Exercise 1. *$\forall\epsilon>0$, setting $N=[\frac{1}{\epsilon}]$, for $n>m>N$,*

$$\begin{aligned}&\frac{\sin(m+1)x}{(m+1)((m+1)+\sin(m+1)x)}+\cdots+\frac{\sin nx}{n(n+\sin nx)}\\=&\frac{1}{m+1}-\frac{1}{m+1+\sin(m+1)x}+\cdots\frac{1}{n}-\frac{1}{n+\sin nx}\\\leq&\frac{1}{m+1}-\frac{1}{m+2}+\cdots\frac{1}{n}-\frac{1}{n+1}\\=&\frac{1}{m+1}-\frac{1}{n+1}<\frac{1}{m+1}<\frac{1}{N+1}\\=&\frac{1}{[\frac{1}{\epsilon}]+1}\leq\epsilon\end{aligned}$$

Exercise 2. *$\forall\epsilon>0$, setting $\delta=\min\{1,\frac{\epsilon}{19}\}$, for x satisfying $0<|x-2|<\delta$, we have*

$$|x^2+2x+4|\leq|x|^2+2|x|+4\leq 3^2+2*3+4=19$$

So

$$|x^3-8|=|x-2||x^2+2x+4|<19\delta\leq 19\frac{\epsilon}{19}=\epsilon,$$

as desired.

Exercise 3. *Let $x_0\in\mathbb{R}$. We need to prove that f is divergent at x_0. We can select $\{a_n\}\subset\mathbb{Q}$ and $\{b_n\}\subset\mathbb{R}\backslash\mathbb{Q}$ so that both of them converge to x_0. Then we know $f(a_n)\to 1$ while $f(b_n)\to 0$. By Henie's theorem, the limit does not exist.*

In the previous exercise we have secretly applied the density of $\mathbb{Q}$ in $\mathbb{R}$. More generally, we have

Proposition 1. *$\forall a,b\in\mathbb{R}$, $\exists q\in\mathbb{Q}$ and $r\in\mathbb{R}\backslash\mathbb{Q}$ such that $a<q<b$ and $a<r<b$.*

We will regard this proposition as an axiom because it involves the establishment of $\mathbb{R}$ from $\mathbb{Q}$.

Exercise 4. *(1) From $\lim_{x\to\infty}f(x)=+\infty$ we have*

$$\forall G>0,\exists X>0,\forall x>X,f(x)>G \tag{1}$$

Since $\lim_{n\to\infty}x_n=+\infty$, for the previous $X>0$, $\exists N\in\mathbb{N}$, $\forall n>N$, $x_n>X$. Hence by (1), $f(x_n)>G$. It follows that $\lim_{n\to\infty}f(x_n)=+\infty$.

For the other side, we argue this by contradiction. Assume

$$\exists G>0,\forall X>0,\exists x>X,f(x)\leq G$$

Letting $X=1$, $\exists x_1>1$, so that $f(x_1)\leq G$;
Letting $X=2$, $\exists x_2>2$, so that $f(x_2)\leq G$;
...
Letting $X=n$, $\exists x_n>n$, so that $f(x_n)\leq G$;
...
Hence we have obtained a sequence $\{x_n\}$ satisfying $x_n>n$, from which we can deduce that $\lim_{n\to\infty}x_n=+\infty$, and $f(x_n)\leq G$, which implies the boundedness of $\{f(x_n)\}$, a contradiction.

(2) We only need to discuss the sufficient side. Assume again that

$$\exists G > 0, \forall X > 0, \exists x > X, f(x) \leq G$$

Letting $X = 1$, $\exists x_1 > 1$, *so that* $f(x_1) \leq G$;
Letting $X = \max\{2, x_1\}$, $\exists x_2 > 2$ *and* $x_2 > x_1$, *so that* $f(x_2) \leq G$;
$\cdots$
Letting $X = \max\{n, x_{n-1}\}$, $\exists x_n > n$ *and* $x_n > x_{n-1}$, *so that* $f(x_n) \leq G$;
$\cdots$
Hence we have obtained a sequence $\{x_n\}$ *satisfying* $x_n > n$, *from which we can deduce that* $\lim_{n\to\infty} x_n = +\infty$, $x_n > x_{n-1}$, *so that* $\{x_n\}$ *is increasing, and* $f(x_n) \leq G$, *which implies the boundedness of* $\{f(x_n)\}$, *a contradiction.*