MA3077 (DLI) Operational Research
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Problem Sheet 1 - Submission Deadline: 16 October at 6pm (GMT) on Blackboard

1) **[5pts]** Formulate the following problem as a linear programming problem. Write the decision variables, constraints and objective function.

Biocare makes liquid plant food for fruit and vegetables. It makes two types: Growrite (G), a high nitrogen fertiliser for green vegetables, and Tomfood (T), with a high potassium content for tomatoes, cucumbers and so on. Both types need the same basic ingredients - Ammonium Nitrate for Nitrogen (N), Phosphorus Pentoxide for Phosphorus (P), and Potassium Dioxide for Potassium (K) - but in different amounts. One litre of G requires 0.11kg of N, 0.06kg P and 0.02kg of K. One litre of T requires 0.08kg of N, 0.03kg P and 0.08kg of K. There are available each day 600kg of N, 300kg of P and 330kg of K. The selling prices per litre are £2.80 for Growrite and £3.00 for Tomfood. At these prices, Biocare can sell all it produces. Biocare wishes to maximise its daily income. How should it do so?

2) Consider the following linear programming problem.

$$
\begin{array}{cc}
max & 35x + 20y \\
s.t. & 23x + 11y \leq 176 \\
 & 4x + 4y \leq 51 \\
 & x, y \geq 0
\end{array}
$$

a) **[4 pts]** Draw (by hand or using some software) the feasible set. Then determine graphically an optimal solution and indicate which constraints are active.
b) **[4 pts]** Determine an approximate optimal solution using Matlab's function linprog. You can either write the commands by hand or include your m-file in your submission.

3) **[5 pts]** Write the following linear programming problem in standard form.

$$
\begin{array}{cc}
min & 30x + 21y + 18z \\
s.t. & 3x - 7z \leq 176 \\
 & 8z - 2y + x - 6 \geq 12 \\
 & 4x + 3y = 19 \\
 & x, y, z \in \mathbb{R}
\end{array}
$$

4) **[5 pts]** Derive the dual problem of the following linear programming problem, identifying clearly the Lagrangian and the dual function.

$$
\begin{array}{cc}
min & x + 4y - 9z \\
s.t. & x - z = 7 \\
 & x + y + z = 2 \\
 & 4x + 3y = 19 \\
 & x, y, z \geq 0
\end{array}
$$

5) **[5 pts]** Formulate the following optimization problem as a linear programming problem.

$$\max x_1 + x_2 \ \ s.t. \|x\|_1^2 \leq 4, x \in \mathbb{R}^2$$

6) **[5 pts]** Let $m, n \in \mathbb{N}$ be such that $m < n$. Let $A \in \mathbb{R}^{m,n}$ have full rank, and let $y \in \mathbb{R}^m$ be in the column space of A. Let $x^* := (w^*, z^*) \in \mathbb{R}^{2n}$ be the optimal solution to the linear programming problem

$$\begin{array}{ll} min & z_1 + z_2 + \cdots + z_n \\ s.t. & -z \leq w \leq z \\ & Aw = y \\ & w, z \in \mathbb{R}^n \end{array} .$$

Prove that $z^* = |w^*|$.

7) **[5 pts]** Let $c = (\cos(\alpha), \sin(\alpha))^T$. For which values of $\alpha \in [0, 2\pi)$ is $x = (1,1)^T$ an optimal solution to the following linear programming problem?

$$\begin{array}{ll} max & c^T x \\ s.t. & x + 2y \leq 3 \\ & 3x + y \leq 4 \\ & x, y \geq 0 \end{array}$$

8) **[6 pts]** Write the following linear programming problem in standard form. Then, show that this problem is infeasible if and only if there is an $x \geq 0$ such that $Ax = 0$ and $c^T x < 0$.

$$\begin{array}{ll} max & b^T y \\ s.t. & c - A^T y = s \\ & s \geq 0, \ y \in \mathbb{R}^m \end{array}$$

9) **[6 pts]** Solve the following problem by use of the branch-and-bound method.

$$\begin{array}{ll} min & x + y \\ s.t. & 2x + 2y \geq 5 \\ & 12x + 5y \leq 30 \\ & x, y \geq 0, \quad x, y \in \mathbb{N} \end{array}$$

Model solutions to Problem Sheet 1

Note: The following model solutions indicate how the problem may have been solved. Alternative solutions are often also possible.

1)

Decision variables: the volumes (litres) of G and T to produce.

Constraints:

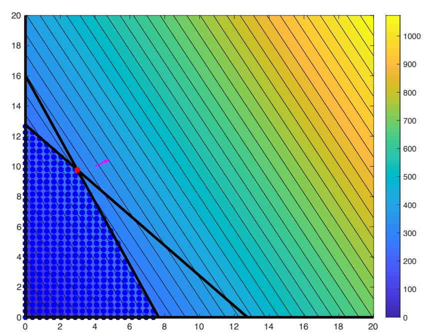
$0.11G + 0.08T \leq 600$
$0.06G + 0.03T \leq 300$
$0.02G + 0.08T \leq 330$
$G, T \geq 0$

Objective function:

Maximise $2.8G + 3T$

2)

a) See `OR_PS1_ex2_feasible_region.m`. The feasible set is in blue.



The following constraints are active: $23x + 11y \leq 176$ and $4x + 4y \leq 51$

The maximal growth of the objective function is in the direction $(35, 20)$, and its level sets sets are hyperplanes with normal vector $(35, 20)$. Since the feasible region is a convex and bounded polygon, the maximum is achieved at the intersection of this polygon with the hyperplane with highest intercept at zero. This happens at the vertex defined by the two equalities

$$23x + 11y = 176$$
$$4x + 4y = 51$$

and whose coordinates are $\left(\frac{143}{48}, \frac{469}{48}\right) \approx (2.98, 9.77)$.

b) See `OR_PS1_ex2_linprog.m`. The approximate solution is $(2.98, 9.77)$.

3) We introduce the variable $v \in \mathbb{R}^8_+$ and write $x = v_1 - v_2, y = v_3 - v_4, z = v_5 - v_6$. Then, the problem becomes

$$\begin{array}{cc} min & 30(v_1 - v_2) + 21(v_3 - v_4) + 18(v_5 - v_6) \\ s.t. & 3(v_1 - v_2) - 7(v_5 - v_6) \le 176 \\ & 8(v_5 - v_6) - 2(v_3 - v_4) + (v_1 - v_2) - 6 \ge 12 \\ & 4(v_1 - v_2) + 3(v_3 - v_4) = 19 \\ & v \in \mathbb{R}^8_+ \end{array}$$

Multiplying the second inequality with -1 and moving the -6 to the righthand side, we obtain

$$\begin{array}{cc} min & 30(v_1 - v_2) + 21(v_3 - v_4) + 18(v_5 - v_6) \\ s.t. & 3(v_1 - v_2) - 7(v_5 - v_6) \le 176 \\ & -8(v_5 - v_6) + 2(v_3 - v_4) - (v_1 - v_2) \le -18 \\ & 4(v_1 - v_2) + 3(v_3 - v_4) = 19 \\ & v \in \mathbb{R}^8_+ \end{array}$$

Finally, using v_7 and v_8 as slack variables in the two inequalities, we obtain

$$\begin{array}{cc} min & 30(v_1 - v_2) + 21(v_3 - v_4) + 18(v_5 - v_6) \\ s.t. & 3(v_1 - v_2) - 7(v_5 - v_6) + v_7 = 176 \\ & -8(v_5 - v_6) + 2(v_3 - v_4) - (v_1 - v_2) + v_8 = -18 \\ & 4(v_1 - v_2) + 3(v_3 - v_4) = 19 \\ & v \in \mathbb{R}^8_+ \end{array}$$

which is in standard form.

4) Let $c^T = (1, 4, -9)$, $b^T = (7, 2, 19)$, and $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 4 & 3 & 0 \end{pmatrix}$. Then, the Lagrangian is

$$L:\ \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3_+ \to \mathbb{R}, \qquad L(v, w, s) = c^T v + w^T(b - Av) - s^T v.$$

The dual function is

$$g: \mathbb{R}^3 \times \mathbb{R}^3_+ \to \mathbb{R}, \qquad g(y, s) \coloneqq \min_x L(x, y, s) = \begin{cases} b^T y, & if \;\; c - A^T y - s = 0, \\ -\infty, & if \;\; c - A^T y - s \neq 0. \end{cases}$$

The dual problem is

$$\begin{array}{ll} max & b^T y \\ s.t. & c - A^T y \geq 0 \\ & s \geq 0,\ y \in \mathbb{R}^3 \end{array}$$

5) To formulate the optimisation problem as a linear programming problem, we need to express the objective function and constraints in linear form. The objective function is already linear. The constraint is equivalent to $\|x\|_1 \leq 2, x \in \mathbb{R}^2$, which we can reformulate as a set of linear constraints by introducing new variables, as follows :

$$-z_1 \leq x_1 \leq z_1$$
$$-z_2 \leq x_2 \leq z_2$$
$$z_1 + z_2 = 2$$

Hence, the optimisation problem is equivalent to the following linear programming problem:

$$\begin{array}{l} \max x_1 + x_2 \\ s.t. \\ -z_1 \leq x_1 \leq z_1 \\ -z_2 \leq x_2 \leq z_2 \\ z_1 + z_2 = 2 \end{array}$$

6) Note that $z^* \geq |w^*|$ otherwise x^* is not feasible. Let $a := z^* - |w^*|$. Then, $a \geq 0$. Let $v := |w^*|$. Then the vector $(w^*, v) \in \mathbb{R}^{2n}$ is feasible and

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} z_i - a_i = \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} z_i.$$

Because $x^* := (w^*, z^*) \in \mathbb{R}^{2n}$ is an optimal solution, $\sum_{i=1}^{n} a_i = 0$, that is, $a = 0$, (otherwise $(w^*, v) \in \mathbb{R}^{2n}$ would be a strictly better solution).

7) Since the solution $x = (1,1)^T$ is determined by the intersection of the lines $x + 2y = 3$ and $3x + y = 4$, the solution $x = (1,1)^T$ remains optimal as long as $c = (\cos(\alpha), \sin(\alpha))^T$ remains between the upward pointing normals to these lines. The normal of the first line is $(1,2)^T/\sqrt{5}$, which gives $\alpha \leq \text{acos}\,(1/\sqrt{5}) = \text{asin}\,(2/\sqrt{5})$. The normal of the first line is $(3,1)^T/\sqrt{10}$, which gives $\alpha \geq \text{acos}\,(3/\sqrt{10}) = \text{asin}\,(1/\sqrt{10})$.

8) Let $y = v - w$, with $v, w \geq 0$, and let $z^T := (s, v, w)$. Then, the problem is equivalent to

$$-\min(0, -b^T, b^T)z \;\; s.t. [1, A^T, -A^T]z = c, z \geq 0,$$

which (except for the minus sign in front of the min) is in standard form (here 1 denotes the identity matrix). By Farkas' lemma, this problem is infeasible iff there is a vector n such that $n^T[1, A^T, -A^T] \leq 0$ and $n^T c > 0$. Splitting $n^T[1, A^T, -A^T] \leq 0$ into three inequalities we obtain

$$n \leq 0, \qquad An \leq 0, \qquad -An \leq 0,$$

which implies $An = 0$. Finally, defining $x: = -n$ implies that the original programming problem is infeasible iff there is $x \geq 0$ such that $Ax = 0$ and $c^T x < 0$.

9) A first approximation to this program is $x^* = 2.5$ and $y^* = 0$, with $z^* = x^* + y^* = 2.5$. Rounding x^* up, thereby remaining feasible, we have $x^* = 3$, $y^* = 0$ with , $z^* = 3$, as an estimate of the optimal solution to the original program. Observe, however, that for integral values of the variables, the objective function must itself be integral. The z-value for the first approximation, $z^* = 2.5$, provides a lower bound for the optimal objective; consequently, the optimal objective cannot be smaller than 3. Since we have an estimate which attains the value 3, the estimate must be optimal; i.e., $x^* = 3$, $y^* = 0$ with z^*=3 .