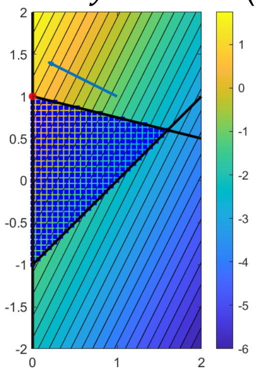
Question 1

a) **[2 Marks]** This optimization problem is not a linear programming problem because the constraint $x^2 + y^2 \leq 2$ is not linear.

b) Consider the following linear programming problem.

i. **[6 Marks]** The vertices of the feasible region are $(0,-1), (\frac{8}{5}, \frac{3}{5})$ and $(0,1)$ **[+3].** The solution is denoted in red in the picture $(x = 0, y = 1)$ **[+1]**. The active constraints are $4y = 4 - x$ (the second constraint) and $x = 0$ (the third constraint). **[+2]**



ii. **[4 Marks]** First, set $y = u - v$, with $u, v \geq 0$. **[+1]** Then, add two slack variables $a, b \geq 0$ to turn the inequality constraints into equality constraints, that is,

$$x - y \leq 1 \text{ becomes } x - u + v + a = 1$$

and

$$4y \leq 4 - x \text{ becomes } x + 4u - 4v + b = 4$$

[+2] Finally, multiply by -1 the objective function to turn it into a minimisation problem. **[+1]** We obtain the standard form

$$-\min 2x - u + v \quad subject\ to \quad \begin{cases} x - u + v + a = 1 \\ x + 4u - 4v + b = 4 \\ x, u, v, a, b \geq 0 \end{cases}$$

iii. **[5 Marks]** Using the standard form and denoting $z^T = (x, u, v, a, b)$, let **[+2]**

$$c^T = (2, -1, 1, 0, 0), \qquad b^T = (1, 4), \qquad and, \qquad A = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 4 & -4 & 0 & 1 \end{pmatrix}.$$

The Lagrangian is **[+1]** $L : \mathbb{R}^5 \times \mathbb{R}^2 \times \mathbb{R}^5_+ \to \mathbb{R}$, $L(z, w, s) = c^T z + w^T(b - Az) - s^T z$.

The dual function is **[+1]** $g : \mathbb{R}^2 \times \mathbb{R}^5_+ \to \mathbb{R}$,

$$g(w, s) := \min_z L(z, w, s) = \begin{cases} b^T w, & if \quad c - A^T w - s = 0, \\ -\infty, & if \quad c - A^T w - s \neq 0. \end{cases}$$

The dual problem is **[+1]**

$$\begin{array}{ll} max & b^T w \\ s.t. & c - A^T w = s \\ & s \geq 0,\ w \in \mathbb{R}^2 \end{array}$$

c) i. **[5 Marks]** From the weak duality, any feasible solution of the dual provides a lower bound on the primal. So if the primal is unbounded, that can't happen -there can't be any feasible solutions to the dual.

In other words, suppose the dual was feasible. Then there exists a y that satisfies the constraints for the dual. But in that case, for every x in the primal, ≤$b^T y \leq c^T x$ But we said that the primal was unbounded, so that $c^T x \to -\infty$. This is a contradiction. Thus, the dual must be infeasible.

ii. **[3 Marks]** Let $A \in \mathbb{R}^{(m.n)}$ and $b \in \mathbb{R}^m$. Then, exactly one of the following is true:

1) There exists $x \geq 0$ such that $Ax = b$.
2) There exists y such that $y^T A \leq 0$ and $y^T b > 0$,

Question 2

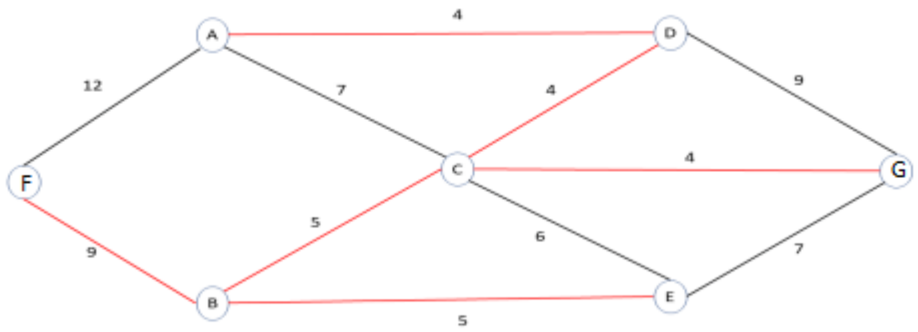
a)

i. **[2 Marks]** This network is not a tree because it contains a cycle: CB-BE-EC.

ii. **[8 Marks]** First, we set $K_0 = \{C\}$, $T = (K_0, \emptyset)$. Then, in each step we identify the cut-set $C(N, K_k)$, identify an edge e in $C(N, K_k)$ with minimal weight, expand K_k by adding a new node from this edge, and add the edge to the tree. Here is the generated sequence and the resulting graph. To shorten the notation, let $A_i \coloneqq argmin\,\{w(e) : e \in C(N, K_i)\}$

$$K_0 = \{C\},\ T = (K_0,\ \emptyset), CD, CG \in A_0, n_1 = D$$
$$K_1 = \{CD\},\ T = (K_1,\ \{CD\}), CG, AD \in A_1, n_2 = G$$
$$K_2 = \{CDG\},\ T = (K_2,\ \{CD, CG\}), AD \in A_2, n_3 = A$$
$$K_3 = \{CDGA\},\ T = (K_3,\ \{CD, CG, AD\}), BC \in A_3\ , n_4 = B$$
$$K_4 = \{CDGAB\},\ T = (K_4,\ \{CD, CG, AD, BC\}), BE \in A_4, n_5 = E$$
$$K_5 = \{CDGABE\},\ T = (K_5,\ \{CD, CG, AD, BC, BE\}), FB \in A_5, n_6 = F$$

So, one spanning tree is $\{CD, CG, AD, BC, BE, FB\}$. **[+5]** To generate the other spanning trees, we revisit the previous steps and identify where we have made an arbitrary decision. This was after defining K_0 and K_1. So, the alternative spanning trees will be repeated $\{CD, AD, CG, BC, BE, FB\}$ and $\{CG, CD, AD, BC, BE, FB\}$. **[+3]**



iii. **[7 Marks]** Let $d = (0, \infty, \ldots, \infty)$,$p = (\emptyset, \ldots, \emptyset)$, and $v = (0, \ldots, 0)$, denote the distance, previous-node, and visited-node vectors, respectively. At each iteration, we pick a not-yet-visited node with shortest distance, mark it as visited, and update the distance of its neighbors if passing through this node is a shorter path. Here is the evolution of these vectors as the algorithm proceeds.

(1)

	F	B	A	C	E	D	G
d	0	9	12	∞	∞	∞	∞
p	$\emptyset$	F	F	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	1	0	0	0	0	0	0

(5)

	F	B	A	C	E	D	G
d	0	9	12	14	14	16	18
p	$\emptyset$	F	F	B	B	A	C
v	1	1	1	1	1	0	0

(6)

	F	B	A	C	E	D	G
d	0	9	12	14	14	16	18
p	$\emptyset$	F	F	B	B	A	C
v	1	1	1	1	1	1	0

(2)

	F	B	A	C	E	D	G
d	0	9	12	∞	∞	∞	∞
p	$\emptyset$	F	F	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	1	1	0	0	0	0	0

(3)

	F	B	A	C	E	D	G
d	0	9	12	14	14	16	∞
p	$\emptyset$	F	F	B	B	A	$\emptyset$
v	1	1	1	0	0	0	0

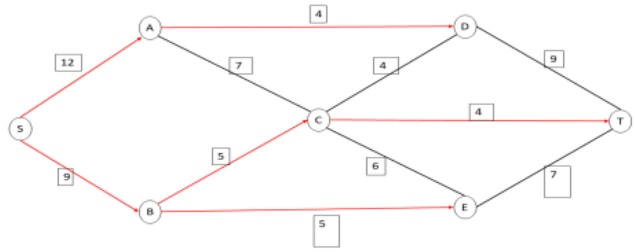
(4)

	F	B	A	C	E	D	G
d	0	9	12	14	14	16	18
p	$\emptyset$	F	F	B	B	A	C
v	1	1	1	1	0	0	0

(7)

	F	B	A	C	E	D	G
d	0	9	12	14	14	16	18
p	$\emptyset$	F	F	B	B	A	C
v	1	1	1	1	1	1	1

You start from F, then solve B (dist. 9), then A (dist. 12), then C and E (dist. 14), then D (dist. 16), then T (dist. 18)



b) **[8 Marks]** One must show that the expected payoff of the optimal mixed-strategies is zero. **[+2]** To show that $p^* = -d^*$, compute **[+4]**

$$p^* = \max_x \min_y x^T A y = -\left(-\max_x \min_y x^T A y\right) = -(\min_x(-\min_y x^T A\, y))$$
$$= -(\min_x \max_y -x^T A\, y) = -(\min_x \max_y x^T(-A)y) = -(\min_x \max_y x^T A^T y)$$
$$= -(\min_x \max_y y^T A\, x) = -d^*,$$

where the last equality follows by noting that we can rename x and y, given they have the same number of entries (because A is a square matrix). Finally, strong duality implies that $p^* = d^*$. Therefore, the equality $p^* = d^* = -p^*$ implies that $p^* = 0$. **[+2]**

Question 3

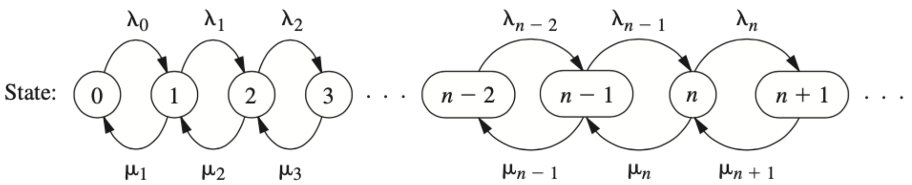
a)

i. **[1 mark]** A queue is in a steady state if the probability of the queue being in a given state (number of customers waiting etc) remains constant over time.

ii. **[5 Marks]** Little's theorem states that $L = \bar{\lambda} W$ **[+2]**, where L the expected number of customers in the queueing system, $\bar{\lambda}$ is the average arrival rate over the long run, and W is the expected stay in the system for each individual customer. **[+3]**

b)

i. **[3 marks]**



[+1] The values λ_i and μ_i are the parameters of the exponentially distributed interarrival and service times when there are i-many customers in the system, respectively. This mean that, if there are i-many customers, the expected waiting time before a new customer arrives is λ_i^{-1} and the expected service time is μ_i^{-1}. **[+2]**

ii. **[5 marks]** We first compute $c_1 = \frac{\lambda_0}{\mu_1} = 3,\ c_2 = 3\frac{\lambda_1}{\mu_2} = 9,\ c_3 = 9\frac{\lambda_2}{\mu_3} = 27,\ c_n = 27\left(\frac{1}{2}\right)^{n-3}$.
[+2] Therefore, the (steady state) probability that the shop is empty is **[+3]**

$$p_0 = \left(1 + \sum_{n=1}^{\infty} c_n\right)^{-1} = \left(1 + 3 + 9 + 27\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^{n-3}\right)^{-1}$$

$$= \left(13 + 27\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}\right)^{-1} = (13+54)^{-1} = (67)^{-1} \cong 0.015$$

iii. **[2 marks]** The queue is empty if there are at most three customers. Since $p_n = c_n p_0$, the probability of this is

$$p_0 + p_1 + p_2 + p_3 = \frac{(1+3+9+27)}{67} \cong 0.597$$

iv. **[4 marks]** The mean number of customers is **[+2]**

$$L = \sum_{n=0}^{\infty} n\, p_n = p_0\left(3 + 2*9 + 3*27 + 27*(2)^3 \sum_{n=4}^{\infty} n\left(\frac{1}{2}\right)^{n}\right)$$

$$= p_0\left(102 +\ 216\left(2 - \frac{1}{2} - 2\frac{1}{4} - 3\frac{1}{8}\right)\right) = \left(102 + 216\frac{5}{8}\right)/67 \cong 3.54$$

and the mean length of the queue is **[+2]**

$$L_q = \sum_{n=3}^{\infty}(n-3)\, p_n = \frac{27}{67}\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n} = \frac{27}{67}\,2 \cong 0.806$$

v. **[5 marks]** The mean arrival rate is **[+3]**

$$\bar{\lambda} := \sum_{n=0}^{\infty} \lambda_n p_n = 3(p_0 + p_1 + p_2) + 1 \sum_{n=3}^{\infty} p_n = \frac{3(1+3+9)}{67} + \frac{27}{67} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{39+54}{67}$$

$$= \frac{93}{67} \approx 1.39$$

Then, by Little's formula, **[+2]**

$$W = \frac{\mathrm{L}}{\bar{\lambda}} = \frac{237/67}{93/67} = \frac{237}{93} \approx 2.55,$$

$$W_q = \frac{L_q}{\bar{\lambda}} = \frac{54}{93} \approx 0.58,$$

$$W_s = W - W_q = \frac{183}{93} \approx 1.97$$

Question 4

a) **[5 Marks]** Statements i. and ii. are not true. A counterexample for both is $f(x) = |x|$.

b)

i. **[5 Marks]** The derivative of f is $f'(x) = x(2 - 4\exp(-x^2))$. **[+2]** The stationary points satisfy $f'(x) = 0$. **[+1]** This means that either $x = 0$ or $\exp(-x^2) = 0.5$, that is $-x^2 = \log(0.5)$, that is, $x = \pm\sqrt{\log(2)}$ **[+2]**

ii. **[5 Marks]** The second derivative of f is **[+2]**

$$f''(x) = 2\exp(-x^2)\,(4x^2 + \exp(x^2) - 2).$$

Therefore, $x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 1 - \frac{2-4\exp(-1)}{2\exp(-1)(2+\exp(1)).} = 1 - \frac{e-2}{2+e} = \frac{4}{2+e}$. **[+3]**

c) **[10 Marks]** The quadratic penalty function is **[+2]**

$$Q(x,p) \coloneqq f(x) + \frac{p}{2}\sum_{i\in E} c_i^2(x) = (x_1 + x_2)^2 + \frac{p}{2}(x_1 + x_2 - 1)^2$$

Performing one step of the quadratic penalty methods means solving **[+2]**

$$\min_x Q(x,p_0) = (x_1 + x_2)^2 + \frac{p_0}{2}(x_1 + x_2 - 1)^2 = (x_1 + x_2)^2 + (x_1 + x_2 - 1)^2\,.$$

The gradient of $Q(x,2)$ is $\nabla_x Q(x,2) = (4x_1 + 4x_2 - 2, 4x_1 + 4x_2 - 2)^T$. Therefore, the steepest descent direction at $x_0 = (0,0)^T$ is $d = -\nabla_x Q(x_0,2) = (2,2)^T$. **[+2]** To compute the optimal step size, we need to solve **[+1]**

$$\min_\alpha Q(x_0 + \alpha d, p_0) = (4\alpha)^2 + (4\alpha - 1)^2 = 32\alpha^2 - 8\alpha + 1\,.$$

Since $32\alpha^2 - 8\alpha + 1$ is a parabola, and hence convex, is suffices to solve $\frac{d}{d\alpha}Q(x_0 - \alpha d, p_0) = 0$, that is $64\alpha - 8 = 0$, which gives $\alpha = \frac{1}{8}$. Therefore, $x_1 = x_0 + \frac{1}{8}d = \left(\frac{1}{4},\frac{1}{4}\right)$. **[+3]**

END OF PAPER