Assignment 7

Dontha Aarthi-CS20BTECH11015

Download latex-tikz codes from

https://github.com/Dontha−Aarthi/AI1103/blob/main/Assignment7/assignment7.tex

1 CSIR UGC NET EXAM (Dec 2014), Q.108

$N, A_1, A_2 \cdots$ are independent real valued random variables such that

$$\Pr(N = k) = (1-p)p^k, k = 0, 1, 2, 3 \cdots \tag{1.0.1}$$

where $0 < p < 1$ and $\{A_i : i = 1, 2, \cdots\}$ is a sequence of independent and identically distributed bounded random variables. Let

$$X(w) = \begin{cases} 0 & \text{if } N(w) = 0 \\ \sum_{j=1}^{k} A_j & \text{if } N(w) = k, k = 1, 2, 3 \cdots \end{cases} \tag{1.0.2}$$

Which of the following are necessarily correct?

1) X is a bounded random variable.
2) Moment generating function m_X of X is
$$m_X(t) = \frac{1-p}{1-pm_A(t)}, t \in \mathbb{R}, \tag{1.0.3}$$
where m_A is moment generating function of A_1.
3) Characteristic function φ_X of X is
$$\varphi_X(t) = \frac{1-p}{1-p\varphi_A(t)}, t \in \mathbb{R}, \tag{1.0.4}$$
where φ_A is the characteristic function of A_1.
4) X is symmetric about 0.

2 Solution

1) Since k is not bounded, X cannot be a bounded random variable necessarily.
$\therefore$ **option 1 is incorrect**.

2) The moment generating function of a random variable is defined as
$$M_X(t) = E[e^{tX}] \tag{2.0.1}$$
So,
$$M_X(t) = \sum_{k=0}^{\infty} \left(E[e^{t\sum_{j=1}^{k} A_j} | N = k] \times \Pr(N = k)\right) \tag{2.0.2}$$
$$= E[e^{t\times 0}] \times \Pr(N = 0) + \sum_{k=1}^{\infty} \left(E[e^{t\sum_{j=1}^{k} A_j} | N = k] \times \Pr(N = k)\right) \tag{2.0.3}$$
$$= E[1] \times (1-p) + \sum_{k=1}^{\infty} \left(E[e^{t\sum_{j=1}^{k} A_j} | N = k] \times \Pr(N = k)\right) \tag{2.0.4}$$
Since $A_1, A_2 \cdots$ are identical and independent, (2.0.4) can be written as
$$= (1-p) + \sum_{k=1}^{\infty} \left(\Pr(N = k) \times \prod_{j=1}^{k} E[e^{tA_j}]\right) \tag{2.0.5}$$
$$= (1-p) + \sum_{k=1}^{\infty} \left(\Pr(N = k) \times \prod_{j=1}^{k} M_{A_j}(t)\right) \tag{2.0.6}$$
$$= (1-p) + \sum_{k=1}^{\infty} \left((1-p)p^k \times (M_A(t))^k\right) \tag{2.0.7}$$
where, $M_A(t)$ is moment generating function of A_1.
On simplifying sum of infinite terms in geometric progression in (2.0.7), we get
$$= (1-p) \times \left(1 + \frac{p \times M_A(t)}{1 - p \times M_A(t)}\right) \tag{2.0.8}$$
$$= \frac{1-p}{1 - p \times M_A(t)} \tag{2.0.9}$$
$\therefore$ **option 2 is correct**.

3) The characteristic function of a random vari-

able is defined as

$$\varphi_X(t) = E[e^{itX}] \tag{2.0.10}$$

And using this,

$$E[X] = \sum_y E[X|Y = y] \times \Pr(Y = y) \tag{2.0.11}$$

We get

$$\varphi_X(t) = \sum_{k=0}^{\infty} \left(E[e^{it\sum_{j=1}^{k} A_j}|N = k] \times \Pr(N = k) \right) \tag{2.0.12}$$

$$= E[e^{it\times 0}] \times \Pr(N = 0) + \sum_{k=1}^{\infty} \left(E[e^{it\sum_{j=1}^{k} A_j}|N = k] \times \Pr(N = k) \right) \tag{2.0.13}$$

$$= E[1] \times (1 - p) + \sum_{k=1}^{\infty} \left(E[e^{it\sum_{j=1}^{k} A_j}|N = k] \times \Pr(N = k) \right) \tag{2.0.14}$$

$$\tag{2.0.15}$$

Since $A_1, A_2 \cdots$ are independent, (2.0.14) can be written as

$$= (1 - p) + \sum_{k=1}^{\infty} \left(\Pr(N = k) \times \prod_{j=1}^{k} E[e^{itA_j}] \right) \tag{2.0.16}$$

$$= (1 - p) + \sum_{k=1}^{\infty} \left(\Pr(N = k) \times \prod_{j=1}^{k} \varphi_{A_j}(t)] \right) \tag{2.0.17}$$

Since $A_1, A_2 \cdots$ are identical, (2.0.17) can be written as

$$= (1 - p) + \sum_{k=1}^{\infty} \left((1 - p)p^k \times (\varphi_A(t))^k \right) \tag{2.0.18}$$

where, $\varphi_A(t)$ is characteristic function of A_1. On simplifying sum of infinite terms in geometric progression in (2.0.18), we get

$$= (1 - p) \times \left(1 + \frac{p \times \varphi_A(t)}{1 - p \times \varphi_A(t)} \right) \tag{2.0.19}$$

$$= \frac{1 - p}{1 - p \times \varphi_A(t)} \tag{2.0.20}$$

$\therefore$ **option 3 is correct**.

4) Now, lets find mean of X.

$$\text{Mean(X)} = \frac{\sum_{j=1}^{k} A_j}{k} \tag{2.0.21}$$

For a distribution to be symmetric about 0, the mean and the median should be equal to 0, and with the given conditions it is not necessary for the mean(X) and median of X to be equal to 0, therefore X is not necessarily symmetric about 0.

$\therefore$ **option 4 is incorrect**.

$\therefore$ The correct options are (2) and (3).