MA3077 DLI
All Candidates

Semester 1 Examinations 2023

School	School of Computing and Mathematical Sciences
Module Code	MA3077 DLI
Module Title	Operational Research
Exam Duration (in words)	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	5 (including cover page)
Number of Questions	4
Instructions to Candidates	Please answer all questions and motivate your answers.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Yes
Books/Statutes provided by the University	No
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	No

Question 1

a) **[2 marks]** Explain why the following optimization problem is not a linear programming problem.

$$\min x + y \; subject\ to\ x^2 + y^2 \leq 2.$$

b) Consider the following linear programming problem.

$$\max y - 2x \quad subject\ to \begin{cases} x - y \leq 1 \\ 4y \leq 4 - x \\ x \geq 0 \end{cases} \tag{1}$$

i. **[6 marks]** Draw the feasible set, determine an optimal solution, and indicate which constraints are active.

ii. **[4 marks]** Write the linear programming problem (1) in standard form.

iii. **[5 marks]** Derive the dual problem of the linear programming problem (1), identifying clearly the Lagrangian and the dual function.

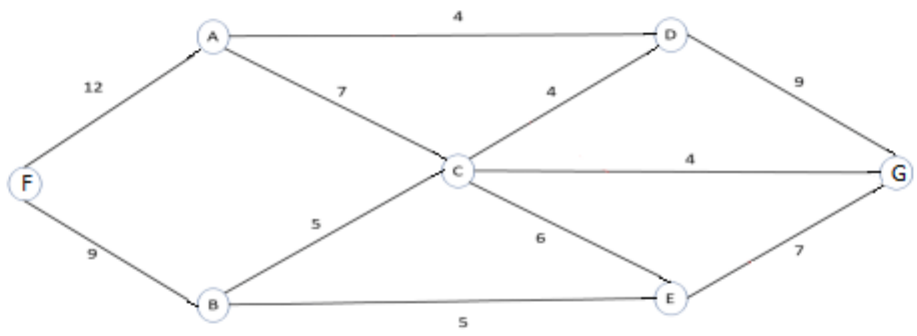
c) i. **[5 marks]** Given the following primal and dual linear programming, show that if the primal problem is unbounded, then the dual problem must be infeasible.

$$\begin{array}{ll} min & c^T x \\ s.t. & Ax = b \\ & x \geq 0 \end{array} \quad \text{and} \quad \begin{array}{ll} max & b^T y \\ s.t. & c - A^T y = s \\ & s \geq 0,\ y \in \mathbb{R}^m \end{array}$$

ii. **[3 marks]** State the Farkas' Lemma.

Question 2

a) Consider the following network with weights as distances between the nodes.



i. **[2 marks]** Explain whether this network is a tree or not.

ii. **[8 marks]** Determine and draw all minimal spanning tree starting at node C using the version of Prim's algorithm presented in class. Show all intermediate steps.

iii. **[7 marks]** Determine and draw a shortest path tree starting at node F using the version of Dijkstra's algorithm presented in class. Show all intermediate steps and indicate the capacity of each edge.

b) **[8 marks]** Prove that if a payoff matrix A is skew-symmetric, that is, $A^T = -A$, and if both players employ mixed-strategies, then the two-person zero-sum game with payoff matrix A is fair.

[*Hint: The optimal expected payoffs of players 1 and 2 are*

$$p^* = \max_x \min_y x^T A y \; and \; d^* = \min_y \max_x x^T A\, y\,,$$

respectively.]

Question 3

a) Answer the following questions.

i. **[1 mark]** What is meant by a steady state queue?
ii. **[5 marks]** State the formula known as Little's Law (also known as Little's Theorem) and explain the meaning of all the variables involved.

b) Let a queueing system with 3 servers be described by a birth-and-death process with exponential distribution parameters

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \qquad \lambda_i = 2 \text{ for } i \geq 3,$$
$$\mu_1 = \mu_2 = \mu_3 = 1, \qquad \mu_i = 4 \text{ for } i \geq 4.$$

i. **[3 marks]** Provide a sketch to describe this birth-and-death process and describe in plain words the meaning of the parameters involved.

ii. **[5 marks]** Compute the probability that, in a steady state scenario, the queueing system is empty.

iii. **[2 mark]** Compute the probability that, in a steady state scenario, the queue is empty.

iv. **[4 marks]** Compute the mean number of customers and the mean length of the queue.

v. **[5 marks]** Compute the mean waiting times in the system and the mean service times.

Question 4

a) **[5 marks]** Prove or disprove the following statements.

i. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be a continuous function. Then, every minimum of* f *is a stationary point.*

ii. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be a convex function. Then, every minimum of* f *is a stationary point.*

b) Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) \coloneqq x^2 + 2\exp(-x^2)$. The objective of this exercise is to minimise (at least approximately) the function f.

i. **[5 marks]** Determine all stationary points of f in $\mathbb{R}$.

ii. **[5 marks]** Perform one step of Newton's method starting from $x_0 = 1$.

c) **[10 marks]** Consider the constrained optimisation problem

$$\min_{x \in \mathbb{R}^2} (x_1 + x_2)^2 \quad \text{subject to} \quad x_1 + x_2 - 1 = 0$$

Perform one step of the quadratic penalty method using the penalty parameter $p_0 = 2$. To solve the internal iteration, use one step of the steepest descent method with initial guess $x_0 = (0,0)^T \in \mathbb{R}^2$ and exact line search.

END OF PAPER