# Computational Project

### Aims and Intended Learning Outcomes

The aims of the Project are to describe methods for solving given computational problems, develop and test Matlab code implementing the methods, and demonstrate application of the code to solving a specific computational problem. In this Project, you be will be required to demonstrate

- ability to investigate a topic through guided independent research, using resources available on the internet and/or in the library;
- understanding of the researched material;
- implementation of the described methods in Matlab;
- use of the implemented methods on test examples;
- ability to present the studied topic and your computations in a written Project Report.

### Plagiarism and Declaration

- This report should be your independent work. You should not seek help from other students or provide such help to other students. All sources you used in preparing your report should be listed in the References section at the end of your report and referred to as necessary throughout the report.
- Your Project Report must contain the following Declaration (after the title page):

> **DECLARATION**
>
> All sentences or passages quoted in this Project Report from other people's work have been specifically acknowledged by clear and specific cross referencing to author, work and page(s), or website link. I understand that failure to do so amounts to plagiarism and will be considered grounds for failure in this module and the degree as a whole.
>
> Name:
>
> Signed: (name, if submitted electronically)
>
> Date:

### Project Report

The report should be about 6-8 pages long, written in Word or Latex. Equations should be properly formatted and cross-referenced, if necessary. All the code should be included in the report. Copy and paste from MATLAB Editor or Command Window and choose 'Courier New' or another fixed-width font. The Report should be submitted via Blackboard in a single file (Word document or Adobe PDF) and contain answers to the following questions:

## Part 0: Context

Let $f(x)$ be a periodic function. The goal of this project is to implement a numerical method for solving the following family of ordinary differential equations (O.D.E):

$$a_n \frac{d^n u(x)}{dx^n} + a_{n-1} \frac{d^{n-1} u(x)}{dx^{n-1}} + \ldots + a_0 u(x) = f(x), \tag{1}$$

where $a_k$, $k = 0, \cdots, n$, are real-valued constants. The differential equation is complemented with periodic boundary conditions:

$$\frac{d^k u(-\pi)}{dx^k} = \frac{d^k u(\pi)}{dx^k}$$

for $k = 0, \cdots, n-1$.

We aim to solve this problem using a trigonometric function expansion.

## Part 1: Basis of trigonometric functions

Let $u(x)$ be a periodic function with period $2\pi$. There exist coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots$, and $\beta_1, \beta_2, \ldots$ such that

$$u(x) = \sum_{k=0}^{\infty} \alpha_k \cos(kx) + \sum_{1}^{\infty} \beta_k \sin(kx).$$

The coefficients $\alpha_k$ and $\beta_k$ can be found using the following orthogonality properties:

$$\int_{-\pi}^{\pi} \cos(kx) \sin(nx)\, dx = 0, \text{ for any } k, n$$

$$\int_{-\pi}^{\pi} \cos(kx) \cos(nx)\, dx = \begin{cases} 0 \text{ if } k \neq n \\ \pi \text{ if } k = n \neq 0 \\ 2\pi \text{ if } k = n = 0. \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(kx) \sin(nx)\, dx = \begin{cases} 0 \text{ if } k \neq n \\ \pi \text{ if } k = n \neq 0. \end{cases}$$

1. **[5 pts]** Implement a function that takes as an input two function handles $f$ and $g$, and an array $x$, and outputs the integral

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx,$$

using your own implementation of the Simpson's rule scheme. Corroborate numerically the orthogonality properties above for different values of $k$ and $n$.

2. **[4 pts]** Show that

$$\alpha_k = \begin{cases} \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \cos(kx)\, dx \text{ if } k \neq 0 \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\, dx \text{ if } k = 0 \end{cases}$$

$$\beta_k = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \sin(kx)\, dx.$$

3. **[4 pts]** Using question 1 and 2, write a function that given a function handle $u$ and an integer $m$, outputs the array $[\alpha_0,\, \alpha_1\, \dots, \alpha_m, \beta_1,\, \dots, \beta_m]$.

4. **[4 pts]** Write a function that given an array $[\alpha_0,\, \alpha_1\, \dots, \alpha_m, \beta_1,\, \dots, \beta_m]$, outputs (in the form of an array) the truncated series

$$u_m(x) := \sum_{k=0}^{m} \alpha_k \cos(kx) + \sum_{k=1}^{m} \beta_k \sin(kx), \tag{2}$$

where $x$ is a `linspace` array on the interval $[-\pi, \pi]$.

5. **[5 pts]** Using the function from question 3, compute the truncated series $u_m(x)$ of the following functions:

   - $u(x) = \sin^3(x)$
   - $u(x) = |x|$
   - $u(x) = \begin{cases} x + \pi, & \text{for } x \in [-\pi, 0] \\ x - \pi, & \text{for } x \in (0, \pi] \end{cases}$ ,

   and using question 4, plot $u(x)$ and $u_m(x)$ for different values of $m$.

6. **[5 pts]** Carry out a study of the error between $u(x)$ and $u_m(x)$ for $\|u(x) - u_m(x)\|_p$ with $p = 2$ and then with $p = \infty$. What do you observe?

## Part 2: Solving the O.D.E

Any given periodic function $u(x)$ can be well approximated by its truncate series expansion (2) if $m$ is large enough. Thus, to solve the ordinary differential equation (1) one can approximate $u(x)$ by $u_m(x)$:

$$u(x) \approx \sum_{k=0}^{m} \alpha_k \cos(kx) + \sum_{k=1}^{m} \beta_k \sin(kx),$$

Since $u_m(x)$ is completely determined by its coefficients $[\alpha_0,\, \alpha_1\, \dots, \alpha_m, \beta_1,\, \dots, \beta_m]$, to solve (1) numerically, one could build a system of equations for determining these coefficients.

7. **[4 pts]** Explain why under the above approximation, the boundary conditions of (1) are automatically satisfied.

8. **[5 pts]** We have that

$$\frac{du_m(x)}{dx} = \sum_{k=0}^{m} \gamma_k \cos(kx) + \sum_{k=1}^{m} \eta_k \sin(kx)$$

Write a function that takes as input the integer $m$, and outputs a square matrix $D$ that maps the coefficients $[\alpha_0,\, \dots, \alpha_m, \beta_1,\, \dots, \beta_m]$ to the coefficients $[\gamma_0,\, \dots, \gamma_m, \eta_1,\, \dots, \eta_m]$.

9. **[5 pts]** Write a function that given a function handler $f$, an integer $m$, and the constants $a_k$, solves the O.D.E. (1). Note that some systems might have an infinite number of solutions. In that case your function should be able identify such cases.

10. **[5 pts]** $u(x) = \cos(\sin(x))$ is the exact solution for

$$f(x) = \sin(x)\sin(\sin(x)) - \cos(\sin(x))\left(\cos^2(x) + 1\right),$$

with $a_2 = 1$, $a_0 = -1$ and $a_k = 0$ otherwise. Plot the $p = 2$ error between your numerical solution and $u(x)$ for $m = 1, 2, \ldots$. Use a log-scale for the $y$-axis. At what rate does your numerical solution converge to the exact solution?

11. **[4 pts]** Show your numerical solution for different $f(x)$ and different $a_k$ of your choice.