UNIVERSITY OF LEICESTER

MA1014
All candidates

Final Test 2022

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
School	COMPUTING AND MATHEMATICAL SCIENCES
Module Code	MA1014
Module Title	Calculus and Analysis
Exam Duration	2 hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	3
Number of Questions	4
Instructions to Candidates	This paper contains 4 questions. Full marks are 100 marks. Please attempt all questions.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	No
Books/Statutes provided by the University	No
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	Yes

In this exam, you are free to use properties of limit, continuity of elementary functions, the facts

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

and

$$\lim_{x\to+\infty}(1+\frac{1}{x})^x=e$$

without proof unless explicitly stated.

You may use any results from the course that you state correctly.

1. (a) Evaluate

$$\int \ln(1+x^2)dx$$

[5 marks]

(b) Let n be an integer such that $n>2$,

$$I_n=\int\frac{\sin(nx)}{\sin x}dx$$

Prove that

$$I_n=\frac{2}{n-1}\sin((n-1)x)+I_{n-2}$$

[6 marks]

(c) Define $f:[0,1]\to\mathbb{R}$ by $f(x)=\frac{1}{1+x}$. For each integer $n\geq 1$, define a partition $P_n=\left(0,\frac{1}{n},\dots,\frac{n-1}{n},1\right)$ of the interval $[0,1]$. Write an expression for each of the Darboux sums $L(f,P_n)$ and $U(f,P_n)$. NO simplification of the sums required. **[6 marks]**

(d) Using (c) above, or otherwise, evaluate

$$\lim_{n\to\infty}\frac{1}{\sqrt{n(n+1)}}+\frac{1}{\sqrt{(n+1)(n+2)}}+\cdots+\frac{1}{\sqrt{(2n-1)2n}}$$

[8 marks]

2. In this question you may use any results from the course that you state correctly.

(a) Find the general solution of

$$y'x=y\ln y$$

[7 marks]

(b) Solve

$$y''+2y'+y=e^{-x}$$

[8 marks]

(c) Let $\varphi:\mathbb{R}\to\mathbb{R}$ be differentiable, and

$$\varphi(x)\cos x+2\int_0^x\varphi(t)\sin t\,dt=x+1$$

Solve $\varphi(x)$. **[10 marks]**

3. (a) State the comparison test for series. **[3 marks]**

   (b) Prove that

$$\lim_{n\to\infty} \frac{n^2}{(\ln n)^{\ln n}} = 0$$

[6 marks]

   (c) By using the comparison test and (b) above, or otherwise, determine whether the series is convergent and justify your answer:

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

[4 marks]

   (d) Determine the range of x values for which the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n(n-1)}$$

converges. You should also clarify whether the convergence is absolute or conditional. **[6 marks]**

   (e) Compute

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n(n-1)}$$

for x in the convergent range. **[6 marks]**

4. (a) Give the definition of an open set in $\mathbb{R}^n$. **[2 marks]**

   (b) Give the definition of a two-variable function $f(x,y)$ being differentiable at a point (x_0, y_0) in the domain. **[2 marks]**

   (c) Prove that the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$, but not differentiable at $(0,0)$. **[8 marks]**

   (d) Let $u = f(x,y)$ be a two-variable function such that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous on $\mathbb{R}^2$. Prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

where $x = \rho\cos\theta, \quad y = \rho\sin\theta$ is the transformation of polar coordinate. Hint: you may consider u as a function of ρ and θ, and both ρ and θ are functions of x and y.

[8 marks]

   (e) Let $f(x,y) = 4(x-y) - x^2 - y^2$. Compute all the local maxima and local minima of f on $\mathbb{R}^2$. **[5 marks]**