Linear Algebra II (MATH1049) — Coursework Sheet 3 — 2020/21

Submit a single pdf with scans of your work to Blackboard by Monday, 1 March 2021, 17:00.

Exercise 1

The set $\mathbb{R}^2$ together with the usual vector addition forms an abelian group. For $a \in \mathbb{R}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ we put $a \otimes \mathbf{x} := \begin{pmatrix} ax_1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$; this defines a scalar multiplication

$$\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (a, \mathbf{x}) \mapsto a \otimes \mathbf{x},$$

of the field $\mathbb{R}$ on $\mathbb{R}^2$. Determine which of the axioms defining a vector space hold for the abelian group $\mathbb{R}^2$ with this scalar multiplication. (*Proofs or counterexamples are required.*)

Exercise 2

The set $\mathbb{R}_{>0}$ of positive real numbers together with multiplication forms an abelian group. Let $\mathbb{R}^n_{>0}$ denote the n-fold cartesian product of $\mathbb{R}_{>0}$ with itself (cf. Exercise 1 on Sheet 2). (*You may find it convenient to use the symbol $\oplus$ for the binary operation in the abelian group $\mathbb{R}^n_{>0}$, that is $(b_1, \ldots, b_n) \oplus (c_1, \ldots, c_n) = (b_1c_1, \ldots, b_nc_n)$ for $b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{R}_{>0}$.*) Furthermore, for $a \in \mathbb{Q}$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n_{>0}$ we put $a \otimes \mathbf{b} := (b_1^a, \ldots, b_n^a)$. Show that the abelian group $\mathbb{R}^n_{>0}$ together with the scalar multiplication

$$\mathbb{Q} \times \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}, \quad (a, \mathbf{b}) \mapsto a \otimes \mathbf{b},$$

is a vector space over $\mathbb{Q}$.

Exercise 3

Let V be a vector space over the field F and let $a \in F$ and $x, y \in V$.

(a) Show that $a(x - y) = ax - ay$ in V.

(b) If $ax = 0_V$ show that $a = 0_F$ or $x = 0_V$.

(*Remember to give a reason for each step.*)

Exercise 4

Let S be a set and let V be a vector space over a field F. Let V^S denote the set of all maps from S to V. We define an addition on V^S and a scalar multiplication of F on V^S as follows: let $f, g \in V^S$ and let $a \in F$; then

$$(f + g)(s) := f(s) + g(s) \text{ and } (af)(s) := a\,(f(s)) \text{ (for any } s \in S).$$

Show that V^S is a vector space over F. (*For a complete proof many axioms need to be checked. In order to save you some writing, your solution will be considered complete, if you check that there exists an additive identity element in V^S, that every element in V^S has an additive inverse and that the second distributivity law holds.*)