Linear Algebra II (MATH1049) — Coursework Sheet 3 — 2020/21

Exercise 1

Define $a \otimes \mathbf{x} \equiv \binom{ax_1}{0}$

$$\otimes : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 : (a, \mathbf{x}) \mapsto a \otimes \mathbf{x} \text{ where } a \in \mathbb{R}$$

and determine which of the properties of a vector space hold when $\otimes$ is used as scalar multiplication in $\mathbb{R}^2$

solution

1. (**1st distribuitivity law**) $\forall a, b \in \mathbb{R}, \forall x \in \mathbb{R}^2,\ (a+b) \otimes x = a \otimes x + b \otimes x \in \mathbb{R}^2$

$$(a+b) \otimes x = a \otimes x + b \otimes x$$
$$x \in \mathbb{R}^2 \Rightarrow x \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ where } x_1, x_2 \in \mathbb{R} \Rightarrow$$
$$\begin{aligned}
\begin{pmatrix} (a+b)x_1 \\ 0 \end{pmatrix} &= \begin{pmatrix} ax_1 \\ 0 \end{pmatrix} + \begin{pmatrix} bx_1 \\ 0 \end{pmatrix} && (\text{Definition of } \otimes) \\
\begin{pmatrix} (a+b)x_1 \\ 0 \end{pmatrix} &= \begin{pmatrix} ax_1 + bx_1 \\ 0 \end{pmatrix} && (\text{Addition of } \mathbb{R}^2) \\
\begin{pmatrix} (a+b)x_1 \\ 0 \end{pmatrix} &= \begin{pmatrix} (a+b)x_1 \\ 0 \end{pmatrix} && (\text{Distributivity of } \mathbb{R}) \\
&\checkmark
\end{aligned}$$

Since $\mathbb{R}$ is closed under addition and multiplication $(a+b)x_1, 0 \in \mathbb{R} \Rightarrow \binom{(a+b)x_1}{0} \in \mathbb{R}^2$ (since $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$)

2. (**2nd distribuitivity law**) $\forall a \in \mathbb{R}, \forall x, y \in \mathbb{R}^2,\ a \otimes (x+y) = a \otimes x + a \otimes y \in \mathbb{R}^2$

$$a \otimes (x+y) = a \otimes x + a \otimes y$$
$$x, y \in \mathbb{R}^2 \Rightarrow x \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y \equiv \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ where } x_1, x_2, y_1, y_2 \in \mathbb{R} \Rightarrow$$
$$\begin{aligned}
a \otimes \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) &= a \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + a \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
a \otimes \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} &= a \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + a \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} && (\text{Addition of } \mathbb{R}^2) \\
\begin{pmatrix} a(x_1 + y_1) \\ 0 \end{pmatrix} &= \begin{pmatrix} ax_1 \\ 0 \end{pmatrix} + \begin{pmatrix} ax_2 \\ 0 \end{pmatrix} && (\text{Definition of } \otimes) \\
\begin{pmatrix} a(x_1 + y_1) \\ 0 \end{pmatrix} &= \begin{pmatrix} ax_1 + ay_1 \\ 0 \end{pmatrix} && (\text{Addition of } \mathbb{R}^2) \\
\begin{pmatrix} a(x_1 + y_1) \\ 0 \end{pmatrix} &= \begin{pmatrix} a(x_1 + y_1) \\ 0 \end{pmatrix} && (\text{Distributivity of } \mathbb{R}) \\
&\checkmark
\end{aligned}$$

Since $\mathbb{R}$ is closed under addition and multiplication $a(x_1+y_1), 0 \in \mathbb{R}^2 \Rightarrow \binom{a(x_1+y_1)}{0} \in \mathbb{R}^2$

3. (**Associativity**) $\forall a, b \in \mathbb{R}, \forall x \in \mathbb{R}^2,\ (ab)x = a(bx) \in \mathbb{R}^2$

$$\begin{aligned}
(ab) \otimes x &= a \otimes (b \otimes x) \\
x \in \mathbb{R}^2 \Rightarrow x &\equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ where } x_1, x_2 \in \mathbb{R} \Rightarrow \\
\begin{pmatrix} (ab)x_1 \\ 0 \end{pmatrix} &= a \otimes \begin{pmatrix} bx_1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} (ab)x_1 \\ 0 \end{pmatrix} &= \begin{pmatrix} a(bx_1) \\ 0 \end{pmatrix} && \text{(Definition of } \otimes \text{)} \\
\begin{pmatrix} (ab)x_1 \\ 0 \end{pmatrix} &= \begin{pmatrix} (ab)x_1 \\ 0 \end{pmatrix} && \text{(Associativity of } \mathbb{R}\text{)} \\
&\checkmark
\end{aligned}$$

Since $\mathbb{R}$ is closed under addition and multiplication $(ab)x_1, 0 \in \mathbb{R}^2 \Rightarrow \binom{(ab)x_1}{0} \in \mathbb{R}^2$

4. (**Identity**) $\forall x \in \mathbb{R}^2, 1 \otimes x = x \in \mathbb{R}^2$
   There must exist sum $a \in \mathbb{R}$ for which the property holds

$$\begin{aligned}
a \otimes x &= x \\
x \in \mathbb{R}^2 \Rightarrow x &\equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ where } x_1, x_2 \in \mathbb{R} \Rightarrow \\
\begin{pmatrix} ax_1 \\ 0 \end{pmatrix} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \\
a = 1,\ x_2 &= 0 \qquad (1.1) \\
\Rightarrow \text{does not hold for all } x &\in \mathbb{R}^2
\end{aligned}$$

Counter example:

$$\begin{aligned}
\text{Let } x &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
(1.1) \Rightarrow a &= 1 \\
\Rightarrow 1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 0 \end{pmatrix} &\neq \begin{pmatrix} 1 \\ 1 \end{pmatrix} && \text{(Definition of } \otimes \text{)}
\end{aligned}$$

Exercise 2

Show that $\otimes$ (as the scalar) and $\oplus$ in $\mathbb{R}^n_{>0}$ form a vector space over $\mathbb{Q}$ where $a \otimes b \equiv (b_1^a, \ldots, b_n^a)$ and $a \oplus b \equiv (a_1 b_1, \ldots, a_n b_n)$

$$\otimes : \mathbb{Q} \times \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0} : (a, \mathbf{b}) \mapsto a \otimes \mathbf{b}$$
$$\oplus : \mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0} : (\mathbf{a}, \mathbf{b}) \mapsto a \oplus b$$

solution

1. (**1st distribuitivity law**) $\forall a, b \in \mathbb{Q}, \forall x \in \mathbb{R}^n_{>0},\ (a \oplus b) \otimes x = a \otimes x \oplus b \otimes x \in \mathbb{R}^n_{>0}$

$$(a \oplus b) \otimes x = a \otimes x \oplus b \otimes x$$
$$x \in \mathbb{R}^n_{>0} \Rightarrow x \equiv (x_1, \ldots, x_n) \text{ where } \forall i \in [n],\ x_i \in \mathbb{R}_{>0}$$
$$(ab) \otimes x = (x_1^a, \ldots, x_n^a) \oplus (x_1^b, \ldots, x_n^b)$$
$$(x_1^{ab}, \ldots, x_n^{ab}) = (x_1^a x_1^b, \ldots, x_n^a x_n^b)$$
$$(x_1^{ab}, \ldots, x_n^{ab}) = (x_1^{ab}, \ldots, x_n^{ab})$$
$$\checkmark$$

Closure

$$\forall ab \in \mathbb{Q}, \forall x \in \mathbb{R}_{>0}\ x^{ab} \in \mathbb{R}_{>0} \Rightarrow$$
$$(x_1^{ab}, \ldots, x_n^{ab}) \in \mathbb{R}^n_{>0} \text{ where } \forall i \in [n],\ x_i \in \mathbb{R}_{>0}$$

2. (**2nd distribuitivity law**) $\forall a \in \mathbb{Q}, \forall x, y \in \mathbb{R}^n_{>0},\ a \otimes (x \oplus y) = a \otimes x \oplus a \otimes y \in \mathbb{R}^n_{>0}$

$$a \otimes (x \oplus y) = a \otimes x \oplus a \otimes y$$
$$x, y \in \mathbb{R}^n_{>0} \Rightarrow x \equiv (x_1, \ldots, x_n), y \equiv (y_1, \ldots, y_n) \text{ where } \forall i \in [n],\ x_i, y_i \in \mathbb{R}_{>0}$$
$$a \otimes (x_1 y_1, \ldots, x_n y_n) = (x_1^a, \ldots, x_n^a) \oplus (y_1^a, \ldots, y_n^a)$$
$$((x_1 y_1)^a, \ldots, (x_n y_n)^a) = (x_1^a y_1^a, \ldots, x_n^a y_n^a)$$
$$((x_1 y_1)^a, \ldots, (x_n y_n)^a) = ((x_1 y_1)^a, \ldots, (x_n y_n)^a)$$
$$\checkmark$$

Closure

$$x, y \in \mathbb{R}_{>0} \Rightarrow xy \in \mathbb{R}_{>0}$$
$$\forall a \in \mathbb{Q},\ \forall xy \in \mathbb{R}_{>0},\ (xy)^a \in \mathbb{R}_{>0} \Rightarrow$$
$$((x_1 y_1)^a, \ldots, (x_n y_n)^a) \in \mathbb{R}^n_{>0} \text{ where } \forall i \in [n],\ x_i, y_i \in \mathbb{R}_{>0}$$

3. (**Associativity**) $\forall a, b \in \mathbb{Q}, \forall x \in \mathbb{R}^n_{>0},\ (a \otimes b) \otimes x = a \otimes (b \otimes x) \in \mathbb{R}^n_{>0}$

$$(a \otimes b) \otimes x = a \otimes (b \otimes x)$$
$$x \in \mathbb{R}^n_{>0} \Rightarrow x \equiv (x_1, \ldots, x_n) \text{ where } \forall i \in [n],\ x_i \in \mathbb{R}_{>0}$$
$$(b^a) \otimes x = a \otimes (x_1^b, \ldots, x_n^b)$$
$$(x_1^{b^a}, \ldots, x_n^{b^a}) = \left(\left(x_1^b\right)^a, \ldots, \left(x_n^b\right)^a\right)$$
$$(x_1^{b^a}, \ldots, x_n^{b^a}) = (x_1^{b^a}, \ldots, x_n^{b^a})$$
$$\checkmark$$

Closure

$$a, b \in \mathbb{Q} \Rightarrow b^a \in \mathbb{R}$$
$$\forall x \in \mathbb{R}_{>0},\ \forall b^a \in \mathbb{R},\ x^{b^a} \in \mathbb{R}_{>0} \Rightarrow$$
$$(x_1^{b^a}, \ldots, x_n^{b^a}) \in \mathbb{R}^n_{>0} \text{ where } \forall i \in [n],\ x_i \in \mathbb{R}_{>0} \text{ and } a, b \in \mathbb{Q}$$

4. (**Identity**) $\forall x \in \mathbb{R}^n_{>0}, 1 \otimes x = x \in \mathbb{R}^n_{>0}$

$$1 \otimes x = x$$
$$x \in \mathbb{R}^n_{>0} \Rightarrow x \equiv (x_1, \ldots, x_n) \text{ where } \forall i \in [n],\ x_i \in \mathbb{R}_{>0}$$
$$\Rightarrow (x_1^1, \ldots, x_n^1) = (x_1, \ldots, x_n)$$
$$(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$$
$$\checkmark$$

Exercise 3

Given V is a vector space over F show:

1. $\forall a \in F,\ \forall x, y \in V,\ a(x-y) = ax - ay \in V$
2. $\forall a \in F,\ \forall x \in V, (ax = 0_V \Rightarrow a = 0_F \cup x = 0_V)$

solution

1. $\forall a \in F,\ \forall x, y \in V,\ a(x-y) = ax - ay \in V$

$$\begin{aligned}
a(x-y) &= ax - ay & \\
ax + a(-y) &= ax - ay & \text{(2nd Distributivity Law)} \\
ax + a(-y) + 0_V &= ax - ay & \text{(Definition of neutral element)} \\
ax + a(-y) + ay - ay &= ax - ay & \text{(Defintion of inverse)} \\
ax + a(-y + y) - ay &= ax - ay & \text{(2nd Distributivity Law)} \\
ax + a0_V - ay &= ax - ay & \text{(Definition of inverse)} \\
ax - ay &= ax - ay & \text{(Definition of neutral element)} \\
&\checkmark &
\end{aligned}$$

2. $\forall a \in F,\ \forall x \in V, (ax = 0_V \Rightarrow a = 0_F \cup x = 0_V)$

Assume the opposite $\Rightarrow \exists a \in F,\ \exists x \in V, (ax = 0_V \Rightarrow a \neq 0_F \cup x \neq 0_V)$
which is the same as $\exists a \in F \backslash \{0_F\},\ \exists x \in V \backslash \{0_V\}, ax = 0_V$

$$\begin{aligned}
\text{Let } b \in F \backslash \{0_F\},\ x \in V \backslash \{0_V\} & \\
bx - bx = 0_V & \quad \text{(Definition of inverse)} \\
0_V + bx - bx = 0_V & \quad \text{(Definition of neutral element)} \\
ax + bx - bx = 0_V & \quad \text{(Assumption)} \\
(a+b)x - bx = 0_V & \quad \text{(1st Distributivity Law)} \\
\text{(from assumption) } a \neq 0_F \Rightarrow a + b \neq b &
\end{aligned}$$

$\Rightarrow (a+b)x$ has two different inverses $-(a+b)x$ and $-bx$
(Prop. 1.3) $\Rightarrow$ Contradiction, a group's element cannot have two inverse

Exercise 4

Show that V^S is a vector space over F by checking: the additive inverse, the additive identity and 2nd distributivity law. Addition and scalar multiplication are defined as:

$$\forall f, g \in V^S,\ \forall s \in S,\ (f+g)(s) \equiv f(s) + g(s)$$
$$\forall f, g \in V^S,\ \forall s \in S, \forall a \in F,\ (af)(s) \equiv a(f(s))$$

Note:S is a set and V is a vector space over a field F.
Note 2: $+$ and multiplication are the operations in vector space V

solutions

1. (**Additive identity**) $\forall f \in V^S,\ \forall s \in S,\ (f + \bar{0})(s) = (\bar{0} + f)(s) = f(s)$ where $\bar{0}(s)$ is the identity

$$\begin{aligned}
&V \text{ is a vector space over } F \\
&\quad \Rightarrow \exists 0_V \in V,\ \forall a \in V, a + e = e + a = a \quad \text{(Identity existence for groups)} \\
&\quad \Rightarrow \text{Let } \forall s \in S, \bar{0}(s) = 0_V \\
&\quad \Rightarrow \bar{0} : S \to V \\
&\quad \Rightarrow \bar{0} \in V^S
\end{aligned}$$

$$
\begin{aligned}
(f+\bar{0})(s) = (\bar{0}+f)(s) &= f(s) \\
f(s)+\bar{0}(s) = \bar{0}(s)+f(s) &= f(s) && \text{(Definition of } +) \\
f: S \to V, \bar{0}: S \to V \Rightarrow f(s)+\bar{0}(s) &= f(s) && \text{(Associativity in groups)} \\
f(s)+0_V &= f(s) && \text{(Definition of } \bar{0}) \\
f: S \to V \Rightarrow f(s) &= f(s) && \text{(Definition of Identity)} \\
&\checkmark
\end{aligned}
$$

2. (**Additive inverse**) $\forall f \in V^S,\ \exists g \in V^S,\ \forall s \in S,\ (f+g)(s) = \bar{0}(s)$

$$
\begin{aligned}
& V \text{ is a vector space over } F \\
& \quad \Rightarrow \forall v \in V, \exists(-v) \in V, v+(-v) = 0_V && \text{(Inverese existence for groups)} \\
& \quad \Rightarrow \text{Let } \forall f \in V^S, \forall s \in S, (-f)(s) = -f(s) \\
& \quad \Rightarrow (-f): S \to V \\
& \quad \Rightarrow (-f) \in V^S
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } g &= (-f) \\
\Rightarrow (f+(-f))(s) &= \bar{0}(s) \\
f(s)+(-f)(s) &= \bar{0}(s) && \text{(Definition of } +) \\
\forall s \in S, f(s)-f(s) &= \bar{0}(s) && \text{(Definition of } (-f)) \\
\bar{0}(s) &= \bar{0}(s) \\
&\checkmark
\end{aligned}
$$

3. (**2nd Distributivity Law**)$\forall a \in F,\ \forall f, g \in V^S,\ \ \forall s \in S,\ (a(f+g))(s) = (af+ag)(s) \in V^S$

$$
\begin{aligned}
(a(f+g))(s) &= (af+ag)(s) \\
(a(f+g))(s) &= (af)(s)+(ag)(s) && \text{(Definition of } +) \\
a((f+g)(s)) &= a(f(s))+a(g(s)) && \text{(Definition of scalar mult.)} \\
a(f(s)+g(s)) &= a(f(s))+a(g(s)) && \text{(Definition of } +) \\
a(f(s))+a(g(s)) &= a(f(s))+a(g(s)) && \text{(2nd Distributivity Law for } V) \\
&\checkmark
\end{aligned}
$$