Linear Algebra II (MATH1049) — Coursework Sheet 4 — 2020/21

Submit a single pdf with scans of your work to Blackboard by Monday, 15 March 2021, 17:00.

Exercise 1

Let $n \geq 2$. Which of the conditions defining a subspace are satisfied for the following subsets of the vector space $M_{n\times n}(\mathbb{R})$ of real $(n \times n)$-matrices? (*Proofs or counterexamples are required.*)

$$\begin{aligned} U &:= \{A \in M_{n\times n}(\mathbb{R}) \,|\, \mathrm{rank}(A) \leq 1\} \\ V &:= \{A \in M_{n\times n}(\mathbb{R}) \,|\, \det(A) = 0\} \\ W &:= \{A \in M_{n\times n}(\mathbb{R}) \,|\, \mathrm{trace}(A) = 0\} \end{aligned}$$

(*Recall that rank*(A) *denotes the number of non-zero rows in a row-echelon form of* A *and trace*(A) *denotes the sum* $\sum_{i=1}^{n} a_{ii}$ *of the diagonal elements of the matrix* $A = (a_{ij})$*.*)

Exercise 2

Which of the following subsets of the vector space $\mathbb{R}^{\mathbb{R}}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ are subspaces? (*Proofs or counterexamples are required.*)

$$\begin{aligned} U &:= \{f \in \mathbb{R}^{\mathbb{R}} \,|\, f \text{ is differentiable and } f'(-5) = 0\} \\ V &:= \{f \in \mathbb{R}^{\mathbb{R}} \,|\, f \text{ is polynomial of the form } f = at^2 \text{ for some } a \in [0,\infty)\} \\ &= \{f \in \mathbb{R}^{\mathbb{R}} \,|\, \exists a \in [0,\infty] : \forall s \in \mathbb{R} : f(s) = as^2\} \\ W &:= \{f \in \mathbb{R}^{\mathbb{R}} \,|\, f \text{ is polynomial of the form } f = at^3 \text{ or } f = at^5 \text{ for some } a \in \mathbb{R}\} \\ &= \{f \in \mathbb{R}^{\mathbb{R}} \,|\, \exists i \in \{3,5\}\ \exists a \in \mathbb{R} : \forall s \in \mathbb{R} : f(s) = as^i\} \\ X &:= \{f \in \mathbb{R}^{\mathbb{R}} \,|\, f \text{ is even}\} \end{aligned}$$

(*Recall that a function* $f : \mathbb{R} \to \mathbb{R}$ *is called* even *if* $f(-s) = f(s)$ *for all* $s \in \mathbb{R}$*.*)

Exercise 3

Let $\mathbb{F}_2 = \{0, 1\}$ denote the field with 2 elements.

(a) Let V be a vector space over $\mathbb{F}_2$. Show that every non-empty subset W of V which is closed under addition is a subspace of V.

(b) Show that $\{(0,0),(1,0)\}$ is a subspace of the vector space $\mathbb{F}_2^2$ over $\mathbb{F}_2$.

(c) Write down all subsets of $\mathbb{F}_2^2$ and underline those subsets which are subspaces. (*No explanations are required.*)

Exercise 4 (optional, not marked)

Let V be a vector space over a field F. Putting $S = V$ in Example 2.7 we obtain the vector space F^V consisting of *all* functions from V to F. Consider the subset

$$V^* := \{L : V \to F \mid L \text{ is a linear transformation}\},$$

consisting of all linear transformations from the vector space V to the (one-dimensional) vector space F. Show that V^* is a subspace of F^V. (*To get you started, at the end of this sheet you'll find a detailed proof of the first of the three conditions that need to be verified for a subspace.*)

Extra question (not marked, do not submit)

Let V be a vector space over a field F and let X, Y and Z be subspaces of V, such that $X \subseteq Y$. Show that $Y \cap (X + Z) = X + (Y \cap Z)$. (*Note: this is an equality of sets, so you need to show that every vector in the LHS also belongs to RHS, and vice versa.*)

Verification of the first condition of being a subspace, for V^* from Exercise 4

(You don't need to reproduce this in your solution, just say that the first condition is proved.)

The first condition for a subspace asserts that the zero vector of the "big" vector space F^V belongs to set V^* that we are showing to be a subspace.

The zero vector (= the additive identity element for vector addition) of F^V is the zero function $\underline{0} : V \to F$, defined by $\underline{0}(v) = 0_F$ for all $v \in V$, that is, it maps *every* vector v from V to the additive identity element 0_F in the field F.

We need to show that this function $\underline{0}$ belongs to the set V^*, in other words, that it is a linear transformation from V to F. This entails checking two conditions:

(a) $\underline{0}$ is compatible with addition: take arbitrary vectors $x, y \in V$. We need to check that $\underline{0}(x + y) = \underline{0}(x) + \underline{0}(y)$ in F:
LHS $= 0_F$ (by definition of $\underline{0}$)
RHS $= 0_F + 0_F = 0_F$ (by definition of $\underline{0}$ and the field axioms)
So LHS = RHS.

(b) $\underline{0}$ is compatible with scalar multiplication: take a vector $x \in V$ and a scalar $a \in F$. We need to check that $\underline{0}(ax) = a(\underline{0}(x))$ in F:
LHS $= 0_F$ (by definition of $\underline{0}$)
RHS $= a0_F = 0_F$ (by definition of $\underline{0}$ and Prop. 2.3(a))
So again LHS = RHS.