MA1014 Examination — Draft Questions and Solutions

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Title of paper	MA1014 — Calculus and Analysis
Version	1
Candidates	All candidates
School	COMPUTING AND MATHEMATICAL SCIENCES
Examination Session	Final Test 2022
Time allowed	2 hours
Instructions	This paper contains 4 questions. Full marks are 100 marks. Please attempt all questions.
Calculators	No
Books/statutes	No
Own Books/statutes/notes	No
Additional Stationery	Yes
Number of questions	4

In this exam, you are free to use properties of limit, continuity of elementary functions, the facts

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

and

$$\lim_{x\to+\infty}(1+\frac{1}{x})^x=e$$

without proof unless explicitly stated.

You may use any results from the course that you state correctly.

Question 1.

(a) Compute

$$\int \frac{dx}{x(x+2)}$$

[6 marks]

(b) Compute

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos x dx$$

[7 marks]

(c) State the definition of $f(x)$ being continuous at a point x_0. **[5 marks]**

(d) Let $g(x)$ be defined on $[0,1]$ such that for all $f(x)$ being continuous on $[0,1]$,

$$\int_0^1 f(x)g(x)dx = 0$$

Prove that $\forall x \in [0,1]$, $g(x) = 0$. **[7 marks]**

Answer — total marks for this question: 25

(a) Since

$$\frac{1}{x(x+2)} = \frac{1}{2x} - \frac{1}{2(x+2)}$$

then

$$\int \frac{dx}{x(x+2)} = \frac{1}{2}\ln|\frac{x}{x+2}| + C$$

(b)

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos x dx = \int_0^{\frac{\pi}{2}} \sin^5 x d(\sin x) = \frac{1}{6}$$

(c) Let $f(x)$ be defined by in a neighborhoud of x_0. If

$$\lim_{x \to x_0} f(x) = f(x_0)$$

then we say that f is continous at x_0.

(d) We argue this by contradiction. Suppose $\exists x_0 \in [0,1]$ such that $g(x_0) \neq 0$. Without loss of generality, we assume that $g(x_0) > 0$. Then since g is continous, $\exists \delta > 0$ such that $\forall x \in (x_0 - \delta, x_0 + \delta)$, $g(x) > \frac{g(x_0)}{2}$. Define f on $[0,1]$ such that $f(x) = 1$ on $(x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2})$; $f(x) = 0$ for $x \notin (x_0 - \delta, x_0 + \delta)$, and linear on the gaps to make it continuous. Then

$$\begin{aligned} \int_0^1 f(x)g(x)dx &= \int_{x_0-\delta}^{x_0+\delta} f(x)g(x)dx \\ &> \frac{g(x_0)}{2} \int_{x_0-\delta}^{x_0+\delta} f(x)dx \\ &= \frac{g(x_0)}{2}\delta > 0, \end{aligned}$$

a contradiction.

Question 2.

(a) Prove that for $x, y \in \mathbb{R}$,

$$2\cos x \cos y = \cos(x+y) - \cos(x-y)$$

[3 marks]

(b) Using (a) above, or otherwise, solve

$$y' + \cos(x+y) = \cos(x-y)$$

You may use the fact

$$\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C$$

without proof where $\sec x = \frac{1}{\cos x}$. **[7 marks]**

(c) Solve

$$y'' - 3y' + 2y = 2e^{-x}$$

with initial conditions

$$y(0) = 2 \text{ and } y'(0) = -1$$

[10 marks]

(d) Find an ODE with solutions $y = x$ and $y = x^2$. Check briefly that the functions satisfy your ODE. **[5 marks]**

Answer — total marks for this question: 25

(a)

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

and

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

(b) Using (b) and seperating the variables, we have

$$\sec y dy = -2\cos x dx$$

So the general solution is

$$\ln|\sec y + \tan y| = -2\sin x + C$$

(c) The general solution to the homogenous equation is

$$y = C_1 e^x + C_2 e^{2x}$$

Assume that

$$y_p = Ae^{-x}$$

is one of the particular solutions. Inserting back to the equation, we have $A = \frac{1}{3}$. Thus the general solution to the original ODE is

$$y = C_1 e^x + C_2 e^{2x} + \frac{1}{3}e^{-x}$$

Inserting the initial conditions

$$y(0) = 2 \text{ and } y'(0) = -1$$

We have the particular solution

$$y = 4e^x - \frac{7}{3}e^{2x} + \frac{1}{3}e^{-x}$$

(d) $y^{(3)} = 0$, or any other equations having the solutions.

Question 3.

(a) State the comparison test for series. **[3 marks]**

(b) By using the comparison test, or otherwise, determine whether the series is convergent and justify your answer:

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n-1}$$

[5 marks]

(c) Let $\{a_n\}$ be a non-negative sequence such that

$$\sum_{n=1}^{\infty} a_n$$

is convergent. Prove that

$$\sum_{n=1}^{\infty} a_n^2$$

is convergent. **[5 marks]**

(d) State the ratio test for series. **[5 marks]**

(e) Using the ratio test, or otherwises, determine whether the series is convergent and justify your answer:

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$$

[7 marks]

Answer — total marks for this question: 25

(a) Let $\{a_n\}$, $\{b_n\}$ be non-negative sequences such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then
(1) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent;
(2) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

(b) Since

$$\lim_{n\to\infty} \frac{n(n+1)}{n^2+2n-1} = 1,$$

$\exists N$, $\forall n > N$,

$$\frac{n+1}{n^2+2n-1} > \frac{1}{2n}$$

Since $\sum \frac{1}{n}$ is divergent, by comparison test,

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n-1}$$

is divergent.

(c) Since $\sum_{n=1}^{\infty} a_n$ is convergent,

$$\lim_{n\to\infty} a_n = 0$$

Hence $\exists N \in \mathbb{N}$, for $n > N$, $a_n^2 \leq a_n$. Since a_n is non-negative, by comparison test,

$$\sum_{n=1}^{\infty} a_n^2$$

is convergent.

(d) Let $\{a_n\}$ be a non-negative sequence and assume the limit

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \alpha$$

exists. Then if the series $\sum a_n$ is convergent when $\alpha < 1$, divergent when $\alpha > 1$, and can go either ways when $\alpha = 1$.

(e) The limit is $\frac{2}{e} < 1$, so convergent.

Question 4.

(a) State the definition of $f(x,y)$ being continous at the point (x_0, y_0). **[3 marks]**

(b) Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} \quad , & (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0). \end{cases}$$

Compute $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$, then prove that $f(x,y)$ is continous at the point $(0,0)$. **[7 marks]**

(c) Compute the tangent plane of the surface $z = 2x^4 + 3y^3$ at the point $(2,1,35)$. **[5 marks]**

(d) Let $f(x,y) = x^2 + 2y^2 - 2x - 12y + 6$. Compute all the local maxima and local minima of f on $\mathbb{R}^2$. **[10 marks]**

Answer — total marks for this question: 25

(a) Let $f(x,y)$ be defined in a neighborhood of (x_0,y_0). If

$$\lim_{x\to x_0, y\to y_0} f(x,y) = f(x_0,y_0)$$

then we say that $f(x,y)$ is continous at the point (x_0,y_0).

(b) By definition, $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Since

$$\lim_{x\to x_0, y\to y_0} \frac{x^2y}{x^2+y^2} = 0 = f(0,0)$$

the function is continous.

(c) The normal vector at the point $(2,1,35)$ is $(64,9,-1)$. Then the tangent plane is

$$64(x-2)+9(y-1)-(z-35) = 0$$

(d) We have $\frac{\partial f}{\partial x} = 2x-2$, $\frac{\partial f}{\partial y} = 4y-12$. Letting they be equal to 0, we have $x=1$ and $y=3$. At $(1,3)$, $\frac{\partial^2 f}{\partial x^2} = 2$ and $\frac{\partial^2 f}{\partial y^2} = 4$, and $\frac{\partial^2 f}{\partial x \partial y} = 0$. So the point $(2,-2)$ is a local minimal point.