MA2252 Introduction to Computing

Lecture 11
Solving System of Linear Equations
Part 1

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Learning outcomes

At the end of lecture, students will be able to

- understand basic theory of system of linear equations
- use MATLAB to find solutions to the system

A system of linear equations is represented as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_2 \end{aligned}\right\} \tag{1}$$

The aim of this lecture is to find solution to the above system.

Introduction (contd.)

Consider first this simple equation:

$$ax = b, \quad a, b \in \mathbb{R}. \tag{2}$$

Find a solution in the following cases.

1. $a \neq 0$
2. $a = 0$ and $b \neq 0$
3. $a = 0$ and $b = 0$

Introduction (contd.)

The solutions in the three cases:

1. x=b/a (unique solution)
2. $x = \emptyset$ (no solution
3. $x \in \mathbb{R}$ (infinitely many solutions)

Introduction (contd.)

Now consider this matrix equation:

$$Ax = b \tag{3}$$

where A is a n x n matrix. Find a solution in the following cases.

1. $|A| \neq 0$
2. $|A| = 0$ and $b \neq \underline{0}$
3. $|A| = 0$ and $b = \underline{0}$

Introduction (contd.)

The solutions in the three cases:

1. $x = A^{-1}b$ (unique solution)
2. $x = \emptyset$ (no solution)
3. $x = N(A)$ (infinitely many solutions)

Here, $N(A)$ means nullspace of matrix A.

Question: What is the solution to (3) when $|A| \neq 0$ and $b = 0$?

Backslash operator

$x = A\backslash b$ solves the system (1) of linear equations $Ax = b$.

Example: Solve the system of equations:

$$\begin{aligned} 2x + y &= 4 \\ x - y &= -1 \end{aligned} \tag{4}$$

Here, $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

So, $x = A\backslash b$ gives $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Question: Can you find this solution using MATLAB's inv() function?

Backslash operator (contd.)

Demo

Backslash operator (contd.)

- If A is a square matrix then $A\backslash b$ and $inv(A) * b$ are equivalent.
- For scalars a and b, $a\backslash b$ solves the equation $ax = b$. So, $a\backslash b$ and b/a are equivalent.

Backslash operator (contd.)

Let us return back to system of equations (1).

For solving this system, $x = A \backslash b$ gives unexpected results when

1. the system has no solution.
2. the system has infinitely many solutions. In this case, a particular solution may be found using $x = pinv(A) * B$. Here, $pinv(A)$ computes the 'pseudo-inverse' of A.

Backslash operator (contd.)

Demo

Rank of a matrix

To find out if the system (1) has a unique or infinitely many solutions, we need to understand 'rank' of a matrix.

Definition

Rank of a matrix A is defined as the maximum number of linearly independent rows/columns of A.

Question: Find out the rank of these matrices.

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$$

MATLAB's rank() function finds the rank of a matrix.

Rank of a matrix (contd.)

Demo

Rank method

Rank can be used to determine if the system (1) has no solution, unique solution or infinitely many solutions.

- Non-homogeneous equations ($Ax = b,\ b \neq 0$)
    1. $rank(A) < rank([A \quad b]) \implies$ No solution
    2. $rank(A) = rank([A \quad b]) = n \implies$ Unique solution
    3. $rank(A) = rank([A \quad b]) = k < n \implies$ Infinitely many solutions

- Homogeneous equations ($Ax = 0$)
    1. $rank(A) = n \implies$ Unique solution (the trivial solution)
    2. $rank(A) = k < n \implies$ Infinitely many solutions

Finding solutions

For non-homogeneous equations:

- First, check the existence of solution using Rank method.
- If the solution exists and is unique, find the solution using $x = A\backslash b$.
- If there are infinitely many solutions, first find the particular solution (say x^*) using $x^* = pinv(A) * b$. The general solution is given by $x = x^* + N(A)$ where $N(A)$ is the nullspace of A.

Finding solutions (contd.)

Follow these steps to find the nullspace $N(A)$:

- Use MATLAB's null(A) function to create a matrix containing orthonormal basis of N as column vectors.
- Let $P =$ null(A) and $p =$nullity of A. Then $p = n - rank(A)$, (**Why?**)
- The nullspace of A is then given by

$$N(A) = c_1 * P(:,1) + c_2 * P(:,2) + \cdots + c_p P(:,p)$$

where the constants $c_1, c_2, \cdots, c_p \in \mathbb{R}$.

Finding solutions (contd.)

Example: Solve the system of equations:

$$
\begin{aligned}
x + y + z + w &= 6 \\
x + 2y - 3z - w &= -4 \\
y - 4z - 2w &= -10 \\
2x + 3y - 2z &= 2
\end{aligned}
\tag{5}
$$

Demo

Finding solutions (contd.)

For homogeneous equations:

- Find $rank(A)$. If $rank(A) = n$, then the trivial solution $x = \underline{0}$ is the only solution.
- Otherwise, the solution is $x = N(A)$.

End of Lecture 11

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