Question 1

Consider the two-dimensional vector space spanned by the basis

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by

$$|\alpha\rangle = i|0\rangle - 2|1\rangle = \begin{pmatrix}1\\-2i\end{pmatrix}, \quad |\beta\rangle = |0\rangle + i|1\rangle = \begin{pmatrix}1\\i\end{pmatrix}$$

1a

We know that:

$$\begin{aligned}
|0\rangle &= \begin{pmatrix}1\\0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \\
|\alpha\rangle &= i|0\rangle - 2|1\rangle, \quad |\beta\rangle = |0\rangle + i|1\rangle, \\
|\alpha\rangle &= i\begin{pmatrix}1\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}i\\-2\end{pmatrix}, \\
|\beta\rangle &= \begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\i\end{pmatrix}.
\end{aligned}$$

$$\langle\alpha| = \begin{pmatrix}i\\-2\end{pmatrix}^\dagger = \begin{pmatrix}-i & -2\end{pmatrix}, \quad \langle\beta| = \begin{pmatrix}1\\i\end{pmatrix}^\dagger = \begin{pmatrix}1 & -i\end{pmatrix}.$$

b

$$\begin{aligned}
\langle\alpha|\beta\rangle &= (-i, -2)\begin{pmatrix}1\\i\end{pmatrix} \\
&= -i(1) + (-2)(i) \\
&= -i - 2i \\
&= -3i, \\
\langle\beta|\alpha\rangle &= \begin{pmatrix}1\\i\end{pmatrix}^\dagger \begin{pmatrix}i\\-2\end{pmatrix} \\
&= (1, -i)\begin{pmatrix}i\\-2\end{pmatrix} \\
&= 1 \cdot i + (-i)(-2) \\
&= i + 2i \\
&= 3i.
\end{aligned}$$

Since $\langle\alpha|\beta\rangle = -3i$, we have $\langle\beta|\alpha\rangle = (\langle\alpha|\beta\rangle)^* = (-3i)^* = 3i$, confirming the conjugate symmetry.

c

$$
\begin{aligned}
|\alpha\rangle\langle\beta| &= \begin{pmatrix} i \\ -2 \end{pmatrix} \begin{pmatrix} 1 & -i \end{pmatrix} \\
&= \begin{pmatrix} i & -i^2 \\ -2 & 2i \end{pmatrix} \\
&= \begin{pmatrix} i & 1 \\ -2 & 2i \end{pmatrix}.
\end{aligned}
$$

d

To satisfy Hermitian condition, $A^\dagger = A$. For the matrix A defined as:

$$
A = \begin{pmatrix} 2i & -1 \\ 2 & i \end{pmatrix},
$$

we compute its conjugate transpose $A^\dagger$:

$$
A^\dagger = \begin{pmatrix} -2i & 2 \\ -1 & -i \end{pmatrix} \neq A.
$$

Question 3

(b) Recall that the exponential of a matrix A is defined by its Taylor series expansion

$$e^A = \mathbb{I} + A + \frac{1}{2!}A^2 + \dots$$

Show that

$$e^{i\beta\hat{r}\cdot\vec{\sigma}} = \cos(\beta)\mathbb{I} + i\sin(\beta)\hat{r}\cdot\vec{\sigma}, \quad (3)$$

where $\hat{r}$ is an arbitrary 3 -dimensional unit vector ($\hat{r}\cdot\hat{r} = 1$), and β is a real number. Hint: use (2) to compute $(\hat{r}\cdot\vec{\sigma})^2$.

Solution

It is established that:

$$e^A = \mathbb{I} + A + \frac{1}{2!}A^2 + \dots$$

Let A be defined as $i\beta\hat{r}\cdot\vec{\sigma}$. This allows us to rewrite the expression as:

- $$e^A = \mathbb{I} + (i\beta\hat{r}\cdot\vec{\sigma}) + \frac{1}{2!}(i\beta\hat{r}\cdot\vec{\sigma})^2 + \frac{1}{3!}(i\beta\hat{r}\cdot\vec{\sigma})^3 + \frac{1}{4!}(i\beta\hat{r}\cdot\vec{\sigma})^4 + \frac{1}{5!}(i\beta\hat{r}\cdot\vec{\sigma})^5 + \frac{1}{6!}(i\beta\hat{r}\cdot\vec{\sigma})^6 + \dots$$
- From the equation below, we set $a = b = \hat{r}$, yielding:
- $$(\hat{r}\cdot\vec{\sigma})(\hat{r}\cdot\vec{\sigma}) = (\hat{r}\cdot\hat{r})\mathbb{I} + i(\hat{r}\times\hat{r})\cdot\vec{\sigma}$$

  This simplifies using the facts that $\hat{r}\cdot\hat{r} = 1$, β is a real number, and $\hat{r}\times\hat{r} = 0$.
- $$(\hat{r}\cdot\vec{\sigma})^2 = (1)\mathbb{I} + i(0) = \mathbb{I}$$

Now we can expand $e^{(i\beta\hat{r}\cdot\vec{\sigma})}$ as follows:

$$\begin{aligned}
e^{(i\beta\hat{r}\cdot\vec{\sigma})} &= \mathbb{I} + i\beta\hat{r}\cdot\vec{\sigma} - \frac{\beta^2}{2!}(\hat{r}\cdot\vec{\sigma})^2 + \cdots \\
&= \mathbb{I} + i\beta\hat{r}\cdot\vec{\sigma} - \frac{\beta^2}{2!}\mathbb{I} + \cdots \\
&= \sum_{n=0}^{\infty}\frac{(i\beta\hat{r}\cdot\vec{\sigma})^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(i\beta\hat{r}\cdot\vec{\sigma})^{2n+1}}{(2n+1)!} \\
&= \cos(\beta)\mathbb{I} + i\sin(\beta)\hat{r}\cdot\vec{\sigma}.
\end{aligned}$$

Even powers $= \mathbb{I} - \frac{\beta^2}{2!}\mathbb{I} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} + \dots = \mathbb{I}\cos(\beta)$

Odd powers $= i\hat{r}\sigma\left(\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \dots\right. = i\hat{r}\vec{\sigma}\sin(\beta)$

Hence combining the two we get

$\mathbb{I}\cos\beta + i\hat{r}\cdot\vec{\sigma}\sin\beta$

Question 4

Solution a: Verify that the state (4) is indeed normalized.

$$|\psi(\theta,\phi)\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$\langle\psi(\theta,\phi)|\psi(\theta,\phi)\rangle = \left(\cos\left(\frac{\theta}{2}\right) e^{i\phi}\sin\left(\frac{\theta}{2}\right)\right)^{\dagger} \times \left(\cos\left(\frac{\theta}{2}\right) e^{i\phi}\sin\left(\frac{\theta}{2}\right)\right)$$

Since $e^{-i\phi \times i\phi} = 1$

$$\langle\psi(\theta,\phi)|\psi(\theta,\phi)\rangle = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)$$

Applying the identity, we get:

$$\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$$

Thus, $|\psi(\theta,\phi)\rangle$ is normalized.

4 b) Show that

$$(\hat{r}\cdot\vec{\sigma})\,|\psi(\theta,\phi)\rangle = |\psi(\theta,\phi)\rangle$$

This is to show that a state and multiplied by a phase will not change the state measurements

Where: $(\hat{r}\cdot\vec{\sigma}) = \sin\theta\cos\phi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \sin\theta\sin\phi\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$$\psi(\theta,\phi) = \cos\left(\frac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix} + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow (\hat{r}\cdot\vec{\sigma})\,|\psi(\theta,\phi)\rangle = \left[\sin\theta\cos\phi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \sin\phi\sin\theta\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]$$

$$\left[\begin{pmatrix}1\\0\end{pmatrix}\cos\left(\frac{\theta}{2}\right) + \begin{pmatrix}0\\1\end{pmatrix}e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\right]$$

$$\Rightarrow \left[\begin{bmatrix}0 & \sin\theta\cos\phi\\ \sin\theta\cos\phi & 0\end{bmatrix} + \begin{bmatrix}0 & -i\sin\theta\sin\phi\\ i\sin\theta\sin\phi & 0\end{bmatrix} + \begin{bmatrix}\cos\theta & 0\\ 0 & -\cos\theta\end{bmatrix}\right]$$

$$\begin{bmatrix}\cos(\theta/2)\\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi\\ \sin\theta\sin\phi + \cos\phi\sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ e^{i\phi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

<u>Using Euler's Identity</u>

$$\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta\cos\left(\frac{\theta}{2}\right) + 1.\sin\theta\sin\left(\frac{\theta}{2}\right) \\ \sin\theta\cos(\theta/2)e^{i\phi} - \cos\theta\sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\left(\frac{\theta}{2}\right) + \sin\theta\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\left[\sin\theta\cos(\theta/2) - \cos\theta\sin\left(\frac{\theta}{2}\right)\right] \end{bmatrix}$$

<u>Using the Trigonometric Identities</u>

$\cos(A-B) = \cos A\cos B + \sin A\sin B$, $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$\begin{bmatrix} \cos(\theta - \theta/2) \\ e^{i\phi}\sin(\theta - \theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{bmatrix}$$

Hence we have proved that $\hat{r}\cdot\vec{\sigma}\,|\psi(\theta,\phi)\rangle = |\psi(\theta,\phi)\rangle$

$\square$

4 c) Determining θ, and ϕ on the Bloch sphere

$$|\psi\rangle \begin{pmatrix} -\frac{i\sqrt{3}}{2} \\ 1/4 + \frac{i\sqrt{3}}{4} \end{pmatrix}$$

We know that given a measurement and you multiply it by a phase the measurements of $|\alpha\rangle$ will not change given multiplication by a phase.

$$i|\psi\rangle = \begin{pmatrix} \sqrt{3}/2 \\ \frac{-\sqrt{3}}{4} + i/4 \end{pmatrix}$$

Now, ~~we~~ have

$$|\psi(\theta,\phi\rangle = i|\psi\rangle$$

$$\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin(\theta/2) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{4} + \frac{i}{4} \end{bmatrix}$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} \Rightarrow \frac{\theta}{2} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi/6$$

$$= \theta/2 = \pi/6$$

$$\boxed{\theta = \frac{\pi}{3}}$$

Now calculating ϕ

$$e^{i\phi}\sin\left(\pi/3\right) = \frac{\sqrt{3}}{4} + i/4$$

$$\Rightarrow \frac{1}{2}e^{i\phi} = \frac{-\sqrt{3}}{4} + i/4$$

Using the Euler's identity we get

$$\cos\phi + i\sin\phi = \frac{-\sqrt{3}}{2} + i/2$$

$$\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \phi$$

$$\boxed{\phi = \frac{\pi}{3}}$$

Question 5

We have $R^{(k)}(\alpha) = e^{i\frac{\alpha}{2}\sigma^k}$ $\quad k=1,2,3$

RTS

$$R^{(k)}(\alpha) = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} + i\sin\left(\alpha/2\right)\sigma^k \text{ for } k=1,2,3$$

We know that from Question 3

$$e^A = \mathbb{I} + A + \frac{1}{2!}A^2 + \cdots$$

Now we expand

$$e^{i\frac{\alpha}{2}\sigma^k} = \mathbb{I} + \left(i\,\alpha/_2\,\sigma^k\right) + \frac{\left(i\,\alpha/2\,\sigma^k\right)^2}{2!} + \frac{\left(i\,\alpha/2\,\sigma^k\right)^3}{3!} + \cdots$$

$\Rightarrow$ Even powers

$$\mathbb{I} - \frac{\left(\alpha/_2\,\sigma^k\right)^2}{2!} + \frac{\left(\alpha/_2\,\sigma^k\right)^4}{4!} - \frac{\left(\alpha/_2\,\sigma^k\right)^6}{6!} + \cdots$$

But we know that matrices σ^k are Hamitian

so $\left(\sigma^k\right)^{2c}$ for $c=1,2,\cdots = \mathbb{I}$

$\Rightarrow$ Even powers simplify to

$$\Rightarrow \mathbb{I}\left[1 - \frac{(\alpha/_2)^2}{2!} + \frac{(\alpha/_2)^4}{4!} - \frac{(\alpha/2)^6}{6!} + \frac{(\alpha/_2)^8}{8!} + \cdots\right]$$

$$\Rightarrow \mathbb{I}\cos\left(\alpha/_2\right)$$

Now doing the same for Odd powers, we have

$$\Rightarrow \quad \alpha/_2 i\sigma^k + \frac{(i\alpha/_2\sigma^k)^3}{3!} + \frac{(i\alpha/_2\sigma^k)^5}{5!} + \frac{(i\alpha/_2\sigma^k)^7}{7!} + \cdots$$

$$\Rightarrow \quad i\sigma^k\left[\alpha/_2 - \frac{(\alpha/_2)^3 \mathbb{I}}{3!} + \frac{(\alpha/_2)^5}{5!} + \cdots\right]$$

$$\Rightarrow \quad i\sigma^k \sin(\alpha/_2)$$

$\Rightarrow$ Combining the two, we get :

$$\underline{\underline{\mathbb{I}\cos(\alpha/_2) + i\sigma^k \sin(\alpha/_2) = e^{i\frac{\alpha}{2}\sigma^k}}}$$

$\square$

b) RTS

Showing that $R^{(k)}(\alpha)$ is Unitary

We know that $R^{k}(\alpha) = \mathbb{I}\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\sigma^{k}$

$$R^{k}(\alpha)^{+} = \mathbb{I}\cos\left(\alpha/2\right) - i\sin\left(\alpha/2\right)\sigma^{k}$$

$$R^{k}(\alpha)^{+}R^{k}(\alpha) = \left[\mathbb{I}\cos\left(\alpha/2\right) - i\sin\left(\alpha/2\right)\sigma^{k}\right]\left[\mathbb{I}\cos\left(\alpha/2\right) + i\sin\left(\alpha/2\right)\sigma^{k}\right]$$

we know that $(\sigma^{k})^{2} = \mathbb{I}$

$$= \mathbb{I}\cos^{2}\left(\alpha/2\right) + \mathbb{I}\cos\left(\alpha/2\right)i\sin\left(\alpha/2\right) - \mathbb{I}\cos\left(\alpha/2\right)i\sin\left(\alpha/2\right) + \sin^{2}\left(\alpha/2\right)\mathbb{I}$$

$$= \mathbb{I}\cos^{2}\left(\alpha/2\right) + \mathbb{I}\sin^{2}\left(\alpha/2\right)$$

$$= \mathbb{I}\left[\cos^{2}\left(\alpha/2\right) + \sin^{2}\left(\alpha/2\right)\right]$$

$$= \mathbb{I}$$

We have shown that $R^{k}(\alpha)$ is Unitary

c) Required to show that

$$R^{k}(\alpha)\,\sigma^{i}\,R^{k}(\alpha)^{+} = \sum_{j=1}^{3}\sigma^{j}R^{k}_{ji}(\alpha)$$

for $i,k,j = 1,2,3$

for $k=3$

We have

$$R^{3}(\alpha)\,\sigma^{i}\,R^{3}(\alpha)^{+} = \sum_{j=1}^{3}\sigma^{j}R^{3}_{ji}(\alpha)$$

Calculating for $i=1$

$$\Rightarrow R^3(\alpha) = \mathbb{I}\cos(\alpha/2) + i\sin(\alpha/2)\sigma^3 \quad \text{where } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha/2) & 0 \\ 0 & \cos(\alpha/2) \end{pmatrix} + \begin{pmatrix} i\sin(\alpha/2) & 0 \\ 0 & -i\sin(\alpha/2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha/2) + i\sin(\alpha/2) & 0 \\ 0 & \cos(\alpha/2) - i\sin(\alpha/2) \end{pmatrix}$$

By Euler's Identity we get

$$= \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

Using the above result, we get

$$R^3(\alpha)\,\sigma^i R^3(\alpha)^+ = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-\alpha/2 i} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}\begin{pmatrix} 0 & e^{i\alpha/2} \\ e^{-\alpha/2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{2i\alpha/2} \\ e^{-2i\alpha/2} & 0 \end{pmatrix}$$

$$\text{LHS} = \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-\alpha i} & 0 \end{pmatrix}$$

Now calculating the RHS

$$\sum_{j=1}^{3} \sigma^j R^3_{ji}(\alpha) = \sigma^1 R^3_{11}(\alpha) + \sigma^2 R^3_{21}(\alpha) + \sigma^3 R^3_{31}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cos(\alpha) + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (-\sin\alpha) + 0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cos(\alpha) \\ \cos\alpha & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i\sin(\alpha) \\ i\sin(\alpha) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cos(\alpha) + i\sin(\alpha) \\ \cos(\alpha) - i\sin(\alpha) & 0 \end{pmatrix}$$

Using Euler's Identity

$$\text{RHS} = \begin{bmatrix} 0 & e^{\alpha} \\ e^{-\alpha} & 0 \end{bmatrix}$$

For $i = 1$, RHS = LHS, so the identity is true now, we investigate for the other instances

<u>for $i = 2$</u>

The identity is

$$\text{LHS} = R^3(\alpha)\, \sigma^2\, R^3(\alpha)^{\dagger} \quad \text{where} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\begin{pmatrix} e^{i(\alpha/2)} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -ie^{i\alpha/2} \\ -ie^{-\alpha/2} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & -ie^{i\alpha} \\ ie^{-i\alpha} & 0 \end{pmatrix}}}$$

Now calculating the RHS of the Identity :

$$RHS = \sum_{j=1}^{3} \sigma^j R^{3}_{j2}(\alpha) = \sigma^1 R^3_{12}(\alpha) + \sigma^2 R^3_{22}(\alpha) + \sigma^3 R^3_{32}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin(\alpha) + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cos(\alpha) + 0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 0 & \sin(\alpha) - i\cos(\alpha) \\ \sin(\alpha) + i\sin\alpha & 0 \end{pmatrix}$$

We know the Euler's Identity is

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \qquad --- (11)$$

* If we multiply both sides of (11) we get

$$\underline{\underline{ie^{i\alpha} = -\sin(\alpha) + i\cos(\alpha)}}$$

* Using the above result we get

$$RHS = \begin{pmatrix} 0 & -ie^{i\alpha} \\ ie^{i\alpha} & 0 \end{pmatrix}$$

For the case where $i = 2$, the Identity holds true and now we want to verify for the last case $(i=3)$

Verification for the case $i=3$

$$LHS = R^3(\alpha)\,\sigma^3\,R^3(\alpha)^+ \quad \text{where} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & -e^{-i\alpha/2} \end{pmatrix}\begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

$$LHS = \begin{pmatrix} e^{i(\alpha/2-\alpha/2)} & 0 \\ 0 & -e^{i(-\alpha/2+\alpha/2)} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}$$

Now calculating the RHS for the case $i=3$

$$RHS = \sum_{j=1}^{3} \sigma^j R^3_{j3}(\alpha) = \sigma^1 R^3_{13}(\alpha) + \sigma^2 R^3_{23}(\alpha) + \sigma^3 R^3_{33}(\alpha)$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} 0 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} 0 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 1$$

$$\underline{\underline{RHS = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}$$

Since, the RHS = LHS for $i=3$

Conclusion

The identity is proved true for the case where $k=3$

Hence $$\underline{\underline{R^3(\alpha)\,\sigma^i\,R^3(\alpha)^+ = \sum_{j=1}^{3} \sigma^j R^3_{ji}(\alpha)}}$$ □

d) $\hat{r}' = R^{(k)}(\alpha)\,\hat{r}(\theta,\phi)$ whose components $r_i' = \sum_{j=1}^{3} R_{ij}^{k}\,\alpha\, r_j(\theta,\phi)$

Required to show that $\hat{r}$ is a unit vector and is in rotation of the vector $\hat{r}(\theta,\phi)$ by angle $-\alpha$

Using the Case $k=3$

$$\hat{r}' = R^{k}(\alpha)\,\hat{r}(\theta,\phi) \quad \text{which with } \theta = \pi/2 \qquad k=3$$

$$= R^{k}(\alpha)\,\hat{r}(\pi/2,\phi)$$

but we have $\hat{r} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$ from Eq 5 from Assignment

$$\Rightarrow \hat{r}' = R^{k}(\alpha)\begin{pmatrix} \sin\pi/2\cos\phi \\ \sin\pi/2\sin\phi \\ \cos\pi/2 \end{pmatrix}$$

But we know that
(i) $\sin\pi/2 = 1$
(ii) $\cos\pi/2 = 0$

$$= R^{k}(\alpha)\cdot\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

But we have $R^{3}(\alpha)$ from the question

$$R^{3}(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows that

$$\hat{r}' = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

$\boxed{3\times 3 \quad 3 \times 1}$

$$= \begin{pmatrix} \cos\alpha \cos\phi + \sin\alpha \sin\phi \\ -\sin\alpha \cos\phi + \cos(\alpha) \sin\phi \\ 0 \end{pmatrix}$$

Using the trigonometry identities ~~as~~ shown earlier.

we get.

$$\hat{r}' = \begin{pmatrix} \cos(\alpha - \phi) \\ \sin(-\alpha + \phi) \\ 0 \end{pmatrix} \equiv \begin{bmatrix} \cos(\phi - \alpha) \\ \sin(\phi - \alpha) \\ 0 \end{bmatrix}$$

We can rewrite $\hat{r}'$ as

$$\hat{r}' = \cos(\phi - \alpha)\,\hat{x} + \sin(\phi - \alpha)\,\hat{y} + 0\,\hat{z}$$

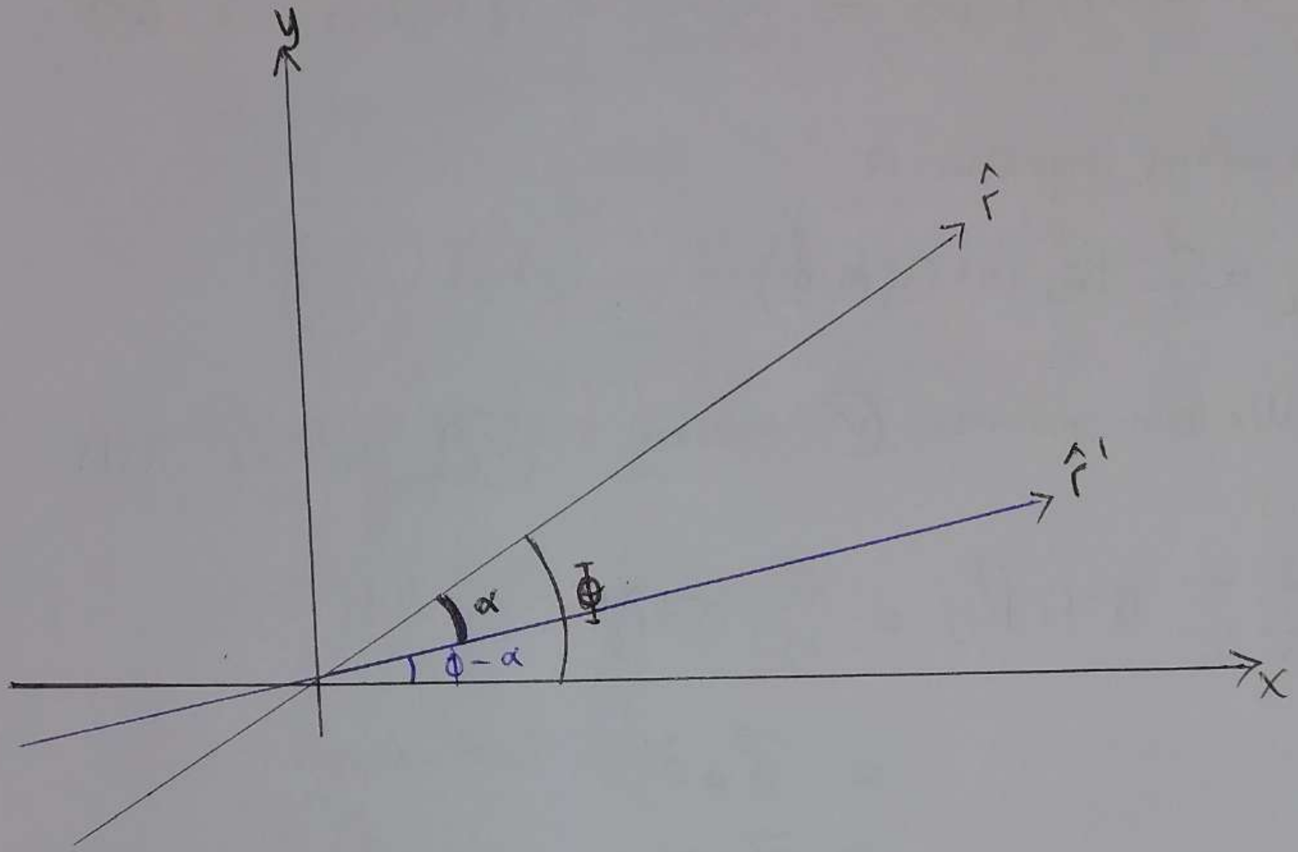
Hence shown ~~that~~. $\square$

Now showing that $\hat{r}'$ is a unit vector we need $|\hat{r}'|$

$$= \sqrt{\cos^2(\phi - \alpha) + \sin^2(\phi - \alpha) + 0^2} = \sqrt{1} = \underline{1}$$

Hence, shown that $\hat{r}_1'$ is a unit vector

Sketch of $\hat{r}'$



5e Rotation Operations

Using (10) and (12) to show that

$$R^k(\alpha)\left(\vec{\sigma}\cdot\hat{r}\right)R^k(\alpha)^\dagger = \vec{\sigma}\cdot\hat{r}'$$

$$(10) \Rightarrow R^k(\alpha)\,\sigma^i\,R^k(\alpha)^\dagger = \sum_{j=1}^{3}\sigma^j R^k_{ji}(\alpha)$$

$$(12) \Rightarrow \hat{r}_i' = \sum_{j=1}^{3} R^k(\alpha)\, r_j(\theta,\phi)\,\sigma$$

$$\text{and } \vec{\sigma}\cdot\hat{r} = \sum_{j=1}^{3}\sigma^j r_j$$

Let' us multiply equation (10) by r_i

$$\sum_{l=1}^{3} R^k(\alpha)\,\sigma^i r_j^i\, R^k_j(\alpha)^\dagger = \sum_{l=1}^{3}\sum_{j=1}^{3} \sigma^j r_i R_{ji}(\alpha) \quad \text{---} \; (\checkmark)$$

Rewriting equation 12

$$\hat{r}_i = \sum_{l=1}^{3} R^k_{ij}(\alpha)\, r_i(\theta, \phi) \quad \text{---} \; (\checkmark\checkmark)$$

$\Rightarrow$ We can rewrite $(\checkmark)$ using $(\checkmark\checkmark)$ on the RHS becomes

$$\sum_{l=1}^{3}\sum_{j=1}^{3} \sigma^j r_i R_{ij} = \sum_{j=1}^{3} \sigma^j \hat{r}_j'$$

$$= \vec{\sigma} \cdot \hat{r}'$$

$$= \vec{\sigma} \cdot r'$$

Finally, we get $(\checkmark)$ as

$$R^k(\alpha)\,(\vec{\sigma} \cdot \hat{r})\, R^k(\alpha)^\dagger = \vec{\sigma} \cdot \hat{r}' \quad \text{(Shown!)}$$

Required to show that the state $|\psi\rangle$ if multiplied by a phase $(\hat{r} \cdot \vec{\sigma})$ it remains the same.

Proof: Since we have $(\hat{r} \cdot \vec{\sigma} \,|\psi(\theta, \Phi)\rangle$, ~~which~~ $= |\psi'\rangle$ where ψ' is an eigen state of $(r' \cdot \vec{\sigma}')$

Now multiplying $(\checkmark)$ by $R^k(\alpha)$ we get.

$$R^k(\hat{r} \cdot \vec{\sigma})|\psi(\theta, \phi)\rangle = R^k|\psi(\theta, \phi)\rangle$$

$$R^{(k)}(\alpha)\,(\vec{\sigma}\circ\hat{r}')\,R^{k}(\alpha)^{+}\,R^{k}(\alpha)\,|\psi\rangle = R^{k}(\alpha)|\psi\rangle$$

But we know

$R^{k}(\alpha)\,(\vec{\sigma}\cdot\hat{r}')\,R^{k}(\alpha)^{+}$ from 13

$R^{k}(\alpha)\,|\psi\rangle$ ~~from~~ the equation 15

$R^{k}(\alpha)|\psi\rangle$ from equation 15 $= |\psi'\rangle$

Hence we get:

$$(\vec{\sigma}\cdot\hat{r}')\,|\psi'\rangle = |\psi'\rangle$$

Hence, shown that

(i) $R^{(k)}(\alpha)\,(\vec{\sigma}\circ\hat{r})\,R^{k}(\alpha)^{+} = \vec{\sigma}\cdot\hat{r}'$

and (ii) $(\hat{r}'\circ\vec{\sigma})\,|\psi'\rangle = |\psi'\rangle$