Exercise 1 (10 marks)**.** *(1) Since $\{a_n\}$ is bounded, from Bolzano-Weiestrass Theorem we know there exists a subsequence $\{a_{n_i}\}$ being convergent. [1 mark] Since $\{b_n\}$ is bounded, its subsequence $\{b_{n_i}\}$ is also bounded. [1 mark] Hence from Bolzano-Weiestrass Theorem there exists a subsequence $\{b_{n_k}\}$ of $\{b_{n_i}\}$ being convergent. [1 mark] On the other hand, $\{a_{n_k}\}$, as a subsequence of $\{a_{n_i}\}$, is also convergent, as desired. [1 mark]*

(The following proof is incoreect: from Bolzano-Weiestrass Theorem we know there exists $\{n_i\}$ and $\{n_j\}$ such that $\{a_{n_i}\}$ and $\{b_{n_j}\}$ are convergent. Setting $\{n_k\} = \{n_i\} \cap \{n_j\}$ as desired.)

(2) $\Rightarrow$: Since $f(x)$ is uniformaly continuous on I, then

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in I \text{ satisfying } |x_1 - x_2| < \delta, |f(x_1) - f(x_2)| < \epsilon$$

Since $\{a_n\}$ is Cauchy, for the previous $\delta > 0$, $\exists N \in \mathbb{N}$, $\forall m, n > N$,

$$|a_n - a_m| < \delta$$

Thus

$$|f(x_n) - f(x_m)| < \epsilon$$

that is, $\{f(x_n)\}$ is Cauchy. [2 marks]

$\Leftarrow$: We argue this by contradiction. Assume that $f(x)$ is not uniformly continuous on I, then

$$\exists \epsilon > 0, \forall \delta > 0, \exists x, y \in I \text{ satisfying } |x - y| < \delta, |f(x) - f(y)| \geq \epsilon$$

Letting $\delta = \frac{1}{n}$, then $\exists \{x_n\}, \{y_n\} \subset I$ such that $|x_n - y_n| < \frac{1}{n}$ and

$$|f(x_n) - f(y_n)| \geq \epsilon$$

Since I is bounded, we know both $\{x_n\}$ and $\{y_n\}$ are bounded. By (1) we know there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that they are both convergent to a pair of real numbers, say x_0 and y_0, respectively. By $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$ and squeeze theorem, we know $x_0 = y_0$. Now let us construct a new sequence

$$z_n = \begin{cases} x_{\frac{n+1}{2}}, & n \text{ is odd} \\ y_{\frac{n}{2}}, & n \text{ is even} \end{cases}$$

then $\{z_n\}$ *is convergent because both its odd subsequence and even subsequence are convergent to the same limit. But we know*

$$|f(z_{n+1}) - f(z_n)| \geq \epsilon$$

so $\{f_{z_n}\}$ *is not Cauchy, a contradiction. [4 marks]*

Exercise 2. *The answer is True. We have the example:*

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

[1 mark]

On the one hand, $f(x)$ *is continuous at* $x_0 = 0$*. In fact, we know that* $f(0) = 0$*. Also,* $\forall\epsilon > 0$*, setting* $\delta = \epsilon$*,* $\forall x : |x| < \delta$*,*

if $x \in \mathbb{Q}$*, then*

$$|f(x) - f(0)| = |x - 0| < \delta = \epsilon$$

if $x \in \mathbb{R}\backslash\mathbb{Q}$*, then*

$$|f(x) - f(0)| = |0 - 0| < \epsilon$$

thus

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

[2 marks]

On the other hand, $f(x)$ *is not continuous anywhere else. Let* $x_0 \in \mathbb{R}\backslash\{0\}$*, and w.l.o.g we assume that* $x_0 > 0$*.*

If $x_0 \in \mathbb{Q}$*: letting* $\epsilon = \frac{x_0}{2}$*,* $\forall\delta \in (0, \epsilon)$*, select* $x \in (x_0 - \delta, x_0 + \delta) \cap \mathbb{R}\backslash\mathbb{Q}$*, then*

$$|f(x) - f(x_0)| = |x - 0| = x > x_0 - \frac{x_0}{2} = \epsilon$$

If $x_0 \in \mathbb{R}\backslash\mathbb{Q}$*: letting* $\epsilon = \frac{x_0}{2}$*,* $\forall\delta \in (0, \epsilon)$*, select* $x \in (x_0 - \delta, x_0 + \delta) \cap \mathbb{Q}$*, then*

$$|f(x) - f(x_0)| = |x - 0| = x > x_0 - \frac{x_0}{2} = \epsilon$$

In both cases, we know the function is not continuous at x_0*. [2 marks]*

Exercise 3. *Firstly we claim that* $\{b_n\}$ *is bounded with* $0 < b_n < \frac{1}{A}$*. We prove this by induction. We know* $0 < b_0 < \frac{1}{A}$*. Assume that* $0 < b_n < \frac{1}{A}$*. Since the functionj* $f(x) = x(2 - Ax)$ *is increasing on* $[0, \frac{1}{A}]$*, we know*

$$0 < b_{n+1} < \frac{1}{A}(2 - A\frac{1}{A}) = \frac{1}{A}$$

[2 marks]

Then we claim that $\{b_n\}$ *is increasing. In fact we have*

$$\frac{b_{n+1}}{b_n} = 2 - Ab_n > 2 - A\frac{1}{A} = 1$$

[1 mark]

By monotonic convergence theorem, we know the sequence $\{b_n\}$ *is convergent. We denote the limit by* b^**. Then we know*

$$b^* = b^*(2 - Ab^*)$$

We claim that $b^* \neq 0$*, because* $b^* \geq b_0 > 0$ *and the order-preserving property. So* $b^* = \frac{1}{A}$*. [2 marks]*