The hyper volume of an infinite dimensional ball is 0 when radius r is finite

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First, let's start from the volume of a 3 dimensional ball:

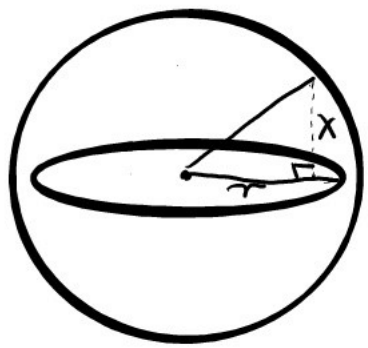

Figure 1: graph 1.1

$$\begin{aligned}
V_{hs} &= \int_0^r \pi\left(r^2 - x^2\right) dx. && (1)\\
&= \pi \int_0^r r^2 - x^2 dx && (2)\\
&= \pi \left(r^2 x - \frac{1}{3} x^3 \right)\Big|_0^r && (3)\\
&= \pi \left[\left(r^3 - \frac{1}{3} r^3 \right) - 0 \right] && (4)\\
&= \frac{2}{3} \pi r^3 && (5)\\
V \text{ sphere } &= 2V_{hs} = \frac{4}{3} \pi r^3 && (6)
\end{aligned}$$

(V_{hs} represents the volume of the hemisphere, while V sphere represents the volume of the sphere)
According to the calculation above, we know that the volume of a sphere is equal to 2 times of the integral of a 2d circle plane in the domain of [0, r].

Thus, the volume of a hyperball should be:

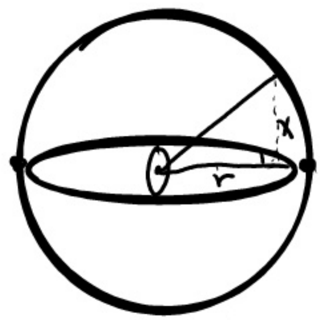

Figure 2: graph 1.2

$$\begin{aligned}
&Vs = \tfrac{4}{3}\pi R^3\\
&R^2 = r^2 - x^2\\
&\therefore R^3 = \left(r^2 - x^2\right)^{\frac{3}{2}}\\
&Hv\left(hHs\right) = \textstyle\int_0^r \tfrac{4}{3}\pi\left(r^2 - x^2\right)^{\frac{3}{2}}\,dx\\
&\qquad\qquad = \tfrac{1}{4}\pi^2 r^4\\
&Hv(Hyper - s) = 2Hv\left(hHs\right) = \tfrac{1}{2}\pi^2 r^4
\end{aligned} \tag{7}$$

(Vs is the volume of a sphere, R is the radius of any slice of sphere from hyperball in [0, r], hHs is a demi-hyperball, $Hyper-s$ is a hyperball, function $Hv(x)$ is hyper volume function)

According to the proof above, we know,
The volume of a sphere is equal to 2 times of the integral of a 2d circle plane in the domain of [0, r].

According to the conclusions above, we can conclude that,
the hyper volume of a **n-ball** satisfies the recursion formula:

$$Hv\left(Hs^{(n)}, r\right) = 2 \cdot \int_0^r Hv\left(Hs^{(n-1)}, \left(r^2 - x^2\right)^{\frac{1}{2}}\right) dx \tag{8}$$

($Hs^{(n)}$ represents n-ball, $Hv(x, r)$ is a binary function of hyper volume)

Using that function to traverse all 1 to 10 dimensional balls, we can get volumes of them:

Dim	1	2	3	4	5	6
	r	πr^2	$\frac{4}{3}\pi r^3$	$\frac{1}{2}\pi^2 r^4$	$\frac{8}{15}\pi^2 r^5$	$\frac{1}{6}\pi^3 r^6$
7	8	9	10	...		
$\frac{16}{105}\pi^3 r^7$	$\frac{1}{24}\pi^4 r^8$	$\frac{32}{945}\pi^4 r^9$	$\frac{1}{120}\pi^5 r^{10}$	...		

Figure 3: table 2.1

According to the following table, we can have 2 assumptions:

$$\forall x, k \quad x = 2k \quad k \in \mathbb{Z}^+ \quad \forall r \in \mathbb{R}^{\geqslant 0}$$
$$Hv\left(Hs^{(x)}, r\right) = \frac{\pi^{\frac{x}{2}} r^x}{\left(\frac{x}{2}\right)!} = \frac{\left(\pi r^2\right)^{\frac{x}{2}}}{\left(\frac{x}{2}\right)!} \tag{9}$$

and,

$$\forall x, k \quad x = 2k - 1 \quad k \in \mathbb{Z}^+ \quad \forall r \quad r \in \mathbb{R}^{\geq 0}$$
$$Hv\left(Hs^{(x)}, r\right) = \frac{2^{\frac{x+1}{2}}}{\prod_{k=1}^{\frac{x-1}{2}} 2k+1} \pi^{\frac{x-1}{2}} r^x \tag{10}$$

Recall the taylor series of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad -\infty < x < \infty \tag{11}$$

We substitute the variable x to πr^2,

$$e^{\pi r^2} = 1 + \frac{\pi r^2}{1!} + \frac{(\pi r^2)^2}{2!} + \frac{(\pi r^2)^3}{3!} + \cdots, \quad 0 < r < \infty \tag{12}$$

Begin from the second term, terms can be substitute as $Hv(Hs^{(n)})$,

$$\therefore e^{\pi r^2} = 1 + Hv(Hs^{(2)}) + Hv(Hs^{(4)}) + Hv(Hs^{(6)}) + \cdots, \quad 0 < r < \infty \tag{13}$$

Since $e^{\pi r^2}$ is a finite number when r is finite, so we can conclude that the series on the right side is approaching a finite number,
Hence, $\lim_{n\to\infty} Hv(Hs^{(n)}) = C$

Of course, according to the theorm of converge taylor series, if a term of the series starts to satisfy the condition $n > \pi r^2$ from a specific value n',
in other words,

$$\forall n \quad n \in \mathbb{R}^{>n'} \quad \frac{(\pi r^2)^{n-1}}{(n-1)!} > \frac{(\pi r^2)^n}{n!} \tag{14}$$

and if the series is approaching a finite number, we can conclude that

$$\lim_{n\to\infty} Hv(Hs^{(n)}) = 0 \tag{15}$$

under condition $\forall n, k \quad n = 2k \quad k \in \mathbb{Z}^+$.

Next, we need to proof for all odd number x,

$$\lim_{x\to\infty} \quad \frac{2^{\frac{x+1}{2}}}{\prod_{k=1}^{\frac{x-1}{2}} 2k+1} = 0 \tag{16}$$

Obiviously, according to the last proof for all odd numbers, we know that

$$\lim_{x\to\infty} \quad \frac{2^x}{x!} = 0 \tag{17}$$

Also, we know,

$$\lim_{x\to\infty} \quad \frac{2^x}{(\frac{x}{2})!} = \lim_{x\to\infty} \quad \frac{(2^2)^{\frac{x}{2}}}{(\frac{x}{2})!} = 0 \tag{18}$$

Since,

$$\frac{2^x}{(\frac{x}{2})!} < \frac{2^{\frac{x+1}{2}}}{\prod_{k=1}^{\frac{x-1}{2}} 2k+1} < \frac{2^x}{x!} \tag{19}$$

According to the squeeze theorem, we know

$$\lim_{x\to\infty} \quad \frac{2^{\frac{x+1}{2}}}{\prod_{k=1}^{\frac{x-1}{2}} 2k+1} = 0 \tag{20}$$

under condition $\forall n, k \quad n = 2k-1 \quad k \in \mathbb{Z}^+$.
Therefore, the following assumption has been proven.

According to everything above, we can conclude,

$$\lim_{x\to\infty} Hv(Hs^{(x)}) = 0 \tag{21}$$

whenever r is **finite**.
Hence, the hyper volume of an infinite dimensional ball is 0 when r is finite.