MA2252 Introduction to computing

lectures 25-26

Series and root finding

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Taylor polynomials

A Taylor polynomial of a function $f(x)$ centered at $x = a$ is a polynomial approximation of $f(x)$.

Figure: Brook Taylor

Question: Find a quadratic polynomial to approximate $f(x)$ near $x = a$.

Taylor polynomials (contd.)

A *nth* order Taylor Polynomial is given as

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n. \quad (1)$$

- $p_0(x) = f(a)$ (**constant function**)
- $p_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$ (**linear approximation**)
- $p_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$

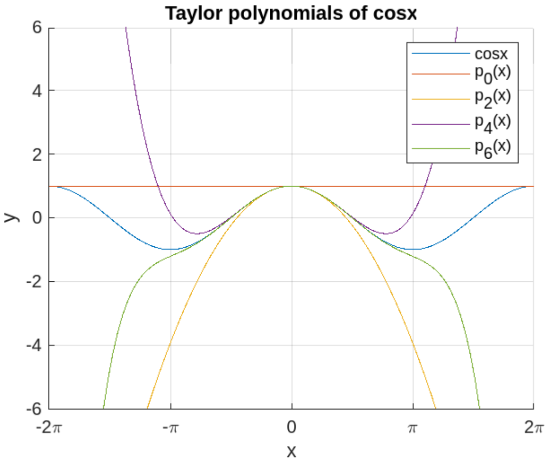
(**quadratic approximation**)

Example: The Taylor polynomials of $f(x) = \cos x$ upto degree 6:

- $n = 0: \quad p_0(x) = 1$
- $n = 2: \quad p_2(x) = 1 - \frac{x^2}{2}$
- $n = 4: \quad p_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$
- $n = 6: \quad p_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$.

Note: It is possible that a *nth* order Taylor polynomial is not of degree n.

Taylor polynomials (contd.)



Taylor series

A Taylor series is an infinite series expansion of a function at a given point in its domain.

Taylor series of a real-valued function $f(x)$ at $x = a$ is defined as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \tag{2}$$

or

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \tag{3}$$

When $a = 0$, we obtain a **Maclaurin series**.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots \tag{4}$$

Taylor's theorem

Theorem

Suppose a function $f(x)$ has $n+1$ continuous derivatives in an open interval I containing $x = a$ then $\forall n$ and $\forall x \in I$,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f''(n)}{2!}(x-a)^n + R_n(x) \quad (5)$$

where

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} \quad (6)$$

for some c between a and x.

Here, $R_n(x)$ is called the Taylor remainder or error term.

Error estimation

We don't know the exact value of $R_n(x)$ since exact value of c is unknown. However, an upper bound on the error term can still be found.

If $|f^{n+1}(x)| \leq M$ for all t between a and x, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{7}$$

Example: Find the maximum error if $p_2(x) = 1 - \frac{x^2}{2}$ is used to estimate the value of $cos(x)$ at $x = 0.3$. Verify that the error estimate in MATLAB is less than the maximum error.

Applications of Taylor series

Some applications:

- Small-angle approximations e.g. $\sin\theta \approx \theta$ for small values of θ.
- Finding non-elementary definite integrals e.g. $\int_1^2 \frac{sinx}{x} dx$
- Deriving formulas for numerical differentiation and integration (we'll study later in this course)

Root finding

Finding roots of functions is something we've been doing since school.

A root of a function $f(x)$ is the value of x for which $f(x) = 0$.

Example: The roots of $f(x) = x^2 - 3x + 2$ are 1 and 2.

Introduction (contd.)

There are many ways to find the roots of a function:

- Try a guess value!
- Use a mathematical formula
- Sketch the graph of the function
- Apply a root-finding method/algorithm

The last approach is what we'll study in this lecture.

Sometimes, finding roots is not that easy! For example, how can we find roots of transcendental equations?

Consider the function $f(x) = e^x - x$.

Does f(x) has a root?

Introduction (contd.)

Now consider the function $f(x) = e^x + x$. Does f(x) has a root?

Introduction (contd.)

Some root-finding methods:

- Bisection Method
- Newton-Raphson Method

Activity

Let f(x) be a function defined in the interval [a,b]. Suppose $f(a)f(b)<0$. Choose the correct statement.

1. f(x) always has a root in (a,b).
2. f(x) has exactly one root in (a,b) if f is continuous on [a,b].
3. f(x) has at least one root in (a,b) if f is continuous on [a,b].
4. f(x) doesn't have a root in (a,b).

To answer, please go to the mentimeter link provided in the chat.

Bisection Method

Steps:

1. Choose a guess interval which may contain the root.
2. Approximate the root by the mid-point of this interval.
3. Bisect the subsequent sub-intervals containing the root until the error is less than tolerance.
4. The root is given by the midpoint of the last bisected interval.

Bisection Method (contd.)

Consider again the function $f(x) = e^x + x$. Find its root using bisection method.

Bisection Method (contd.)

Write a function in MATLAB which takes the values of end points a and b of the guess interval, the function f, the tolerance tol and outputs the root of function f.

Limitations of bisection method:

- Requires knowledge of interval containing the root
- Cannot detect multiple roots
- Fails if the function is discontinuous on the interval [a,b]

Newton-Raphson Method

Steps:

1. Choose a guess value x_0 for the root.
2. Find a linear approximation of f(x) around $x = x_0$.
3. Find the root (say x_1) of this linear approximation. The obtained x_1 is an improvement on the guess value x_0.
4. Repeat the steps 2 and 3 to find improved guess values x_i ($i = 2, 3, ..., n$) until the error is less than tolerance.

The iterative formula for Newton-Raphson method is derived as

$$x_i = x_{i-1} - \frac{f(x_i)}{f'(x_{i-1})} \tag{1}$$

Newton-Raphson Method (contd.)

Find the root of the function $f(x) = e^x + x$ using Newton-Raphson method.

Newton-Raphson Method (contd.)

Write a function in MATLAB which takes the guess value x0, the function f and its derivative df, the tolerance tol and outputs the root of function f.

Limitations of Newton-Raphson method:

- Fails if $f'(x_i) = 0$ for some x_i.
  Example: $f(x) = x^3 - x^2 - x - 1$ with $x_0 = 1$.

- Fails if $f'(x_i)$ gets closer to zero for successive x_i values.
  Example: Consider $f(x) = \frac{1}{x} - e^x$ for $x_0 < 0$.

- In case of multiple roots, a guess values may converge to a different root than the one which is required.
  Example: $f(x) = \tan^{-1} x - x^2$ has two roots but for $x_0 < 0$, the method only gives the root $x = 0$.