MA3077 DLI
All Candidates

Semester 1 Mock Exam 2022

School	School of Computing and Mathematical Sciences
Module Code	MA3077 DLI
Module Title	Operational Research
Exam Duration (in words)	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	5 (including cover page)
Number of Questions	4
Instructions to Candidates	Please answer all questions and motivate your answers.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Yes
Books/Statutes provided by the University	No
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	No

Question 1

a) Consider the following linear programming problem.

$$\begin{array}{cc} max & 6x - y \\ s.t. & 4y - x \geq 10 \\ & 2x + y \leq 10 \\ & 2y - x \leq 5 \\ & x, y \in \mathbb{R} \end{array}$$

i. **[3 Marks]** Draw by hand the feasible set.

ii. **[2 Marks]** Determine graphically an optimal solution and indicate which constraints are active.

iii. **[3 Marks]** Write a Matlab's code based on the function `linprog` to determine an approximate solution.

iv. **[4 Marks]** Write this linear programming problem in standard form.

v. **[5 Marks]** Derive the dual problem of this linear programming problem, identifying clearly the Lagrangian and the dual function.

b) Consider the following optimization problem,

$$\begin{array}{cc} max & \cos(\alpha)\, x + \sin(\alpha)\, y \\ s.t. & (\|(x, y)^T\|_1)^2 \leq 1 \\ & x, y \in \mathbb{R} \end{array}$$

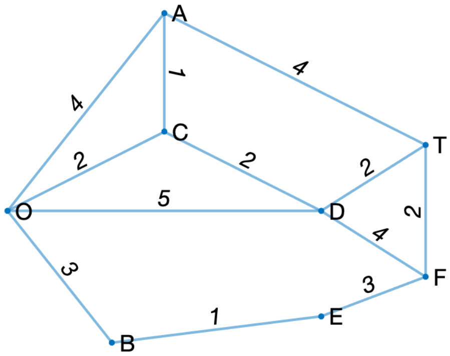
where $\alpha \in [0, 2\pi)$ is a parameter.

i. **[4 Marks]** Reformulate this optimization problem as a linear programming problem.

ii. **[4 Marks]** Determine for which values of $\alpha \in [0, 2\pi)$ does this optimization problem have multiple solutions.

Question 2

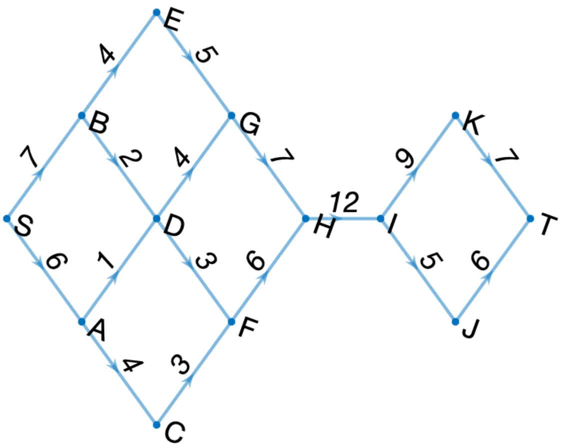
a) Consider the network on the right.



i. **[5 Marks]** Determine and draw a minimal spanning tree starting at node O using Prim's algorithm. Show all intermediate steps.

ii. **[5 Marks]** Determine and draw the shortest path tree starting at node O using Dijkstra's algorithm. Show all intermediate steps.

b) zConsider the directed network N with source S and sink T on the right. Let $f\colon \mathcal{E}(N) \to \mathbb{R}_+$ be defined by



$$f(e) = \begin{cases} 6, & e \in \{SB, GH, HI\} \\ 4, & e \in \{BE, EG, IK, KT\} \\ 2, & e \in \{BD, DG, IJ, JT\} \\ 0, & otherwise \end{cases}$$

i. **[3 pts]** Verify that f is a flow and compute its value.

ii. **[2 pts]** Compute the capacity of the cut induced by the set $\{S, A, B, D\}$ and its complement. Does this result imply that f is not a maximal flow?

iii. **[2 pts]** Identify an f-augmenting path and compute its capacity.

iv. **[4 pts]** Formulate the task of determining a maximal flow of this network as a linear programming problem.

v. **[4 pts]** Prove that any flow $g\colon \mathcal{E}(N) \to \mathbb{R}_+$ satisfies $|g| \le g(HI)$.

Question 3

a) Let the random variable T have exponential distribution with parameter $\alpha = 2$.

i. **[2 marks]** Compute the expected value and the variance of T.

ii. **[3 marks]** Let S be an independent copy of T. Compute the quantities $P[T > 4 | S \leq 3]$ and $P[T + S \leq 5]$.

Hint: if X is a random variable with Erlang distribution with parameters $(n + 1,\ \alpha)$, then

$$P[X \leq x] = 1 - \sum_{k=0}^{n} \frac{(\alpha x)^k \exp(-\alpha x)}{k!}.$$

b) The owner of a local barber shop employs three skilled barbers who work rapidly and provide a range of different haircuts, shaves, and head massages. The owner estimates that, on average, a barber spends 12 minutes with a customer, and that the service times have an exponential distribution. In general, the number of employees seems to be sufficient to deal with the current frequency of customers, because on average there are 12 new customers every hour, and they arrive following a Poisson distribution. However, at times the barber shop can get busy, in which case the owner steps in to take care of an additional customer. However, since the owner works only seldomly, on average it takes them 20 minutes to take care of a customer.

i. **[3 marks]** This queueing system can be modelled as a birth-and-death process. Provide a sketch to describe it and specify the values of the parameters involved.

ii. **[4 marks]** Compute the probability that, in a steady state scenario, the shop is empty.

iii. **[2 marks]** Compute the probability that, in a steady state scenario, the queue is empty.

iv. **[3 marks]** Compute the mean number of customers and the mean length of the queue.

v. **[2 marks]** Compute the mean waiting times in the system and the mean service times.

c) **[6 marks]** A red telephone box can be modelled as a queueing system that can host at most one customer at a time and that has only one server. Assuming that the customers' interarrival time and the service time have both exponential distribution with potentially parameters λ and μ, respectively, and that at time $t = 0$ the telephone box is empty, derive an analytic formula of the probability that the telephone box is empty at time t.

Question 4

a) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) \coloneqq \exp(\cos(x))$. The objective of this exercise is to minimize (at least approximately) the function f.

i. **[3 Marks]** Determine all stationary points of f in $\mathbb{R}$.

ii. **[3 Marks]** Perform one step of Newton's method starting from $x_0 = 2$.

iii. **[4 Marks]** Perform one step of the steepest descent method starting from $x_0 = 2$ and using the smallest positive optimal step size.

b) **[7 Marks]** For a fixed number $n \in \mathbb{N}$, consider the function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) \coloneqq (Ax - b)^T(Ax - b)$, where $A \in \mathbb{R}^{n,n}$ is an invertible matrix and $b \in \mathbb{R}^n$ is a given vector. Perform one step of Newton's method using a generic starting point $x_0 \in \mathbb{R}^n$.

c) **[8 Marks]** Consider the constrained optimization problem

$$\min_{x \in \mathbb{R}^2} x_1^2 + x_2 \ \text{ subject to } \ x_2 - \cos(x_1) = 0$$

Perform one step of the quadratic penalty method using the penalty parameter $p_0 = 2$. To solve the internal iteration, use one step of the steepest descent method with initial guess $x_0 = (0,0)^T \in \mathbb{R}^2$ and step size $\alpha = 1$.

END OF PAPER