Coursework Sheet 2

1. $\overset{id}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}}, \overset{\pi_1}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}}, \overset{\pi_2}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}, \overset{\pi_3}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}},$

$\overset{\pi_4}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}, \overset{\pi_5}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}$

$\pi_1 = \langle 2,3 \rangle \quad \pi_2 = \langle 1,3 \rangle \quad \pi_3 = \langle 1,2 \rangle \quad \pi_4 = \langle 1,3,2 \rangle \quad \pi_5 = \langle 1,2,3 \rangle$

$\circ$	id	π_1	π_2	π_3	π_4	π_5
id	id	π_1	π_2	π_3	π_4	π_5
π_1	π_1	id	π_5	π_4	π_3	π_2
π_2	π_2	π_4	id	π_5	π_1	π_3
π_3	π_3	π_5	π_4	id	π_2	π_1
π_4	π_4	π_2	π_3	π_1	π_5	id
π_5	π_5	π_3	π_1	π_2	id	π_4

$\pi_1 \circ \pi_2 = \pi_5 \qquad \pi_1 \circ \pi_2 \neq \pi_2 \circ \pi_1 \Rightarrow$ not an abelian group

$\pi_2 \circ \pi_1 = \pi_4$

2. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 6 & 1 & 8 & 9 & 4 & 2 & 5 \end{pmatrix} \in S_9 \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 6 & 3 & 1 \end{pmatrix} \in S_6$

$\eta = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix} \in S_n$

a) $\sigma = \langle 1,3,6,9,5,8,2,7,4, \rangle$; $sgn(\sigma) = 1$

$\tau = \langle 1,4,6 \rangle \circ \langle 3,5 \rangle$; $sgn(\tau) = -1$

$\eta = \langle 1,n \rangle \circ \langle 2,n-1 \rangle \circ \dots \circ \langle \frac{n}{2}, \frac{n}{2}+1 \rangle$; $sgn(\eta) = (-1)^{(1+\frac{n}{2})\frac{n}{4}}$

b) $\sigma^2 = \langle 6,5,2,4,3,9,8,7 \rangle \qquad \tau^2 = \langle 1,6,4 \rangle$

$\sigma^{-1} = \langle 1,4,7,2,8,5,9,6,3 \rangle \qquad \tau^{-1} = \langle 6,4,1 \rangle \circ \langle 3,5 \rangle$

$\eta^2 = id \qquad \eta^{-1} = \eta$

2. c)

using (b) from exercise 2:

$sgn(\sigma^2) = (-1)^{9-1} = 1$; $sgn(\tau^2) = (-1)^{3-1} = 1$

$sgn(\eta^2) =$

$= sgn(id) = 1$

using Thm. 1.10. :

$sgn(\sigma^2) = sign(\sigma)\, sgn(\sigma) = 1$

$sgn(\tau^2) = sgn(\tau)\, sgn(\tau) = (-1)(-1) = 1$

$sgn(\eta^2) = sgn(\eta)\, sgn(\eta) = (-1)^{(1+\frac{n}{2})\frac{n}{4}} \cdot (-1)^{(1+\frac{n}{2})\frac{n}{4}} = (-1)^{2(1+\frac{n}{2})\frac{n}{4}}$

$(1+\frac{n}{2})\frac{n}{4} \in \mathbb{N} \Rightarrow k = (1+\frac{n}{2})\frac{n}{4} \Rightarrow sgn(\eta^2) = (-1)^{2k} = 1$

n is even $\in \mathbb{N}$

[side note: the square of any permutation will be +1 since $(1)\cdot(1) = 1$, $(-1)(-1) = 1$]
from Thm. 1.10

3.

$\langle a_1, \dots, a_s \rangle \in S_n$; $\sigma \in S_n$

show: $\sigma \circ \langle a_1, \dots, a_s \rangle \circ \sigma^{-1} = \langle \sigma(a_1), \dots, \sigma(a_s) \rangle$

case $b \in \{\sigma(a_1), \dots, \sigma(a_n)\}$:

$\Rightarrow$ let $b = \sigma(a_k)$

$\langle \sigma(a_1), \dots, \sigma(a_n) \rangle(b) = \sigma(a_{k+1})$

$\sigma \circ \langle a_1, \dots, a_s \rangle \circ \sigma^{-1}(b) = \sigma \circ \langle a_1, \dots, a_s \rangle \sigma^{-1}(\sigma(a_k))$

$= \sigma \circ \langle a_1, \dots, a_s \rangle (a_k)$

$= \sigma \circ (a_{k+1}) = \sigma(a_{k+1})$

$\Rightarrow \langle \sigma(a_1), \dots, \sigma(a_n) \rangle(b) = \sigma \circ \langle a_1, \dots, a_s \rangle \circ \sigma^{-1}(b)$

case $b \notin \{\sigma(a_1), \dots, \sigma(a_n)\}$:

$\sigma^{-1}(b) \notin \{a_1, \dots, a_n\}$

$\langle \sigma(a_1), \dots, \sigma(a_s) \rangle(b) = b$

$\sigma \circ \langle a_1, \dots, a_s \rangle \circ \sigma^{-1}(b) \overset{\sigma^{-1}(b) \notin \{a_1, \dots, a_n\}}{\Rightarrow} = \sigma \circ \sigma^{-1}(b) = b$

$\Rightarrow \langle \sigma(a_1), \dots, \sigma(a_s) \rangle(b) = \sigma \circ \langle a_1, \dots, a_s \rangle \circ \sigma^{-1}(b)$

4.

$\mathbb{Q}(\sqrt{5}) = \{ z \in \mathbb{R} \mid a + b\sqrt{5}, \ a, b \in \mathbb{Q} \}$

Show : $\mathbb{Q}(\sqrt{5})$ under "+" and "$\times$" is a field

$\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{R} \Rightarrow$ distributivity, commutativity & associativity

additive closure : $\overset{\text{show}}{w + z \in \mathbb{Q}}, \ w, z \in \mathbb{Q}(\sqrt{5})$

$w, z \in \mathbb{Q}(\sqrt{5}) \Rightarrow w = a + b\sqrt{5} \qquad z = c + d\sqrt{5}$

$$\left.\begin{aligned} w + z = a + b\sqrt{5} + c + d\sqrt{5} = a + c + (b+d)\sqrt{5} \\ a + c \in \mathbb{Q}, \ b + d \in \mathbb{Q} \end{aligned}\right\} \Rightarrow w + z \in \mathbb{Q}(\sqrt{5})$$

multiplicative closure : $\overset{\text{show}}{wz \in \mathbb{Q}}, \ w, z \in \mathbb{Q}(\sqrt{5})$

$$wz = (a + b\sqrt{5})(c + d\sqrt{5}) = ac + 5bd + ad\sqrt{5} + cb\sqrt{5} =$$

$$\left.\begin{aligned} = ac + 5bd + (ad + cb)\sqrt{5} \\ ac + 5bd \in \mathbb{Q}, \ (ad + cb) \in \mathbb{Q} \end{aligned}\right\} \Rightarrow wz \in \mathbb{Q}$$

$0 \in \mathbb{Q}(\sqrt{5})$:

$$\left.\begin{aligned} 0 = 0 + 0\sqrt{5} \\ 0 \in \mathbb{Q} \end{aligned}\right\} \Rightarrow 0 \in \mathbb{Q}(\sqrt{5})$$

$1 \in \mathbb{Q}(\sqrt{5})$:

$$\left.\begin{aligned} 1 = 1 + 0\sqrt{5} \\ 1, 0 \in \mathbb{Q} \end{aligned}\right\} \Rightarrow 1 \in \mathbb{Q}(\sqrt{5})$$

additive inverse : $\overset{\text{show}}{-z \in \mathbb{Q}(\sqrt{5})}$

$z + (-z) = 0 \qquad a + b\sqrt{5} + (-z) = 0$

$$\left.\begin{aligned} -z = -a - b\sqrt{5} \\ a, b \in \mathbb{Q} \Rightarrow -a, -b \in \mathbb{Q} \end{aligned}\right\} \Rightarrow -z \in \mathbb{Q}(\sqrt{5})$$

multiplication inverse: $z^{-1} \in \mathbb{Q}$

$z z^{-1} = 1$

$$\left.\begin{aligned} z^{-1} = \frac{1}{z} = \frac{1}{a + b\sqrt{5}} = \frac{a - b\sqrt{5}}{a^2 - 5b^2} = \frac{a}{a^2 - 5b^2} - \frac{b\sqrt{5}}{a^2 - 5b^2} \\ a^2 - 5b^2, a \in \mathbb{Q} \Rightarrow \frac{a}{a^2 - 5b^2} \in \mathbb{Q} \\ b, a^2 - 5b^2 \in \mathbb{Q} \Rightarrow \frac{b}{a^2 - 5b^2} \in \mathbb{Q} \end{aligned}\right\} \Rightarrow z^{-1} \in \mathbb{Q}(\sqrt{5})$$

5. F is a field, $a, b \in F$, $\frac{a}{b} = ab^{-1}$

$a, a' \in F$, $b, b' \in F \setminus \{0\}$

i) Show: $\frac{a}{b} + \frac{a'}{b'} = \frac{ab' + a'b}{bb'}$

$ab^{-1} + a'b'^{-1} = 1 \cdot ab^{-1} + 1a'b'^{-1} = (b'b'^{-1}) \cdot ab^{-1} + (bb^{-1})a'b'^{-1} =$

$= (b'^{-1}b') \cdot ab^{-1} + (b^{-1}b)a'b'^{-1} = (b'^{-1}b')(b^{-1}a) + (b^{-1}b)(b'^{-1}a') = b'^{-1}(b'b^{-1})a + b^{-1}(b'^{-1}b)a' =$

$= (b'^{-1}b^{-1})(b'a) + (b'^{-1}b^{-1})(ba') = (b'^{-1}b^{-1})(b'a + ba') = \frac{b'a + ba'}{bb'}$

$[(b'^{-1}b^{-1})bb' = b'^{-1}(b^{-1}b)b' = b'^{-1}b' = 1 \Rightarrow (bb')^{-1} = b'^{-1}b^{-1}] \Rightarrow$

ii) Show: $\frac{a}{b} \cdot \frac{a'}{b'} = \frac{aa'}{bb'}$

$\frac{a}{b} \cdot \frac{a'}{b'} = (ab^{-1})(a'b'^{-1}) = a(a'b^{-1})b'^{-1} = (aa')(b^{-1}b'^{-1}) = (aa')(bb')^{-1} = \frac{aa'}{bb'}$

$b^{-1}b'^{-1}b'b = b^{-1}(b'^{-1}b')b = b^{-1}b = 1 \Rightarrow (b^{-1}b'^{-1}) = (b'b)^{-1} = (bb')^{-1}$