Assignment 2

Dontha Aarthi-CS20BTECH11015

Download all python codes from

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https://github.com/Dontha−Aarthi/AI1103/blob
/main/Assignment2/Codes/assignment2.py
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and latex-tikz codes from

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https://github.com/Dontha−Aarthi/AI1103/blob
/main/Assignment2/main.tex
```

1 Gate Problem 43

If calls arrive at a telephone exchange such that the time of arrival of any call is independent of the time of arrival of earlier or future calls, the probability distribution function of the total number of calls in a fixed time interval will be
(A) Poisson
(B) Gaussian
(C) Exponential
(D) Gamma

2 Solution

Symbol	Description	Property	Random
T	Total time period	$T = n\Delta t$	No
n	Total Number of intervals		No
Δt	One time interval	Δt=T/n	No
k	Number of calls arrived during the time interval (0,T)		Yes
t_i	Denotes the time of arrival of each call in interval $(0, T)$		Yes
p	Probability of receiving a call at time t_i		No
λ	Average number of calls $(0, T)$	$\lambda = np$	No
e	Euler's number		No

Lets denote the fixed time interval by [0,T]. To find the probability of k number of calls during this time interval, lets divide the interval into n parts of equal length Δt. Let us denote the probability of receiving a call at a particular time t_i by p. Suppose the telephone exchange receives an average of λ calls in time interval of length T.
Hence, we have

$$np = \lambda \tag{2.0.1}$$

$$\implies p = \frac{\lambda}{n} \tag{2.0.2}$$

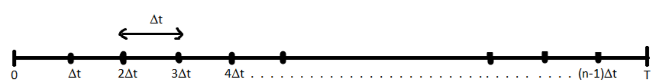

Fig. 0: Figure showing division of time intervals

In Fig. 0, the interval (0,T) has been divided into n equal parts, where length of each interval is Δt and the number of calls in each interval is a random variable.

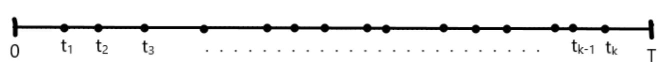

Fig. 0: Figure showing times of arrival of k calls

t_i where $i = \{1, 2, 3 \cdots k\}$ are the time of arrival of k calls in the interval $(0, T)$.

A call has probability p for arriving at $t_i, \forall i = \{1, 2, \cdots k\}$ and the probability of 1-p for not arriving at that instant.

In Binomial distribution we have certain number of intervals, i.e. n, with probability of arrival of each call as p and for a binomial random variable $X = \{0, 1 \cdots n\}$, the probability of call arriving in any k intervals is

$$\Pr(X = k) = {}^{n}\mathrm{C}_k \cdot p^k \cdot (1 - p)^k \tag{2.0.3}$$

But in Poisson distribution, we essentially have infinite intervals, so $n \to \infty$. Thus, the probability expression changes to:

$$\lim_{n\to\infty} \Pr(X = k) = \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \tag{2.0.4}$$

$$\lim_{n\to\infty} \Pr(X = k) = \left(\frac{\lambda^k}{k!}\right) \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \tag{2.0.5}$$

Now lets take the limit of right-hand side one term at a time. We'll do this in three steps. The first step is to find the limit of

$$\begin{aligned}\lim_{n\to\infty} \frac{n!}{(n-k)!n^k} &= \lim_{n\to\infty} \frac{n(n-1)(n-2)..(n-k+1)}{n^k} \\ &= \lim_{n\to\infty} \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)....\left(\frac{n-k+1}{n}\right) \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)...\left(1 - \frac{k-1}{n}\right) \\ &= 1 \cdot 1 \cdot 1........1 \\ &= 1\end{aligned} \tag{2.0.6}$$

Now we have to find the limit of

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n \tag{2.0.7}$$

We know that the definition e is given as

$$e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x \tag{2.0.8}$$

So, lets replace the value of $-\frac{n}{\lambda}$ by x in (2.0.7), we get

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x(-\lambda)} = e^{-\lambda} \tag{2.0.9}$$

And the third part is to find the limit of

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} \tag{2.0.10}$$

As n approaches infinity, this term becomes 1^{-k} which is equal to one. So,

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} = 1 \tag{2.0.11}$$

Now on substituting (2.0.6), (2.0.9) and (2.0.11) in equation (2.0.5), we get

$$\left(\frac{\lambda^k}{k!}\right) \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} = \left(\frac{\lambda^k}{k!}\right)(1)\left(e^{-\lambda}\right)(1) \tag{2.0.12}$$

This just simplifies into

$$\Pr(X = k) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right) \tag{2.0.13}$$

(2.0.13) is equal to probability density function of Poisson distribution, which gives us probability of k successes per period, with given parameter of λ.

$\therefore$The probability distribution function of the total number of calls in a fixed time interval will be **Poisson** distribution.
Answer: Option(A)