Assignment 5

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Download latex-tikz codes from

https://github.com/Dontha−Aarthi/AI1103/blob/main/Assignment5/assignment5.tex

1 GATE 2021 (ST), Q.48 (statistics section)

Let $\{X_n\}_{n\geq1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on [0,3]. Let Y be a random variable, independent of $\{X_n\}_{n\geq1}$, having probability mass function

$$\Pr(Y=k) = \begin{cases} \frac{1}{(e-1)k!} & k = 1,2,3\cdots \\ 0 & otherwise \end{cases} \tag{1.0.1}$$

Then $\Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1)$ equals ....

2 Solution

Given that $\{X_n\}_{n\geq1}$ is having a uniform distribution on [0,3], so probability can be written as

$$\Pr(X_n)_{n\geq1} = \begin{cases} \frac{1}{3} & 0 \leq X_n \leq 3 \\ 0 & otherwise \end{cases} \tag{2.0.1}$$

So,

$$\Pr(X_n \leq 1)_{n\geq1} = \frac{1}{3} \tag{2.0.2}$$

Required probability

$$= \Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1) \tag{2.0.3}$$

Since, $\{X_n\}_{n\geq1}$ is a sequence of independent variables and Y is also independent of $\{X_n\}_{n\geq1}$.
And also in (2.0.3), the index of $X_i's$ depends on Y, so number of terms depends on Y, like if $Y=1$, then there is only X_1, if $Y=2$, then there's X_1, X_2, so required probability

$$\begin{aligned} &= \Pr(max\{X_1\} \leq 1|Y=1)\cdot\Pr(Y=1) + \\ &\quad \Pr(max\{X_1,X_2\} \leq 1|Y=2)\cdot\Pr(Y=2) + \cdots\infty \end{aligned} \tag{2.0.4}$$

$$\begin{aligned} &= \Pr(max\{X_1\} \leq 1)\cdot\Pr(Y=1) + \\ &\quad \Pr(max\{X_1,X_2\} \leq 1)\cdot\Pr(Y=2) + \cdots\infty \end{aligned} \tag{2.0.5}$$

$$\begin{aligned} &= \Pr(X_1 \leq 1)\cdot\Pr(Y=1) + \\ &\quad \Pr(X_1,X_2 \leq 1)\cdot\Pr(Y=2) + \cdots\infty \end{aligned} \tag{2.0.6}$$

$$\begin{aligned} &= \Pr(X_1 \leq 1)\cdot\Pr(Y=1) + \\ &\quad \Pr(X_1 \leq 1)\cdot\Pr(X_2 \leq 1)\cdot\Pr(Y=2) + \cdots\infty \end{aligned} \tag{2.0.7}$$

$$\begin{aligned} &= \frac{1}{3}\cdot\frac{1}{(e-1)\cdot1!} + \\ &\quad \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{(e-1)\cdot2!} + \cdots\infty \end{aligned} \tag{2.0.8}$$

$$= \sum_{p=1}^{\infty}\left(\frac{1}{e-1}\right)\left(\frac{1}{3}\right)^p\left(\frac{1}{p!}\right) \tag{2.0.9}$$

$$= \left(\frac{1}{e-1}\right)\left[\sum_{p=0}^{\infty}\left(\frac{1}{3}\right)^p\left(\frac{1}{p!}\right) - 1\right] \tag{2.0.10}$$

Using Taylor's Series of e^x in (2.0.10),
Required probability

$$= \frac{e^{1/3}}{e-1} - \frac{1}{e-1} \tag{2.0.11}$$

$$= 0.23 \tag{2.0.12}$$