$Solution\ to\ Coursework\ 3$

$Exercise\ 1(1)$

$f(x)$ *is an integrable odd function defined on* $[-a,a]$, *so we have*:

$$f(-x) = -f(x) \text{ on } [-a,a]$$

$$\text{hence} \quad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} [-f(x)]\,dx + \int_{0}^{a} f(x)\,dx = 0$$

$$\text{that} \quad \int_{-a}^{a} f(x)\,dx = 0$$

$Exercise\ 1(2)$

$f(x)$ *is an integrable oven function defined on* $[-a,a]$, *so we have*:

$$f(-x) = f(x) \text{ on } [-a,a]$$

$$\text{hence} \quad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\text{that} \quad \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$Exercise\ 2$

Let $f(x) \in C[a,b]$ *satisfy* $\forall \phi(x) \in C[a,b], \int_a^b f(x)\phi(x) = 0$

Hence when $\phi(x) = f(x) \in C[a,b], \int_a^b f(x)\phi(x)\,dx = \int_a^b [f(x)]^2\,dx = 0$

Let $g(x) = [f(x)]^2$, *so* $g(t) \in C[a,b]$ *and* $g(t) \geq 0$ *on*$[a,b]$,

for $\forall x \in (a,b)$, *we have* $0 \leq \int_a^x g(t)\,dt \leq \int_a^b g(t)\,dt = 0 \Longrightarrow \int_a^x g(t)\,dt \equiv 0$

$$g(x) = \left(\int_a^x g(t)\,dt\right)' \equiv 0 \Longrightarrow g(x) \equiv 0$$

$g(x) \in C[a,b]$ *and* $g(x) \equiv 0$ *on* (a,b), *that* $g(x) \equiv 0$ *on* $[a,b] \Longrightarrow f(x) \equiv 0$ *on* $[a,b]$

Hence $\forall x \in [a,b], f(x) = 0$

$Exercise\ 3$

Let $M = \max\limits_{[a,b]} |f(x)|$.

(1) *Let* $S_\delta := \{x \in [a,b], |f(x)| \geq M - \delta$ *for any* $\delta > 0$.

Then we have for all p

$$M \cdot (b-a)^{\frac{1}{p}} \geq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \geq \left(\int_{S_\delta} |f(x)|^p dx\right)^{\frac{1}{p}} \geq (M-\delta)\big(\lambda(S_\delta)\big)^{\frac{1}{p}}.$$

Since $f(x) \in C[a,b]$, *the measure of* S_δ *is positive and taking the* $\liminf$ *and* $\limsup$ *for all* $\delta > 0$, *we have that*

$$M \geq \lim_{p\to\infty} sup(\int_a^b |f(x)|^p dx)^{\frac{1}{p}} \geq M - \delta \ and \ M \geq \lim_{p\to\infty} inf(\int_a^b |f(x)|^p dx)^{\frac{1}{p}} \geq M - \delta$$

$Hence\ , M = \max_{[a,b]}|f(x)| = \lim_{p\to\infty}\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$

(2) $for\ any\ \varepsilon > 0, there\ exists\ the\ [c,d]\ , [c,d] \in [a,b]\ and\ \forall x \in\ [c,d]$

$satisfies\ |f(x)| \geq M - \frac{\varepsilon}{2}\ .So\ we\ have\ that$

$$M \cdot (b-a)^{\frac{1}{p}} \geq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \geq \left(\int_c^d |f(x)|^p dx\right)^{\frac{1}{p}} \geq (M - \frac{\varepsilon}{2})(d-c)^{\frac{1}{p}}$$

$$hence\ \exists N > 0, when\ p > N, we\ have\ (b-a)^{\frac{1}{p}} < 1 + \frac{\varepsilon}{M}\ , (b-a)^{\frac{1}{p}} < 1 + \frac{\varepsilon}{M}$$

$$that\ M - \varepsilon < M - \varepsilon + \frac{\varepsilon}{4M} < \left(M - \frac{\varepsilon}{2}\right)(d-c)^{\frac{1}{p}} < \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} < M \cdot (b-a)^{\frac{1}{p}} < M + \varepsilon$$

$$we\ get\ \left|\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}\right| < \varepsilon\ , that\ \lim_{p\to\infty}\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} = M = \max_{[a,b]}|f(x)|$$

$Exercise\ 4$

(1) $If\ g'(0) \equiv 0\ , then\ obviously\ 0 = |g'(0)| \leq K\left(\int_0^1 |g''(t)|\, dt + \int_0^1 |g(t)|\, dt\right)$

(2) $If\ g'(0) \neq 0\ \ on\ [0,\varepsilon]\ (\varepsilon > 0)$

① $If\ g''(t) \equiv 0, then\ g'(t)\ is\ constant\ on\ [0,\varepsilon]. Hence\ g(t) \not\equiv 0\ on\ [0,\varepsilon]\ ,$

$which\ means\ \int_0^\varepsilon |g(t)|\, dt > 0. So\ we\ must\ can\ find\ a\ constant\ K$

$big\ enough\ to\ satisfy\ |g'(0)| \leq K\left(\int_0^1 |g''(t)|\, dt + \int_0^1 |g(t)|\, dt\right)$

② $Assume\ g(t) \equiv 0\ . According\ to\ the\ problem\ description\ , g(t) \in C^2[0,1]$

$so\ g'(t)\ must\ be\ continuous\ . Hence\ \exists t_0 \in [0,\varepsilon]\ , that\ g'(t_0) \neq 0.$

$which\ means\ that\ the\ assumption\ is\ incorrect.$

$Hence\ there\ exists\ a\ constant\ K\ such\ that:$

$$|g'(0)| \leq K\left(\int_0^1 |g''(t)|\, dt + \int_0^1 |g(t)|\, dt\right)$$