
O
G
D
O
O
O
O
D

2023.10.18.

Cuts

- For an undirected graph / network (V, E) each edge has a weight.

  a cut is a position $K, \bar{K} \subset V$ s.t. $K \cap \bar{K} = \emptyset$, $K \cup \bar{K} = V$ } separate V into two parts.

- cut-set $C(N, K) = \{(a,b) \in E : a \in K, b \in \bar{K}\}$

  edges between K and $\bar{K}$

Why do we study cuts? Theoratical tool for:

① Shortest path problem

When we reformulate the shortest path problem as an LP problem, we use a property of cuts.

The first constraint $\boxed{\sum_{e \in C(N,K)} x_e \geq 1}$ $\forall K \in \{V' \subseteq V : O \in V', T \notin V'\}$

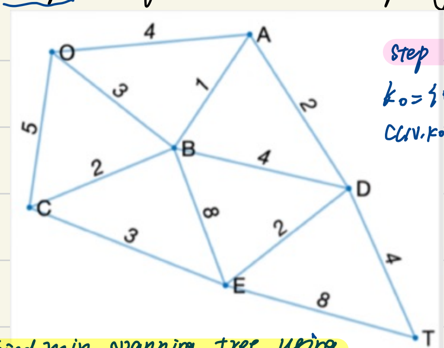
$x_e = \begin{cases} 0 & \text{if } e \text{ is not chosen.} \\ 1 & \text{if } e \text{ is chosen} \end{cases}$

cut K with $O \in K$, $T \in K'$

At least one edge in the cut-set is chosen. for all possible cuts K that separate O and T.

② Maximal spanning tree problem

In Prim's algorithm, we used a property of cuts: in a cut, an edge in the cut-set with minimal weight belongs to some minimal spanning tree.



Step 0

$K_0 = \{O\}$ $E_0' = \emptyset$

$C(N, K_0) = \{OA, OB, OC\}$
4 ③ 5
Smallest.

$\Rightarrow$ "Add this to the cut"

Step 1. $K_1 = \{O, B\}$. $E_1' = \{OB\}$

$C(N, K_1) = \left\{ \begin{matrix} \overset{4}{OA}, \overset{5}{OC} \\ \underset{①}{BA}, \underset{4}{BD}, \underset{8}{BE}, \underset{2}{BC} \end{matrix} \right\}$

Smallest

$\Rightarrow$ step 2.

$K_2 = \{O, B, A\}$ $E_2' = \{OB, OA\}$

$C(N, K_2) = \left\{ \begin{matrix} OC\,5 \\ \overset{4}{BD}\, \overset{8}{BE}\, \overset{2}{BC} \\ AD\, 2 \end{matrix} \right\}$.

You can freely choose BC or AD to add to the cut to K_3.

Find min spanning tree using

- Prim's algorithm
- keep updating a cut
- $K_0 = \{O\}$, $E_0' = \emptyset$

Find shortest path using Dijkstra's algorithm

- keep updating 3 vectors

d p v.

	d	p	v
0	0		1
A	∞	ϕ	0
B	∞	ϕ	0
C	∞	ϕ	0
D	∞	ϕ	0
E	∞	ϕ	0
T	∞	ϕ	0

① At 0

- A: old: ∞
  new: 4
- B old: ∞
  new: 3
- C old: ∞
  new: 5.

$\Rightarrow$.

	d	p	v
0	0		1
A	4	0	0
B	3	0	1
C	5	0	0
D	∞	ϕ	0
E	∞	ϕ	0
T	∞	ϕ	0

① cont'd

Among all unvisited nodes (ie v=0) the one with min d is B with d=3.

Mark B as visited.

(Tip: Change the color for B into yellow) to indicate its information is FINAL.

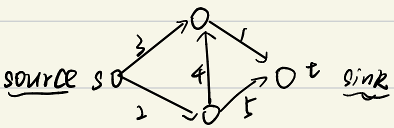
② At B

- A: old: 4
  new: 3+1=4 } No need to update.
- C old: 5
  new 3+2=5 } No need to update.
- D old: ∞
  new: 3+4=7 $\Rightarrow$ update
- E old: ∞
  new: 3+8=11 $\Rightarrow$ update.

	d	p	v
0	0		1
A	4	0	1
B	3	0	1
C	5	0	0
D	7	B	0
E	11	B	0
T	∞	ϕ	0

2023.10.19. Directed network.

eg. roads water pipes, electrical circuit.



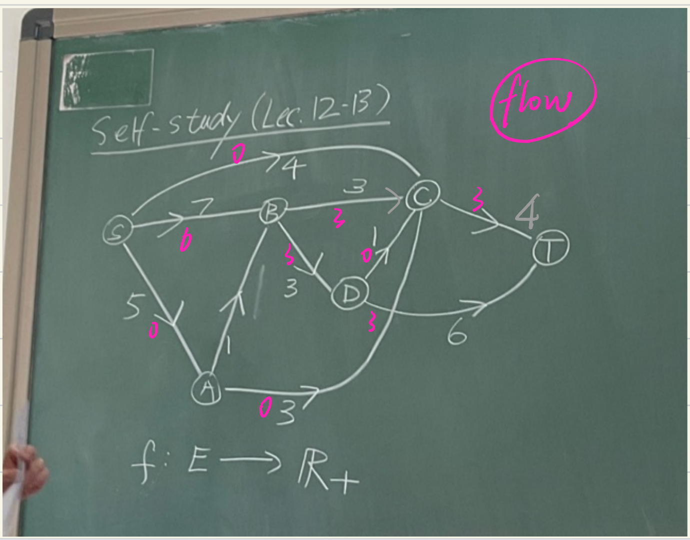
Weight $w(e)$ represent the capacity of an edge e.

flow: $f(e)$ represents the amount that is transported along the edge e.

Let f^* be a maximal flow and K^* a minimal cut.

Δ $|f| \le |f^*| \le c(N, K^*) \le c(N, K)$ for any flow f and any cut K.

2023.10.20. Self-study.



$|f| = SC + SB + SA = 6$ flow.

$|f| = CT + DT = 3+3 = 6$.

To verify f is a flow:

① capacity constraint: $f(e) \le W(e)$ for all edges e
(Check all the edges)

② conservation: for every node v that is not S, T, we want to check: "total incoming flow at v" = "total outgoing flow at v"

At A In $f(SA) = 0$
Out $f(AB) + f(AC) = 0+0 = 0$ ✓

At B In $f(SB) + f(AB) = 6+0 = 6$
Out $f(BC) + f(BD) = 3+3 = 6$ ✓

At C In: $f(BC) + f(DC) = 3+0 = 3$
Out: $f(CT) = 3$ ✓

At D In: $f(BD) = 3$
Out: $f(DC) + f(DT) = 0+3 = 3$ ✓

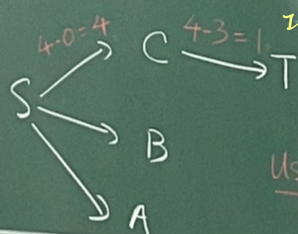
Therefore f is a flow.

Remark Value $|f| = f(SA) + f(SB) + f(SC) = 0+6+0 = 6$
or $f(CT) + f(DT) = 3+3 = 6$

Want to make this bigger! Use an f-augmenting path.

Find an augmenting path (from S to T)

$$\varepsilon(p) = \min_{\text{edges in } P} \begin{cases} W(e) - f(e) & \text{if } e \text{ is in the correct direction} \\ f(e) & \text{if wrong direction} \end{cases}$$



If go to another node, the result won't be better than 1. $\underbrace{\varepsilon(\text{path } SCT) = 1}$ → $\min(4,1)$

So $\{SCT\}$ is augmenting

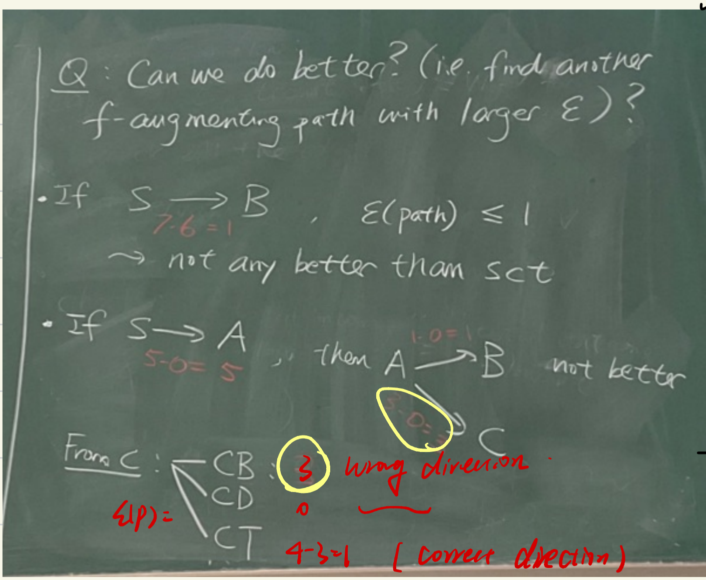
Using $\{SCT\}$ to improve f

$\hat{f}(SC) = 0+1 = 1$
$\hat{f}(CT) = 3+1 = 4$ } correct dir. → Correct direction.
$\hat{f}$(other edges) unchanged

$f(SC) = 0+6+0 = 6$
$= 3+3 = 6$

$\hat{f}$ is still a flow

$|\hat{f}| = |f| + \varepsilon(p) = 6+1 = 7$
4



Summary.

S
A B C
B C A D T.
B D T.
3 0 1

(20ix 4).

$\varepsilon = \min\{0, 1\} = 1$.

V.

$\varepsilon = \min(1, 0) = 0$.

X

(回到前面的路).

$\varepsilon = \min\{3, 0, 1\} = 0$

X

$\hat{f} = |f| + \varepsilon$

$= 6 + 1 = 7$

→ to D = BD : ε = 3 - 1 = 0 not augn

Not augmenting

not better.

Self study (Lec14)

Abstract game		Player 2			
	Strategy	~~1~~	2	3	~~4~~
Player 1	1	~~2~~	-2	-5	~~1~~
	2	~~4~~	-2	-3	~~-3~~
	3	~~0~~	-1	2	~~3~~
	~~4~~	~~-3~~	~~-3~~	~~-3~~	~~-4~~

Reduction by dominance.

- Minimax, maximin

① Find any dominated strategy.

For player 1

Str 1	2	-2	-5	1
	^			V
Str 2	4	-2	-3	-3

Neither str 1 nor str 2 dominates the other ⇒ keep them both.

For player 1

Str 2	4	-2	-3	-3
	V	V	V	V
~~Str 4~~	~~-3~~	~~-3~~	~~-3~~	~~-4~~

So Str 2 dominates Str 4.

Remove str 4 for player 1

For player 2 (1胜为正. 2胜为负 则.Smaller means better!)

Str1		Str2		Str3		Str4
~~2~~	>	-2		-5	<	~~1~~
~~4~~	>	-2		-3	≤	~~-3~~
~~0~~	>	-1		2	<	~~3~~

Note: For player 2, Str 3 dominates Str 4 only after play 1's Str 4 has been eliminated.

$\Rightarrow$

		P2	
P1		~~-2~~	~~-5~~
		~~-2~~	~~-3~~
		-1	2

Str 3 dominates Str 2.

Str 2 dominates Str 1

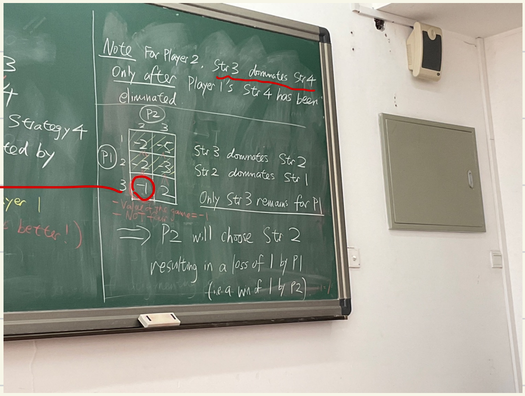
Only Str 3 remains for P1

$\Rightarrow$ P_2 will choose str 2. resulting in a loss of 1 by P1 (a win of 1 by P2).

(1 有利方胜)

value of the game = -1

Not fair



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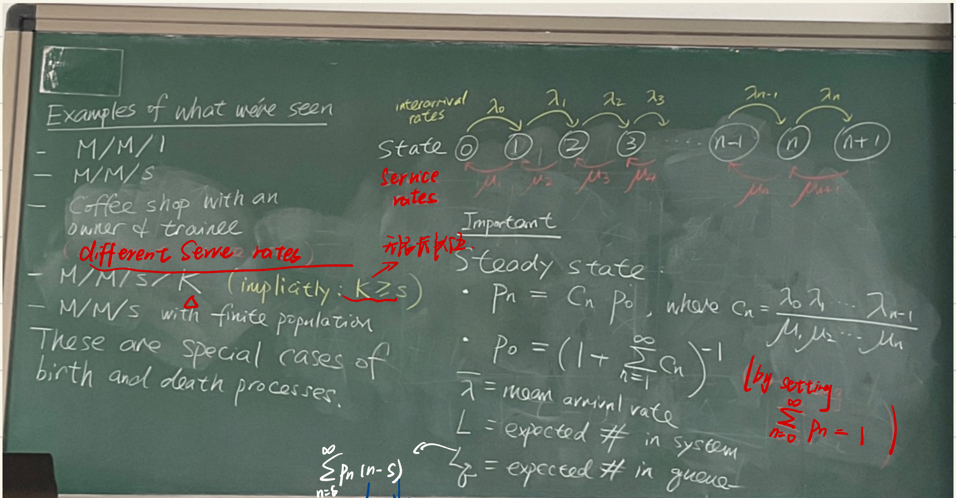
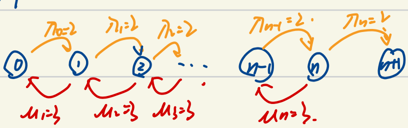


$\sum_{n=s}^{\infty} P_n (n-s)$ ↓ total ↓ serve ⇒ in the queue

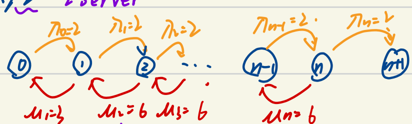
Lec19 of 20 M/M/1 & M/M/2

$\lambda = 2 \quad \mu = 3$

M/M/1



M/M/2 2 server



⇓ 2个server 一起工作 ⇒ service rate = 3×2 = 6.

Steady rate for M/M/2

- $c_1 = \frac{2}{3}$
- $c_2 = \frac{2}{3} \times \frac{2}{6} = \frac{2}{9}$
- $c_3 = \frac{2}{3} \times \frac{2}{6} \times \frac{2}{6} = \frac{2}{27}$

$$\boxed{C_n = \frac{2}{3}\left(\frac{2}{6}\right)^{n-1} \quad \text{for } n \geq 1}$$

△ $P_0 = (1 + \sum_{n=1}^{\infty} C_n)^{-1}$ $\quad P_n = C_n \cdot P_0$

$\bar{\lambda} = \sum_{n=0}^{\infty} P_n \lambda_n \quad L = \sum_{n=0}^{\infty} P_n \cdot n \quad L_q = \sum_{n=2}^{\infty} P_n (n-2)$.

✓

查S为2. Next: ① $P_0 = (1 + \sum_{n=1}^{\infty} C_n)^{-1}$

$$= \left(1 + \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{6}\right)^{n-1}\right)^{-1}$$

$$= \left(1 + \frac{2}{3}\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}\right)^{-1}$$

$$= \left(1 + \frac{2}{3}\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n}\right)^{-1} = \left(1 + \frac{2}{3} \cdot \frac{1}{1-\frac{1}{3}}\right)^{-1} = \frac{1}{2}$$

⇓

②

$p_n = C_n \cdot p_0 = \frac{2}{3}\left(\frac{1}{3}\right)^{n-1} \cdot \frac{1}{2} = \frac{1}{3^n}$ (for all $n \geq 1$)

③

$\bar{\pi} = \sum_{n=0}^{\infty} p_n \pi_n = p_0 \pi_0 + \sum_{n=1}^{\infty} p_n \pi_n = \frac{1}{2} \cdot 2 + \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot 2 = 1 + 1 = 2$

or all $\pi_n = 2 \Rightarrow \bar{\pi} = 2$ ✓

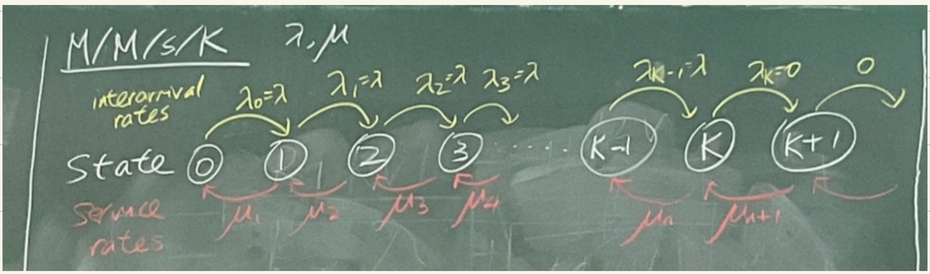
④

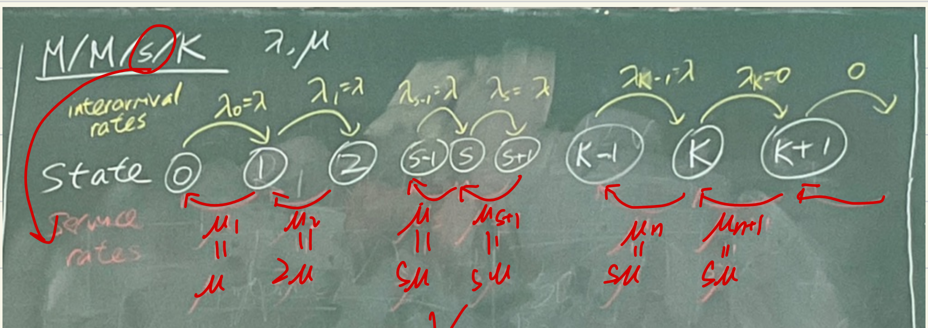
$L = \sum_{n=0}^{\infty} p_n \cdot n$

$= \sum_{n=1}^{\infty} p_n \cdot n = \sum \frac{1}{3^n} \cdot n = \frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = \frac{3}{4}$

$L_q = \sum_{n=2}^{\infty} p_n (n-2) = \sum_{n=2}^{\infty} \frac{1}{3^n}(n-2) = \sum_{m=0}^{\infty} \frac{1}{3^{m+2}} \cdot m = \frac{1}{3^2} \sum_{m=0}^{\infty} \frac{m}{3^m} = \frac{1}{9} \times \frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = \frac{1}{9} \times \frac{3}{4} = \frac{1}{12}$

Little Law ⇒ W, W_q.





共s个 server

Lec 21

Formula for L (in terms of L_q) and L_q

$$L = \sum_{n=1}^{\infty} p_n n$$

$$L_q = \sum_{n=s}^{\infty} p_n (n-s)$$

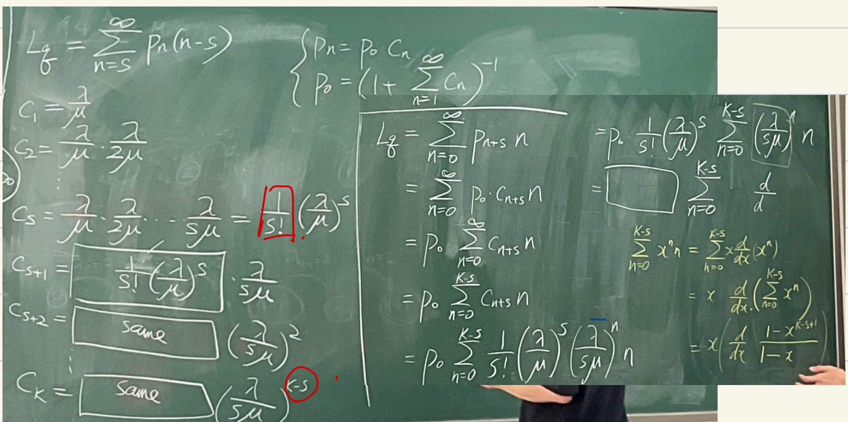
$$L = \sum_{n=1}^{s-1} p_n n + \sum_{n=s}^{\infty} p_n n$$

$$= \sum_{n=1}^{s-1} p_n n + \sum_{n=s}^{\infty} p_n (n-s) + \sum_{n=s}^{\infty} p_n s$$

$$= \sum_{n=1}^{s-1} p_n n + L_q + s \sum_{n=s}^{\infty} p_n \qquad \left(1 - \sum_{n=0}^{s-1} p_n\right)$$

$$= \sum_{n=1}^{s-1} p_n n + L_q + s\left(1 - \sum_{n=0}^{s-1} p_n\right)$$

$$L_q = \sum_{n=s}^{\infty} p_n (n-s)$$



$$\left(\frac{\lambda}{\mu}\cdot\frac{\lambda}{2\mu}\cdots\frac{\lambda}{s\mu}\right) = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s$$

$$\frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \cdot \left(\frac{\lambda}{s\mu}\right)^{k-s}$$

$\Rightarrow$

$$L_q = \sum_{n=s}^{\infty} p_n (n-s) = \sum_{n=0}^{\infty} p_{n+s}(n+s-s) = \sum_{n=0}^{\infty} p_{n+s}\cdot n.$$

$$x \cdot n \cdot x^{n-1}$$

$$\Rightarrow L_q = \sum_{n=0}^{\infty} p_{n+s}\cdot n$$

$$= \sum_{n=0}^{\infty} p_0 \cdot C_{n+s}\cdot n$$

$$= p_0 \sum_{n=0}^{\infty} C_{n+s}\cdot n$$

$$= p_0 \sum_{n=0}^{k-s} C_{n+s}\cdot n$$

$$= p_0 \sum_{n=0}^{k-s} \cdot \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \left(\frac{\lambda}{s\mu}\right)^n \cdot n$$

$$\boxed{\begin{aligned} p_c - \sum_{n=0}^{k-s} x^n \cdot n &= \sum_{n=0}^{k-s} x \frac{d}{dx}(x^n) \\ &= x \cdot \frac{d}{dx}\left(\sum_{n=0}^{k-s} x^n\right). \\ &= x\left(\frac{d}{dx}\frac{1-x^{k-s+1}}{1-x}\right)\end{aligned}}$$

$$\frac{a_1(1-q^n)}{1-q}$$

$$\therefore L_q = p_0 \cdot \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \cdot \frac{\lambda}{s\mu}\cdot \frac{d}{d\left(\frac{\lambda}{s\mu}\right)} \cdot \frac{1-\left(\frac{\lambda}{s\mu}\right)^{k-s+1}}{1-\left(\frac{\lambda}{s\mu}\right)}.$$

$$= \frac{p_0}{s}\cdot\frac{1}{s!}\cdot\left(\frac{\lambda}{\mu}\right)^{s+1}\cdot\frac{(-(k-s+1)\cdot x^{k-s})(1-x)+(1-x^{k-s+1})}{(1-x)^2}.$$

$= \checkmark$ 可求

2023. 10.31

Lect 22. $\begin{cases} \lambda_1 = 0.2 & \text{interarrived rate.} \\ \lambda_2 = 0.6 & \text{fever } 1^{st} \text{ class customers} \\ \mu = 3. & \to \text{service rate.} \end{cases}$

Q Compute W_2.

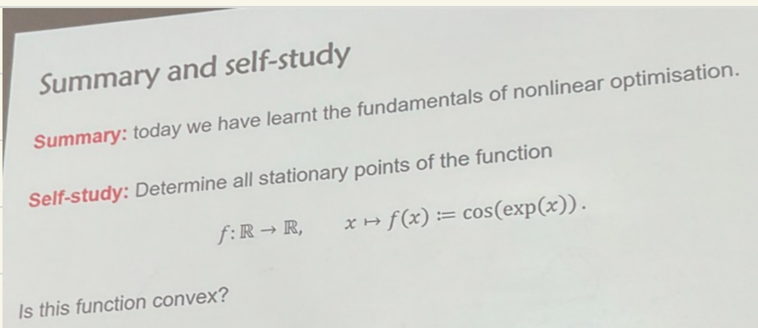
A Ideas:

- 1^{st} class can ignore 2^{st} class: λ_1
- Generic class (from 1^{st} to 2^{st} classes).: like a single group with interarrival rate: $\lambda_1 + \lambda_2$
- Isolate W_2 (Splitting Poission)

M/M/1 $\quad W = (\mu - \lambda)^{-1}$

Now $\quad W_1 = (\mu - \lambda_1)^{-1} = \frac{1}{2.8} = \frac{5}{14} \qquad W_{12} = (\mu - \lambda_1 - \lambda_2)^{-1} = \frac{1}{3-0.8} = \frac{5}{11}$

Splitting Poisson $\quad W_{12} = \frac{\lambda_1}{\lambda_1+\lambda_2} W_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} W_2 = \frac{1}{4} \cdot \frac{5}{14} + \frac{3}{4} \cdot W_2 = \frac{5}{11} \Rightarrow W_2 = \checkmark$



2023.11.2.

Optimization

· Unconstrained optimization

- Line Search (iterative process)

① Starting at x_k, find a

descent direction p_k.

(i.e. p_k s.t $p_k^T \nabla f(x_k) < 0$).

Examples. - Guess. or given by question.

- Steepest descent : $p_k = -\nabla f(x_k)$

- Newton (need $\nabla^2 f(x_k) \succ 0$: $p_k = -(\nabla^2 f(x_k))^{-1} \nabla f(x_k)$

- Quasi-Newton : Similar to Newton. but without matrix inverse.

② Find step length $s_k > 0$

长度

p_k 方向

$s_k = \underset{s>0}{\arg\min} f(x_k + s p_k)$ → 找出使 f minimize 的变量 s 的值.

② (Cont'd)

Methods. - Solve exactly (single-variable calculus)

- Wolf conditions (Armijo + curvature):

Use any value of s satisfy both conditions; no need to minimize $f(x_k + s p_k)$

- Backtracking

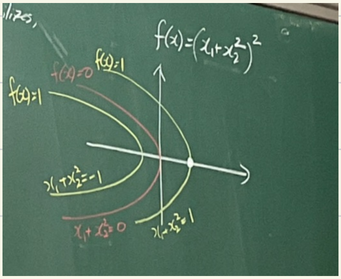
No need to minimize $f(x_k + s p_k)$

Try a value of s; keep reducing s (by a factor $t \in (0,1)$) until it satisfies Armijo's rule.

③ Update $x_{k+1} = x_k + s_k p_k$

Stopping criteria (e.g. $|\nabla f(x_k)|$ is small. or $\{x_k\}$ stablizes or $\{f(x_k)\}$ stablizes, or after N iterations, etc)

eg. (Exercise 2.9) $\min_{x \in \mathbb{R}^2} (x_1+x_2^2)^2$



$x_0 = \begin{pmatrix}1\\0\end{pmatrix}$, $p_0 = \begin{pmatrix}-1\\1\end{pmatrix}$

$f(x) = (x_1+x_2^2)^2$ $\quad \nabla f(x) = \begin{pmatrix}2(x_1+x_2^2)\\4x_2(x_1+x_2^2)\end{pmatrix}$

(a). Using the given direction p_0

· Check p_0 is a descent direction $\nabla f(x_0) = \begin{pmatrix}2\\0\end{pmatrix}$

$$p_0^T \nabla f(x_0) = (-1)\times 2 + 1\times 0 = -2 < 0$$

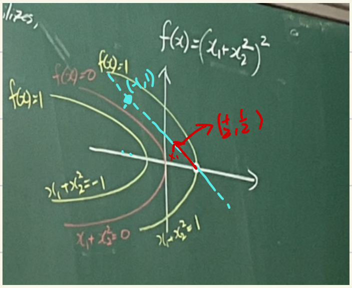
· Find step length

minimize $f(x_0+sp_0) = f\left(\begin{pmatrix}1\\0\end{pmatrix} + s\begin{pmatrix}-1\\1\end{pmatrix}\right) = f\begin{pmatrix}1-s\\s\end{pmatrix} = (1-s+s^2)^2$

· use calculus

· complete the squence $[(s-\frac{1}{2})^2+\frac{3}{4}]^2$ min when $s=\frac{1}{2}$ $\Rightarrow s_0 = \frac{1}{2}$

· $x_1 = x_0 + s_0 p_0 = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$



⇒ Repeat.

(b) Steepest descent at $x_0 = \begin{pmatrix}1\\0\end{pmatrix}$

$\nabla f(x_0) = \begin{pmatrix}2\\0\end{pmatrix} \Rightarrow p_0 = -\begin{pmatrix}2\\0\end{pmatrix}$

· step length: minimize $f(x_0+sp_0)$

$f(x_0+sp_0) = f\begin{pmatrix}1-2s\\0\end{pmatrix} = (1-2s)^2$ minimize when $s=\frac{1}{2}$ $\Rightarrow s_0=\frac{1}{2}$

· $x_1 = x_0 + s_0 p_0 = \begin{pmatrix}0\\0\end{pmatrix}$ - global min of f

Remark: If we had normalized p_0 to be a unit vector, then s_0 would work out to be 1. So $s_0 p_0$ remains unchanged

(C) Newton at $x_0 = \begin{pmatrix}1\\0\end{pmatrix}$

$$\nabla f(x) = \begin{pmatrix} 2(x_1+x_2^2) \\ 4x_2(x_1+x_2^2) \end{pmatrix} \qquad \nabla f(x_0) = \begin{pmatrix}2\\0\end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 4x_2 \\ 4x_2 & 4x_1+12x_2^2 \end{pmatrix} \qquad \nabla^2 f(x_0) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$ positive def

$$p_0 = -(\nabla^2 f(x_0))^{-1} \nabla f(x_0) = -\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} = -\begin{pmatrix}1\\0\end{pmatrix}$$

$s_0 = 1$ (no need to change).

- $x_1 = x_0 + s_0 p_0 = \begin{pmatrix}0\\0\end{pmatrix}$. $\leftarrow$ global min STOP

- Constrained optimization.

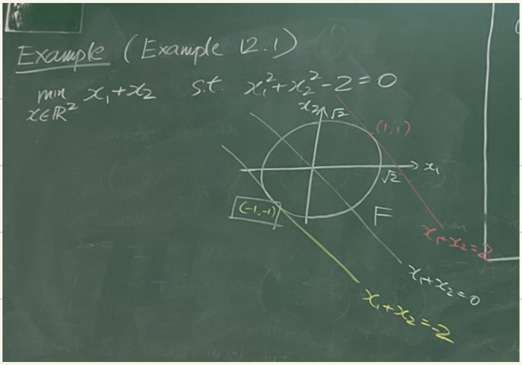
  - KKT (by studying Lagrange)
  - other methods for turning the problem into an unconstrained one
    - Quadratic penalty
    - Augumented Lagrangian.

$\Rightarrow$ Process.
- Iterate over p_k
- For each k, solve the unconstrained problem for $Q(x, p_k)$ or $L_A(x, y_k, p_k)$

  eg. by line search.

  $x_0^{(k)} \to x_1^{(k)} \to x_2^{(k)} \to \dots \; x_*^{(k)}$ (actual or approximate solution)
- Update p_{k+1} (y_{k+1})

  Usually $x_0^{(k+1)} = x_*^{(k)}$
- Solutions $x_*^{(0)}, x_*^{(1)}, x_*^{(2)}, \dots \to x_*$ for any constrained problem.



KKT check LICQ.

$$\nabla c(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 2x$$

For one vector, LI means exactly it's nonzero.

$\begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ unless $x=0$ but this is not in F

so for $x \in F$, $\nabla c(x) \neq 0$

Lagrange $\mathcal{L}: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ ← one constraint.

$$\mathcal{L}(\underset{\mathbb{R}^2}{x}, \underset{\mathbb{R}}{y}) = x_1 + x_2 - y(x_1^2 + x_2^2 - 2)$$

$$\nabla_x \mathcal{L}(x, y) = \begin{pmatrix} 1 - 2x_1 y \\ 1 - 2x_2 y \end{pmatrix}$$

KKT At x^*, $\exists$ a unique y^* s.t.

$$\nabla_x \mathcal{L}(x^*, y^*) = \begin{pmatrix} 1 - 2x_1^* y^* \\ 1 - 2x_2^* y^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

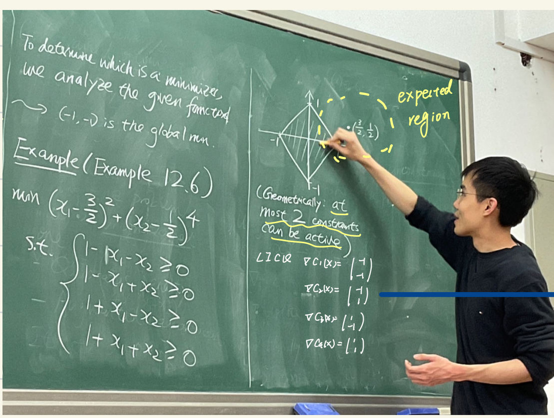
$$c(x^*) = (x_1^*)^2 + (x_2^*)^2 - 2 = 0$$

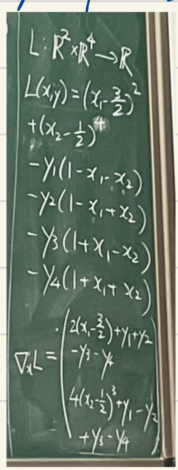
$y^* c(x^*) = y^* [(x_1^*)^2 + (x_2^*)^2 - 2] = 0$ (This complementarity condition is redundant)

$\Rightarrow x_1^* = 1/2y^*$ $\quad x_2^* = 1/2y^*$ $\quad (\frac{1}{2y^*})^2 \times 2 - 2 = 0$ $\Rightarrow y^* = \pm\frac{1}{2}$ $\quad x^* = (1,1)$ or $(-1,-1)$

To determine which is a minimizer, we analyze the given functions. $(-1,-1)$ is the global min

Example.





KKT

- $\nabla_x \mathcal{L}(x^*, y^*) = 0$ (gives 2 equations)
- $c_i(x^*) \geq 0$
- $y_i^* \geq 0$ (for ineq constraints)
- $y_i^* c_i(x^*) = 0$. (complementarity)

Solve this: $x^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad y^* = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix}$ active constraints

Return to LICQ $\nabla c_1(x^*) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ $\quad \nabla c_2(x^*) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are LI.

eg. minimize x_1+x_2 s.t. $x_1^2+x_2^2-2=0$

Quadratic penalty (unconstrained)

$$Q(x,\pi)=x_1+x_2+\frac{\pi}{2}(x_1^2+x_2^2-2)^2$$

· k^{th} iteration π_k $x_0^{(k)}$

minimize $Q(x,\pi_k)$ in x

- Solve exactly
- line search.

$$x_0^{(k)}\to x_1^{(k)}\to x_2^{(k)}\to\cdots\ \boxed{x_*^{(k)}}$$

solution

update parameter

· $x_*^{(1)}, x_*^{(2)}, x_*^{(3)}\ \ldots\ x_*^{(k)}$

$\pi_0=1$ $x_0=\begin{pmatrix}0\\0\end{pmatrix}$

$$Q(x,\pi_0)=x_1+x_2+\frac{1}{2}(x_1^2+x_2^2-2)^2$$

want to minimize

Line search (internal iteration)

$\boxed{0}$ $x_0^{(0)}=\begin{pmatrix}0\\0\end{pmatrix}$ $\xrightarrow{\text{代入}}$ $\Rightarrow \nabla_x Q(x_0^{(0)},\pi_0)=\begin{pmatrix}1\\1\end{pmatrix}$.

$$\nabla_x Q(x,\pi_0)=\begin{pmatrix}1+2x_1(x_1^2+x_2^2-2)\\1+2x_2(x_1^2+x_2^2-2)\end{pmatrix}$$

· Steepest descent $p_0=-\begin{pmatrix}1\\1\end{pmatrix}$

· Step length: $\min Q(x_0+sp_0,\pi_0)$.

$$=-2s+\frac{1}{2}(2s^2-2)^2$$

$$=-2s+2(s^2-1)^2$$

min when $s=1.107$

$$\Rightarrow x^{(0)}=x_0^{(0)}+sp_0=\begin{pmatrix}-1.107\\-1.107\end{pmatrix}$$

Augumented Lagrangian (unconstrained)

$$\mathcal{L}_A(x,y,\pi)=x_1+x_2-y(x_1^2+x_2^2-2)+\frac{\pi}{2}(x_1^2+x_2^2-2)^2$$

· k^{th} iteration π_k y_k $x_0^{(k)}$

minimize $\mathcal{L}_A(x,y_k,\pi_k)$ in x

- Solve exactly
- line search.

$$x_0^{(k)}\to x_1^{(k)}\to x_2^{(k)}\to\cdots\ \boxed{x_*^{(k)}}$$

solution

update parameter

$y_{k+1,j}=y_{k,j}-\pi_k c_j(x_*^{(k)})$

· $x_*^{(1)}, x_*^{(2)}, x_*^{(3)}\ \ldots\ x_*^{(k)}$

For one constraint. $y_{k+1}=y_k-\pi_k c(x_*^{(k)})$

$\pi_0=1$ $x_0=\begin{pmatrix}0\\0\end{pmatrix}$

$$Q(x,\pi_0)=x_1+x_2+\frac{1}{2}(x_1^2+x_2^2-2)^2$$

want to minimize

Line search (internal iteration)

$\boxed{0}$ $x_0^{(0)}=\begin{pmatrix}0\\0\end{pmatrix}$ $\xrightarrow{\text{代入}}$ $\Rightarrow \nabla_x Q(x_0^{(0)},\pi_0)=\begin{pmatrix}1\\1\end{pmatrix}$.

$$\nabla_x Q(x,\pi_0)=\begin{pmatrix}1+2x_1(x_1^2+x_2^2-2)\\1+2x_2(x_1^2+x_2^2-2)\end{pmatrix}$$

· Steepest descent $p_0=-\begin{pmatrix}1\\1\end{pmatrix}$

· Step length: $\min Q(x_0+sp_0,\pi_0)$.

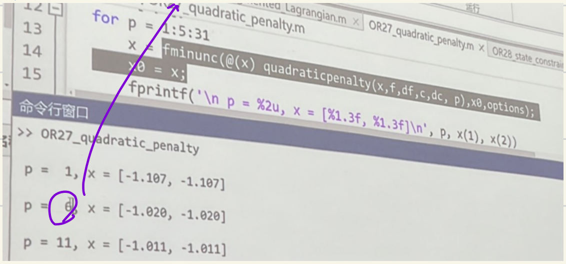
$$=-2s+\frac{1}{2}(2s^2-2)^2$$

$$=-2s+2(s^2-1)^2$$

min when $s=1.107$

$$x^{(0)}=x_0^{(0)}+sp_0=\begin{pmatrix}-1.107\\-1.107\end{pmatrix}$$

Next $\lambda_1 = 6$



$Q(x,\lambda_1) = x_1+x_2+3(x_1^2+x_2^2-2)^2$

start at $x_0^{(1)} = \begin{pmatrix}-1.107\\-1.107\end{pmatrix}$