**Exercise 1.** *We have the classical triangle inequality* $|a+b| \le |a|+|b|$.

*(1) We have* $\forall x, y \in \mathbb{R}$

$$|x| = |x-y+y| \le |x-y| + |y|$$

*where we have applied the classical triangle inequality with* $a = x-y$ *and* $b = y$. *This yields*

$$|x| - |y| \le |x-y|$$

*Similarly we have*

$$|y| - |x| \le |x-y|$$

*Together this implies*

$$||x| - |y|| \le |x-y|$$

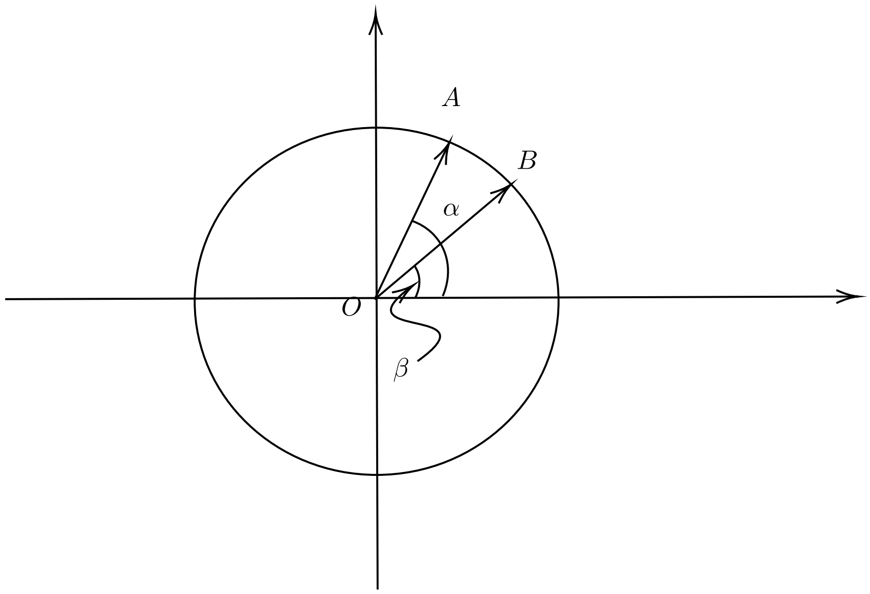
*For the other side, simply subsitute* $a = x$ *and* $b = -y$ *in the classical triangle inequality and it is done.*

*(2) Without of loss generality we assume that* $|x| \le |y|$. *Then we have*

$$\begin{aligned} |x+y|^p \le &(|x|+|y|)^p \\ \le &(2|y|)^p \\ \le &2^p(|x|^p + |y|^p) \end{aligned}$$

**Exercise 2.** *(1) and (2) can be proven by cases of quaduants.*
*(3) As the graph below, we denote by* $\overrightarrow{\alpha} = \overrightarrow{OA} = (\cos\alpha, \sin\alpha)$ *and* $\overrightarrow{\beta} = \overrightarrow{OB} = (\cos\beta, \sin\beta)$. *Then*

$$\cos(\alpha-\beta) = \cos(\alpha-\beta)|\overrightarrow{\alpha}||\overrightarrow{\beta}| = \overrightarrow{\alpha} \cdot \overrightarrow{\beta} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$




*(4) Proving by cases of quaduants yields* $\sin(\frac{\pi}{2}-\alpha)=\cos\alpha$ *and* $\cos(\frac{\pi}{2}-\alpha)=\sin\alpha$*. Then*

$$\begin{aligned}\sin(\alpha+\beta) &= \cos(\frac{\pi}{2}-(\alpha+\beta))\\ &= \cos((\frac{\pi}{2}-\alpha)-\beta)\\ &= \cos(\frac{\pi}{2}-\alpha)\cos\beta+\sin(\frac{\pi}{2}-\alpha)\sin\beta\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\end{aligned}$$

*(5) We prove this by induction. When* $n=1$*, the result is true. Assume it holds* $n$ *so that* $|\sin(nx)|\leq n|\sin x|$*. Then by (4),*

$$\begin{aligned}|\sin((n+1)x)| =& |\sin(nx)\cos x+\cos(nx)\sin x|\\ \leq& |\sin(nx)||\cos x|+|\cos(nx)||\sin x|\\ \leq& n|\sin x|+1*|\sin x|\\ =& (n+1)|\sin x|,\end{aligned}$$

*as desired.*