

# Assignment 7

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Download latex-tikz codes from

<https://github.com/Dontha-Aarthi/AI1103/blob/main/Assignment7/assignment7.tex>

## 1 CSIR UGC NET EXAM (DEC 2014), Q.108

$N, A_1, A_2 \dots$  are independent real valued random variables such that

$$\Pr(N = k) = (1 - p)p^k, k = 0, 1, 2, 3 \dots \quad (1.0.1)$$

where  $0 < p < 1$  and  $\{A_i : i = 1, 2, \dots\}$  is a sequence of independent and identically distributed bounded random variables. Let

$$X(w) = \begin{cases} 0 & \text{if } N(w) = 0 \\ \sum_{j=1}^k A_j & \text{if } N(w) = k, k = 1, 2, 3 \dots \end{cases} \quad (1.0.2)$$

Which of the following are necessarily correct?

- 1)  $X$  is a bounded random variable.
- 2) Moment generating function  $m_X$  of  $X$  is

$$m_X(t) = \frac{1 - p}{1 - pm_A(t)}, t \in \mathbb{R}, \quad (1.0.3)$$

where  $m_A$  is moment generating function of  $A_1$ .

- 3) Characteristic function  $\varphi_X$  of  $X$  is

$$\varphi_X(t) = \frac{1 - p}{1 - p\varphi_A(t)}, t \in \mathbb{R}, \quad (1.0.4)$$

where  $\varphi_A$  is the characteristic function of  $A_1$ .

- 4)  $X$  is symmetric about 0.

## 2 SOLUTION

- 1) Since  $k$  is not bounded,  $X$  cannot be a bounded random variable necessarily.  
 $\therefore$  **option 1 is incorrect.**
- 2) The moment generating function of a random variable is defined as

$$M_X(t) = E[e^{tX}] \quad (2.0.1)$$

So,

$$M_X(t) = \sum_{k=0}^{\infty} (E[e^{t \sum_{j=1}^k A_j} | N = k] \times \Pr(N = k)) \quad (2.0.2)$$

$$= E[e^{t \times 0}] \times \Pr(N = 0) + \sum_{k=1}^{\infty} (E[e^{t \sum_{j=1}^k A_j} | N = k] \times \Pr(N = k)) \quad (2.0.3)$$

$$= E[1] \times (1 - p) + \sum_{k=1}^{\infty} (E[e^{t \sum_{j=1}^k A_j} | N = k] \times \Pr(N = k)) \quad (2.0.4)$$

Since  $A_1, A_2 \dots$  are identical and independent, (2.0.4) can be written as

$$= (1 - p) + \sum_{k=1}^{\infty} \left( \Pr(N = k) \times \prod_{j=1}^k E[e^{tA_j}] \right) \quad (2.0.5)$$

$$= (1 - p) + \sum_{k=1}^{\infty} \left( \Pr(N = k) \times \prod_{j=1}^k M_{A_j}(t) \right) \quad (2.0.6)$$

$$= (1 - p) + \sum_{k=1}^{\infty} ((1 - p)p^k \times (M_A(t))^k) \quad (2.0.7)$$

where,  $M_A(t)$  is moment generating function of  $A_1$ .

On simplifying sum of infinite terms in geometric progression in (2.0.7), we get

$$= (1 - p) \times \left( 1 + \frac{p \times M_A(t)}{1 - p \times M_A(t)} \right) \quad (2.0.8)$$

$$= \frac{1 - p}{1 - p \times M_A(t)} \quad (2.0.9)$$

$\therefore$  **option 2 is correct.**

- 3) The characteristic function of a random vari-

able is defined as

$$\varphi_X(t) = E[e^{itX}] \quad (2.0.10)$$

And using this,

$$E[X] = \sum_y E[X|Y = y] \times \Pr(Y = y) \quad (2.0.11)$$

We get

$$\varphi_X(t) = \sum_{k=0}^{\infty} \left( E[e^{it \sum_{j=1}^k A_j} | N = k] \times \Pr(N = k) \right) \quad (2.0.12)$$

$$= E[e^{it \times 0}] \times \Pr(N = 0) + \sum_{k=1}^{\infty} \left( E[e^{it \sum_{j=1}^k A_j} | N = k] \times \Pr(N = k) \right) \quad (2.0.13)$$

$$= E[1] \times (1 - p) + \sum_{k=1}^{\infty} \left( E[e^{it \sum_{j=1}^k A_j} | N = k] \times \Pr(N = k) \right) \quad (2.0.14)$$

$$(2.0.15)$$

Since  $A_1, A_2 \dots$  are independent, (2.0.14) can be written as

$$= (1 - p) + \sum_{k=1}^{\infty} \left( \Pr(N = k) \times \prod_{j=1}^k E[e^{it A_j}] \right) \quad (2.0.16)$$

$$= (1 - p) + \sum_{k=1}^{\infty} \left( \Pr(N = k) \times \prod_{j=1}^k \varphi_{A_j}(t) \right) \quad (2.0.17)$$

Since  $A_1, A_2 \dots$  are identical, (2.0.17) can be written as

$$= (1 - p) + \sum_{k=1}^{\infty} \left( (1 - p)p^k \times (\varphi_A(t))^k \right) \quad (2.0.18)$$

where,  $\varphi_A(t)$  is characteristic function of  $A_1$ .

On simplifying sum of infinite terms in geometric progression in (2.0.18), we get

$$= (1 - p) \times \left( 1 + \frac{p \times \varphi_A(t)}{1 - p \times \varphi_A(t)} \right) \quad (2.0.19)$$

$$= \frac{1 - p}{1 - p \times \varphi_A(t)} \quad (2.0.20)$$

$\therefore$  **option 3 is correct.**

4) Now, let's find mean of  $X$ .

$$\text{Mean}(X) = \frac{\sum_{j=1}^k A_j}{k} \quad (2.0.21)$$

For a distribution to be symmetric about  $a$ , the mean and the median should be equal to  $a$ , and with the given conditions it is not necessary for the mean( $X$ ) and median of  $X$  to be equal to 0, therefore  $X$  is not necessarily symmetric about 0.

$\therefore$  **option 4 is incorrect.**

$\therefore$  The correct options are (2) and (3).