MA2252 Introduction to computing

lectures 17-18

Review of linear algebra

Matias Ruiz

November 2023

Sets

A **set** or (**space**) is a collection of objects.

Table 12.1 Various Sets of Numbers and Corresponding Notations Used to Denote Them

Set Name	Symbol	Description
Naturals	$\mathbb{N}$	$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$.
Wholes	$\mathbb{W}$	$\mathbb{W} = \mathbb{N} \cup \{0\}$
Integers	$\mathbb{Z}$	$\mathbb{Z} = \mathbb{W} \cup \{-1, -2, -3, \cdots\}$
Rationals	$\mathbb{Q}$	$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}\backslash\{0\}\}$
Irrationals	$\mathbb{I}$	$\mathbb{I}$ is the set of real numbers not expressible as a fraction of integers.
Reals	$\mathbb{R}$	$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$
Complex Numbers	$\mathbb{C}$	$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$

Vector spaces

Let S be a set. S is a vector space if it satisfies some extra properties:

- if $u \in S$ and $v \in S$, then $u + v \in S$.
- if $u \in S$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), then $\alpha u \in S$

$$\Rightarrow \alpha u + \beta v \in S, \text{ where } \beta \in \mathbb{R} \text{ (or } \mathbb{C}\text{)}.$$

Excerise: Which of the following sets are or aren't vector spaces

1. $\{(x, y)$ with $x, y \in \mathbb{R}$ such that $x + y = 0\}$
2. $\{(x, y)$ with $x, y \in \mathbb{R}$ such that $x + y = 1\}$
3. The space of differentiable functions
4. The space of polynomials of degree n.

Vector spaces

Let S be a set. S is a vector space if it satisfies some extra properties:

- if $u \in S$ and $v \in S$, then $u + v \in S$.
- if $u \in S$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), then $\alpha u \in S$

$$\Rightarrow \alpha u + \beta v \in S, \text{ where } \beta \in \mathbb{R} \text{ (or } \mathbb{C}).$$

Excerise: Which of the following sets are or aren't vector spaces

1. $\{(x, y)$ with $x, y \in \mathbb{R}$ such that $x + y = 0\}$
2. $\{(x, y)$ with $x, y \in \mathbb{R}$ such that $x + y = 1\}$
3. The space of differentiable functions
4. The space of polynomials of degree n.
5. The space $\mathbb{P}_n$ of polynomials of degree **equal or less** than n.

The vector space $\mathbb{R}^n$

$$\boxed{\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) \text{ with } x_1, x_2, \ldots, x_n \in \mathbb{R}\}}$$

If $u \in \mathbb{R}^n$ one typically writes

$$u = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- if $\alpha \in \mathbb{R}$ then

$$\alpha u := \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

- if $v = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

$$u + v := \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

The vector space $\mathbb{R}^n$

Some standard operations on $\mathbb{R}^n$:

- transpose:

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad , \quad u' = (x1, \cdots, x_n)$$

- norm: $\|u\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$
- p-norm of a vector: $\|u\|_p = (x_1^p + x_2^p + \ldots + x_n^p)^{1/p}$
- dot product: $u \cdot v = x_1 y_1 + \cdots + x_n y_n = \|u\| \|v\| \cos(\theta)$
- cross product: $u \times v = \|u\| \|v\| \sin(\theta) \hat{n}$; for $u, v \in \mathbb{R}^3$

Linear combinations and linear independence

Definition (linear combination)

For $\alpha_k \in \mathbb{R}$ and $u_k \in \mathbb{R}^n$

$$\sum_{k=1}^{m} \alpha_k u_k$$

is called a linear combination.

Definition (linear independence)

$$\sum_{k=1}^{m} \alpha_k u_k = 0$$

only if $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$

Dimension of a vector space

Definition (Dimension of a vector space)

The dimension of S is the maximum number of linear independent vectors.

Definition (Basis of a vector space)

A set of n linearly independent vectors of a space of dimension n.

$\Rightarrow$ for any $u \in S$ there exist unique $\alpha_k \in \mathbb{R}$ such that

$$u = \sum_{k=1}^{n} \alpha_k u_k$$

- The dimension of a subspace of S is less or equal than the dimension of S.
- Vector spaces of the same dimension are "similar".

Linear transformations

Definition (Linear operator)

An operator L acting from a vector space S to a vector space E is a linear operator if

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$

- A linear operator is completely determined by its action over a basis. Why?
- $\Rightarrow$ linear operators can be represented by a **matrix**.

Matrices

A **matrix** of size $n \times m$ is a collection of m column vectors of $\mathbb{R}^n$

$$A = \begin{pmatrix} u_{1,1} & u_{1,2} & \dots & u_{1,m} \\ \vdots & & & \vdots \\ u_{n,1} & u_{n,2} & \dots & u_{n,m} \end{pmatrix} = (u_1 \mid u_2 \mid \cdots \mid u_m)$$

- The traspose $A' = \begin{pmatrix} u_1' \\ \vdots \\ u_n' \end{pmatrix}$
- The space of matrices of size $n \times m$ is a vector space
- Multiplication with a column vector $\alpha \in \mathbb{R}^m$

$$A\alpha := \sum_{k=1}^{m} \alpha_k u_k$$

- rank(A) = dimension of the "span" of its column (or row) vectors
- the null space of A: $N(A) = \{v \in \mathbb{R}^m \text{ such that } Av = 0\}$

Square matrices

A **matrix** of size $n \times n$.

- $|A|$ = determinant of A.
- Inverse A^{-1} is such that $AA^{-1} = A^{-1}A = I$ with I = identity matrix

Linear systems of equations

Introduction

A system of linear equations is represented as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_2 \end{aligned}\right\} \tag{1}$$

The aim of this lecture is to find solution to the above system.

Introduction (contd.)

Consider first this simple equation:

$$ax = b, \quad a, b \in \mathbb{R}. \tag{2}$$

Find a solution in the following cases.

1. $a \neq 0$
2. $a = 0$ and $b \neq 0$
3. $a = 0$ and $b = 0$

Introduction (contd.)

The solutions in the three cases:

1. x=b/a (unique solution)
2. $x = \emptyset$ (no solution
3. $x \in \mathbb{R}$ (infinitely many solutions)

Introduction (contd.)

Now consider this matrix equation:

$$Ax = b \tag{3}$$

where A is a n x n matrix. Find a solution in the following cases.

1. $|A| \neq 0$
2. $|A| = 0$ and $b \neq \underline{0}$
3. $|A| = 0$ and $b = \underline{0}$

Introduction (contd.)

The solutions in the three cases:

1. $x = A^{-1}b$ (unique solution)
2. $x = \emptyset$ (no solution)
3. $x = N(A)$ (infinitely many solutions)

Here, $N(A)$ means nullspace of matrix A.

Question: What is the solution to (3) when $|A| \neq 0$ and $b = 0$?

Backslash operator

$x = A \backslash b$ solves the system (1) of linear equations $Ax = b$.

Example: Solve the system of equations:

$$\begin{aligned} 2x + y &= 4 \\ x - y &= -1 \end{aligned} \tag{4}$$

Here, $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

So, $x = A \backslash b$ gives $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Question: Can you find this solution using MATLAB's inv() function?

Backslash operator (contd.)

Demo

Backslash operator (contd.)

- If A is a square matrix then $A\backslash b$ and $inv(A) * b$ are equivalent.
- For scalars a and b, $a\backslash b$ solves the equation $ax = b$. So, $a\backslash b$ and b/a are equivalent.

Backslash operator (contd.)

Let us return back to system of equations (1).

For solving this system, $x = A\backslash b$ gives unexpected results when

1. the system has no solution.
2. the system has infinitely many solutions. In this case, a particular solution may be found using $x = pinv(A) * B$. Here, $pinv(A)$ computes the 'pseudo-inverse' of A.

Rank method

Rank can be used to determine if the system (1) has no solution, unique solution or infinitely many solutions.

- Non-homogeneous equations ($Ax = b,\ b \neq 0$)
  1. $rank(A) < rank([A \quad b]) \implies$ No solution
  2. $rank(A) = rank([A \quad b]) = n \implies$ Unique solution
  3. $rank(A) = rank([A \quad b]) = k < n \implies$ Infinitely many solutions

- Homogeneous equations ($Ax = 0$)
  1. $rank(A) = n \implies$ Unique solution (the trivial solution)
  2. $rank(A) = k < n \implies$ Infinitely many solutions

Finding solutions

For non-homogeneous equations:

- First, check the existence of solution using Rank method.
- If the solution exists and is unique, find the solution using $x = A\backslash b$.
- If there are infinitely many solutions, first find the particular solution (say x^*) using $x^* = pinv(A) * b$. The general solution is given by $x = x^* + N(A)$ where $N(A)$ is the nullspace of A.

Finding solutions (contd.)

Follow these steps to find the nullspace $N(A)$:

- Use MATLAB's null(A) function to create a matrix containing orthonormal basis of N as column vectors.
- Let $P =$ null(A) and $p =$nullity of A. Then $p = n - rank(A)$, (**Why?**)
- The nullspace of A is then given by

$$N(A) = c_1 * P(:,1) + c_2 * P(:,2) + \cdots + c_p P(:,p)$$

where the constants $c_1, c_2, \cdots, c_p \in \mathbb{R}$.

Finding solutions (contd.)

Example: Solve the system of equations:

$$\begin{aligned} x + y + z + w &= 6 \\ x + 2y - 3z - w &= -4 \\ y - 4z - 2w &= -10 \\ 2x + 3y - 2z &= 2 \end{aligned} \tag{5}$$

Finding solutions (contd.)

Demo

Finding solutions (contd.)

For homogeneous equations:

- Find $rank(A)$. If $rank(A) = n$, then the trivial solution $x = \underline{0}$ is the only solution.
- Otherwise, the solution is $x = N(A)$.