Linear Algebra II (MATH1049) — Coursework Sheet 2 — 2020/21

Submit a single pdf with scans of your work to Blackboard by Monday, 22 February 2021, 17:00.

Exercise 1

Write down the group table for the permutation group S_3 and show that S_3 is not abelian. (*You may find it more convenient to write all elements of S_3 in cycle notation.*)

Exercise 2

$$\text{Let} \quad \sigma := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 6 & 1 & 8 & 9 & 4 & 2 & 5 \end{pmatrix} \in S_9, \qquad \tau := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 6 & 3 & 1 \end{pmatrix} \in S_6$$

$$\text{and} \quad \eta := \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix} \in S_n \quad \text{(for any even } n \in \mathbb{N}\text{)}.$$

(a) Determine the sign of σ, τ and η.

(b) Write σ^2, σ^{-1}, τ^2, τ^{-1}, η^2 and η^{-1} as a composition of cycles.

(c) Determine the sign of σ^2, τ^2 and η^2 in two ways, firstly using (b) and secondly using (a) and Theorem 1.10 (b).

Exercise 3

Let $n \geq 1$. Let $\langle a_1, \dots, a_s \rangle \in S_n$ be a cycle and let $\sigma \in S_n$ be arbitrary. Show that

$$\sigma \circ \langle a_1, \dots, a_s \rangle \circ \sigma^{-1} = \langle \sigma(a_1), \dots, \sigma(a_s) \rangle \text{ in } S_n.$$

(*Note this is an equality between maps. Hence, in order to show this equality you need to show that both sides are equal after applying them to an arbitrary element b of $\{1, 2, \dots, n\}$. To do so you will need to distinguish whether b belongs to $\{\sigma(a_1), \dots, \sigma(a_s)\}$ or not.*)

Exercise 4

Let $\mathbb{Q}(\sqrt{5})$ denote the set of real numbers z of the form $z = a + b\sqrt{5}$ where $a, b \in \mathbb{Q}$. Show that $\mathbb{Q}(\sqrt{5})$ together with the usual addition and multiplication of real numbers is a field. (*Hint: You need to show that for any $w, z \in \mathbb{Q}(\sqrt{5})$ also $w + z, wz, -z$ and z^{-1} (if $z \neq 0$) are in $\mathbb{Q}(\sqrt{5})$ and that 0 and 1 are in $\mathbb{Q}(\sqrt{5})$. Distributivity, commutativity and associativity for addition and multiplication hold in $\mathbb{Q}(\sqrt{5})$ because they hold in $\mathbb{R}$.*)

Exercise 5

Let F be a field. For any $a, b \in F$, $b \neq 0$, we write $\frac{a}{b}$ for ab^{-1}. Prove the following statements for any $a, a' \in F$ and $b, b' \in F \backslash \{0\}$:

(i) $\dfrac{a}{b} + \dfrac{a'}{b'} = \dfrac{ab' + a'b}{bb'}$; (ii) $\dfrac{a}{b}\dfrac{a'}{b'} = \dfrac{aa'}{bb'}$.

Extra items for Exercise 5 (not assessed, do not submit):

(iii) $\dfrac{a}{b} = \dfrac{a'}{b'}$ if and only if $ab' = a'b$;

(iv) $\dfrac{\frac{a}{b}}{\frac{a'}{b'}} = \dfrac{ab'}{a'b}$ (if in addition $a' \neq 0$).

Extra problems to think about (do not submit)

The solutions for this will not be provided (but possible to find in a book or google). Not necessary for the rest of the module at all. Feel free to ignore.

Task 1. The aim is to prove Thm 1.10 from the notes, about the sign function on the symmetric groups S_n. Here's one possible path to a proof.

- Every cycle of length k can be written as a product of $k-1$ transpositions.
- Thus, every permutation can be written as a product of transpositions.
- Let σ be a permutation, and write it as a product of transpositions. Define the number $\mathrm{nsgn}(\sigma)$ (for "new sign' ') to be equal to 1 if the number of transpositions is even, and -1 if the number of transpositions is odd. Again, apriori nsgn depends on *how* do we write σ as a product of transpositions. However, by the first point above, $\mathrm{nsgn}(\sigma) = \mathrm{sgn}(\sigma)$, since every cycle decomposition of σ gives also a way to write σ as a product of transpositions. So the goal now is to prove that nsgn is well defined, and that it's multiplicative.
- A way to prove the above is to find a way to characterise nsgn to be something *intrinsic* to a permutation. Here's such a thing: Given a permutation $\sigma \in S_n$, we say that σ *reverses the pair* (i, j), if $i, j \in \{1, \ldots, n\}$, $i < j$ and $\sigma(i) > \sigma(j)$. Let $\mathrm{isgn}(\sigma)$ be 1 if σ reverses even number of pairs, and -1 if σ reverses odd number of pairs.
- Prove that if σ is a permutation and τ is a transposition, then $\mathrm{isgn}(\sigma \circ \tau) = -\mathrm{isgn}(\sigma) = \mathrm{isgn}(\tau \circ \sigma)$.
- From the previous point, conclude that $\mathrm{isgn}(\sigma) = \mathrm{nsgn}(\sigma)$ (thus the sign is well defined).
- From the definition of nsgn, show that $\mathrm{nsgn}(\sigma \circ \tau) = \mathrm{nsgn}(\sigma)\mathrm{nsgn}(\tau)$ for any two permutations σ and τ.

Task 2. Prove that the number of elements of S_n (i.e. the order of the symmetric group S_n) is $n!$.