Assignment 5

Dontha Aarthi-CS20BTECH11015

Download latex-tikz codes from

https://github.com/Dontha−Aarthi/AI1103/blob/main/Assignment5/assignment5.tex

1 GATE 2021 (ST), Q.48 (statistics section)

Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on [0,3]. Let Y be a random variable, independent of $\{X_n\}_{n\geq 1}$, having probability mass function

$$\Pr(Y = k) = \begin{cases} \frac{1}{(e-1)k!} & k = 1, 2, 3\cdots \\ 0 & otherwise \end{cases} \quad (1.0.1)$$

Then $\Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1)$ equals ....

2 Solution

Given that $\{X_n\}_{n\geq 1}$ is having a uniform distribution on [0,3], so probability can be written as

$$\Pr(X_n)_{n\geq 1} = \begin{cases} \frac{1}{3} & 0 \leq X_n \leq 3 \\ 0 & otherwise \end{cases} \quad (2.0.1)$$

So,

$$\Pr(X_n \leq 1)_{n\geq 1} = \frac{1}{3} \quad (2.0.2)$$

Required probability

$$= \Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1) \quad (2.0.3)$$

Since, $\{X_n\}_{n\geq 1}$ is a sequence of independent variables and Y is also independent of $\{X_n\}_{n\geq 1}$.
And also in (2.0.3), X_Y depends on Y, like if $Y = 1$, then there is only X_1, if $Y = 2$, then there's X_1, X_2, so required probability

$$= \Pr(max\{X_1\} \leq 1|Y = 1) \cdot \Pr(Y = 1) + \Pr(max\{X_1, X_2\} \leq 1|Y = 2) \cdot \Pr(Y = 2) + \cdots\infty \quad (2.0.4)$$

$$= \Pr(max\{X_1\} \leq 1) \cdot \Pr(Y = 1) + \Pr(max\{X_1, X_2\} \leq 1) \cdot \Pr(Y = 2) + \cdots\infty \quad (2.0.5)$$

$$(2.0.6)$$

$$= \Pr(X_1 \leq 1) \cdot \Pr(Y = 1) + \Pr(X_1, X_2 \leq 1) \cdot \Pr(Y = 2) + \cdots\infty \quad (2.0.7)$$

$$= \Pr(X_1 \leq 1) \cdot \Pr(Y = 1) + \Pr(X_1 \leq 1) \cdot \Pr(X_2 \leq 1) \cdot \Pr(Y = 2) + \cdots\infty \quad (2.0.8)$$

$$= \frac{1}{3} \cdot \frac{1}{(e-1) \cdot 1!} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{(e-1) \cdot 2!} + \cdots\infty \quad (2.0.9)$$

$$= \sum_{p=1}^{\infty} \left(\frac{1}{e-1}\right)\left(\frac{1}{3}\right)^p\left(\frac{1}{p!}\right) \quad (2.0.10)$$

$$= \left(\frac{1}{e-1}\right)\left[\sum_{p=0}^{\infty}\left(\frac{1}{3}\right)^p\left(\frac{1}{p!}\right) - 1\right] \quad (2.0.11)$$

Using Taylor's Series of e^x in (2.0.11),
Required probability

$$= \frac{e^{1/3}}{e-1} - \frac{1}{e-1} \quad (2.0.12)$$

$$= 0.23 \quad (2.0.13)$$