Assignment 7

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Download latex-tikz codes from

https://github.com/Dontha−Aarthi/AI1103/blob/main/Assignment7/assignment7.tex

1 CSIR UGC NET EXAM (Dec 2014), Q.108

$N, A_1, A_2\cdots$ are independent real valued random variables such that

$$\Pr(N = k) = (1-p)p^k, k = 0, 1, 2, 3\cdots \tag{1.0.1}$$

where $0 < p < 1$ and $\{A_i : i = 1, 2, \cdots\}$ is a sequence of independent and identically distributed bounded random variables. Let

$$X(w) = \begin{cases} 0 & \text{if } N(w) = 0 \\ \sum_{j=1}^{k} A_j & \text{if } N(w) = k, k = 1, 2, 3\cdots \end{cases} \tag{1.0.2}$$

Which of the following are necessarily correct?

1) X is a bounded random variable.
2) Moment generating function m_X of X is
$$m_X(t) = \frac{1-p}{1-pm_A(t)}, t \in \mathbb{R}, \tag{1.0.3}$$
where m_A is moment generating function of A_1.
3) Characteristic function φ_X of X is
$$\varphi_X(t) = \frac{1-p}{1-p\varphi_A(t)}, t \in \mathbb{R}, \tag{1.0.4}$$
where φ_A is the characteristic function of A_1.
4) X is symmetric about 0.

2 Solution

Since k is not bounded, X cannot be a bounded random variable necessarily.
$\therefore$ **option 1 is incorrect**.

The characteristic function of a random variable is defined as

$$\varphi_X(t) = E[e^{itX}] \tag{2.0.1}$$

And using this,

$$E[X] = \sum_y E[X|Y = y] \cdot \Pr(Y = y) \tag{2.0.2}$$

We get

$$\varphi_X(t) = \sum_{k=0}^{\infty} \left(E[e^{it\sum_{j=1}^{k} A_j}|N = k] \cdot \Pr(N = k) \right) \tag{2.0.3}$$

$$= E[e^{it\cdot 0}] \cdot \Pr(N = 0) + \sum_{k=1}^{\infty} \left(E[e^{it\sum_{j=1}^{k} A_j}|N = k] \cdot \Pr(N = k) \right) \tag{2.0.4}$$

$$= E[1] \cdot (1-p) + \sum_{k=1}^{\infty} \left(E[e^{it\sum_{j=1}^{k} A_j}|N = k] \cdot \Pr(N = k) \right) \tag{2.0.5}$$

$$\tag{2.0.6}$$

Since $A_1, A_2\cdots$ are independent, (2.0.5) can be written as

$$= (1-p) + \sum_{k=1}^{\infty} \left(\Pr(N = k) \cdot \prod_{j=1}^{k} E[e^{itA_j}] \right) \tag{2.0.7}$$

$$= (1-p) + \sum_{k=1}^{\infty} \left(\Pr(N = k) \cdot \prod_{j=1}^{k} \varphi_{A_j}(t)] \right) \tag{2.0.8}$$

Since $A_1, A_2\cdots$ are identical, (2.0.8) can be written as

$$= (1-p) + \sum_{k=1}^{\infty} \left((1-p)p^k \cdot (\varphi_A(t))^k \right) \tag{2.0.9}$$

where, $\varphi_A(t)$ is characteristic function of A_1.
On simplifying sum of infinite terms in geometric progression in (2.0.9), we get

$$= (1-p) \cdot \left(1 + \frac{p \cdot \varphi_A(t)}{1 - p \cdot \varphi_A(t)} \right) \tag{2.0.10}$$

$$= \frac{1-p}{1 - p \cdot \varphi_A(t)} \tag{2.0.11}$$

$\therefore$ **option 3 is correct**.

The moment generating function of a random variable is defined as

$$M_X(t) = E[e^{tX}] \tag{2.0.12}$$

So,

$$M_X(t) = \sum_{k=0}^{\infty} \left(E[e^{t \sum_{j=1}^{k} A_j} | N = k] \cdot \Pr(N = k) \right) \tag{2.0.13}$$

$$= E[e^{t \cdot 0}] \cdot \Pr(N = 0) + \sum_{k=1}^{\infty} \left(E[e^{t \sum_{j=1}^{k} A_j} | N = k] \cdot \Pr(N = k) \right) \tag{2.0.14}$$

$$= E[1] \cdot (1 - p) + \sum_{k=1}^{\infty} \left(E[e^{t \sum_{j=1}^{k} A_j} | N = k] \cdot \Pr(N = k) \right) \tag{2.0.15}$$

Since $A_1, A_2 \cdots$ are identical and independent, (2.0.15) can be written as

$$= (1 - p) + \sum_{k=1}^{\infty} \left(\Pr(N = k) \cdot \prod_{j=1}^{k} E[e^{tA_j}] \right) \tag{2.0.16}$$

$$= (1 - p) + \sum_{k=1}^{\infty} \left(\Pr(N = k) \cdot \prod_{j=1}^{k} M_{A_j}(t) \right) \tag{2.0.17}$$

$$= (1 - p) + \sum_{k=1}^{\infty} \left((1 - p)p^k \cdot (M_A(t))^k \right) \tag{2.0.18}$$

where, $M_A(t)$ is moment generating function of A_1. On simplifying sum of infinite terms in geometric progression in (2.0.18), we get

$$= (1 - p) \cdot \left(1 + \frac{p \cdot M_A(t)}{1 - p \cdot M_A(t)} \right) \tag{2.0.19}$$

$$= \frac{1 - p}{1 - p \cdot M_A(t)} \tag{2.0.20}$$

$\therefore$ **option 2 is correct**.

Now, lets find mean of X.

$$\text{Mean(X)} = \frac{\sum_{j=1}^{k} A_j}{k} \tag{2.0.21}$$

With the given conditions it is not necessary for the mean(X) to be equal to 0, therefore X is not necessarily symmetric about 0.
$\therefore$ **option 4 is incorrect**.