Question 1

We have Identities $A_0 = \mathbb{I} \otimes A$ and $A_1 = A \otimes \mathbb{I}$ and we want to prove that

$$A_1 = S_{10} A_0 S_{10} \quad \text{and} \quad A_0 = S_{10} A_1 S_{10}$$

(i) $A_1 = S_{10} A_0 S_{10}$, Applying basis both sides we get

$$A_1 |x\rangle|y\rangle = S_{10} A_0 S_{10} |x\rangle|y\rangle$$

RHS : $A_1 |x\rangle|y\rangle$ but $A_1 = A \otimes \mathbb{I}$

$$= (A \otimes \mathbb{I}) |x\rangle \otimes |y\rangle$$

$$= A|x\rangle \otimes \mathbb{I}|y\rangle$$

$$= A|x\rangle \otimes |y\rangle$$

LHS : $S_{10} A_0 S_{10} |x\rangle|y\rangle$

$= S_{10} A_0 |y\rangle|x\rangle$ but $A_0 = \mathbb{I} \otimes A$

$= S_{10} (\mathbb{I} \otimes A) |y\rangle|x\rangle$

$= S_{10} [\mathbb{I}|y\rangle \otimes A|x\rangle]$

$= S_{10} [|y\rangle \otimes A|x\rangle]$

$= A|x \otimes |y\rangle$

Since LHS == RHS, it proves that the identity holds

(ii) $A_0 = S_{10} A_1 S_{10}$ applying basis $|x\rangle|y\rangle$ both sides

$A_0 |x\rangle |y\rangle = S_{10} A_1 S_{10} |x\rangle |y\rangle$

RHS: $A_0 |x\rangle |y\rangle$ but $A_0 = \mathbb{I} \otimes A$

$= (\mathbb{I} \otimes A) |x\rangle \otimes |y\rangle$

$= \mathbb{I}|x\rangle \otimes A|y\rangle$

$= |x\rangle \otimes A|y\rangle$

LHS : $S_{10} (A \otimes \mathbb{I}) S_{10} |x\rangle |y\rangle$

$= S_{10} (A \otimes \mathbb{I}) |y\rangle |x\rangle$

$= S_{10} (A|y\rangle \otimes |x\rangle)$

$= |x\rangle \otimes A|y\rangle$

Since, ~~te~~ the LHS = RHS, Hence proved the identity hold's true $A_0 = S_{10} A_1 S_{10}$

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c) We need to verify (2) Using $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

we have the identity

$$B_0 A_1 = A_1 B_0 = A \otimes B$$

Note : Matrices applied / multiplied by $\mathbb{I}$ (Identity) from the left or right is that Matrice

$\mathbb{I} A = A$ and $A \mathbb{I} = A$

We need to prove that

$$B_0 A_1 = A_1 B_0 = A \otimes B$$

Proving $B_0 A_1 \stackrel{?}{=} A \otimes B$

But we know that $B_0 = \mathbb{I} \otimes B$ and $A_1 = A \otimes \mathbb{I}$

$$B_0 A_1 = (\mathbb{I} \otimes B)(A \otimes \mathbb{I})$$

by the identity

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

$$= \mathbb{I}A \otimes B\mathbb{I}$$

$$= A \otimes B \text{, by multiplication by Identity}$$

Proving $A_1 B_0 \stackrel{?}{=} A \otimes B$

$$A_1 B_0 = (A \otimes \mathbb{I})(\mathbb{I} \otimes B)$$

$$= (A\mathbb{I}) \otimes (\mathbb{I} B)$$

$$= A \otimes B \text{, by multiplication by the identity}$$

Since, $B_0 A_1 = A_1 B_0 = A \otimes B$, Hence, proved that the identity is true and holds.

2a) (4) $\Rightarrow$ $S_{10}|x\rangle|y\rangle = |y\rangle|x\rangle$

(6) $\Rightarrow$ $C^x_{10}|x\rangle|y\rangle = |x\rangle|y \oplus x\rangle$

$C^x_{01}|x\rangle|y\rangle = |y\oplus x\rangle|x\rangle$

Required to prove

$S_{10} = C^x_{10} C^x_{01} C^x_{10}$

LHS $S_{10}|x\rangle|y\rangle = |y\rangle|x\rangle$

RHS: $C^x_{10} C^x_{01} C^x_{10} = C^x_{10} C^x_{01} \overset{1}{|x\rangle}\overset{0}{|y\oplus x\rangle}$

$= C^x_{10}|x\oplus y \oplus x\rangle|y\oplus x\rangle = C^x_{10}|y\rangle y \oplus x\rangle$

$= |y\rangle|y\oplus x \oplus y\rangle$

RHS $= |y\rangle|x\rangle$

Hence, since the RHS == LHS, we conclude that

$S_{10} = C^x_{10} C^x_{01} C^x_{10}$

2c) $S_{10} C^x_{10} S_{10} = C^x_{01}$

RHS : $C^x_{01} = |x\oplus y\rangle|y\rangle$

LHS :

$S_{10} C^x_{10} S_{10} |x\rangle|y\rangle$

$= S_{10} C^x_{10}|y\rangle|x\rangle$

$= S_{10}[|y\rangle|y\oplus x\rangle]$

LHS $= |y\oplus x\rangle|y\rangle$

Since, LHS == RHS, then swap operator can transform C^x_{10} to C^x_{01}

2c)

$$S_{10} C^X_{01} S_{10} = C^X_{10}$$

RHS : $C^X_{10} = \underline{|x\rangle|y \oplus x\rangle}$

LHS : $S_{10} C^X_{01} S_{10} |x\rangle|y\rangle$

$$= S_{10} C^X_{01} |y\rangle|x\rangle$$

$$= S_{10} |y \oplus x\rangle|x\rangle$$

$$= \underline{|x\rangle|y \oplus x\rangle}$$

Hence, we can conclude that the SWAP Operator transforms C^X_{01} to C^X_{10}

2d) $C^X_{10} (|1\rangle \otimes |\phi\rangle) \stackrel{?}{=} |1\rangle \otimes X|\phi\rangle$

<u>Proving the LHS is equal to RHS ?</u>

But we know $C^X_{10} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes X$

LHS :

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes X\right](|1\rangle \otimes |\phi\rangle)$$

Using the identity $(A \otimes b)(c \otimes d) = (Ac) \otimes (bd)$

$$\underbrace{\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I}\right)(|1\rangle \otimes |\phi\rangle)}_{1} + \underbrace{\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes X\right)(|1\rangle \otimes |\phi\rangle)}_{2}$$

Solving 1 first, we get:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \mathbb{I}(|\phi\rangle)$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \otimes \phi$$

$$= \hat{0}$$

(So, this part is not going to add anything to the sum of expression)

Solving 2 Now, we get:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes X|\phi\rangle$$

$$LHS = |1\rangle \otimes X|\phi\rangle$$

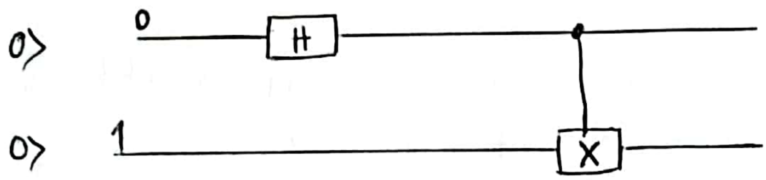
Since, the ~~left~~ LHS = RHS, we conclude that the Identity

$$C^{x}_{10}(|1\rangle \otimes |\phi\rangle) = |1\rangle \otimes X|\phi\rangle$$

4 a)

$$|\psi\rangle = C_{01}^{x} H_0 |00\rangle$$

Circuit Diagram of $|\psi\rangle$



b)

$$|\psi\rangle = C_{01}^{x} H_0 |00\rangle$$

Initial state $= (1 \ 0 \ 0 \ 0)^T$

Applying Hadamard gate, we get

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H_0|00\rangle = \cancel{\frac{1}{\sqrt{2}}(|00\rangle + |[illegible]\rangle)} \ \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle$$

$$H_0|00\rangle = \cancel{\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)} \ \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle$$

Now, applying CNOT gate

$$C_{01}^{x} H_0|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + \underbrace{C_{01}^{x}}_{\text{control}} \underbrace{|\overset{1\downarrow}{0}\overset{0\downarrow}{1}\rangle}_{\text{target}})$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

4b Proof Using Matrices and tensors

$$|\psi\rangle = C^X_{01} H_0 |00\rangle \quad \text{where } |00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$H_0 = \mathbb{I} \otimes H \quad \text{where } H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$H_0|00\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

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then $C^X_{01} H_0 |00\rangle =$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C^X_{01} H_0 |00\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\underline{\underline{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)}}$$

c) Suppose that we want to make a measurement of 2-qubits

Probabilities are given as

$$P(ij) = |\alpha_{ij}|^2 \quad \text{where } ij \in 0,1$$

We get :

$$P(00) = \left|\left(\frac{1}{\sqrt{2}}\right)\right|^2 = \frac{1}{2}$$

$$P(11) = \left|\left(\frac{1}{\sqrt{2}}\right)\right|^2 = \frac{1}{2}$$

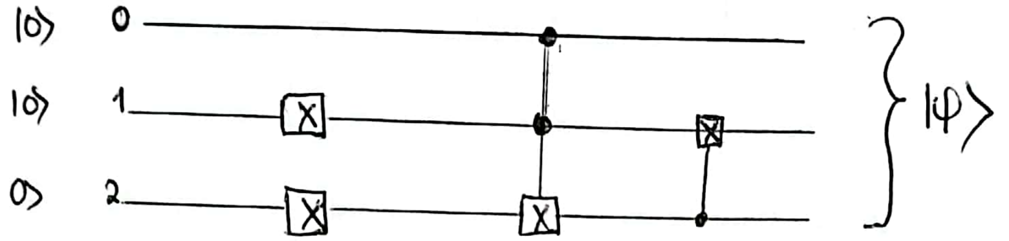
$$P(01) = P(10) = 0$$

$$\begin{array}{cl} \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} & \begin{array}{l} \leftarrow \text{probability associated with state } 00 \\ \leftarrow \text{probability associated with state } 01 \\ \leftarrow \text{probability associated with state } 10 \\ \leftarrow \text{probability associated with state } 11 \end{array} \end{array}$$

6a) $|\psi\rangle = C^{X}_{21} \, (C^2X)_{210} \, X_2 X_1 |000\rangle$

where $(C^2X)_{210} |x\rangle|y\rangle|z\rangle = |x\rangle|y\rangle|z \oplus xy\rangle$

Toffoli Gate Circuit Diagram



b) $|\psi\rangle = C^{X}_{21} \, (C^2X)_{210} \, X_2 X_1 |\overset{2\,1\,0}{\downarrow\downarrow\downarrow}\rangle$ $|000\rangle$

Applying the NOT gate X_1 and X_2 respectively

$= C^{X}_{21} \, (C^2X)_{210} X_2 |010\rangle$

$= C^{X}_{21} \, (C^2X)_{210} |110\rangle$

Applying the double NOT

$= C^{X}_{21} |1\rangle|1\rangle|0 \oplus (1)(1)\rangle$

$= C^{X}_{21} |111\rangle$

$|\psi\rangle = |101\rangle \equiv |5\rangle_3$

c) The final state $|\psi\rangle = |10\phi\rangle$ is a computational basis state. When we measure the 3 qubits, the possible outcomes are:

(i) $|101\rangle$ with probability 1

(ii) All other 7 possible outcomes with probability 0