MA2252 Introduction to Computing

Lecture 18
Numerical Integration

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Learning outcomes

At the end of lecture, students will be able to

- apply numerical methods to evaluate integrals
- understand geometrical interpretation of these methods
- implement these methods in MATLAB
- use MATLAB built-in integration functions

Introduction

Why study numerical integration?

- The anti-derivatives of many functions cannot be represented in terms of elementary functions. **Examples:** $\frac{\sin x}{x}$, e^{-x^2} and $\frac{1}{\ln x}$
- Analytical form of the integrand function(say $f(x)$) may be unknown. **Example:** The values of $f(x)$ are only known at a set of data points x_i.

Problem statement

Consider a function $f(x)$ defined over a interval $[a, b]$. We want to evaluate

$$I = \int_a^b f(x)dx. \tag{1}$$

This integral can be geometrically seen as area under the curve $y = f(x)$ for $x \in [a, b]$.

Problem statement (contd.)

Steps to evaluate (1) numerically:

- Create a numerical grid $x_i \quad (i = 0, 1, 2, \cdots, n)$ such that $x_0 = a, x_n = b$ and $x_{i+1} - x_i = h$(say).
- Using some appropriate method, calculate the area A_i under $f(x)$ for each sub-interval $[x_i, x_{i+1}] \quad (i = 0, 1, 2, \cdots, n-1)$.
- Compute the sum of the areas A_i over the interval $[a, b]$ i.e.

$$I \approx \sum_{i=0}^{n-1} A_i \tag{2}$$

Numerical integration methods

- Midpoint rule
- Trapezoidal rule
- Simpson's rule

Midpoint rule

Steps:

- The value of function in a subinterval $[x_i, x_{i+1}]$ is interpolated by a constant function with the value $f(\frac{x_i+x_{i+1}}{2})$.
- The area A_i is calculated by area of rectangle under the constant function.

$$A_i = h * f(\frac{x_i + x_{i+1}}{2}) \tag{3}$$

Midpoint rule (contd.)

Example: Write a script file which uses Midpoint rule to approximate $\int_0^{\pi} \sin x \, dx$.

Midpoint rule (contd.)

Demo

Trapezoidal rule

Steps:

- Here, the function in the subinterval $[x_i, x_{i+1}]$ is approximated using a straight line joining points $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ (linear interpolation).
- The area A_i is calculated by the area of trapezium formed under this straight line.

$$A_i = \frac{1}{2}(f(x_i) + f(x_{i+1}))h \tag{4}$$

Trapezoidal rule (contd.)

Example: Write a script file which uses Trapezoidal rule to approximate $\int_0^{\pi} \sin x \, dx$.

Trapezoidal rule (contd.)

Demo

Simpson's rule

Steps:

- Here, the function $f(x)$ is approximated on two subintervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ taken together. The interpolating function is a quadratic passing through points $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$.

- The area B_i over interval $[x_{i-1}, x_{i+1}]$ is derived as

$$B_i = \frac{h}{3}(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})) \tag{5}$$

- The integral I is given by

$$I \approx \sum_{i=1, i=odd}^{n-1} B_i \tag{6}$$

Simpson's rule (contd.)

(6) can also be expressed in the form:

$$I \approx \frac{h}{3}\left[f(x_0) + 4\left(\sum_{i=1, i=odd}^{n-1} f(x_i)\right) + 2\left(\sum_{i=2, i=even}^{n-2} f(x_i)\right) + f(x_n)\right] \quad (7)$$

Note: Since B_i is calculated for two consecutive subintervals taken together, Simpson's rule requires even number of subintervals i.e. n should be even.

Simpson's rule (contd.)

Example: Write a script file which uses Simpson's rule to approximate $\int_0^\pi \sin x \, dx$.

Demo

MATLAB's built-in integration functions

Two useful functions are trapz() and integral().

- trapz(x,f) takes of numerical grid x and function f as vector arguments and computes the value of integral I using trapezoidal rule.
- integral(fun,xmin,xmax) integrates the function fun from lower limit xmin to upper limit xmax.

MATLAB's built-in integration functions (contd.)

Write a script file using trapz() and integral() functions to approximate $\int_0^{\pi} \sin x \, dx$.

MATLAB's built-in integration functions (contd.)

Demo

End of Lecture 18

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