Assignment 5

Dontha Aarthi-CS20BTECH11015

Download latex-tikz codes from

https://github.com/Dontha−Aarthi/AI1103/blob/main/Assignment5/assignment5.tex

1 GATE 2021 (ST), Q.48 (statistics section)

Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on [0,3]. Let Y be a random variable, independent of $\{X_n\}_{n\geq 1}$, having probability mass function

$$\Pr(Y = k) = \begin{cases} \frac{1}{(e-1)k!} & k = 1, 2, 3\cdots \\ 0 & otherwise \end{cases} \tag{1.0.1}$$

Then $\Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1)$ equals ....

2 Solution

Given that $\{X_n\}_{n\geq 1}$ is having a uniform distribution on [0,3], so probability can be written as

$$\Pr(X_n)_{n\geq 1} = \begin{cases} \frac{1}{3} & 0 \leq X_n \leq 3 \\ 0 & otherwise \end{cases} \tag{2.0.1}$$

So,

$$\Pr(X_n \leq 1)_{n\geq 1} = \frac{1}{3} \tag{2.0.2}$$

Required probability

$$= \Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1) \tag{2.0.3}$$

Since, $\{X_n\}_{n\geq 1}$ is a sequence of independent variables and Y is also independent of $\{X_n\}_{n\geq 1}$.
And also in (2.0.3), the index of $X_i's$ depends on Y, so number of terms depends on Y, like if $Y = 1$, then there is only X_1, if $Y = 2$, then there's X_1, X_2, so required probability

$$= \sum_{p=1}^{\infty} \Pr\left(max\{X_1, X_2, \cdots X_p\} \leq 1 | Y = p\right) \cdot \Pr(Y = p) \tag{2.0.4}$$

$$= \sum_{p=1}^{\infty} \Pr\left(max\{X_1, X_2, \cdots X_p\} \leq 1\right) \cdot \Pr(Y = p) \tag{2.0.5}$$

$$= \sum_{p=1}^{\infty} \Pr\left(X_1, X_2, \cdots X_p \leq 1\right) \cdot \Pr(Y = p) \tag{2.0.6}$$

$$= \sum_{p=1}^{\infty} \Pr(X_1 \leq 1) \cdot \Pr(X_2 \leq 1) \cdots \Pr\left(X_{p-1} \leq 1\right) \cdot \Pr\left(X_p \leq 1\right) \cdot \Pr(Y = p) \tag{2.0.7}$$

$$= \sum_{p=1}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{1}{e-1}\right)\left(\frac{1}{p!}\right) \tag{2.0.8}$$

$$= \left(\frac{1}{e-1}\right)\left[\sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{1}{p!}\right) - 1\right] \tag{2.0.9}$$

Using Taylor's Series of e^x in (2.0.9),
Required probability

$$= \frac{e^{1/3}}{e-1} - \frac{1}{e-1} \tag{2.0.10}$$

$$= 0.23 \tag{2.0.11}$$