Exercise 1. *Let $a_n = n$, then a_n is increasing but not convergent.*

Exercise 2. *Since $\{a_n\}$ is unbounded,*

$$\forall M > 0, \exists n \in \mathbb{N}, |a_n| \geq M$$

Let $M = 1$, then $\exists n_1$, such that $|a_{n_1}| \geq 1$;
Let $M = 2$, then we claim that $\exists n_2 > n_1$ such that $|a_{n_2}| \geq 2$.
Otherwise, if $\forall n > n_1$ we have $|a_n| \leq 2$. Then setting

$$M = \max\{2, |a_1|, \cdots, |a_{n_1}|\}$$

would yield the boundedness of the sequence $\{a_n\}$, a contradiction.
$\cdots$
Let $M = k$, then we claim that $\exists n_k > n_{k-1}$ such that $|a_{n_k}| \geq k$.
Otherwise we have $\forall n > n_{k-1}$, $|a_n| \leq k$. Then setting

$$M = \max\{k, |a_1|, \cdots, |a_{n_{k-1}}|\}$$

would yield the boundedness of the sequence $\{a_n\}$, a contradiction.

Hence we have obtained a subsequence $\{a_{n_k}\}$ such that

$$|a_{n_k}| \geq k$$

holds $\forall k \in \mathbb{N}$. Next we will prove that $\lim_{n\to\infty} a_{n_k} = +\infty$. $\forall G > 0$, Setting $K = [G] + 1$, then $\forall k > K$,

$$|a_{n_k}| \geq k > K = [G] + 1 \geq G$$

as desired.

Exercise 3. *(1) Firstly $\forall n \in \mathbb{N}$, $0 \in (-\frac{1}{n}, \frac{1}{n})$. On the other side, if $a \in \bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$ and $a \neq 0$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we know that $\exists n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{a}{2}$. Hence $a \notin (-\frac{1}{n_0}, \frac{1}{n_0})$, a contradiction.*

(2) Similar as (1).

Exercise 4. *We know that* $a_n = (1+\frac{1}{n})^n$ *is increasing to* e*. So* $e \geq a_2 > a_1 = 2$*. Also from the proof of convergence of* $a_n = (1+\frac{1}{n})^n$*, we know that*

$$a_n < 1 + \frac{1}{2!} + \cdots \frac{1}{n!} < 3$$

so $e < 3$*.*

Exercise 5. *(1) By induction we can see that* $x_n > 0$ *for all* $n \in \mathbb{N}$*. We claim that* x_n *is bounded above with an upper bound* 2*. We know that* $x_1 = \sqrt{2} \leq 2$*. Assume* $x_n \leq 2$*, then*

$$x_{n+1} = \sqrt{2 + x_n} \leq \sqrt{2+2} = 2$$

By the principle of induction, we know that $x_n \leq 2$ *holds* $\forall n \in \mathbb{N}$*.*

Next we will prove that $\{x_n\}$ *is increasing. In fact we have*

$$\begin{aligned} x_{n+1} - x_n =& \sqrt{2+x_n} - x_n \\ =& \frac{(\sqrt{2+x_n} - x_n)(\sqrt{2+x_n} + x_n)}{\sqrt{2+x_n} + x_n} \\ =& -\frac{(x_n+1)(x_n-2)}{\sqrt{2+x_n} + x_n} \geq 0 \end{aligned}$$

So $\lim_{n\to\infty} x_n = x$ *exists. Take limit on both sides of* $x_{n+1} = \sqrt{2+x_n}$*, we have* $x = \sqrt{2+x}$*. By sign-preserving of convergence, we know* $x \geq 0$*, so* $x = 2$*.*

(2) By induction we can prove that $0 < y_n < 1$ *for* $n \in \mathbb{N}$*. Also*

$$\frac{y_{n+1}}{y_n} = 2 - y_n > 2 - 1 = 1$$

so $\{y_n\}$ *is increasing. Thus* $\lim_{n\to\infty} y_n = y$ *exists. Take limit on both sides of* $y_{n+1} = y_n(2 - y_n)$*, we have* $y = y(2-y)$*. Since* $y_n \geq y_1 > 0$*,* $y \geq y_1 > 0$ *by the order-preserving property. Thus* $y = 1$*.*

Exercise 6. *(1) We have*

$$0 \leq \frac{a \cdot a \cdots a}{1 \cdot 2 \cdots [a]} \cdot \frac{a \cdots a}{([a]+1) \cdots n} < \frac{a^{[a]}}{[a]!} \cdot \frac{a}{n} \to 0$$

as $n \to \infty$*. By squeeze theorem, we have*

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

(2) We have

$$0 \le \frac{1\cdot 2\cdots n}{n\cdot n\cdots n} < \frac{1}{n} \to 0$$

as $n \to \infty$*. By squeeze theorem, we have*

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$