ELEC 221 Formula Sheet

Continuous Time Signals

Even and Odd Components:

$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

A signal is periodic with fundamental period T_0 if

$x(t + kT_0) = x(t) \quad \forall t \in (-\infty, \infty)$

Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Power:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

Causality:

- Causal: $x(t) = 0$ for $t < 0$.
- Anti-Causal: $x(t) = 0$ for $t \geq 0$.
- A-Causal or Non-Causal: Both of the above.

Continuous Time Systems

Dynamic systems have memory. Active systems can deliver energy to the outside world.

Linearity:

$\mathcal{S}[\alpha x(t) + \beta y(t)] = \alpha \mathcal{S}[x(t)] + \beta \mathcal{S}[y(t)]$

Time Invariance:

If $\mathcal{S}[x(t)] = y(t)$ then $\mathcal{S}[x(t \pm \tau)] = y(t \pm \tau)$.

Zero-State Response:

Due to the input as the initial conditions are zero.

Zero-Input Response:

Due to the initial conditions as the input is zero.

Convolution Integral:

$$[x * y](t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\, d\tau$$

Total Response from Impulse Response:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Causality:

A continuous time system $\mathcal{S}$ is causal if whenever $x(t) = 0$ and there are no initial conditions, $y(t) = 0$ and the output $y(t)$ does not depend on future inputs.

Bounded-Input Bounded-Output Stability:

If an input $x(t)$ bounded then the output of an BIBO system is also bounded.

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Laplace Transform

$s = \sigma + j\omega$

Eigenfunction Property:

$\mathcal{S}[e^{s_0 t}] = H(s_0)e^{s_0 t}$

One Sided Laplace Transform:

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0^-}^{\infty} f(t)e^{-st}\, dt$$

Region of Convergence:

- Finite support $\rightarrow$ Entire s-plane.
- Causal function $\rightarrow \sigma > \max(\sigma_i)$, $-\infty < \omega < \infty$.
- Anti-causal $\rightarrow \sigma < \min(\sigma_i)$, $-\infty < \omega < \infty$.
- Non-causal $\rightarrow \mathcal{R} = \mathcal{R}_{\text{causal}} \cap \mathcal{R}_{\text{anti-causal}}$.

Initial Value Theorem:

$f(0+) \Leftrightarrow \lim_{s\to\infty} sF(s)$

Final Value Theorem:

$\lim_{t\to\infty} f(t) \Leftrightarrow \lim_{s\to 0} sF(s)$

Bounded-Input Bounded-Output Stability:

If the region of convergence contains the $j\omega$-axis, then the system is BIBO stable.

Fourier Series

Fourier analysis in the steady state.

Eigenfunction Property:

$\mathcal{S}[e^{j\omega_0 t}] = H(j\omega_0)e^{j\omega_0 t}$

$$x(t) = \sum_k X_k e^{j\omega_k t} \implies y(t) = \sum_k X_k H(j\omega_k) e^{j\omega_k t}$$
$$= \sum_k X_k |H(j\omega_k)| e^{j(\omega_k t + \angle H(\omega_k))}$$

Fourier Series Coefficients (for any t_0):

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)e^{-jk\omega_0 t}\, dt$$

Parseval's Power Relation (for any t_0):

$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Symmetry of Line Spectra:

$|X_k| = |X_{-k}|$

$\angle X_k = -\angle X_{-k}$

Trigonometric Fourier Series:

$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \Theta_k)$

$x(t) = c_0 + 2\sum_{k=1}^{\infty} [c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)]$

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\cos(k\omega_0 t)\, dt \qquad k = 0,1,2\ldots$$

$$d_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\sin(k\omega_0 t)\, dt \qquad k = 1,2,3\ldots$$

$\Theta_k = -\arctan(d_k/c_k)$

Fourier Coefficients from Laplace Transform:

If $x_1(t)$ is a single period of $x(t)$, then

$$X_k = \frac{1}{T_0} \mathcal{L}[x_1(t)]\Big|_{s=jk\omega_0}$$

Response of LTI Systems to Periodic Signals:

If the input to an LTI system has Fourier Series $x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega t + \angle X_k)$, then the steady state response is $y(t) = X_0|H(j0)| + 2\sum_{k=1}^{\infty} |X_k||H(jk\omega_0)| \cos(k\omega_0 t + \angle X_k + \angle H(jk\omega_0))$

Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

Fourier Transform from Laplace Transform (if $X(s)$ contains the $j\omega$-axis):

$$\mathcal{F}[x(t)] = \mathcal{L}[x(t)]\Big|_{s=j\omega} = X(s)\Big|_{s=j\omega}$$

Duality:

$$X(t) \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi x(-\omega)$$

Fourier Transform of Periodic Signals:

$$\sum_k X_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} \sum_k 2\pi_k \delta(\omega - k\omega_0)$$

Parseval's Energy Relation:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} |X(\omega)|^2\, d\omega$$

Symmetry of Spectral Representations:

$|X(\omega)| = |X(-\omega)|$

$\text{Re}[X(\omega)] = \text{Re}[X(-\omega)]$

$\angle X(\omega) = -\angle X(-\omega)$

$\text{Im}[X(\omega)] = -\text{Im}[X(-\omega)]$

Sampling Theory

$x_s(t) = x(nT_s) = x(t)|_{t=nT_s} = x(t)\sum_n \delta(t - nT_s)$

$X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$

Nyquist-Shannon Sampling Rate:

$\omega_s = \frac{2\pi}{T_s} \geq 2\omega_{\max}$

Aliasing occurs if $\omega_s < 2\omega_{\max}$.

Reconstruction $X(\omega) = X_s(\omega)H_{\text{lp}}(\omega)$:

$$H_{\text{lp}}(\omega) = \begin{cases} T_s & -\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

Signal Reconstruction from Sinc Interpolation:

$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin(\pi(t - nT_s))/T_s}{\pi(t - nT_s)/T_s}$

Discrete Time Signals

Define $x[n] = x(nT_0)$.

$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Periodicity:

$x[n + kN] = x[n] \qquad \forall k \in \mathbb{Z}$

When sampling an analog sinusoid of fundamental period T_0, we obtain a periodic discrete sinusoid provided that m, N not divisible by each-other:

$\frac{T_s}{T_0} = \frac{m}{N}$

Aliasing occurs if:

$T_s > \frac{T_0}{2}$

Energy:

$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$

Power:

$P_x = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$

Convolutional Sum:

$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Bounded-Input Bounded-Output Stability:

$\sum_k |h[n]|^2 < \infty$

Z-Transform

$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n}$

Convergence:

$|X(z)| = \sum_n |x[n]|r^{-n} < \infty$

Updated December 1, 2018
https://github.com/DonneyF/formula-sheets