Exercise 1. *This is because*

$$\begin{aligned} &\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - 0| < \epsilon \\ \iff &\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n| < \epsilon \\ \iff &\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, ||a_n| - 0| < \epsilon \end{aligned}$$

Exercise 2. $\forall \epsilon > 0$, *setting* $N = [\frac{1}{\epsilon}]$, $\forall n > N$,

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \frac{1}{N+1} = \frac{1}{[\frac{1}{\epsilon}]+1} < \epsilon$$

Exercise 3. *No. We have the counterexample* $a_n = (-1)^n$. *It is bounded, but not convergent.*

Exercise 4. *(1) No. Setting* $a_n = \frac{1}{n}$. *Then* $a_n > 0$, *but we know*

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

(2) We can sey that $\{a_n\}$ *is eventually positive if* $a > 0$. *Rigorously speaking, this means*

$$\exists N, \forall n > N, a_n > 0$$

No information of the sign property can be obtained if $a = 0$.

Exercise 5. *(1) We argue this by contradiction. Assume that* $\{a_n + b_n\}$ *is convergent, then by the arithmetic property,*

$$b_n = a_n + b_n - a_n$$

would be convergent, a contradiction.
(2) We cannot say anything about the convergence or divergence. In fact, If

$$a_n = (-1)^n \quad b_n = -(-1)^n$$

then both are divergent while the sum is convergent. If

$$a_n = (-1)^n \quad b_n = n$$

then both are divergent while the sum is divergent (why?)

Exercise 6. *(1) By the arithmetic property,*

$$\lim_{n\to\infty} \frac{n^2+2n-1}{2n^2-4n-6} = \lim_{n\to\infty} \frac{1+\frac{2}{n}-\frac{1}{n^2}}{2-\frac{4}{n}-\frac{6}{n^2}} = \frac{1}{2}$$

(2) $\forall \epsilon > 0$, *setting* $N = \max\{6, \left[\frac{4}{\epsilon}\right]+1\}$, $\forall n > N$,

$$\begin{aligned}\left|\frac{n^2+2n-1}{2n^2-4n-6} - \frac{1}{2}\right| &= \left|\frac{2n+1}{n^2-2n-3}\right| \\ &= \frac{2n+1}{n^2-2n-3} \\ &< \frac{2n+2}{n^2-2n-3} \\ &= \frac{2}{n-3} \\ &< \frac{4}{n} < \frac{4}{N} \\ &= \frac{4}{\left[\frac{4}{\epsilon}\right]+1} < \epsilon\end{aligned}$$

Exercise 7. *We have*

$$\begin{aligned}&\frac{n}{n+\sqrt{n}} \\ \leq &\frac{1}{n+\sqrt{1}} + \cdots + \frac{1}{n+\sqrt{n}} \\ \leq &\frac{n}{n+\sqrt{1}}\end{aligned}$$

Applying a similar argument as the previous exercise, we know

$$\lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{n}{n+\sqrt{1}} = 1$$

By squeeze theorem, we know

$$\lim_{n\to\infty} \left(\frac{1}{n+\sqrt{1}} + \cdots + \frac{1}{n+\sqrt{n}}\right) = 1$$

Exercise 8. *We prove this by case.*

Case 1: $a = 1$. *This is obvious because the sequence becomes a constant sequence under this case.*

Case 2: $a > 1$. Define $\beta_n = a^n - 1$, then $\beta_n > 0$. Hence

$$a = (1+\beta_n)^n = 1 + n\beta_n + \cdots > n\beta_n$$

Thus

$$0 < \beta_n < \frac{a}{n}$$

By squeeze theorem, $\lim_{n\to\infty} \beta_n = 0$, hence

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1$$

Exercise 9. *We assume $a = 0$ for the moment. Then $\forall \epsilon > 0, \exists N_1, \forall n > N_1, |a_n| < \epsilon$. Since*

$$\lim_{n\to\infty} \frac{a_1 + \cdots a_{N_1}}{n} = 0,$$

$\exists N > N_1, \forall n > N, \frac{a_1+\cdots a_{N_1}}{n} < \epsilon$. Hence

$$\begin{aligned}\left|\frac{a_1 + \cdots\cdots a_{N_1} + a_{N_1+1} + \cdots a_n}{n}\right| &\le \left|\frac{a_1 + \cdots\cdots a_{N_1}}{n}\right| + \left|\frac{a_{N_1+1} + \cdots a_n}{n}\right| \\ &< \epsilon + \frac{n - N_1 + 1}{n}\epsilon \le 2\epsilon,\end{aligned}$$

as desired. If $a \neq 0$, then we define $b_n = a_n - a$.