Linear Algebra II (MATH1049)

Coursework 5

Exercise 1

Which of the following span the vector space $\mathbb{P}_2$?

1. $U = \{\frac{1}{2}, t^2 + t, t^2 - 1\}$
2. $V = \{1, 2t, t^2, 3t^2 + 5\}$
3. $W = \{t + 1, t^2 + t\}$

Solution

1. U is a spanning set of $\mathbb{P}_2$

- Proof of $\mathrm{Span}(U) \subset \mathbb{P}_2$
  Since $\frac{1}{2}, t^2 + t, t^2 - 1 \in \mathrm{Span}(\{t^0, t^1, t^2\})$ it follows that $\mathrm{Span}(U) \subset \mathbb{P}_2$

- Proof of $\mathbb{P}_2 \subset \mathrm{Span}(U)$

$$\begin{aligned}
t^0 &= 2 \cdot \frac{1}{2} \\
t^1 &= (t^2 + t) + (-1) \cdot (t^2 - 1) + (-2) \cdot \frac{1}{2} \\
t^2 &= (t^2 - 1) + 2 \cdot \frac{1}{2} \\
&\Rightarrow \{t^0, t^1, t^2\} \subset \mathrm{Span}(U) \Rightarrow \mathbb{P}_2 \subset \mathrm{Span}(U)
\end{aligned}$$

$\Rightarrow \mathbb{P}_2 = \mathrm{Span}(U)$

2. V is a spanning set of $\mathbb{P}_2$

- Proof of $\mathrm{Span}(V) \subset \mathbb{P}_2$
  Since $V \subset \mathrm{Span}(\{t^0, t^1, t^2\})$ it follows that $\mathrm{Span}(V) \subset \mathbb{P}_2$

- Proof of $\mathbb{P}_2 \subset \mathrm{Span}(V)$

$$\begin{aligned}
t^0 &\in V \\
t^1 &= \frac{1}{2} \cdot 2t \\
t^2 &\in V \\
&\Rightarrow \{t^0, t^1, t^2\} \subset \mathrm{Span}(V) \Rightarrow \mathbb{P}_2 \subset \mathrm{Span}(V)
\end{aligned}$$

$\Rightarrow \mathbb{P}_2 = \mathrm{Span}(V)$

3. W is not a spanning set of $\mathbb{P}_2$
   Assume the opposite: $\mathrm{Span}(W) = \mathbb{P}_2$

$$\Rightarrow \exists a_0, a_1 \in \mathbb{R},\ a_0(t+1) + a_1(t^2+t) = t^0$$
$$a_1 t^2 + (a_0 + a_1)t + a_0 = 1$$
$$\text{Since the RHS has no } t^2 \Rightarrow a_1 = 0$$
$$\Rightarrow a_0 t = 1$$
$$\text{Contradiction}$$

Exercise 2

Which of the following are linearly independent?

1. $U = \{1+t,\ 1+t+t^2,\ 1+t+t^2+t^3,\ 1+t+t^2+t^4\}$
2. $V = \{\sin,\ \cos^2,\ \sin^3\}$
3. $W = \{1,\ \sin^2,\ \cos^2\}$

Solution

1. U is linearly independent
   Assume the opposite: U is linearly dependent

$$\Rightarrow \exists a_0, a_1, a_2, a_3 \in \mathbb{R}, a_0(1+t) + a_1(1+t+t^2) + a_2(1+t^2+t^3) + a_3(1+t+t^2+t^4) = 0$$
$$a_3 t^4 + a_2 t^3 + (a_1 + a_2 + a_3)t^2 + (a_0 + a_1 + a_3)t^+(a_0 + a_1 + a_2 + a_3) = 0$$
$$\text{Since the RHS has no } t^4 \Rightarrow a_3 = 0$$
$$\Rightarrow a_2 t^3 + (a_1 + a_2)t^2 + (a_0 + a_1)t + (a_0 + a_1 + a_2) = 0$$
$$\text{Since the RHS has no } t^3 \Rightarrow a_2 = 0$$
$$\Rightarrow a_1 t^2 + (a_0 + a_1)t + (a_0 + a_1 + a_2) = 0$$
$$\text{Since the RHS has no } t^2 \Rightarrow a_1 = 0$$
$$\Rightarrow a_1 t + (a_0 + a_1) = 0$$
$$(\ldots) \Rightarrow a_1, a_0 = 0$$
$$\text{Contradiction as not all } a \text{ can be equal to } 0$$

2. V is linearly independent
   Assume the opposite: V is linearly dependent

$$\Rightarrow \exists a_0, a_1, a_2 \in \mathbb{R},\ a_0 \sin x + a_1 \cos^2 x + a_2 \sin^3 x = 0$$
$$a_0 \sin x + a_1\ cos^2 x + a_2(\sin x - \sin x \cos^2 x) = 0$$
$$(a_0 + a_2)\sin x + \cos^2 x(a_1 - a_2 \sin x) = 0$$
$$\text{Since the RHS has no } \sin x \Rightarrow a_0 = -a_2$$
$$\Rightarrow \cos^2 x(a_1 + a_0 \sin x) = 0$$
$$\forall x \in \mathbb{R}, \cos^2 x \neq 0 \Rightarrow a_1 + a_0 \sin x = 0$$
$$\text{Since the RHS has no } \sin x \Rightarrow a_0 = 0$$
$$\Rightarrow a_1 = 0$$
$$\text{Contradiction as not all } a \text{ can be equal to } 0$$

3. W is linearly dependent
   $\sin^2 x + \cos^2 x = 1$

Exercise 3

Find the basis and dimension of $N(A) \subset \mathbb{R}^5$ where

$$A = \begin{pmatrix} 1 & -3 & 3 & -1 & -1 \\ -2 & 6 & -1 & -3 & -8 \\ 3 & -9 & 10 & -4 & -5 \end{pmatrix} \in M_{3\times 5}(\mathbb{R})$$

Solution

Solving for $N(A) : A\mathbf{x} = 0$

$$\left(\begin{array}{ccccc|c} 1 & -3 & 3 & -1 & -1 & 0 \\ -2 & 6 & -1 & -3 & -8 & 0 \\ 3 & -9 & 10 & -4 & -5 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccccc|c} 1 & -3 & 3 & -1 & -1 & 0 \\ 0 & 0 & 5 & -5 & -10 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccccc|c} 1 & -3 & 3 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 5 & -5 & -10 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 5R_2]{R_1 \to R_1 - 3R_2} \left(\begin{array}{ccccc|c} 1 & -3 & 0 & -4 & 5 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & -10 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{10} R_3} \left(\begin{array}{ccccc|c} 1 & -3 & 0 & -4 & 5 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 + 4R_3} \left(\begin{array}{ccccc|c} 1 & -3 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 - 3x_2 + 5x_5 = 0 \\ x_3 - 2x_5 = 0 \\ x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = 3x_2 - 5x_5 \\ x_3 = 2x_5 \\ x_4 = 0 \end{cases}$$

$$\text{Let } x_5 = u, x_2 = v \text{ hence } N(A) = \{u, v \in \mathbb{R} : \begin{pmatrix} 3v - 5u \\ v \\ 2u \\ 0 \\ u \end{pmatrix}\}$$

$$\text{Basis of } N(A) \text{ is } \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \text{ the dimension of } N(A) \text{ is } 2$$

Exercise 4

1. Do the following matrices form a basis of $M_{2\times 2}(\mathbb{R})$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$

2. Find the basis and dimension of $W = \{A \in M_{2\times 2}(\mathbb{R}) |\, \text{trace}(A) = 0\}$

Solution

1. A_1, A_2, A_3, A_4 form a basis of $M_{2\times 2}(\mathbb{R})$

- A_1, A_2, A_3, A_4 are linearly independent
  Assume the opposite:

$$\exists a_1, a_2, a_3, a_4 \in \mathbb{R}, a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 = 0_{2\times 2}$$
$$\begin{pmatrix} a_1 + 2a_2 + 3a_3 + 4a_4 & a_2 + 2a_3 + 3a_4 \\ a_3 + 2a_4 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow a_4 = 0 \Rightarrow \begin{pmatrix} a_1 + 2a_2 + 3a_3 & a_2 + 2a_3 \\ a_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow a_3 = 0 \Rightarrow \begin{pmatrix} a_1 + 2a_2 & a_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow a_2 = 0 \Rightarrow \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$a_1 = 0$$
$$\text{Contradiction as not all } a \text{ can be equal to } 0$$

- $\text{Span}(A_1, A_2, A_3, A_4) = M_{2\times 2}(\mathbb{R})$
  - $\text{Span}(A_1, A_2, A_3, A_4) \subset M_{2\times 2}(\mathbb{R})$
    Since matrix addition and scalar multiplication is closed in that matrix size it follows that $\text{Span}(A_1, A_2, A_3, A_4) \subset M_{2\times 2}(\mathbb{R})$
  - $M_{2\times 2}(\mathbb{R}) \subset \text{Span}(A_1, A_2, A_3, A_4)$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A_1$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2A_1 + A_1$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = A_1 - 2A_1 + A_3$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A_2 - 2A_3 + A_4$$
$$\text{Span}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \subset \text{Span}(A_1, A_2, A_3, A_4)$$
$$\Rightarrow M_{2\times 2}(\mathbb{R}) \subset \text{Span}(A_1, A_2, A_3, A_4)$$

2. Finding the basis and dimension of W

$$\forall A \in W, \ \text{trace}(A) = 0$$
$$\forall A \in W, \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ where } a + d = 0 \Rightarrow d = -a \quad (a, b, c, d \in \mathbb{R})$$
$$\forall A \in W, A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$\Rightarrow \text{Span}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = W$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ are linearly independent as none share an element in any spot}$$
$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ form a basis of } W \text{ hence } \dim(W) = 3$$