Exercise 1. *(1) Let S be the finite subset of real numbers. Then*

$$\exists x_0 \in S, \forall x \in S, x_0 \geq x$$

(2) Let C be the set of all countries, $P(C)$ be the set of all the people in the country C, and $f : P(C) \to \mathbb{R}^+$ be defined such that $\forall p \in P(C)$, $f(p)$ is the age of the person p. Then

$$\forall c \in C, \exists r \in \mathbb{R}, \forall p \in \{p \in P(C) : f(p) \geq r\}, Q(p)$$

where $Q(p)$ means the person p can obtain pension payment in the country c.

There will not be any problem of the type (2) showing up in your exam. The aims of this problem are twofold:

A. To indicate the complexity of our daily language - out of our intuition, a sentence as simple as seven words includes a structure as complex as $\forall, \exists, \forall$;

B. To demonstrate the ways of stating in math by setting this as an example.

Exercise 2. *Setting $M = \max\{|a|, |b|\}$. Then $\forall x \in [a, b)$,*

If $x \geq 0$, then $b > 0$, so

$$|x| = x < b \leq |b| \leq \max\{|a|, |b|\} = M$$

If $x < 0$, then $a \leq x < 0$, so

$$|x| = -x \leq -a = |a| \leq \max\{|a|, |b|\} = M$$

as desired.

Exercise 3. *Since A is bounded, $\exists M > 0$, $\forall x \in A$, $|x| \leq M$. Since $B \subset A$, we have*

$$\forall x \in B, |x| \leq M$$

as desired.

Exercise 4. *For $i = 1, 2$, since S_i is bounded,*

$$\exists M_i > 0, \forall x \in S_i, |x| \leq M_i$$

Setting $M = \max\{M_1, M_2\}$, then applying the similar argument from Exercise 2, we have $\forall x \in S_1 \cup S_2$, $|x| \leq M$.

Exercise 5. *(1) We say $\beta \in \mathbb{R}$ is the **infimum** of the set S, if*
(a) $\forall s \in S$, $\beta \leq s$;
(b) $\forall \epsilon > 0$, $\exists s_0 \in S$, such that $s_0 < \beta + \epsilon$.

(2) Since T has an upper bound, $\alpha = \sup T$ exists according to the Axiom of supremum. By definition,
(a) $\forall t \in T, t \leq \alpha$. From this we have (a) $-\alpha \geq -t$, which means that $-\alpha$ is a lower bound for U;*
(b) $\forall \epsilon > 0$, $\exists t_0 \in T$, such that $t_0 > \alpha - \epsilon$. This implies (b) $-t_0 \in U$ and $-t_0 < -\alpha + \epsilon$.*
It follows that $\sup T = -\inf U$ from (a) and (b*) above.*

(3) Existence: We claim that if a set U has a lower bound, then its infimum exists.

***Proof*.** Applying the similar approach from (2) above, we know that $-U$ has an upper bound and hence $\alpha = \sup U$ exists. By (2) again, $\inf U = -\alpha$ exists. □

Uniqueness: If the infimum of U exists, it must be unique. This is a conclusion symmetric to that of supremum.

Exercise 6. *(1) By the definition of intervals: $1 \in [-1, 1]$ and $\forall x \in [-1, 1]$, $x \leq 1$.*
(2) By contradiction. Assume M is the maximum of S, then by (a), $-1 \leq M < 1$. However, $\frac{M+1}{2} \in S$ but $\frac{M+1}{2} > M$, a contradiction.
(3) Suppose $M = \max S$ exists. Then from (a), M is an upper bound. $\forall \epsilon > 0$, taking $s = M$, then $s > M - \epsilon$. By definition of supremum, $\sup S = M$.