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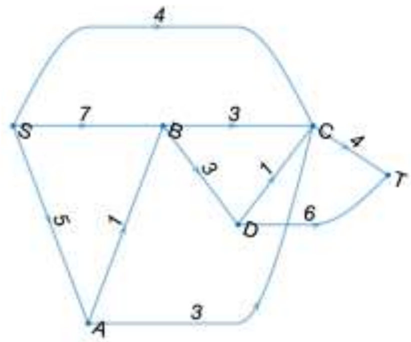
Problem Sheet 2 - Submission Deadline: 20 November at 6pm (GMT)

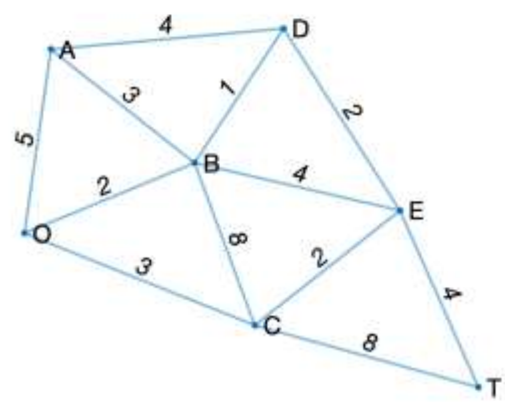
1) Consider the following directed network $N = (V, E)$ with source S and sink T.



Let $f: E \to \mathbb{R}_+$ be defined by $f(e) = \begin{cases} 4, & e \in \{SC, CT\} \\ 1, & e \in \{SA, AB, BD, DT\}, \\ 0, & otherwise. \end{cases}$

a. **[2 Marks]** Verify that f is a flow and compute its value.
b. **[2 Marks]** Identify an f-augmenting path and compute its capacity.
c. **[2 Marks]** Provide a nontrivial upper bound for the value of a maximal flow.

2) Consider the following network.



a. **[3 Marks]** Determine and draw a minimal spanning tree starting at node O using the version of Prim's algorithm presented in class. Show all intermediate steps.
b. **[5 Marks]** Write a Matlab program to compute a minimal spanning tree of the full subgraph induced by $V' = \{O, A, B, C\}$ using Matlab's function intlinprog.

3) Consider a two-person zero-sum game with payoff matrix

$$A = \begin{pmatrix} 0 & 2 & 1 & -1 \\ 3 & 4 & 0 & -5 \\ -1 & 3 & 0 & 2 \\ -2 & -1 & 2 & 1 \end{pmatrix}.$$

a. **[3 Marks]** Assuming that players do not employ mixed-strategies, determine their best strategies and explain in simple terms why this game is not stable.

b. **[4 Marks]** Determine the optimal mixed-strategy for player 1.

4) Let the random variable T have exponential distribution with parameter $\alpha = 1$.

a. **[1 marks]** Compute the expected value and the variance of T.

b. **[3 marks]** Let V and W be independent copies of T. Compute the quantities $P[1 \le T \le 3 | V \le 3]$ and $P[T + V + W \le 1]$.

5) Consider the $M/M/s/K$ queueing model with $s = 2, K = 100, \lambda = 99$ and $\mu = 50$.

a. **[3 marks]** Provide a sketch to describe this birth-and-death process and describe in plain words the meaning of the parameters involved.

b. **[3 marks]** Compute the probability that, in a steady state scenario, the queueing system is empty.

6) **[3 marks]** Determine for which values of μ a birth-and-death process with $\mu_n = \mu$ and $\lambda_n = 2 + \cos(n\pi)$ admits steady state probabilities $\{p_n\}_{n \ge 0}$.

7) Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) \coloneqq x^4 - 2x^3 + x^2$.

a. **[2 Marks]** Determine all stationary points of f in $\mathbb{R}$.

b. **[3 Marks]** Perform one step of Newton's method starting from $x_0 = -1$.

c. **[3 Marks]** Perform one step of the steepest descent method starting from $x_0 = -1$ and using the smallest optimal step size.

8) **[4 Marks]** For a fixed number $n \in \mathbb{N}$, consider the function $f: \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) \coloneqq (Ax - b)^T (Ax - b)$, where $A \in \mathbb{R}^{n,n}$ is an invertible matrix and $b \in \mathbb{R}^n$ is a given vector. Perform one step of Newton's method using a generic starting point $x_0 \in \mathbb{R}^n$.

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9) **[4 Marks]** Consider the state constrained optimisation problem

$$\min_{x\in\mathbb{R}} f(x,u) := u_1 + \cos(u_2) \quad subject\ to \begin{pmatrix} 1+x^2 & x \\ x & 1 \end{pmatrix} u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Use the Lagrange method to compute the total derivative $\frac{d}{dx} f(x,u)$.