

# PHYS 401 Formula Sheet

## Differential Maxwell's Equations

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

## Integral Maxwell's Equations

$$\begin{aligned}\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} & \oint \mathbf{B} \cdot d\mathbf{a} &= 0 \\ \oint \mathbf{E} \cdot d\mathbf{l} &= -\frac{d\Phi_B}{dt} & \oint \mathbf{B} \cdot d\mathbf{l} &= \mu_0 I_{\text{enc}} + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}\end{aligned}$$

## Electromagnetic Waves

Wave Speed:

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}$$

Poynting Vector:

$$\mathbf{S} = \frac{1}{\mu} \mathbf{E} \times \mathbf{B}$$

Momentum Density:

$$\mathbf{P} = \frac{1}{c^2} \mathbf{S}$$

Energy per unit volume:

$$u = \frac{1}{2} \left( \epsilon |\mathbf{E}|^2 + \frac{|\mathbf{B}|^2}{\mu} \right)$$

Intensity:

$$I = \langle \mathbf{S} \cdot \hat{\mathbf{n}} \rangle$$

Gauge Transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla \Lambda \quad V' = V - \frac{\partial \Lambda}{\partial t}$$

Lorentz Gauge:

$$\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0$$

Lorentz Transformation:

$$\mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu \mathbf{J}$$

$$\mu\epsilon \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\epsilon}$$

## Plane Waves

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

Phase Velocity:

$$v_p = \omega/k$$

Group Velocity:

$$v_g = \frac{d\omega}{dk}$$

Magnetic Field from Electric Field:

$$\mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega}$$

Intensity:

$$I = \frac{1}{2} v \epsilon E_0^2$$

Law of Refraction:

$$\theta_I = \theta_B$$

Snell's Law:

$$n_I \sin \theta_I = n_T \sin \theta_T$$

Misc. Constants:

$$\alpha = \frac{\cos \theta_T}{\cos \theta_I} \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

## Boundary Conditions

- $\epsilon_1 \mathbf{E}_1^\perp = \epsilon_2 \mathbf{E}_2^\perp$
- $\mathbf{E}_1^\parallel = \mathbf{E}_2^\parallel$
- $\mathbf{B}_1^\perp = \mathbf{B}_2^\perp$
- $\frac{\mathbf{B}_1^\parallel}{\mu_1} = \frac{\mathbf{B}_2^\parallel}{\mu_2}$

## E Parallel to Plane of Incidence

$$\begin{aligned}\mathbf{k}_I &= k_I (\cos \theta_I \hat{\mathbf{z}} + \sin \theta_I \hat{\mathbf{x}}) \\ \mathbf{E}_I &= E_I e^{i[k_I (\cos \theta_I z + \sin \theta_I x) - \omega t]} (\cos \theta_I \hat{\mathbf{x}} - \sin \theta_I \hat{\mathbf{z}}) \\ \mathbf{k}_R &= k_I (-\cos \theta_R \hat{\mathbf{z}} + \sin \theta_R \hat{\mathbf{x}}) \\ \mathbf{E}_R &= E_R e^{i[k_R (-\cos \theta_R z + \sin \theta_R x) - \omega t]} (\cos \theta_R \hat{\mathbf{x}} + \sin \theta_R \hat{\mathbf{z}}) \\ \mathbf{k}_T &= k_T (\cos \theta_T \hat{\mathbf{z}} + \sin \theta_T \hat{\mathbf{x}}) \\ \mathbf{E}_T &= E_T e^{i[k_T (\cos \theta_T z + \sin \theta_T x) - \omega t]} (\cos \theta_T \hat{\mathbf{x}} - \sin \theta_T \hat{\mathbf{z}})\end{aligned}$$

Field Magnitudes (Fresnel Equations):

$$\frac{E_R}{E_I} = \left( \frac{\alpha - \beta}{\alpha + \beta} \right) \quad \frac{E_T}{E_I} = \left( \frac{2}{\alpha + \beta} \right)$$

Reflection and Transmission Coefficients:

$$R = \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^2 \quad T = \alpha \beta \left( \frac{2}{\alpha + \beta} \right)^2$$

Brewster's Angle (no reflected wave):

$$\tan \theta_B = n_2/n_1$$

## E Perpendicular to Plane of Incidence

$$\begin{aligned}\mathbf{k}_I &= k_I (\cos \theta_I \hat{\mathbf{z}} + \sin \theta_I \hat{\mathbf{x}}) \\ \mathbf{B}_I &= B_I e^{i[k_I (\cos \theta_I z + \sin \theta_I x) - \omega t]} (-\cos \theta_I \hat{\mathbf{x}} + \sin \theta_I \hat{\mathbf{z}}) \\ \mathbf{k}_R &= k_I (-\cos \theta_R \hat{\mathbf{z}} + \sin \theta_R \hat{\mathbf{x}}) \\ \mathbf{B}_R &= B_R e^{i[k_R (-\cos \theta_R z + \sin \theta_R x) - \omega t]} (\cos \theta_R \hat{\mathbf{x}} + \sin \theta_R \hat{\mathbf{z}}) \\ \mathbf{k}_T &= k_T (\cos \theta_T \hat{\mathbf{z}} + \sin \theta_T \hat{\mathbf{x}}) \\ \mathbf{B}_T &= B_T e^{i[k_T (\cos \theta_T z + \sin \theta_T x) - \omega t]} (-\cos \theta_T \hat{\mathbf{x}} + \sin \theta_T \hat{\mathbf{z}})\end{aligned}$$

Field Magnitudes (Fresnel Equations):

$$\frac{E_R}{E_I} = \left( \frac{1 - \alpha\beta}{1 + \alpha\beta} \right) \quad \frac{E_T}{E_I} = \left( \frac{2}{1 + \alpha\beta} \right)$$

Reflection and Transmission Coefficients:

$$R = \left( \frac{1 - \alpha\beta}{1 + \alpha\beta} \right)^2 \quad T = \alpha\beta \left( \frac{2}{1 + \alpha\beta} \right)^2$$

## Electromagnetic Waves in Ohmic Conductors

Wave Number:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \quad \tilde{k} = k + i/\delta$$

$$k = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon\omega} \right)^2} + 1 \right]^{1/2}$$

$$\frac{1}{\delta} = \omega \sqrt{\frac{\mu\epsilon}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon\omega} \right)^2} - 1 \right]^{1/2}$$

Good and Poor Conductors:

$$\delta_{\text{good}} \approx \frac{2}{\sigma} \sqrt{\frac{\epsilon}{\mu}} \quad \delta_{\text{poor}} \approx \sqrt{\frac{2}{\sigma\mu\omega}}$$

Field Magnitudes (Fresnel Equations):

$$\frac{E_R}{E_I} = \left( \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) \quad \frac{E_T}{E_I} = \left( \frac{2}{1 + \tilde{\beta}} \right) \quad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2$$

Reflection and Transmission Coefficients:

$$R = \left| \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right|^2 \quad T = 1 - \left| \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right|^2$$

Complex Index of Refraction:

$$n \approx \sqrt{\epsilon_r} = n_R + in_I$$

## Polarization of Hydrogen

Complex permittivity for N/2 molecules of H<sub>2</sub>.

$$\epsilon = \epsilon_0 \left( 1 + \frac{Nq^2/m\epsilon_0}{\omega_0^2 - \omega^2 - i\gamma\omega} \right)$$

For optical materials (small complex index of refraction):

$$\text{Re}\{n\} = 1 + \frac{Nq^2}{2m\epsilon_0} \left( \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \right)$$

$$\text{Im}\{n\} = \frac{Nq^2}{2m\epsilon_0} \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Cauchy's Formula:

$$n = 1 + A \left( 1 + \frac{B}{\lambda^2} \right)$$

## Dilute Plasmas

Conductivity:

$$\sigma_{\text{plasma}} \approx \frac{iN_e q^2 / m}{\omega}$$

Wave Number:

$$\tilde{k}^2 = \mu_0 \epsilon_0 \omega^2 - \frac{\mu_0 N_e q^2}{m} = \frac{\omega^2 - \omega_p^2}{c^2}$$

Plasma Frequency:

$$\omega_p = \sqrt{\frac{c^2 \mu_0 N_e q^2}{m}}$$

Phase and Group Velocities:

$$v_p = \frac{c\omega}{\sqrt{\omega^2 - \omega_p^2}} \quad v_g = \frac{c\sqrt{\omega^2 - \omega_p^2}}{\omega}$$

Index of Refraction:

$$n = \frac{1}{\omega} \sqrt{\omega^2 - \omega_p^2}$$

Critical Angle (angle for which  $\theta_T = \pi/2$ ):

$$\sin \theta_C = \frac{n_T}{n_I} = \frac{1}{\omega} \sqrt{\omega^2 - \omega_p^2}$$

## Rectangular Wave-Guide

Dispersion Relation:

$$v^2 k^2 = \omega^2 - \omega_{mn}^2$$

Frequencies:

$$\omega_{mn} = v\pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

z-components:

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{i(kz - \omega t)}$$

$$E_B = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{i(kz - \omega t)}$$

x-components:

$$E_x = \frac{iv^2}{\omega_{mn}^2} \left( k \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right)$$

$$B_x = \frac{iv^2}{\omega_{mn}^2} \left( k \frac{\partial B_z}{\partial x} - \frac{\omega}{v^2} \frac{\partial E_z}{\partial y} \right)$$

y-components:

$$E_y = \frac{iv^2}{\omega_{mn}^2} \left( k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right)$$

$$B_y = \frac{iv^2}{\omega_{mn}^2} \left( k \frac{\partial B_z}{\partial y} + \frac{\omega}{v^2} \frac{\partial E_z}{\partial x} \right)$$

## Transmission Lines

Fields of a coaxial-cable:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon s} e^{i(kz - \omega t)} \hat{\mathbf{s}} \quad \mathbf{B} = \frac{\lambda}{2\pi\epsilon s v} e^{i(kz - \omega t)} \hat{\phi}$$

Inductance and Capacitance per unit length:

$$C \frac{\partial V}{\partial t} = -\frac{\partial I}{\partial z} \quad \frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} - RI$$

Impedance of a perfectly conductive transmission line:

$$Z = \sqrt{L/C}$$

Conservation of Charge:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

## Vector Derivatives

### Cartesian

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}} \quad d\tau = dx dy dz$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

### Spherical

$$d\tau = r^2 \sin \theta dr d\theta d\phi$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$$

### Cylindrical

$$d\tau = s ds d\phi dz$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \mathbf{v} = \left[ \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[ \frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

## Fundamental Theorems

Fundamental Theorem of Line Integrals:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence Theorem:

$$\int_V (\nabla \cdot \mathbf{A}) d\tau = \oint_S \mathbf{A} \cdot d\mathbf{a}$$

Stoke's Theorem:

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$

## Vector Identities

$$\nabla \cdot \left( \frac{\hat{\mathbf{z}}}{z^2} \right) = 4\pi \delta^3(\mathbf{z})$$

$$\nabla \left( \frac{1}{z} \right) = -\frac{\hat{\mathbf{z}}}{z}$$

$$\delta(kx) = \frac{1}{|k|} \delta(x)$$

## Spherical Coordinates

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\hat{\mathbf{x}} = \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \left( \sqrt{x^2 + y^2} / z \right)$$

$$\phi = \tan^{-1}(y/x)$$

$$\hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$$

## Cylindrical Coordinates

$$x = s \cos \phi$$

$$y = s \sin \phi$$

$$z = z$$

$$\hat{\mathbf{x}} = \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$s = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}(y/x)$$

$$z = z$$

$$\hat{\mathbf{s}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Updated March 14, 2021

<https://github.com/DonneyF/formula-sheets>