Exercise 1. *Let*

$$f(x) = \begin{cases} x^2 + b, & x > 2 \\ ax + 1, & x \leqslant 2 \end{cases}$$

and assume that f is differentiable at $x_0 = 2$. Find a and b.

Exercise 2. *If $f'(x_0)$ exists. Then evaluate*

$$\lim_{\Delta x \to 0} \frac{f(x_0 - \Delta x) - f(x_0)}{\Delta x}$$

and

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{h}$$

Exercise 3. *Let $f^{(n)}(x)$ and $g^{(n)}(x)$ exist. Then*

$$[f(x) \cdot g(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)$$

Hence try to find $(\sin(3x)x^2)^{(100)}$.

Exercise 4. *(1) Prove the Dauboux Theorem: Let $f \in D[a,b]$ satisfying $f'_+(a)f'_(b) < 0$. Prove that there exists $\eta \in (a,b)$ such that $f'(\eta) = 0$. (2) Let $f \in D(a,b)$ such that $f'(x) \neq 0$. Prove that f is strictly monotonic.*

Exercise 5 (Hard)**.** *Let $f(x) \in C^2[0, \frac{1}{2}]$ satisfying $f(0) = f'(0)$ and $f(\frac{1}{2}) = 0$. Prove that $\exists \eta \in (0, \frac{1}{2})$ such that*

$$f''(\eta) = \frac{3f'(\eta)}{1 - 2\eta}$$

Hint: Construct a proper $F(x)$ and apply Rolle's theorem.

Exercise 6. *Let $f \in D(a, +\infty)$ and $\lim_{x \to \infty} f'(x) = L > 0$. Prove that*

$$\lim_{x \to \infty} f(x) = +\infty$$

Exercise 7. *Evaluate*

$$\lim_{x \to 0} \frac{\cos(\sin x) - \cos(\tan x)}{x^4}$$

Exercise 8. *Prove that $\ln(\frac{b}{a}) < \frac{b-a}{a}$ for $b > a > 0$.*

Exercise 9. *Prove that* $e^x > 1 + x$ *for* $x \in \mathbb{R}$.

Exercise 10. *Evaluate*

$$\lim_{x \to 0^+} \frac{\ln(\tan 7x)}{\ln(\tan 2x)}$$

and

$$\lim_{x \to 0} \frac{x \tan x - \sin^2 x}{x^4}$$

Exercise 11. *Let* $f(x) \in D(a, +\infty)$ *and*

$$\lim_{x \to +\infty} [f(x) + f'(x)] = k$$

Prove that

$$\lim_{x \to +\infty} f(x) = k$$

Exercise 12. *Let* $\delta > 0$, *and* $f : (-\delta, \delta) \to \mathbb{R}$ *and* $f^{(n)}(0)$ *exists. Let*

$$f(x) = \sum_{i=0}^{n} a_i x^i + o(x^n)$$

where each a_i *is a Taylor's coefficent. Prove that*

(1) If $f(x)$ *is an odd function, then* $a_i = 0$ *for* i *being even;*

(2) If $f(x)$ *is an even function, then* $a_i = 0$ *for* i *being odd.*

Applying this result to obtain the Taylor's expansion of $\tan x$ *up to* x^7.

Exercise 13. *Prove that* e *is an irrational number.*