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Mathematical formulation

A scheduling problem of n jobs on two machines arranged in a flow-shop configuration is considered. Each job must be processed first on machine 1 and then on machine 2. Multiple criteria, including makespan, total energy consumption, idle time and total job duration, are considered.

The assumptions made to model the problem as an MILP include that

- there are a fixed number of jobs and machines,
- there are no machine failures,
- all jobs have non-negative processing and set-up times,
- the processing speed factor and the conversion factor for the processing speed are strictly positive, and
- jobs are non-pre-emptive meaning that once a job starts on a machine, it must run to completion without interruptions.

The decision variables are defined by

$$\lambda_j = \begin{cases} 1, & \text{if job } j \text{ is the first in the sequence and} \\ 0, & \text{otherwise,} \end{cases}$$

$$\omega_{jk} = \begin{cases} 1, & \text{if job } j \text{ is executed right before job } k \text{ in the sequence and} \\ 0, & \text{otherwise,} \end{cases}$$

$$u_{jm\ell} = \begin{cases} 1, & \text{if job } j \text{ is processed at speed } \ell \text{ on machine } m \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Futhermore,

- c_{jm} is the completion time of job j on machine m,
- c_{sup} is the makespan (total completion time of the last job on the last machine),
- e is the total energy consumption,
- q is the sum of the job durations,
- t_m is the total idle time on machine m.

The parameters

- p_{jm} is the processing time of job j on machine m,
- s_j is the setup offset time required for job j on the second machine,
- ρ_m is the power consumption rate of machine m when processing jobs,
- n is the total number of jobs in the system,
- r_ℓ is the processing speed factor, where $\ell = 1, 2, 3$ corresponds to fast, normal, and slow speeds respectively,
- d_{jkm} is the sequence-dependent set-up time required when transitioning from job j to job k on machine m,
- β_ℓ is the conversion factor associated with processing speed ℓ,
- γ_m is the conversion factor for idle time on machine m,
- l is a very large number (set to $1\,000\,000$).

The objectives are to minimise

$$\sum_{j=1}^{n}\sum_{m=1}^{2}\sum_{\ell=1}^{3}\frac{p_{jm}}{r_\ell}u_{jm\ell}, \tag{1}$$

$$\sum_{j=1}^{n}\sum_{m=1}^{2}\sum_{\ell=1}^{3}u_{jm\ell}\left(\frac{p_{jm}}{60r_\ell}\beta_\ell\rho_m\right)+\sum_{m=1}^{2}\frac{\gamma_m\rho_m}{60}t_m, \tag{2}$$

$$c_{sup}, \tag{3}$$

$$\sum_{m=1}^{2}t_m. \tag{4}$$

subject to

$$\sum_{j=1}^{n}\lambda_j=1, \qquad j=1,\dots,n, \tag{5}$$

$$\sum_{k=1}^{n}\omega_{jk}=1, \qquad j=1,\dots,n,\ j\neq k, \tag{6}$$

$$\sum_{j=1}^{n}\omega_{jk}=1, \qquad k=1,\dots,n,\ j\neq k, \tag{7}$$

$$\sum_{\ell=1}^{3}u_{jm\ell}=1, \qquad j=1,\dots,n,\ m=1,2, \tag{8}$$

$$l(1-\omega_{jk})+s_j\geq d_{jj2}-c_{j1}, \qquad j=k, \tag{9}$$

$$c_{j1}\geq\frac{p_{j1}}{r_\ell}u_{j1\ell}+d_{jj1}\lambda_j, \qquad j=1,\dots,n,\ \ell=1,2,3, \tag{10}$$

$$c_{j2}\geq c_{j1}+s_j+\frac{p_{j2}}{r_\ell}u_{j2\ell}, \qquad j=1,\dots,n,\ \ell=1,2,3, \tag{11}$$

$$l(\lambda_j)+l(1-\omega_{jk})+c_{km}\geq c_{jm}+\frac{p_{mk}}{r_\ell}u_{mk\ell}+d_{mjk}\omega_{jk}, \qquad j,k,m,\ell \quad | \quad j\neq k, \tag{12}$$

$$c_{sup}\geq c_{j2}, \qquad j=1,\dots,n, \tag{13}$$

$$t_m=c_{sup}-\sum_{j=1}^{n}\sum_{\ell=1}^{3}\frac{p_{jm}}{r_\ell}u_{jm\ell}, \qquad m=1,2, \tag{14}$$

$$c_{jm}\geq 0,\quad t_m\geq 0,\quad q\geq 0,\quad e\geq 0,\quad s_j\geq 0, \tag{15}$$

$$m_j\in\{0,1\},\quad \omega_{jk}\in\{0,1\},\quad u_{jm\ell}\in\{0,1\},$$

$$\beta_\ell>0,\quad r_\ell>0. \tag{16}$$

This optimisation model seeks to improve both performance and sustainability in a two-machine flow shop setting. The first objective function (1) aims to minimise the total processing time of all jobs across both machines, which is the sum of job duration q. The second objective function (2) focuses on sustainability by reducing total energy consumption e. This includes the energy used during job processing, calculated based on machine speed, power rate, and job characteristics, as well as idle energy, which depends on idle durations and machine-specific parameters such γ_m and ρ_m. The third objective function (3) seeks to minimise the makespan c_{sup}, which represents the time the last job finishes processing, and is a common measure of service level. Also, the fourth objective function (4) minimises the total idle time t_m across the two machines, improving resource utilisation. To support these objectives, several constraints are applied. Constraint set (5) ensures that only one job is selected as the first in the sequence. Constraint sets (6) and (7) ensure each job is assigned exactly once in the schedule. Constraint set (8) guarantees that each job is processed at exactly one speed level on each machine. Setup times are handled by constraint set (9), which ensures anticipatory setups are properly considered for the first job. Constraint sets (10) and (11) calculate completion times for each job on machines 1 and 2, respectively. Constraint (12) enforces proper sequencing of jobs, accounting for setup time, processing time, and the sequence order. The makespan is determined in constraint set (13), while constraint set (14) calculates the total idle time for each machine. Finally, constraint sets (15) and (16) define the domains of the decision variables, ensuring non-negativity and binary restrictions, while parameters such as β_ℓ and r_ℓ are assumed to be strictly positive.