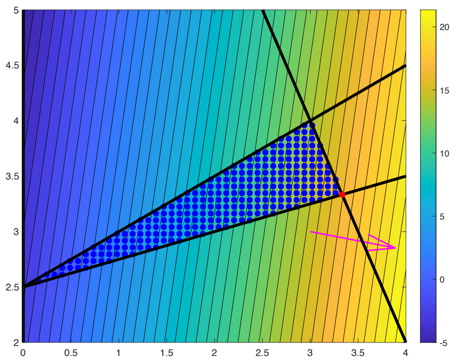
Semester 1 Mock Exam 2022 – MA3077 DLI

Note: The following model solutions indicate how the problem may have been solved. Alternative solutions are often also possible.

Question 1

a)

i. The feasible set is



ii. The solution is denoted in red in the previous picture. The active constraints are $4y - x = 10$ and $2x + y = 10$.

iii.
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f = [-6;1]; A = [1 -4; 2 1;-1 2]; b = [-10 10 5];
x = linprog(f, A, b);
```

iv. Since the feasible set is in the upper right quadrant, we can replace $x, y \in \mathbb{R}$ with $x, y \geq 0$. The, multiplying by -1 the objective function and the first inequality and introducing the slack variables $a, b, c \geq 0$, we obtain the standard form

$$\begin{array}{ll} (-)min & y - 6x \\ s.t. & x - 4y + a = -10 \\ & 2x + y + b = 10 \\ & 2y - x + c = 5 \\ & x, y, a, b, c \geq 0 \end{array}$$

v. Using the standard form and denoting $z^T = (x, y, a, b, c)$, let $c^T = (-6,1)$, $b^T = (-10,10,5)$, and $A = \begin{pmatrix} 1 & -4 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 1 \end{pmatrix}$. Then, the Lagrangian is

$$L : \mathbb{R}^5 \times \mathbb{R}^3 \times \mathbb{R}^5_+ \to \mathbb{R}, \qquad L(z, w, s) = c^T z + w^T(b - Az) - s^T z.$$

The dual function is $g: \mathbb{R}^3 \times \mathbb{R}^5_+ \to \mathbb{R}$,

$$g(w, s) := \min_z L(z, w, s) = \begin{cases} b^T w, & if \ \ c - A^T w - s = 0, \\ -\infty, & if \ \ c - A^T w - s \neq 0. \end{cases}$$

The dual problem is

$$\begin{array}{ll} max & b^T w \\ s.t. & c - A^T w = s \\ & s \geq 0, \ w \in \mathbb{R}^3 \end{array}$$

b)

i. By definition, $\|(x,y)^T\|_1 = |x| + |y| > 0$. Therefore, $(\|(x,y)^T\|_1)^2 \leq 1$ iff $\|(x,y)^T\|_1 \leq 1$. The latter can be modelled by introducing two variables $a, b \in \mathbb{R}$ and setting $-a \leq x \leq a, -b \leq y \leq b$, and $a + b = 1$. Therefore, this optimization problem can be modelled as the following linear programming problem:

$$
\begin{array}{cc}
max & \cos(\alpha)\,x + \sin(\alpha)\,y \\
s.t. & -a \leq x \leq a \\
 & -b \leq y \leq b \\
 & a + b = 1 \\
 & x, y, z, a, b \in \mathbb{R}
\end{array}
$$

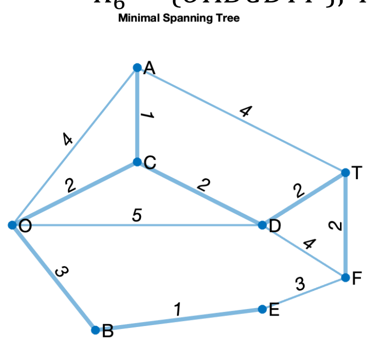
ii. The feasible set is a rhombus with vertices $(\pm 1, 0), (0, \pm 1)$. The optimization problem has multiple solutions when the vector $(\cos(\alpha), \sin(\alpha))^T$ is perpendicular to the edges of the rhombus. Since the normal to its edges are $(\pm 1, \pm 1)^T \backslash \sqrt{2}$, the vector $(\cos(\alpha), \sin(\alpha))^T$ is perpendicular to an edge if $\cos(\alpha) = \sin(\alpha)$ or if $\cos(\alpha) = -\sin(\alpha)$, which happens when $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Question 2

a)

i. First, we set $K_0 = \{O\}$, $T = (K_0, \emptyset)$. Then, in each step we identify the cut-set $C(N, K_k)$, identify an edge e in $C(N, K_k)$ with minimal weight, expand K_k by adding a new node from this edge, and add the edge to the tree. Here is the generated sequence and the resulting graph. Let $A_i := argmin\,\{w(e) : e \in C(N, K_i)\}$

$K_0 = \{O\}$, $T = (K_0, \emptyset), OC \in A_0, n_1 = C$
$K_1 = \{OC\}$, $T = (K_1, \{OC\}), CA \in A_1, n_2 = A$
$K_2 = \{OAC\}$, $T = (K_2, \{OC, CA\}), CD \in A_2, n_3 = D$
$K_3 = \{OACD\}$, $T = (K_3, \{OC, CA, CD\}), DT \in A_3, n_4 = T$
$K_4 = \{OACDT\}$, $T = (K_4, \{OC, CA, CD, DT\}), TF \in A_4, n_5 = F$
$K_5 = \{OACDTF\}$, $T = (K_5, \{OC, CA, CD, DT, TF\}), OB \in A_5\ (FE\ too), n_6 = B$
$K_6 = \{OABCDTF\}$, $T = (K_5, \{OC, CA, CD, DT, TF, OB\}), BE \in A_5\ , n_7 = E$



ii. Let $d = (0, \infty, \dots, \infty)$,$p = (\emptyset, \dots, \emptyset)$, and $v = (0, \dots, 0)$, denote the distance, previous-node, and visited-node vectors, respectively. At each iteration, we pick a not-yet-visited node with shortest distance, mark it as visited, and update the distance of its neighbors if passing through this node is a shorter path. Here is the evolution of these vectors as the algorithm proceeds.

(1)

⬚	O	A	B	C	D	E	F	T
d	0	4	3	2	5	∞	∞	∞
p	∅	O	O	O	O	∅	∅	∅
v	1	0	0	0	0	0	0	0

(2)

⬚	O	A	B	C	D	E	F	T
d	0	3	3	2	4	∞	∞	∞
p	∅	C	O	O	C	∅	∅	∅
v	1	0	0	1	0	0	0	0

(3)

⬚	O	A	B	C	D	E	F	T
d	0	3	3	2	4	∞	∞	7
p	∅	C	O	O	C	∅	∅	A
v	1	1	0	1	0	0	0	0

(4)

⬚	O	A	B	C	D	E	F	T
d	0	3	3	2	4	4	∞	7
p	∅	C	O	O	C	B	∅	A
v	1	1	1	1	0	0	0	0

(5)

⬚	O	A	B	C	D	E	F	T
d	0	3	3	2	4	4	8	6
p	∅	C	O	O	C	B	D	D
v	1	1	1	1	1	0	0	0

(6)

⬚	O	A	B	C	D	E	F	T
d	0	3	3	2	4	4	7	6
p	∅	C	O	O	C	B	E	D
v	1	1	1	1	1	1	0	0

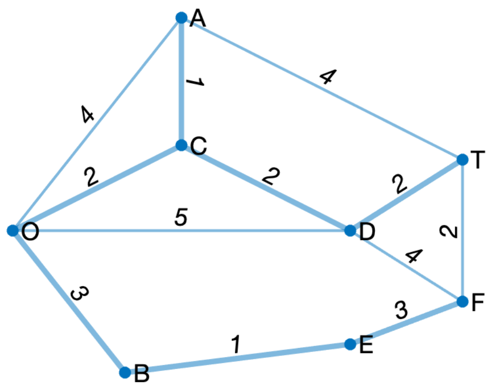
(7)

⬚	O	A	B	C	D	E	F	T
d	0	3	3	2	4	4	7	6
p	∅	C	O	O	C	B	E	D
v	1	1	1	1	1	1	0	1

(8)

⬚	O	A	B	C	D	E	F	T
d	0	3	3	2	4	4	7	6
p	∅	C	O	O	C	B	E	D
v	1	1	1	1	1	1	1	1

Shortest Path Tree



b)

i. The function f is a flow because $f(e) \leq w(e)$ for every edge e, and for every node (excluding the source and the sink), the sum of incoming flows equals the sum of the outgoing flows. The nontrivial nodes are B, D, E,G,H,I,J,K, and we can quickly verify that indeed

$$
\begin{aligned}
&f(SB) = 6 = 4 + 2 = f(BE) + f(BD)\\
&f(BE) = 4 = f(EG), f(BD) = 2 = f(DG)\\
&f(DG) + f(EG) = 2 + 4 = 6 = f(GH)\\
&f(GH) = 6 = f(HI), f(HI) = 6 = 4 + 2 = f(IK) + f(IJ)\\
&f(IK) = 4 = f(KT), f(IJ) = 2 = f(JT)
\end{aligned}
$$

The value of this flow is f$(SB) + f(SA) = 6 + 0 = 6$.

ii. The capacity of this cut is

$$f(BE) + f(DG) + f(DF) + f(AC) = 4 + 4 + 3 + 4 = 15$$

This does not imply that f is not a maximal flow, because we don't know whether this cut is minimal.

iii. The path $\{S, A, C, F, H, I, J, T\}$ is f-augmenting. Indeed, its capacity is

$$\min(6,4,3,6,12 - 6, 5 - 2, 6 - 2) = 3 > 0$$

iv. Let $f \in \mathbb{R}^{17}$ denote the control variable, where $f_1 = f(SB), f_2 = f(SA), f_3 = f(BE), f_4 = f(BD), f_5 = f(AD), f_6 = f(AC), f_7 = f(EG), f_8 = f(DG), f_9 = f(DF), f_{10} = f(CF), f_{11} = f(GH), f_{12} = f(FH), f_{13} = f(HI), f_{14} = f(IK), f_{15} = f(IJ), f_{16} = f(KT), f_{17} = f(JT)$. Define $w \in \mathbb{R}^{17}$ similarly. Then, the maximal flow is the solution to

$$
\begin{array}{cc}
max & f_1 + f_2\\
s.t. & f_3 + f_4 - f_1 = 0\\
 & f_5 + f_6 - f_2 = 0\\
 & f_7 - f_3 = 0\\
 & f_8 + f_9 - f_4 - f_5 = 0\\
 & f_{10} - f_6 = 0\\
 & f_{11} - f_7 - f_8 = 0\\
 & f_{12} - f_9 - f_{10} = 0\\
 & f_{13} - f_{11} - f_{12} = 0\\
 & f_{14} + f_{15} - f_{13} = 0\\
 & f_{16} - f_{14} = 0\\
 & f_{17} - f_{15} = 0\\
 & f \leq w\\
 & f \geq 0
\end{array}
$$

v. In class, we showed that, for any S-T cut P and any flow f, it holds $|f| \leq \sum_{e \in C(N,P)} f(e)$. Since the cut-set of the cut $P = \{S, A, B, C, D, E, F, G, H\}$ is $C(N, P) = \{HI\}$, it follows that $|f| \leq f(HI)$ for any flow f.

Question 3

a)

i. $E[T] = \alpha^{-1} = 0.5$, $Var[T] = \alpha^{-2} = 0.25$.

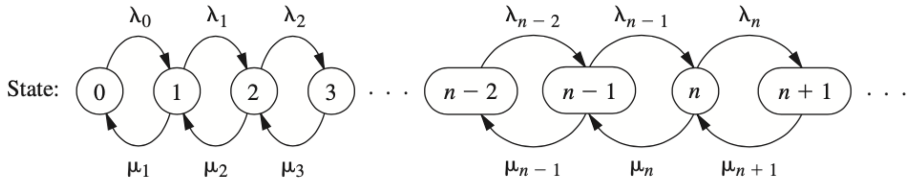
ii. Since T and S are independent,

$$P[T > 4|S \leq 3] = P[T > 4] = \exp(-2 * 4) = \exp(-8)$$

The sum $T + S$ has Erlang distribution with parameters (2,2). Therefore,

$$P[T + S \leq 5] = 1 - \sum_{k=0}^{1} \frac{(2 * 5)^k \exp(-2 * 5)}{k!} = 1 - 11 \exp(-10).$$

b)



i. Since customers arrive following a Poisson distribution with mean rate of 20 customers per hour, and since this does not depend on how many customers are already in, we conclude that $\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \bar{\lambda} = 12$.

When there is only one customer, only one barber works. Hence, $\mu_1 = \frac{60}{12} = 5$.When there are two customers, two barbers work. Hence, $\mu_2 = 10$, with three customers, three barbers work, hence $\mu_3 = 15$. Finally, if there are for or more customer, the service rate becomes $\mu_4 = \mu_5 = \cdots = 15 + \frac{60}{20} = 18$.

ii. We first compute $c_1 = \frac{\lambda_0}{\mu_1} = \frac{12}{5} = 2.4$, $c_2 = 2.4\frac{\lambda_1}{\mu_2} = 2.4\frac{12}{10} = 2.88$, $c_3 = 2.88\frac{\lambda_2}{\mu_3} = 2.88\frac{12}{15} = 2.304$, $c_n = 2.304\left(\frac{12}{18}\right)^{n-3}$. Therefore, the (steady state) probability that the shop is empty is

$$p_0 = \left(1 + \sum_{n=1}^{\infty} c_n\right)^{-1} = \left(1 + 2.4 + 2.88 + 2.304 + 2.304\sum_{n=4}^{\infty}\left(\frac{2}{3}\right)^{n-3}\right)^{-1}$$

$$= \left(6.28 + 2.304\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n}\right)^{-1} = (6.28 + 6.912)^{-1} = (13.192)^{-1} \cong 0.076$$

iii. The queue is empty if there are at most four customers. Since $p_n = c_n p_0$, the probability of this is

$$p_0 + p_1 + p_2 + p_3 + p_4 = (13.192)^{-1}(1 + 2.4 + 2.88 + 2.304 + 1.536) \cong 0.767$$

iv. The mean number of customers is

$$L = \sum_{n=0}^{\infty} n\, p_n = p_0\left(2.4 + 2 * 2.88 + 3 * 2.304 + 2.304\left(\frac{3}{2}\right)^3 \sum_{n=4}^{\infty} n\left(\frac{2}{3}\right)^n\right)$$

$$= p_0\left(15.072 + \ 2.304\left(\frac{3}{2}\right)^3\left(6 - \frac{2}{3} - 2\frac{4}{9} - 3\frac{8}{27}\right)\right) = p_0\left(15.072 + \ 2.304\left(\frac{3}{2}\right)^3\frac{32}{9}\right)$$
$$\cong 3.24$$

and the mean length of the queue is

$$L_q = \sum_{n=4}^{\infty}(n-4)\,p_n = \frac{2.304}{13.192}\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^{n+1} = \frac{2.304}{13.192}\frac{2}{3}\ 6 \cong 0.70$$

v. By Little's formula,

$$W = \frac{\mathrm{L}}{\lambda} \cong \frac{3.24}{12} \cong 0.27 \cong 16^{'}11^{''},$$
$$W_q = \frac{L_q}{\lambda} \cong \frac{0.7}{12} \cong 0.06 \cong 3^{'}30^{''},$$
$$W_s = W - W_q \cong 0.2116 \cong 12'42''$$

c) Let $p(0,t)$ and $p(1,t)$ denote the probability that the box is or is not empty at time t. Then, $p(0,t)$ and $p(1,t)$ satisfy the Kolmogorov equations

$$\partial_t p(0,t) = p(1,t)\mu - p(0,t)\lambda$$
$$\partial_t p(1,t) = p(0,t)\lambda - p(1,t)\mu$$

This is a homogeneous linear differential equation with matrix $A = \begin{pmatrix} -\lambda & \mu \\ \lambda & -\mu \end{pmatrix}$, whose eigenvalues and corresponding eigenvectors are $e = 0, -(\lambda+\mu)$, $v = (\mu,\lambda)^T$,$(-\lambda,\lambda)$, and the general solutions is

$$\begin{pmatrix} p(0,t) \\ p(1,t) \end{pmatrix} = a\exp(-(\lambda+\mu)t)\begin{pmatrix} -\lambda \\ \lambda \end{pmatrix} + b\begin{pmatrix} \mu \\ \lambda \end{pmatrix}$$

If the telephone box is initially empty, then $\begin{pmatrix} -\lambda & \mu \\ \lambda & \lambda \end{pmatrix}\binom{a}{b} = \binom{1}{0}$, that is, $a = -(\mu + \ \lambda)^{-1}$, $b = (\mu + \ \lambda)^{-1}$. Therefore, $p(0,t) = \frac{\mu+\lambda\exp(-(\lambda+\mu)t)}{\mu+\lambda}$.

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Question 4

a)

i. The derivative of f is $f'(x) = -\sin(x)\, f(x)$. The stationary points satisfy $f'(x) = 0$. Since $f(x) > 0$, $f'(x) = 0$ iff $\sin(x) = 0$, that is, iff $x = k\pi, k \in \mathbb{Z}$.

ii. The second derivative of f is $f''(x) = -\sin(x)\, f'(x) - \cos(x)\, f'(x) = f(x)(\sin^2(x) - \cos\,(x))$. Therefore, $x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} \cong\ 2.73$.

iii. To determine an optimal step size, we need to minimize $g(t) = f(x_0 - tf'(x_0))$ with respect to t. The derivative of g is $g'(t) = f'(x_0 - tf'(x_0))(-f'(x_0))$, and $g'(t) = 0$ if $\sin\big(x_0 - tf'(x_0)\big) = 0$, that is, $x_0 - tf'(x_0) = k\pi,$ that is, $t = (x_0 - k\pi)/f'(x_0)$. The smallest positive optimal step size is obtained with $k = 1$ (because $f'(x_0) < 0$), that is $t = (x_0 - \pi)/f'(x_0) \cong 1.9$. Hence, $x_1 = x_0 -\ [(x_0 - \pi)/f'(x_0)]f'(x_0) = \pi$.

b) After deriving the formulas $\nabla f(x) = 2A^TAx - 2A^Tb$ and $Hf(x) = 2A^TA$, we compute $x_1 = x_0 - Hf(x_0)^{-1}\nabla f(x_0) = x_0 - 0.5A^{-1}A^{-T}(\,2A^TAx_0 - 2A^Tb) = A^{-1}b$.

c) The quadratic penalty function is

$$Q(x,p) \coloneqq f(x) + \frac{p}{2}\sum_{i \in E} c_i^2(x) = x_1^2 + x_2 + \frac{p}{2}(x_2 - \cos(x_1))^2$$

Performing one step of the quadratic penalty methods means solving

$$\min_x Q(x, p_0) = (x_1 + x_2)^2 + \frac{p_0}{2}(x_1 + x_2 + 1)^2 = x_1^2 + x_2 + (x_2 - \cos(x_1))^2\,.$$

The gradient of $Q(x,2)$ is

$$\nabla_x Q(x,2) = \big(2x_1 + 2(x_2 - \cos(x_1))\sin(x_1)\,, 1 + 2(x_2 - \cos(x_1))\big)^T.$$

Therefore, the steepest descent direction at $x_0 = (0,0)^T$ is $d = -\nabla_x Q(x_0, 2) = -(0,-1)^T$. Using $\alpha = 1$, we obtain $x_1 = x_0 + \alpha d = (0,1)^T$.

END OF PAPER