Exercise 1. *$\forall\varepsilon > 0$, $\forall x_0 \in \mathbb{R}$, setting $\delta = \varepsilon$. When $x \in (x_0 - \delta, x_0 + \delta)$,*

$$\begin{aligned}|f(x) - f(x_0)| =& |\cos x - \cos x_0| \\ =& |-2\sin\frac{x + x_0}{2}\sin\frac{x - x_0}{2}| \\ \leq& 2|\sin\frac{x - x_0}{2}| \\ \leq& 2|\frac{x - x_0}{2}| \\ =& |x - x_0| < \delta = \varepsilon\end{aligned}$$

Exercise 2. *Since*

$$\lim_{x\to+\infty} f(x) = A$$

setting $\epsilon = 1$, $\exists X > 0$, $\forall x > X$: $|f(x) - A| < 1$. Since f is continuous on $[a, X]$, it is bounded. Hence $\exists M_1 > 0$ such that $|f(x)| \leq M_1$ for $x \in [a, X]$. Thus setting $M = \max\{M_1, |A+1|, |A-1|\}$, then for all $x \in [a, +\infty)$, $|f(x)| \leq M$, as desired.

Exercise 3. *For $\epsilon > 0$. Since $f(x)$ is uniformly continuous on $(a, b]$, $\exists\delta_1 > 0$, such that $\forall x_1, x_2 \in (a, b]$ and $|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| < \epsilon$. Similarly $\exists\delta_2 > 0$, such that $\forall x_3, x_3 \in [b, c)$ and $|x_3 - x_4| < \delta$, $|f(x_3) - f(x_4)| < \epsilon$.*

Now setting $\delta = \min\{\delta_1, \delta_2\}$. $\forall x_5, x_6 \in (a, c)$ and $|x_5 - x_6| < \delta$. W.l.o.g. we assume that $x_5 < x_6$. Then
If $x_5, x_6 \in (a, b]$, then $|f(x_5) - f(x_6)| < \epsilon$;
If $x_5, x_6 \in [b, c)$, then $|f(x_5) - f(x_6)| < \epsilon$;
If $a < x_5 < b < x_6 < c$, then we know

$$b - x_5 < x_6 - x_5 < \delta \leq \delta_1$$

so $|f(b) - f(x_5)| < \epsilon$. Similarly $|f(b) - f(x_6)| < \epsilon$. Hence by triangle inequality,

$$|f(x_5) - f(x_6)| \leq |f(b) - f(x_5)| + |f(b) - f(x_6)| < 2\epsilon$$

as desired.

Exercise 4. *Let $f(x) = x + \frac{15}{1+\cos^2 x} - 5$. Then*

$$f(0) = \frac{15}{2} - 5 = \frac{5}{2} > 0$$

$$f(-100) = -95 + \frac{15}{1 + \cos^2 95} \leq -95 + 15 = -80 < 0$$

By intermediate value theorem, there exists $x_0 \in (-100, 0) \subset \mathbb{R}$ *such that* $f(x_0) = 0$.