UNIVERSITY OF LEICESTER

MA3077 MA4077 MA7077
All Candidates

Semester 1 Examinations 2023

School	School of Computing and Mathematical Sciences
Module Code	MA3077-MA4077-MA7077
Module Title	Operational Research
Exam Duration (in words)	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	5 (including cover page)
Number of Questions	4
Instructions to Candidates	Please answer all questions and motivate your answers.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Permitted calculators are the Casio FX83 and FX85 models
Books/Statutes provided by the University	No
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	No

Question 1

a) **[2 Marks]** Explain why the following optimisation problem is not a linear programming problem.

$$\min e^{x+y} \ \ subject\ to\ x \le y.$$

b) Consider the following linear programming problem.

$$\max y - 2x \quad subject\ to \begin{cases} x - y \le 1 \\ 4y \le 4 - x \\ x \ge 0 \end{cases} \tag{1}$$

i. **[6 Marks]** Draw the feasible set, determine an optimal solution, and indicate which constraints are active.

ii. **[4 Marks]** Write the linear programming problem (1) in standard form.

iii. **[5 Marks]** Derive the dual problem of the linear programming problem you formulated in ii (the previous question), identifying clearly the Lagrangian and the dual function.

c) **[3 Marks]** The Matlab code

```
x = linprog([-1;2], [1, 2; -2, 3; 3, -5], [1; 0; 2]);
```

solves a linear programming problem. Identify the objective function and all constraints of this linear programming problem.

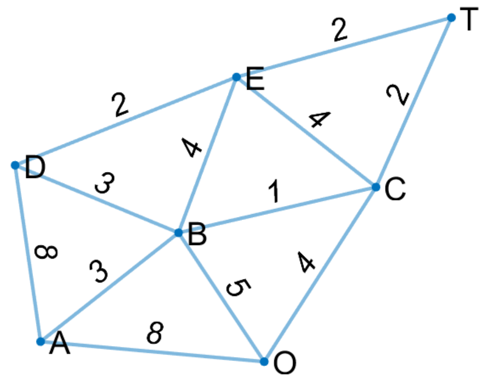
d) **[5 Marks]** Let p^* denote the optimal objective value of the linear programming problem

$$\min x \quad subject\ to\ 0 \le x \le -1,$$

and let d^* denote the optimal objective value of its dual. Prove that $p^* \neq d^*$.

Question 2

a) Consider the following network.



i. **[3 Marks]** Explain why this network is not a tree.

ii. **[6 Marks]** Determine and draw a minimal spanning tree starting at node O using the version of Prim's algorithm presented in class. Show all intermediate steps.

b) Consider the following output of Dijkstra's algorithm:

	O	A	B	C	D	E	F	G	T
d	0	3	1	2	4	4	7	5	6
p	$\emptyset$	C	O	B	C	B	E	A	D
v	1	1	1	1	1	1	1	1	1

i. **[3 marks]** Explain the meaning of the vectors, d, p, and v.

ii. **[6 marks]** Draw the corresponding shortest path three and indicate the capacity of each edge.

c) **[7 Marks]** Prove or disprove the following statement:

If a two-person zero-sum game with payoff matrix M is stable, then the two-person zero-sum game with payoff matrix M^T is also stable.

Question 3

a) **[7 marks]** Write the definition of the queueing model $M/M/s/K$. Then, assuming that $s < K$, represent the $M/M/s/K$ queueing model as a birth-and-death process by drawing the corresponding sketch and by specifying the formulas of the parameters $\lambda_n, n \geq 0$ and $\mu_n, n \geq 1$ involved.

b) Let a queueing system be described by a birth-and-death process with parameters

$$\lambda_i = \max(6 - i, 4) \text{ for } i \geq 0, \qquad and \;\; \mu_1 = \mu_2 = 1, \qquad \mu_i = 5 \text{ for } i \geq 3$$

i. **[4 marks]** Show that in a steady state scenario, the probability that the queueing system is empty is $p_0 = (157)^{-1}$.

ii. **[2 mark]** Compute the probability that, in a steady state scenario, the system contains at least 2 customers.

iii. **[6 marks]** Compute the mean number of customers L and the mean waiting time W.

c) **[6 marks]** Prove that in an M/M/1 model, the following equality

$$\rho + \rho L - L = 0$$

holds, where $\rho = \frac{\lambda}{\mu} < 1$ is the service facility utilisation factor and L is the mean number of customers.

[*Hint:* prove first that $p_0 = 1 - \rho$]

Question 4

a) Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2 + 2\exp(-x^2)$. The objective of this exercise is to minimise (at least approximately) the function f.

i. **[4 Marks]** Determine all stationary points of f in $\mathbb{R}$.

ii. **[4 Marks]** Perform one step of Newton's method starting from $x_0 = 1$.

b) **[7 Marks]** For $n \in \mathbb{N}$ fixed, consider the function $f: \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \cos(x^T x)$. Perform one step of Newton's method using the starting point $x_0 = (1,0, \dots ,0) \in \mathbb{R}^n$.

c) **[10 Marks]** Consider the constrained optimisation problem

$$\min_{x \in \mathbb{R}^2} (x_1 + x_2)^2 \quad \text{subject to} \quad x_1 + x_2 - 1 = 0$$

Perform one step of the quadratic penalty method using the penalty parameter $p_0 = 2$. To solve the internal iteration, use one step of the steepest descent method with initial guess $x_0 = (0,0)^T \in \mathbb{R}^2$ and exact line search.

END OF PAPER