MECH 280 Formula Sheet

Vectors & Matrices

Inner Product (aka scalar or dot product):

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$

Cross Product (aka vector product):

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \, \hat{\mathbf{n}}$

Triple Product:

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$

Outer Product (aka tensor product):

$$\mathbf{v} \otimes \mathbf{w} \begin{bmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_m \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_m \\ \vdots & \vdots & \ddots & \vdots \\ v_n w_1 & v_n w_2 & \cdots & v_n w_m \end{bmatrix}$$

Vector Derivatives

Cartesian

Gradient:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

Spherical

(θ is the polar angle.)

Gradient:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r \right) + \frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(\sin \theta v_\theta) + \frac{1}{r \sin \theta}\frac{\partial v_\theta}{\partial \phi}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r \sin \theta}\left[\frac{\partial}{\partial \theta}(\sin \theta \, v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right]\hat{\mathbf{r}} +$$

$$\frac{1}{r}\left[\frac{1}{\sin \theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(rv_\phi) \right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial \theta} \right]\hat{\boldsymbol{\phi}}$$

Tensor:

$$\nabla \mathbf{u} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_z}{\partial x} \\[2mm] \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_z}{\partial y} \\[2mm] \dfrac{\partial u_x}{\partial z} & \dfrac{\partial u_y}{\partial z} & \dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

Cylindrical

Gradient:

$$\nabla f = \frac{\partial f}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \mathbf{v} =$$
$$\left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial \phi} \right]\hat{\mathbf{z}}$$

Vector Calculus

Divergence Theorem:

$$\int (\nabla \cdot \mathbf{A}) \, dV = \oint \mathbf{A} \cdot d\mathbf{S}$$

Stoke's Theorem:

$$\int (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{l}$$

Directional Derivative (slope of a scalar field in the direction of the vector $\mathbf{u}$):

$$\nabla f \cdot \mathbf{u}$$

Material Derivative:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho \qquad \frac{\mathrm{D}\mathbf{A}}{\mathrm{D}t} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A}$$

Liebniz Rule:

$$\frac{d}{dt}\int_{\mathrm{CV}(t)} \rho dV = \int_{\mathrm{CV}} \frac{\partial \rho}{\partial t} dt + \rho \int_{\mathrm{CS}} \mathbf{u}_{\mathrm{CS}} \cdot d\mathbf{S}$$

Integral Equations of Motion

Flux:

$$\phi = \int_{\mathrm{CS}} \rho \mathbf{u} \cdot d\mathbf{S}$$

Conservation of Mass:

$$\int_{\mathrm{CV}} \frac{\partial \rho}{\partial t} = - \int_{\mathrm{CS}} \rho \mathbf{u} \cdot d\mathbf{S}$$

Conservation of Momentum:

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho \mathbf{u} dV = - \int_{\mathrm{CS}} (\rho \mathbf{u})\hat{\mathbf{u}} \cdot d\mathbf{S} + \int_{\mathrm{CV}} \mathbf{f}_b \, dV + \int_{\mathrm{CS}} \mathbf{t} \, dS$$

Differential Equations of Motion

Conservation of Mass:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{u} = 0$$

Conservation of Momentum:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{f}_b + \mathbf{f}_c$$