\sqrt{27}}a^{-3/2}\left(1 - \frac{2}{3}\frac{r}{a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right)e^{-r/3a}$$

$$R_{31} = \frac{8}{27\sqrt{6}}a^{-3/2}\left(1 - \frac{1}{6}\frac{r}{a}\right)\left(\frac{r}{a}\right)e^{-r/3a}$$

$$R_{32} = \frac{4}{81\sqrt{30}}a^{-3/2}\left(\frac{r}{a}\right)^2 e^{-r/3a}$$

Updated April 10, 2019
https://github.com/DonneyF/formula-sheets