MA1014 Examination — Draft Questions and Solutions

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Title of paper	MA1014 — Calculus and Analysis
Version	1
Candidates	All candidates
School	COMPUTING AND MATHEMATICAL SCIENCES
Examination Session	Final Test 2022
Time allowed	2 hours
Instructions	This paper contains 4 questions. Full marks are 100 marks. Please attempt all questions.
Calculators	No
Books/statutes	No
Own Books/statutes/notes	No
Additional Stationery	Yes
Number of questions	4

In this exam, you are free to use properties of limit, continuity of elementary functions, the facts

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

and

$$\lim_{x\to +\infty}(1+\frac{1}{x})^x=e$$

without proof unless explicitly stated.

You may use any results from the course that you state correctly.

Question 1.

(a) Evaluate

$$\int \ln(1+x^2)dx$$

[5 marks]

(b) Let n be an integer such that $n > 2$,

$$I_n = \int \frac{\sin(nx)}{\sin x}dx$$

Prove that

$$I_n = \frac{2}{n-1}\sin((n-1)x) + I_{n-2}$$

[6 marks]

(c) Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \frac{1}{1+x}$. For each integer $n \geq 1$, define a partition $P_n = \left(0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right)$ of the interval $[0,1]$. Write an expression for each of the Darboux sums $L(f,P_n)$ and $U(f,P_n)$. NO simplification of the sums required. **[6 marks]**

(d) Using (c) above, or otherwise, evaluate

$$\lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} + \frac{1}{\sqrt{(n+1)(n+2)}} + \cdots + \frac{1}{\sqrt{(2n-1)2n}}$$

[8 marks]

Answer — total marks for this question: 25

(a) By integration by parts,

$$\begin{aligned}\int \ln(1+x^2)dx &= x\ln(1+x^2) - 2\int \frac{x^2}{1+x^2}dx \\ &= x\ln(1+x^2) - 2x + 2\arctan x + C\end{aligned}$$

(b)

$$\begin{aligned}I_n &= \int \frac{\sin(n-1)x\cos x + \sin x\cos(n-1)x}{\sin x}\,\mathrm{d}x \\ &= \int \frac{\sin(n-1)x\cos x}{\sin x}\,\mathrm{d}x + \int \cos(n-1)x\,\mathrm{d}x \\ &= \frac{1}{2}\int \frac{\sin nx + \sin(n-2)x}{\sin x}\,\mathrm{d}x + \int \cos(n-1)x\,\mathrm{d}x \\ &= \frac{1}{2}I_n + \frac{1}{2}I_{n-2} + \frac{1}{n-1}\sin(n-1)x\end{aligned}$$

So $I_n = \frac{2}{n-1}\sin(n-1)x + I_{n-2}$.

(c) Since $f(x)$ is decreasing on $[0,1]$, we have

$$L(f,P_n) = \frac{1}{n}\left(\frac{1}{1} + \frac{1}{1+\frac{1}{n}} + \cdots + \frac{1}{1+\frac{n_1}{n}}\right)$$

and

$$U(f,P_n) = \frac{1}{n}\left(\frac{1}{1+\frac{1}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right)$$

(d) We have

$$\begin{aligned}&\frac{1}{n+1} + \cdots \frac{1}{2n} \\ \leq &\frac{1}{\sqrt{n(n+1)}} + \frac{1}{\sqrt{(n+1)(n+2)}} + \cdots + \frac{1}{\sqrt{(2n-1)2n}} \\ \leq &\frac{1}{n} + \cdots + \frac{1}{2n-1}\end{aligned}$$

By (c), both of the limits are $\int_0^1 \frac{1}{1+x}dx = \ln 2$. By squeeze theorem,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} + \frac{1}{\sqrt{(n+1)(n+2)}} + \cdots + \frac{1}{\sqrt{(2n-1)2n}} = \ln 2$$

Question 2.

In this question you may use any results from the course that you state correctly.

(a) Prove that

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln(x+\sqrt{x^2+1})+C$$

[7 marks]

(b) Solve

$$y''+2y'+y=e^{-x}$$

[8 marks]

(c) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be differentiable, and

$$\varphi(x)\cos x + 2\int_0^x \varphi(t)\sin t\,dt = x+1$$

Solve $\varphi(x)$. **[10 marks]**

Answer — total marks for this question: 25

(a) Letting $x = \tan t$, then $dx = \sec t dt$. So

$$\begin{aligned}\int \frac{dx}{\sqrt{1+x^2}} &= \int \sec t dt \\ &= \int \frac{d(\sin t)}{1-\sin^2 t} \\ &= \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right| + C \\ & (\sin t = \frac{x}{\sqrt{1+x^2}}) \\ &= \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right| \\ &= \ln(x+\sqrt{x^2+1}) + C\end{aligned}$$

(b) The characteristic equation to the homogenous equation is

$$\lambda^2 + 2\lambda + 1 = 0$$

Hence the general solution to the homogenous equation is

$$y = (C_1 + C_2 x)e^{-x}$$

Set one of the particular solution is in the form

$$y_p = Ax^2 e^{-x}$$

Inserting back to the equation, we have $2A = 1$, so $A = \frac{1}{2}$. So the general solution to the original equation is

$$y = (C_1 + C_2 x)e^{-x} + \frac{1}{2}x^2 e^{-x}$$

(c) By fundemental theorem of calculus, $(\int_0^x \varphi(t)\sin t dt)' = \varphi(x)\sin x$. By taking derivatives on both sides, we have

$$\varphi'(x)\cos x + \varphi(x)\sin x = 1$$

So the general solution is

$$\varphi(x) = \sin x + C\cos x$$

Since $\varphi(0) = 1$, we have $C = 1$, so the solution is

$$\varphi(x) = \sin x + \cos x$$

Question 3.

(a) State the comparison test for series. **[3 marks]**

(b) By using the comparison test, or otherwise, determine whether the series is convergent and justify your answer:

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

[10 marks]

(c) Determine the range of x values for which the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n(n-1)}$$

converges. You should also clarify whether the convergence is absolute or conditional. **[6 marks]**

(d) Compute

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n(n-1)}$$

for x in the convergent range. **[6 marks]**

Answer — total marks for this question: 25

(a) Let $\{a_n\}$, $\{b_n\}$ be non-negative sequences such that $a_n \le b_n$ for all $n \in \mathbb{N}$. Then
(1) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent;
(2) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

(b) Observe that $(\ln n)^{\ln n} = n^{\ln(\ln n)}$. Since $\lim_{n\to\infty} \ln(\ln n) = +\infty$, there exists N_1, $\forall n > N_1$, $\ln(\ln n) > 3$. Then for $n > N_1$,

$$0 \le \frac{\frac{1}{n^{\ln(\ln n)}}}{\frac{1}{n^2}} < \frac{1}{n}$$

So by squeeze theorem,

$$\lim_{n\to\infty} \frac{\frac{1}{n^{\ln(\ln n)}}}{\frac{1}{n^2}} = 0$$

So $\exists N_2 > N_1$, for $n > N_2$,

$$\frac{1}{n^{\ln(\ln n)}} \le \frac{1}{2}\frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ is convergent, by comparison test,

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

is convergent.

(c) The convergent radius is 1, and the series is convergent when $x = -1$ and $x = 1$, so the range is $[-1, 1]$.

(d)

$$S'(x) = \sum_{n=2}^{\infty} (-1)^n \frac{x^{n-1}}{n-1}$$

$$S''(x) = \sum_{n=2}^{\infty} (-1)^n x^{n-2} = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

Since $S'(0) = 0$, then

$$S'(x) = \int_0^x S''(t)\mathrm{d}t = \int_0^x \frac{\mathrm{d}t}{1+t} = \ln(1+x), x \in (-1, 1)$$

Also $S(0) = 0$,

$$S(x) = \int_0^x \ln(1+t)\mathrm{d}t = (x+1)\ln(1+x) - x, x \in (-1, 1)$$

Since $S(x)$ is continuous on $[-1, 1]$, we have

$$S(x) = \int_0^x \ln(1+t)\mathrm{d}t = (x+1)\ln(1+x) - x, x \in (-1, 1]$$

and $S(-1) = \lim_{x\to -1^+}[(x+1)\ln(1+x) - x] = 1$.

Question 4.

(a) Give the definition of an open set in $\mathbb{R}^n$. **[2 marks]**

(b) Give the definition of a two-variable function $f(x,y)$ being differentiable at a point (x_0, y_0) in the domain. **[2 marks]**

(c) Prove that the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} \quad , & (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$, but not differentiable at $(0,0)$. **[8 marks]**

(d) Let $u = f(x,y)$ be a two-variable function such that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous on $\mathbb{R}^2$. Prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

where $x = \rho\cos\theta, \quad y = \rho\sin\theta$ is the transformation of polar coordinate. **[8 marks]**

(e) Let $f(x,y) = 4(x-y) - x^2 - y^2$. Compute all the local maxima and local minima of f on $\mathbb{R}^2$. **[5 marks]**

Answer — total marks for this question: 25

(a) $\Omega \subset \mathbb{R}^n$ is said to be an open set, if $\forall x \in \Omega$, $\exists \delta > 0$ such that $B(x,\delta) \subset \Omega$.

(b) $f(x,y)$ is differentiable at a point (x_0,y_0) if there exists $A \in \mathbb{R}^2$ such that

$$\lim_{x\to x_0, y\to y_0} \frac{|f(x,y)-f(x_0,y_0)-(x-x_0,y-y_0)\cdot A|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

(c) By Cauchy's inequality, $x^2+y^2 \geq 2|xy|$. Then

$$0 \leq \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{|xy|}{\sqrt{2|xy|}} = \frac{\sqrt{|xy|}}{2} \to 0$$

So $f(x,y)$ is continuous at $(0,0)$. Also we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$$

and similarly $\frac{\partial f}{\partial y}(0,0) = 0$. So

$$\begin{aligned}&\lim_{x\to x_0, y\to y_0} \frac{|f(x,y)-f(x_0,y_0)-(x-x_0,y-y_0)\cdot A|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\\ =&\lim_{x\to 0, y\to 0} \frac{xy}{x^2+y^2}\end{aligned}$$

does not exist, as we can take $y = kx$ and the limit depends on k.

(d) From $x = \rho\cos\theta, \quad y = \rho\sin\theta$, we have $\rho = \sqrt{x^2+y^2}, \quad \theta = \arctan\frac{y}{x}$. By chain rule, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial \rho}\frac{x}{\rho} - \frac{\partial u}{\partial \theta}\frac{y}{\rho^2} = \frac{\partial u}{\partial \rho}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{\rho}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial \rho}\frac{y}{\rho} + \frac{\partial u}{\partial \theta}\frac{x}{\rho^2} = \frac{\partial u}{\partial \rho}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{\rho}$$

Squaring the previous identities and adding them, we have

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(e) We have $\frac{\partial f}{\partial x} = 4-2x$, $\frac{\partial f}{\partial y} = -4-2y$. Letting they be equal to 0, we have $x=2$ and $y=-2$. At $(2,-2)$, $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -2$, and $\frac{\partial^2 f}{\partial x \partial y} = 0$. So the point $(2,-2)$ is a local maximum point.