POISSON DISTRIBUTION

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Introduction

Definition

A **discrete** random variable X is said to have a Poisson distribution, with parameter $\lambda > 0$, then the probability is given by:

$$\Pr(X = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!} \tag{1}$$

where,

1. k denotes the number of occurrences
2. λ is a positive real number which is equal to expected value of X and its variance. That is,

$$\lambda = E(X) = Var(X) \tag{2}$$

Other way of defining

Let λ denote the average number of events in the total time period, and let the time rate of number of events occurring be r, then in a certain time interval t, we can say that

$$\lambda = r \cdot t \tag{3}$$

And the probability of occurrence of k events in time t is

$$\Pr\left(X = k\right) = \frac{(rt)^k \cdot e^{-rt}}{k!} \tag{4}$$

NOTE: k can be any integer in the interval $[0,\infty]$

Assumptions

1. The occurrence of one event does not affect the probability that a second event will occur. That is, events occur independently.
2. The average rate at which events occur is independent of any occurrences. For simplicity, this is usually assumed to be constant, but in practice, it may vary with time.

If the above assumptions are true, then we can say that k is a poisson random variable, and its distribution follows poisson distribution.

Related Distributions

The Poisson distribution is also the limit of a binomial distribution, for which the probability of success for each trial equals λ divided by the number of trials, as the number of trials approaches infinity.
That is

$$p = \frac{\lambda}{n} \tag{5}$$

Probability is :

$$\Pr(X = k) = \lim_{n \to \infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \tag{6}$$

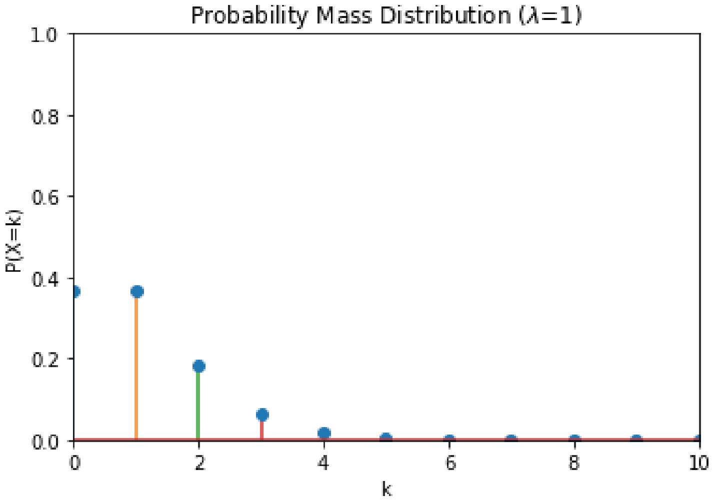
Probability Mass Distribution

Probability Mass Distribution(PMF) is nothing but the probability of k number of events to happen, given that an average of λ events occur in the entire time interval.
It is denoted by

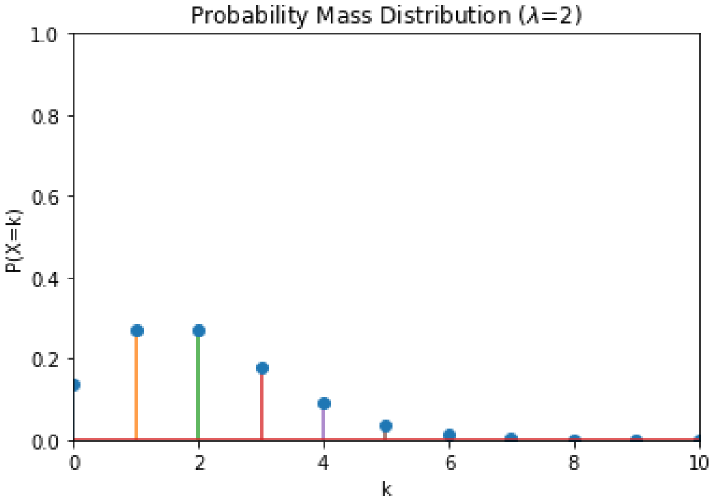
$$f_X(k) = \Pr(X = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!} \tag{7}$$

PMF Graphs

These are few PMF graphs with different values of λ.

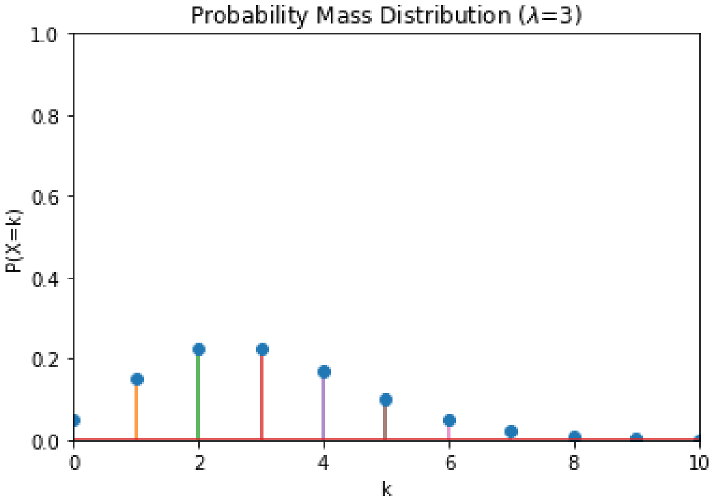


PMF Graphs

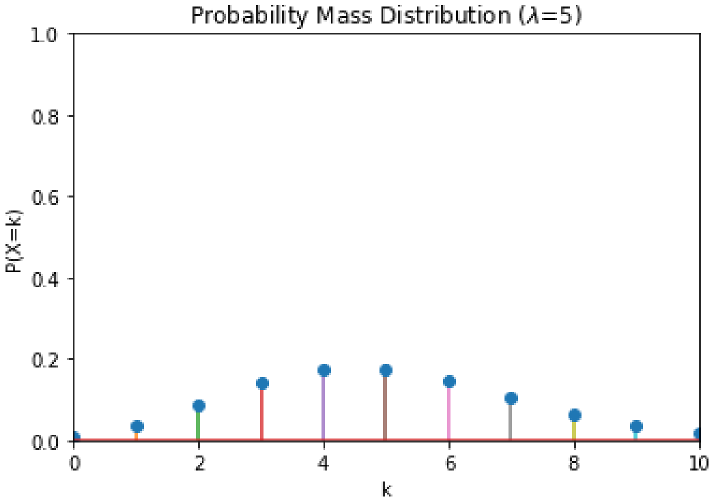


Probability Mass Distribution (λ=2)
1.0
0.8
0.6
0.4
0.2
0.0
P(X=k)
0
2
4
6
8
10
k

PMF Graphs


Probability Mass Distribution (λ=3)
P(X=k)
k
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10

PMF Graphs



Probability Mass Distribution (λ=5)
P(X=k)
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
k

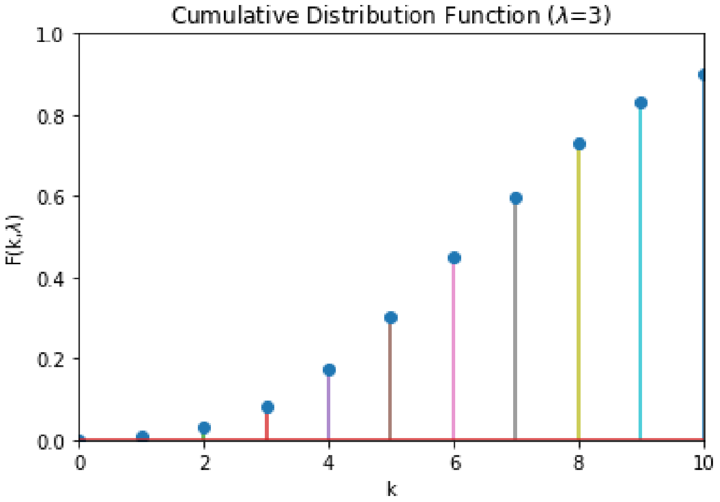
Cumulative Distribution Function

Cumulative Distribution Function(CDF) is given by

$$F_X(k) = \Pr(X \leq k) \tag{8}$$

$$= \sum_{n=0}^{n=k} \Pr(X = n) \tag{9}$$

CDF Graph



Cumulative Distribution Function (λ=3)
1.0
0.8
0.6
0.4
0.2
0.0
F(k,λ)
0
2
4
6
8
10
k

Example Question

Question

Suppose that astronomers estimate that large meteorites (above a certain size) hit the earth on average thrice every 100 years and that the number of meteorite hits follows a Poisson distribution. What is the probability of 2 meteorite hits in the next 50 years?

Solution

Solution

Rate of meteors hitting is 3 per 100 years.

$$rate(r) = \frac{3}{100} \tag{10}$$

So, for 50 years,

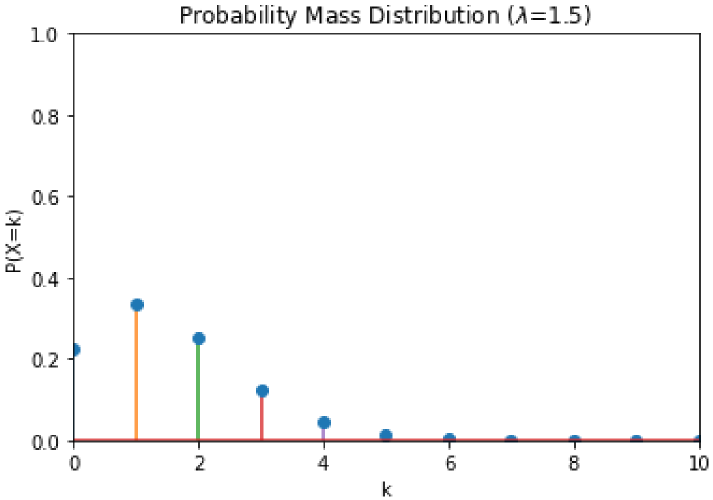
$$\lambda = r \cdot t = \frac{3}{100} \cdot 50 = \frac{3}{2} = 1.5 \tag{11}$$

We have to find the probability of 2 meteors hitting in the next 50 years.
So, we have to find the value of $\Pr(X = 2)$ with $\lambda = 1.5$
By putting the values of k=2 and $\lambda = 1.5$ in (1)

$$\Pr(X = 2) = \frac{1.5^2 \cdot e^{-1.5}}{2!} \tag{12}$$

$$= 0.251 \tag{13}$$

PMF Graph



Probability Mass Distribution (λ=1.5)
P(X=k)
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
k