Linear Algebra II (MATH1049)

Coursework 4

Exercise 1

Which conditions for being a subspace are satisfied for the following subsets of $M_{n\times n}(\mathbb{R})$? ($n \geq 2$)

$$U \equiv \{A \in M_{n\times n}(\mathbb{R}) : rank(A) \leq 1\}$$
$$V \equiv \{A \in M_{n\times n}(\mathbb{R}) : \det(A) = 0\}$$
$$W \equiv \{A \in M_{n\times n}(\mathbb{R}) : \text{trace}(A) = 0\}$$

Solution

a) $U \equiv \{A \in M_{n\times n}(\mathbb{R}) : rank(A) \leq 1\}$

1. Additive identity ✓

$$0_V = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$
$$\text{rank}(0_V) = 0 \Rightarrow 0_V \in U$$

2. Additive closure $\times$
   Take $n = 2$ so $M_{2\times 2}$

$$\text{rank}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = 1 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in U$$
$$\text{rank}(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = 1 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in U$$
$$\text{rank}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = 2 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin U$$

3. Scalar closure ✓

$$\forall \lambda \in \mathbb{R}\backslash\{0\}, \forall A \in U \subset M_{m\times n}(\mathbb{R}),\ \text{rank}(\lambda A) = \text{rank}(A) \quad \text{(Thm. 3.33 (ii) LA 1)}$$
$$A \in U \Rightarrow \text{rank}(\lambda A) = \text{rank}(A) \leq 1$$
$$\Rightarrow \text{rank}(\lambda A) \leq 1 \Rightarrow \lambda A \in U$$
$$\Rightarrow \forall \lambda \in \mathbb{R}\backslash\{0\}, \forall A \in U,\ \lambda A \in U$$

$$\text{Let } \lambda = 0$$
$$\forall A \in U, \lambda A = 0_V \qquad \text{(Additive identity scalar)}$$
$$0_V \in U \Rightarrow \forall A \in U, \lambda A \in U \text{ where } \lambda = 0$$

Since cases $\lambda = 0$ and $\lambda \in \mathbb{R}\backslash\{0\}$ have been proved to be closed we can conclude that U is closed under scalar multiplication

b) $V \equiv \{A \in M_{n\times n}(\mathbb{R}) : \det(A) = 0\}$

1. Additive Identity ✓

$$0_V = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$
$$(\text{two identical rows}) \Rightarrow \det(0_V) = 0 \Rightarrow 0_V \in V$$

2. Additive Closure $\times$

Take $n = 2$ so $M_{2\times 2}$

$$\det(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = 1 * 0 + 0 * 0 = 0 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in V$$
$$\det(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = 0 * 1 + 0 * 0 = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in V$$
$$\det(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = 1 * 1 + 0 * 0 = 1 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin V$$

3. Scalar Closure ✓

$$\begin{aligned} \forall \lambda \in \mathbb{R}, \forall A \in V \subset M_{n\times n},\ \det(\lambda A) &= \lambda^n \det(A) \qquad (\text{Def. 4.1 D1 (i) LA 1}) \\ A \in V \Rightarrow \det(\lambda A) &= \lambda^n \cdot 0 = 0 \\ &\Rightarrow \det(\lambda A) \in V \\ \Rightarrow \forall \lambda \in \mathbb{R}, \forall A \in V \subset M_{n\times n},\ &\det(\lambda A) \in V \end{aligned}$$

c) $W \equiv \{A \in M_{n\times n}(\mathbb{R}) : \text{trace}(A) = 0\}$

1. Additive Identity ✓

$$0_V = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$
$$\text{trace}(0_V) = 0^n = 0 \Rightarrow 0_V \in W$$

2. Additive closure $\times$

Take $n = 2$ so $M_{2\times 2}$

$$\text{trace}(\begin{pmatrix}1&0\\0&0\end{pmatrix}) = 1*0 = 0 \Rightarrow \begin{pmatrix}1&0\\0&0\end{pmatrix} \in W$$
$$\text{trace}(\begin{pmatrix}0&0\\0&1\end{pmatrix}) = 0*1 = 0 \Rightarrow \begin{pmatrix}0&0\\0&1\end{pmatrix} \in W$$
$$\text{trace}(\begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix}) = 1*1 = 1 \Rightarrow \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} \notin W$$

3. Scalar Closure ✓

$$\begin{aligned}
\text{Let } A \in W,\ A = (a_{ij}) \\
\Rightarrow \text{trace}(A) = 0 \\
\Rightarrow a_{11} \times a_{22} \times \cdots \times a_{nn} = 0 \\
\Rightarrow \lambda^n \times na_{11} \times \cdots \times a_{nn} = 0 \text{ where } \lambda \in \mathbb{R} \\
\Rightarrow \text{trace}(\lambda A) = 0 \Rightarrow \lambda A \in W \\
\Rightarrow \forall \lambda \in \mathbb{R}, \forall A \in W,\ \lambda A \in W
\end{aligned}$$

Exercise 2

Which of these are are subspaces of $\mathbb{R}^{\mathbb{R}}$?

$$\begin{aligned}
U &\equiv \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is diff. and } f'(-5) = 0\} \\
V &\equiv \{f \in \mathbb{R}^{\mathbb{R}} : \exists a \in [0, \infty) : \forall s \in \mathbb{R} : f(s) = as^2\} \\
W &\equiv \{f \in \mathbb{R}^{\mathbb{R}} : \exists i \in \{3, 5\}, \exists a \in \mathbb{R} : \forall s \in \mathbb{R} : f(s) = as^i\} \\
X &\equiv \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is even}\}
\end{aligned}$$

Solution

a) $U \equiv \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is diff. and } f'(-5) = 0\}$

1. Additive Identity ✓

$$\begin{aligned}
\forall x \in \mathbb{R}, \underline{0}(x) = 0 \\
\underline{0}'(x) = 0 \Rightarrow \underline{0}(x) \in U
\end{aligned}$$

2. Additive closure ✓

$$\begin{aligned}
\text{Let } f, g \in U & \\
(f+g)(x) = f(x) + g(x) & \\
\Rightarrow (f+g)(x) \text{ is diff.} & \qquad \text{(Calculus)} \\
(f+g)'(x) = f'(x) + g'(x) & \\
(f+g)'(-5) = f'(-5) + g'(-5) = 0 & \qquad \text{(Definition of } U) \\
(f+g)(x) \text{ is diff. and } (f+g)'(-5) = 0 & \\
\Rightarrow (f+g)(x) \in U &
\end{aligned}$$

3. Scalar closure $\times$

$$
\begin{aligned}
\text{Let } f \in U, \lambda \in \mathbb{R} \\
(\lambda f)(x) = \lambda(f(x)) \\
\Rightarrow (\lambda f)(x) \text{ is diff.} && \text{(Calculus)} \\
(\lambda f)'(x) = \lambda f'(x) \\
(\lambda f)'(-5) = \lambda f'(-5) = 0 && \text{(Definition of } U\text{)}
\end{aligned}
$$

Since 1.-3. hold U is a subspace.

b) $V \equiv \{f \in \mathbb{R}^{\mathbb{R}} : \exists a \in [0, \infty) : \forall s \in \mathbb{R} : f(s) = as^2\}$

1. Additive identity ✓

$$\underline{0}(x) = 0 = 0 * x^2 \Rightarrow \underline{0}(x) \in V$$

2. Additive closure ✓

$$
\begin{aligned}
\text{Let } f, g \in V \\
\Rightarrow f(x) = ax^2,\ g(x) = bx^2 \text{ where } a, b \in [0, \infty) \\
(f+g)(x) = f(x) + g(x) = ax^2 + bx^2 = (a+b)x^2 \\
a, b \in [0, \infty) \Rightarrow a \geq 0,\ b \geq 0 \Rightarrow a + b \geq 0 \Rightarrow a + b \in [0, \infty) \\
\Rightarrow (f+g)(x) \in V
\end{aligned}
$$

3. Scalar closure $\times$

$$
\begin{aligned}
\text{Let } f = x^2, \lambda = -1 \\
1 \in [0, \infty) \Rightarrow f = 1 \cdot x^2 \in V \\
(-1 \cdot f)(x) = -x^2 \\
-1 \notin [0, \infty) \Rightarrow (-1 \cdot f)(x) \notin V
\end{aligned}
$$

Since scalar multiplication closure does not hold in V, V is not a subspace of $\mathbb{R}^{\mathbb{R}}$.

c) $W \equiv \{f \in \mathbb{R}^{\mathbb{R}} : \exists i \in \{3, 5\}, \exists a \in \mathbb{R} : \forall s \in \mathbb{R} : f(s) = as^i\}$

Counter-example to additive closure

$$
\begin{aligned}
\text{Let } f(x) = x^3, g(x) = x^5 \\
1 \in \mathbb{R} \Rightarrow f, g \in W \\
(f+g)(x) = f(x) + g(x) = x^3 + x^5 \\
(\text{not in the form } ax^3 \text{ or } ax^5) \\
\Rightarrow (f+g)(x) \notin W
\end{aligned}
$$

Hence addition is not closed in W and W is not a subspace of $\mathbb{R}^{\mathbb{R}}$.

d) $X \equiv \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is even}\}$

1. Additive Identity ✓

$$\underline{0}(x) = 0 = \underline{0}(-x) \Rightarrow \underline{0}(x) \text{ is even}$$

2. Additive closure ✓

$$\begin{aligned}
&\text{Let } f, g \text{ be even functions} \\
&\Rightarrow f, g \in X \\
(f+g)(x) &= f(x) + g(x) \\
(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) &= (f+g)(x) && (f,g \text{ are even}) \\
\Rightarrow (f+g)(-x) &= (f+g)(x) \\
&\Rightarrow (f+g)(x) \text{ is an even function} \\
&\Rightarrow (f+g)(x) \in X
\end{aligned}$$

3. Scalar closure ✓

$$\begin{aligned}
&\text{Let } f \text{ be an even function}, \lambda \in \mathbb{R} \\
&\Rightarrow f \in X \\
(\lambda f)(x) &= \lambda(f(x)) \\
(\lambda f)(-x) = \lambda(f(-x)) = \lambda(f(x)) &= (\lambda f)(x) && (f \text{ is even}) \\
\Rightarrow (\lambda f)(x) &= (\lambda f)(-x) \\
&\Rightarrow \lambda f \text{ is even} \\
&\Rightarrow \lambda f \in X
\end{aligned}$$

Exercise 3

a) Let V be a vector space over $\mathbb{F}_2$. Show that every non-empty subset W of V which is closed under addition is a subspace of V .
b) Show that $\{(0,0),(1,0)\}$ is a subspace of the vector space $\mathbb{F}_2^2$ over F_2.
c) Write down all subsets of $\mathbb{F}_2^2$ and underline those subsets which are subspaces.

Solutions

a) Let V be a vector space over $\mathbb{F}_2$. Show that every non-empty subset W of V which is closed under addition is a subspace of V.

1. Additive identity

$$\begin{aligned}
&\text{Let } a \in W, \\
a + a &\in W && \text{(additive closure)} \\
\Rightarrow (1+1)a &\in W && \text{(Distributivity in V)} \\
0a = 0_V &\in W && \text{(Addition in } \mathbb{F}_2 + \text{identity multip.)}
\end{aligned}$$

3. Scalar multiplication

$$\begin{aligned}
&\text{Let } a \in W, \\
0a &= 0_V && \text{(Identity multiplication)} \\
1a &= a \in W && \text{(Definition of 1 multip.)}
\end{aligned}$$

Since all three conditions hold W is a subspace of V

b) Showing $\{(0,0),(1,0)\}$ is a subspace of $\mathbb{F}_2^2$ over F_2

1. Additive identity

$$(0,0) \in \{(0,0),(1,0)\}$$

2. Additive closure

$$\begin{gathered}(0,0)+(0,1)=(0,1) \in \{(0,0),(1,0)\} \\ (0,1)+(0,1)=(0,0) \in \{(0,0),(1,0)\} \\ (0,0)+(0,0)=(0,0) \in \{(0,0),(1,0)\} \\ (0,1)+(0,0)=(0,1) \in \{(0,0),(1,0)\}\end{gathered}$$

3. Scalar multiplication

$$\begin{gathered}1*(0,0)=(0,0) \in \{(0,0),(1,0)\} \\ 0*(0,0=(0,0) \in \{(0,0),(1,0)\} \\ 1*(0,1)=(0,1) \in \{(0,0),(1,0)\} \\ 0*(0,0)=(0,0) \in \{(0,0),(1,0)\}\end{gathered}$$

c) Write down all subsets of $\mathbb{F}_2^2$ and underline those subsets which are subspaces.

$$\begin{aligned}\mathcal{P}(\mathbb{F}_2^2) = |\emptyset, \boxed{\{(0,0)\}}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}, \\ \boxed{\{(0,0),(0,1)\ \}}, \boxed{\{(0,0),(1,0)\}}, \boxed{\{(0,0),(1,1)\}}, \{(0,1),(1,0)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \\ \{(0,0),(1,0)(0,1)\}, \{(0,0),(1,0),(1,1)\}, \{(0,0),(0,1),(1,1)\}, \{(0,1),(1,0),(1,1)\}, \\ \boxed{\{(0,0),(0,1),(1,0),(1,1)\}}|\end{aligned}$$