# NAIL062 Propositional and Predicate Logic: Lecture Notes

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The lecture is based on the textbook *A. Nerode, R. Shore: Logic for Applications* [3], with some parts taken from the book *M. Ben-ari: Mathematical Logic for Computer Science* [1]. The course structure and the majority of the content are borrowed from the lectures of Petr Gregor from previous years [2]; see also the lecture notes by Martin Pilát [4]. This version was translated from the original Czech version with the assistance of an LLM-based translator. If you find any typos or other errors, or if there are any parts that are difficult to understand, please let me know.

# Contents

# Chapter 1

# Introduction to Logic

The word *logic* is used in two senses:

- A set of principles that underlie the organization of elements within a system (e.g., a computer program, an electronic device, a communication protocol)
- Reasoning conducted according to strict rules that preserve validity

In computer science, these two meanings converge: first, we formally describe the given system, and then we *formally reason* about it (in practical applications, this is done automatically), i.e., we derive valid *inferences* about the system using some *(formal) proof system*.

Practical applications of logic in computer science include software verification, logic programming, SAT solving, automated reasoning, database theory, knowledge-based representation, and many others. Moreover, logic (to a greater extent than mathematics) is a fundamental tool for describing theoretical computer science.

## 1.1 Propositional Logic

Now let's demonstrate logic in action with two real-life examples (from the lives of a treasure hunter and a theoretical computer scientist):

### 1.1.1 Example: Treasure Hunting

*Example* 1.1.1. While searching for treasure in a dragon's lair, we encountered a fork in the path with two corridors. We know that at the end of each corridor, there is either treasure or a dragon, but not both. A dwarf we met at the fork told us, "At least one of these two corridors leads to the treasure," and after some further urging (and a small bribe), she also said, "The first corridor leads to a dragon." It is well known that dwarves you meet in a dragon's lair either always tell the truth or always lie. Which way should we go?

### 1.1.2 Formalization in Propositional Logic

We will begin by formalizing the situation and our knowledge in propositional logic. A *proposition* is a statement to which we can assign a truth value: *True* (1) or *False* (0).

Some propositions can be expressed using simpler propositions and logical connectives, e.g., "(The dwarf is lying) *if and only if* (the second corridor leads to a dragon)" or "(The first corridor leads to treasure) *or* (the first corridor leads to a dragon)." If a proposition cannot be decomposed in this way, it is called a *simple proposition*, *atomic proposition*, or *propositional variable*.

Thus, we will describe the entire situation using *propositional variables*. We can also think of these as simple yes/no questions we need to answer to know everything about the given situation. Let's choose "The treasure is in the first corridor" (denote as $p_1$), and "The treasure is in the second corridor" ($p_2$). Other propositional variables could be considered, such as "There is a dragon in the first corridor" ($d_1$) or "The dwarf is telling the truth" ($t$). However, these can be expressed using $\{p_1, p_2\}$, e.g., $t$ holds if and only if $p_1$ does not hold. That is, if we know the truth values of $p_1, p_2$, the truth values of $d_1, t$ are uniquely determined. A smaller number of propositional variables means a smaller search space.

Next, we will express all our knowledge as (*compound*) propositions and write them in formal notation in the *language* of propositional logic over the set of atomic propositions $\mathbb{P} = \{p_1, p_2\}$, using symbols representing logical connectives: $\neg$ ("not X", *negation*), $\wedge$ ("X and Y", *conjunction*), $\vee$ ("X or Y", *disjunction*), $\to$ ("if X, then Y", *implication*), $\leftrightarrow$ ("X if and only if Y", *equivalence*), and parentheses (, ). It is worth mentioning that disjunction is not exclusive; that is, "X or Y" holds even if both X and Y hold, and implication is purely logical: "if X, then Y" holds whenever X does not hold or Y holds.

The information that the corridor contains either a treasure or a dragon, but not both, is already encoded in our choice of propositional variables: the presence of a dragon is the same as the absence of treasure. The dwarf's statement that "The first corridor leads to a dragon" is thus expressed as "It is not the case that the treasure is in the first corridor," formally $\neg p_1$. The statement that "At least one of these two corridors leads to the treasure" is expressed as "The treasure is in the first corridor or the treasure is in the second corridor," formally $p_1 \vee p_2$. The information that dwarves either always tell the truth or always lie can be understood to mean that either both our propositions hold or the negations of both our propositions hold, formally:

$$(\neg p_1 \wedge (p_1 \vee p_2)) \vee (\neg(\neg p_1) \wedge \neg(p_1 \vee p_2))$$

Let us denote this proposition as $\varphi$ (from the word "formula"; propositions are sometimes also called *propositional formulas*). In our example, all information can be expressed with a single proposition. But in practice, we often need more propositions, sometimes even infinitely many (for example, if we want to describe the execution of a computer program and we do not know in advance how many steps it will take). We then describe the situation using a set of propositions, called a *theory*, here $T = \{\varphi\}$. The propositions in $T$ are also called *axioms* of the theory $T$[1].

### 1.1.3 Models and Consequences

Is our information sufficient to determine whether there is treasure in one particular corridor? In other words, we are asking if one of the propositions $p_1$ or $p_2$ is a logical *consequence* of the proposition $\varphi$ (or the theory $T$). What does this mean?

Let's imagine that there are several "worlds" differing in what is at the end of the first and second corridors. For example, in one of the worlds, there is treasure at the end of the first

[1] Terminology in logic often comes from its application in mathematics.

corridor and a dragon at the end of the second corridor. We can describe this world using a truth valuation of the propositional variables: $p_1 = 1, p_2 = 0$. Such a valuation is called a *model* of the language $\mathbb{P} = \{p_1, p_2\}$ and we write it succinctly as $v = (1, 0)$ ($v$ from the word "valuation"). Thus, we have a total of four different worlds, described by the *models of the language*:

$$\mathrm{M}_{\mathbb{P}} = \{(0,0), (0,1), (1,0), (1,1)\}.$$

Is the world described by the model $v = (1, 0)$ consistent with the information we have, i.e., does the proposition $\varphi$ (or the theory $T$) *hold* in it? We can easily determine the truth value of the (compound) proposition $\varphi$ in the model $v$, denote it $v(\varphi)$: We know that $v(p_1) = 1$ and $v(p_2) = 0$, so $v(\neg p_1) = 0$, and also $v(\neg p_1 \wedge (p_1 \vee p_2)) = 0$ (it is a conjunction of two propositions, and the first conjunct is false in the model $v$). Similarly, $v(p_1 \vee p_2) = 1$ (because $v(p_1) = 1$), so $v(\neg(p_1 \vee p_2)) = 0$, and $v(\neg(\neg p_1) \wedge \neg(p_1 \vee p_2)) = 0$. The proposition $\varphi$ is a disjunction of two propositions, neither of which holds in the model $v$, so $v(\varphi) = 0$.

A keen reader will surely see the tree structure of the proposition $\varphi$ and the step-by-step evaluation of $v(\varphi)$ from the leaves to the root. We will present a formal definition in the next chapter.

Similarly, we determine the truth values of the proposition $\varphi$ in other models. We find that the set of *models of the proposition* $\varphi$ (or the *models of the theory* $T$), i.e., the set of all models of the language in which $\varphi$ (or all axioms of the theory $T$) holds, is

$$\mathrm{M}_{\mathbb{P}}(\varphi) = \mathrm{M}_{\mathbb{P}}(T) = \{(0,1)\}.$$

We see that our information uniquely determines the model $(0, 1)$, the world in which there is a dragon in the first corridor and treasure in the second corridor. In general, there can be more models; we only need to know that in every model of $\varphi$ (or of $T$) the proposition $p_2$ holds, to conclude that $p_2$ is a *consequence* of the theory $T$; we also say that $p_2$ *holds* in the theory $T$.

### 1.1.4 Proof Systems

The approach we have chosen is very inefficient. If we have $n$ propositional variables[2], there are $2^n$ models of the language, and it is practically impossible to check the validity of the theory in each of them. This is where *proof systems* come into play. In a given proof system, a *proof* of a proposition $\psi$ from a theory $T$ is a precisely, formally defined syntactic object that includes an easily (mechanically) verifiable "proof" (reason) that $\psi$ holds in $T$, and which can be searched for (using a computer) purely based on the structure of the proposition $\psi$ and the axioms of the theory $T$ ("syntax"), i.e., without having to deal with models ("semantics").

We want two properties from a proof system:

- *soundness*, i.e., if we have a proof of $\psi$ from $T$, then $\psi$ holds in $T$, and
- *completeness*, i.e., if $\psi$ holds in $T$, then there exists a proof of $\psi$ from $T$,

where soundness is a necessity (without it, searching for proofs makes no sense), and completeness is a desirable property, but an efficient proof system can be useful even if not everything that holds can be proven.

[2]In practice, we commonly have thousands of variables.

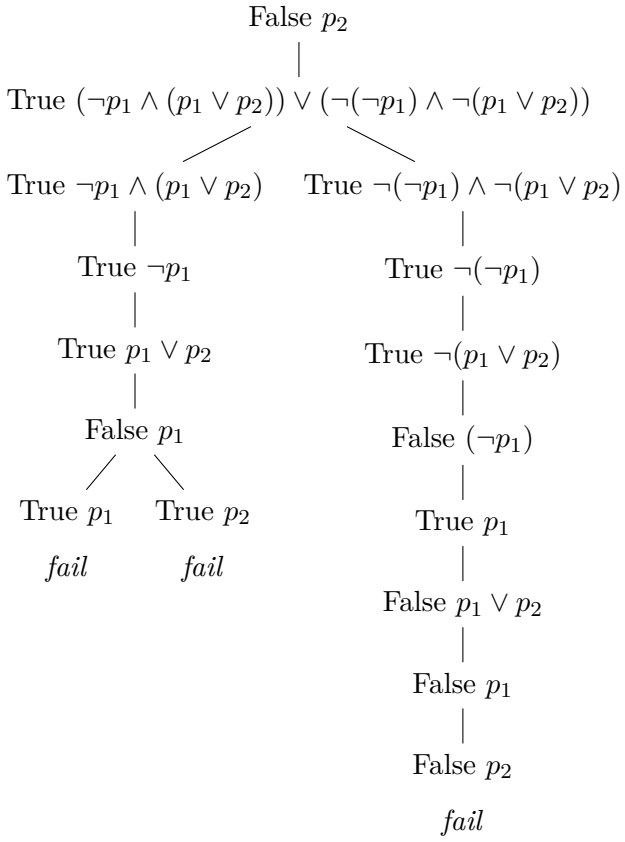



Figure 1.1: Tableau proof of the proposition $p_2$ from the theory $T$

Here we briefly outline two proof systems: the *method of analytic tableaux* and the *resolution method*. They will be formally introduced later, and we will prove soundness and completeness for both of them. Both of these proof systems are based on *proof by contradiction*, i.e., they assume the validity of the axioms from $T$ and the negation of the proposition $\psi$, and try to reach a contradiction.

### 1.1.5 Tableau Method

In the method of analytic tableaux, the 'proof' is a *tableau*: a tree whose nodes are labeled with assumptions about the validity of propositions. Let's look at an example of a tableau in Figure 1.1.

We start with the assumption that the proposition $p_2$ does not hold (because we are proving by contradiction). Then we add the validity of all axioms of the theory $T$ (in our case, there is only one: the proposition $\varphi$ constructed above). We then build the tableau by simplifying the propositions in the assumptions according to certain rules that ensure the following invariant:

> *Every model of the theory $T$ in which $p_2$ does not hold must* agree *with one of the branches of the tableau (i.e., satisfy all assumptions on that branch).*

The proposition $\varphi$ is a disjunction of two propositions, $\varphi = \varphi_1 \vee \varphi_2$. If it holds in some model, then either $\varphi_1$ holds in that model or $\varphi_2$ holds in it. We branch the tree according to these two possibilities. In the next step, we have the assumption that the proposition $\neg p_1 \wedge (p_1 \vee p_2)$ is true. In that case, both $\neg p_1$ and $p_1 \vee p_2$ must hold, so we add both of these assumptions to the end of the branch. The truth of $\neg p_1$ means the falsity of $p_1$, and so on.

We proceed in this manner until it is no longer possible to simplify the propositions in the assumptions, i.e., until they are just propositional variables. If we find a pair of opposite assumptions about some proposition $\psi$ on one branch, i.e., that it both holds and does not hold, we know that no model can agree with this branch. Such a branch is called *contradictory*. Since we are proving by contradiction, a proof is a tableau in which every branch is contradictory. This ensures that there is no model of $T$ in which $p_2$ does not hold. From this, it follows that $p_2$ holds in every model of $T$, in other words, it is a consequence of $T$, which is what we wanted to prove.

For now, we will be satisfied with understanding the basic idea of this method; the details will be presented later in Chapter 4.

### 1.1.6 Resolution Method

It is not difficult to program a systematic search for a tableau proof. In practice, however, another proof system is used that has a much simpler and more efficient implementation: the *resolution method*. This method dates back to 1965 and forms the basis of most *automated reasoning systems*, *SAT solvers*, or Prolog language interpreters (see Subsection **??**).

The resolution method is based on the fact that every proposition can be equivalently expressed in a special form, called *conjunctive normal form (CNF)*. A *literal* is a propositional variable $p$ or its negation $\neg p$ (i.e., literals are propositions that just determine the value of a single propositional variable). A disjunction of several literals, e.g., $p \vee \neg q \vee \neg r$, is called a *clause*. A proposition is in CNF if it is a conjunction of clauses. For every proposition $\psi$, there is an *equivalent* proposition $\psi'$ in CNF. Equivalent means having the same meaning (the same models), which we write as $\psi \sim \psi'$. Later, we will show two methods of conversion to CNF; for now, let's look at our example: in the proposition

$$(\neg p_1 \wedge (p_1 \vee p_2)) \vee (\neg(\neg p_1) \wedge \neg(p_1 \vee p_2))$$

we first replace $\neg(\neg p_1) \sim p_1$ and $\neg(p_1 \vee p_2) \sim (\neg p_1 \wedge \neg p_2)$:

$$(\neg p_1 \wedge (p_1 \vee p_2)) \vee (p_1 \wedge \neg p_1 \wedge \neg p_2)$$

and then repeatedly use the *distributivity* of $\vee$ over $\wedge$ (imagine $\vee$ as multiplication and $\wedge$ as addition):

$$(\neg p_1 \vee p_1) \wedge (\neg p_1 \vee \neg p_1) \wedge (\neg p_1 \vee \neg p_2) \wedge (p_1 \vee p_2 \vee p_1) \wedge (p_1 \vee p_2 \vee \neg p_1) \wedge (p_1 \vee p_2 \vee \neg p_2)$$

This proposition is already in CNF, but we further simplify it: we omit duplicate literals from clauses, and we realize that if a clause contains a pair of opposite literals $p, \neg p$, it is a *tautology* (true in every model), and we can remove it. We obtain the CNF proposition

$$\neg p_1 \wedge (\neg p_1 \vee \neg p_2) \wedge (p_1 \vee p_2)$$

which is equivalent to the original proposition $\varphi$. Since we want to prove $p_2$ by contradiction, we add the clause $\neg p_2$:

$$\neg p_1 \wedge (\neg p_1 \vee \neg p_2) \wedge (p_1 \vee p_2) \wedge \neg p_2$$

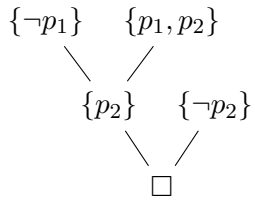
The proposition $p_2$ holds in the theory $T$ if and only if this CNF proposition is *unsatisfiable* (has no model). Propositions in CNF will also be written in *set notation*[3]:

$$\{\{\neg p_1\}, \{\neg p_1, \neg p_2\}, \{p_1, p_2\}, \{\neg p_2\}\}$$

The *resolution rule* states that if we have a pair of clauses, one containing the literal $p$ and the other the opposite literal $\neg p$, then their *resolvent*, that is, the clause formed by removing the literal $p$ from the first clause and $\neg p$ from the second clause and taking the union of the remaining literals, is a logical consequence of the two clauses. For example, from $p \vee \neg q \vee \neg r$ and $\neg p \vee \neg q \vee s$, we can derive the resolvent $\neg q \vee \neg r \vee s$. *Resolution refutation* of a formula in CNF is then a sequence of clauses ending with the *empty clause* $\square$ (indicating a contradiction), where each clause is either from the given CNF formula or it is a resolvent of some two preceding clauses. In our case:

$$\{\neg p_1\}, \{p_1, p_2\}, \{p_2\}, \{\neg p_2\}, \square$$

The third clause is the resolvent of the first and second, and the fifth is the resolvent of the third and fourth. The resolution can also be naturally represented using a *resolution tree*, where the leaves are clauses from the given formula, and the inner nodes are resolvents of their children:



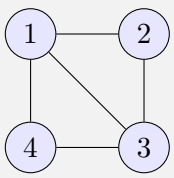

If the CNF formula had a model, the model would have to satisfy all of its clauses, and thus gradually all the resolvents in the sequence, and finally the empty clause. But no model can satisfy the empty clause (a disjunction of zero possibilities). We therefore have a proof by contradiction and thus know that we will find the treasure in the second corridor.

The details of the resolution method will be presented in Chapter 5.

### 1.1.7 Example: Graph Coloring

In the second example, we will get a bit closer to applications. Unlike logical puzzles in the form of word problems, various variants of the graph coloring problem appear in a variety of practical tasks, from scheduling problems to the design of physical and network systems to image processing.

*Example* 1.1.2. Find a vertex coloring of the following graph using three colors, i.e., assign to its vertices the colors R, G, B in such a way that no edge is monochromatic.




[3]In practical implementation, we could use a list of clauses, where each clause is a list of (unique) literals in some chosen order. Again, imagine thousands of propositional variables and clauses.

We represent the graph as a set of vertices and a set of edges, where each edge is a pair of vertices. It is easier to work with ordered pairs, so we choose an (arbitrary) orientation of the edges.

$$\mathcal{G} = \langle V; E\rangle = \langle \{1,2,3,4\}; \{(1,2),(1,3),(1,4),(2,3),(3,4)\}\rangle$$

We start again with formalization in propositional logic. Let the set of colors be $C = \{R, G, B\}$. The natural choice of propositional variables is "vertex $v$ has color $c$", denoted as $p_v^c$, for each vertex $v \in V$ and each color $c \in C$. Our (ordered) set of propositional variables has 12 elements:

$$\mathbb{P} = \{p_v^c \mid c \in C, v \in V\} = \{p_1^R, p_1^G, p_1^B, p_2^R, p_2^G, p_2^B, p_3^R, p_3^G, p_3^B, p_4^R, p_4^G, p_4^B\}$$

We have a total of $|\,\mathrm{M}_{\mathbb{P}}\,| = 2^{12} = 4096$ models of the language (represented by 12-dimensional 0–1 vectors). Most of them cannot be interpreted as a coloring of the graph. For example, $v = (1,1,0,0,\ldots,0)$ indicates that vertex 1 is colored both red and green. We start with a theory expressing that each vertex has at most one color. There are several ways to express this. We will state, for each vertex, that it must not have (at least) one color from each pair of colors. This gives us a theory in CNF:

$$\begin{aligned} T_1 = \{&(\neg p_1^R \vee \neg p_1^G) \wedge (\neg p_1^R \vee \neg p_1^B) \wedge (\neg p_1^G \vee \neg p_1^B),\\ &(\neg p_2^R \vee \neg p_2^G) \wedge (\neg p_2^R \vee \neg p_2^B) \wedge (\neg p_2^G \vee \neg p_2^B),\\ &(\neg p_3^R \vee \neg p_3^G) \wedge (\neg p_3^R \vee \neg p_3^B) \wedge (\neg p_3^G \vee \neg p_3^B),\\ &(\neg p_4^R \vee \neg p_4^G) \wedge (\neg p_4^R \vee \neg p_4^B) \wedge (\neg p_4^G \vee \neg p_4^B)\}\\ = \{&(\neg p_v^R \vee \neg p_v^G) \wedge (\neg p_v^R \vee \neg p_v^B) \wedge (\neg p_v^G \vee \neg p_v^B) \mid v \in V\} \end{aligned}$$

The theory $T_1$ could be called the theory of *partial* vertex colorings of the graph $\mathcal{G}$. The theory $T_1$ has $|\,\mathrm{M}_{\mathbb{P}}(T_1)| = 4^4 = 2^8 = 256$ models. (Why?) If we want complete colorings, we add the condition that each vertex has at least one color.[4]

$$\begin{aligned} T_2 &= T_1 \cup \{p_1^R \vee p_1^G \vee p_1^B, p_2^R \vee p_2^G \vee p_2^B, p_3^R \vee p_3^G \vee p_3^B, p_4^R \vee p_4^G \vee p_4^B\}\\ &= T_1 \cup \{p_v^R \vee p_v^G \vee p_v^B \mid v \in V\}\\ &= T_1 \cup \{\bigvee_{c\in C} p_v^c \mid v \in V\} \end{aligned}$$

The theory $T_2$ has $3^4 = 81$ models. It is an *extension* of the theory $T_1$, as every consequence of $T_1$ also holds in the theory $T_2$. In fact, $\mathrm{M}_{\mathbb{P}}(T_2) \subseteq \mathrm{M}_{\mathbb{P}}(T_1)$.[5] The remaining condition is to forbid monochromatic edges. For each edge and each color, we specify that at least one of the vertices of the edge must not have the given color. For illustration, we will write out the

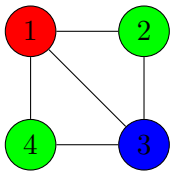
[4]The symbol $\bigvee$ is used similarly to the symbols $\sum$ for summation and $\prod$ for product: to simplify the notation of a proposition in the form of a disjunction. For example, if $v = 1$, then $\bigvee_{c\in C} p_v^c$ represents the proposition $p_1^R \vee p_1^G \vee p_1^B$. Analogously, $\bigwedge$ for conjunction.

[5]Here we see a typical example of the anti-monotonic relationship of the so-called *Galois correspondence*: the more properties (propositions) we require, the fewer objects (models) satisfy these properties.

complete list of propositions one last time; in the future, we will use the abbreviated notation.

$$\begin{aligned} T_3 = T_2 \cup \{&(\neg p_1^R \vee \neg p_2^R) \wedge (\neg p_1^G \vee \neg p_2^G) \wedge (\neg p_1^B \vee \neg p_2^B), \\ &(\neg p_1^R \vee \neg p_3^R) \wedge (\neg p_1^G \vee \neg p_3^G) \wedge (\neg p_1^B \vee \neg p_3^B), \\ &(\neg p_1^R \vee \neg p_4^R) \wedge (\neg p_1^G \vee \neg p_4^G) \wedge (\neg p_1^B \vee \neg p_4^B), \\ &(\neg p_2^R \vee \neg p_3^R) \wedge (\neg p_2^G \vee \neg p_3^G) \wedge (\neg p_2^B \vee \neg p_3^B), \\ &(\neg p_3^R \vee \neg p_4^R) \wedge (\neg p_3^G \vee \neg p_4^G) \wedge (\neg p_3^B \vee \neg p_4^B)\} \\ = T_2 \cup \{ &\bigwedge_{c \in C} (\neg p_u^c \vee \neg p_v^c) \mid (u,v) \in E\} \end{aligned}$$

The resulting theory $T_3$ is *satisfiable* (has a model) if and only if the graph $\mathcal{G}$ is 3-colorable. It has 6 models. The models are in one-to-one correspondence with 3-colorings of the graph $\mathcal{G}$. The model $v = (1,0,0,0,1,0,0,0,1,0,1,0)$ corresponds to the following coloring; other colorings can be obtained by permuting the colors.




Once we have the theory $T_3$ formalizing 3-coloring of the graph $\mathcal{G}$, we can easily solve related questions, such as finding all colorings where vertex 1 is blue and vertex 2 is green: these correspond to the models of the theory $T_3 \cup \{p_1^B, p_2^G\}$. Or we can prove that vertices 2 and 4 must be colored the same color. We can use the tableau method: at the root of the tableau will be the assumption

$$\text{False } (p_2^R \wedge p_4^R) \vee (p_2^G \wedge p_4^G) \vee (p_2^B \wedge p_4^B)$$

Or we can find a resolution refutation of the theory formed by converting the axioms of $T_3$ into CNF and adding the CNF equivalent of the *negation* of the proposition $(p_2^R \wedge p_4^R) \vee (p_2^G \wedge p_4^G) \vee (p_2^B \wedge p_4^B)$ (again, because it is a proof by contradiction, we assume for contradiction that the vertices *do not* have the same color).

## 1.2 Predicate Logic

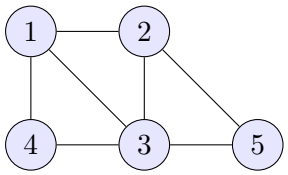
Now we will very briefly and informally introduce *predicate logic*. Predicate logic deals with properties of objects and relationships between objects. For example:

> *All humans are mortal.*
> *Socrates is a human.*
> *Socrates is mortal.*

In fact, propositional logic emerged later (about a century) than Aristotle's predicate logic, and was largely forgotten for a long time.

### 1.2.1 Disadvantages of Formalization in Propositional Logic

The disadvantage of formalizing our graph coloring problem in propositional logic is that the resulting theory $T_3$ is quite large and was created ad hoc for the graph $\mathcal{G}$. Imagine we need to modify the graph $\mathcal{G}$, for example, by adding a new vertex 5 connected by edges to vertices 2 and 3:




To formalize the new problem, we need to add three new propositional variables to our language: $\mathbb{P}' = \mathbb{P} \cup \{p_5^R, p_5^G, p_5^B\}$, and create new theories $T_1', T_2', T_3'$ by adding axioms related to vertex 5 and edges $(2,5), (3,5)$. The issue here is that the structure of the graph $\mathcal{G}$ and natural properties like "there is an edge from vertex $u$ to vertex $v$" or "vertex $u$ is green" have been ('unnaturally') 'hardcoded' into 0–1 variables. This drawback is addressed by *predicate logic*.

### 1.2.2 A Brief Introduction to Predicate Logic

A *model* in predicate logic is not a 0–1 vector but rather a *structure*. Examples of structures are our (oriented) graphs:

$$\mathcal{G} = \langle V^{\mathcal{G}}; E^{\mathcal{G}} \rangle = \langle \{1,2,3,4\}; \{(1,2),(1,3),(1,4),(2,3),(3,4)\} \rangle$$
$$\mathcal{G}' = \langle V^{\mathcal{G}'}; E^{\mathcal{G}'} \rangle = \langle \{1,2,3,4,5\}; \{(1,2),(1,3),(1,4),(2,3),(3,4),(2,5),(3,5)\} \rangle$$

Both graphs consist of a set of vertices and a binary relation on this set. These are structures in the *language of graph theory* $\mathcal{L} = \langle E \rangle$, where $E$ is a binary *relation symbol*. The *language* of predicate logic specifies which relations (how many and of what arity — unary, binary, ternary, etc.) the structures should have, and which symbols we will use for them. Additionally, we use the equality symbol $=$, and structures can also contain functions and constants (such as the functions $+, -, \cdot$ and constants $0, 1$ in the *field* of real numbers), but we will leave these for later.

Predicate logic uses the same logical connectives as propositional logic, but the basic building blocks of *predicate formulas* are not propositional variables, but so-called *atomic formulas*, for example: $E(x,y)$ represents the statement that there is an edge from vertex $x$ to vertex $y$ in the graph. Here $x, y$ are *variables* representing the vertices of the given graph. Additionally, we can use *quantifiers* in the formulas: $(\forall x)$ "for all vertices $x$" and $(\exists y)$ "there exists a vertex $y$".[6]

Now we can formalize statements that make sense for any graph. For example:

- "There are no loops in the graph":

$$(\forall x)(\neg E(x,x))$$

[6]We can think of quantifiers as "conjunction" or "disjunction" over all vertices of the graph.

- "There exists a vertex with out-degree 1":

$$(\exists x)(\exists y)(E(x,y) \wedge (\forall z)(E(x,z) \rightarrow y = z))$$

In the given graph $\mathcal{G}$ and under the assignment of vertex $u$ to the variable $x$ and vertex $v$ to the variable $y$, we evaluate $E(x,y)$ as True if and only if $(u,v) \in E^{\mathcal{G}}$.

### 1.2.3 Formalization of Graph Coloring in Predicate Logic

Let us return to graph coloring. A natural way to formalize our 3-coloring problem is in the language $\mathcal{L}' = \langle E, R, G, B \rangle$, where $E$ is binary and $R, G, B$ are unary relation symbols, so $R(x)$ means "vertex $x$ is red". A structure for this language is an (oriented) graph along with a triple of sets of vertices, e.g.,

$$\begin{aligned}\mathcal{G}_C &= \langle V^{\mathcal{G}_C}; E^{\mathcal{G}_C}, R^{\mathcal{G}_C}, G^{\mathcal{G}_C}, B^{\mathcal{G}_C} \rangle \\ &= \langle \{1,2,3,4\}; \{(1,2),(1,3),(1,4),(2,3),(3,4)\}, \{1\}, \{2,4\}, \{3\} \rangle\end{aligned}$$

represents the graph $\mathcal{G}$ with the valid coloring from the picture above. We will say that $\mathcal{G}_C$ is an *expansion* of the $\mathcal{L}$-structure $\mathcal{G}$ into the language $\mathcal{L}'$.

As in propositional logic, we first need to ensure that our models represent colored graphs. We start with the requirement that each vertex is colored with at most one color:

$$(\forall x)((\neg R(x) \vee \neg G(x)) \wedge (\neg R(x) \vee \neg B(x)) \wedge (\neg G(x) \vee \neg B(x)))$$

We express coloring with at least one color as follows:

$$(\forall x)(R(x) \vee G(x) \vee B(x))$$

And we formalize the edge condition using the predicate $E(x,y)$ as follows:

$$(\forall x)(\forall y)(E(x,y) \rightarrow ((\neg R(x) \vee \neg R(y)) \wedge (\neg G(x) \vee \neg G(y)) \wedge (\neg B(x) \vee \neg B(y))))$$

The models of the resulting theory represent oriented graphs with vertex 3-coloring.

## 1.3 Other Types of Logical Systems

Predicate logic where variables represent individual vertices is called *first-order* logic (abbreviated as *FO* logic). In *second-order* logic (abbreviated as *SO* logic), we also have variables representing sets of vertices or even sets of $n$-tuples of vertices (i.e., relations, functions). For example, the statement that a given graph is bipartite can be formalized in second-order logic with the following formula, where $S$ is a *second-order variable* representing a set of vertices and $S(x)$ expresses that "vertex $x$ is an element of set $S$":

$$(\exists S)(\forall x)(\forall y)(E(x,y) \rightarrow (S(x) \leftrightarrow \neg S(y)))$$

As another important example, consider the statement that every non-empty subset bounded from below has an infimum,[7] which can be formalized in second-order logic as follows:[8]

$$\begin{aligned}&(\forall S)((\exists x)S(x) \wedge (\exists x)(\forall y)(S(y) \rightarrow x \leq y) \rightarrow \\ &(\exists x)((\forall y)(S(y) \rightarrow x \leq y) \wedge (\forall z)((\forall y)(S(y) \rightarrow z \leq y) \rightarrow z \leq x)))\end{aligned}$$

[7] Which holds in the ordered set of real numbers but not in the rational numbers, e.g., $\{x \mid x^2 > 2, x > 0\}$.
[8] Even though this formula is quite complex, try to understand its individual parts.

In *third-order* logic, we even have variables representing sets of sets (which is useful, for example, in topology), etc.

Besides propositional and predicate logic, there are other types of logical systems, such as intuitionistic logic (which only allows constructive proofs), temporal logics (which talk about validity 'always', 'sometime in the future', 'until', etc.), modal logics ('it is possible', 'it is necessary'), and fuzzy logic (where we have statements that are '0.35 true'). These logics have important applications in computer science, e.g., in artificial intelligence (modal logics for reasoning of autonomous agents about their environment), in parallel programming (temporal logics), or in washing machines (fuzzy logics). In this course, we will limit ourselves to propositional logic and first-order predicate logic.

## 1.4 About the Lecture

The lecture is divided into three parts:

The first part deals with propositional logic. First, we introduce syntax and semantics, then the problem of satisfiability of CNF formulas (the well-known NP-complete problem SAT) and its polynomially solvable fragments (2-SAT, Horn-SAT). We will also demonstrate the practical use of a SAT solver. We will continue with the tableau method, proving its soundness and completeness, and several applications, such as the Compactness Theorem. Finally, we will introduce the resolution method in propositional logic.

In the second part, we will introduce predicate logic. Again, we start with syntax and semantics, show how to adapt the tableau method for predicate logic, and end with the resolution method. Additionally, we will mention several practical applications, such as database queries and logic programming (the Prolog language). The structure of the exposition in this part is closely tied to the previous part. Many definitions, theorems, proofs, and algorithms will be very similar to their counterparts in propositional logic. They will mainly differ at the low level, in technical details. Therefore, it is important to have a good understanding of propositional logic before moving on to predicate logic.

The third and smallest part is an introduction to model theory, axiomatizability, and algorithmic decidability. This is a taste of more advanced topics that one will encounter primarily in theoretical computer science and mathematical logic, although some have their place in applied computer science as well. Finally, we will introduce the famous Gödel's incompleteness theorems, which demonstrate the limits of formal methods (formal provability in an axiomatic system).

# Part I

# Propositional logic

# Chapter 2

# Syntax and Semantics of Propositional Logic

*Syntax* is a set of formal rules for creating well-formed sentences consisting of words (in the case of natural languages) or formal expressions consisting of symbols (e.g., statements in a programming language). In contrast, *semantics* describes the meaning of such expressions. The relationship between syntax and semantics is fundamental to all of logic and is therefore the key to its understanding.

## 2.1 Syntax of Propositional Logic

First, we define the formal 'statements' with which we will work in propositional logic.

### 2.1.1 Language

The *language* of propositional logic is determined by a non-empty set of *propositional variables* $\mathbb{P}$ (also called *atomic propositions* or *atoms*). This set can be finite or infinite, but it will usually be countable[1] (unless otherwise specified), and it will have a fixed ordering. For propositional variables, we will typically use the notation $p_i$ (from the word "proposition"), but for better readability, especially if $\mathbb{P}$ is finite, we will also use $p, q, r, \ldots$. For example:

$$\mathbb{P}_1 = \{p, q, r\}$$
$$\mathbb{P}_2 = \{p_0, p_1, p_2, p_3, \ldots\} = \{p_i \mid i \in \mathbb{N}\}$$

In addition to propositional variables, the language also includes *logical symbols*: symbols for logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ and parentheses $(,)$. However, for simplicity, we will talk about the "language $\mathbb{P}$".

*Remark* 2.1.1. If we need to formally express the ordering of the propositional variables in $\mathbb{P}$, we imagine it as a bijection $\iota\colon \{0, 1, \ldots, n-1\} \to \mathbb{P}$ (for a finite, $n$-element language) or $\iota\colon \mathbb{N} \to \mathbb{P}$ (if $\mathbb{P}$ is countably infinite). In our examples, $\iota_1(0) = p$, $\iota_1(1) = q$, $\iota_1(2) = r$, and $\iota_2(i) = p_i$ for all $i \in \mathbb{N}$.[2]

[1] This is important in many applications in computer science, as uncountable sets cannot fit into a (even infinite) computer.

[2] The set of natural numbers $\mathbb{N}$ includes zero, see standard ISO 80000-2:2019.

### 2.1.2 Proposition

The basic building block of propositional logic is a *proposition*, also called *propositional formula*. It is a finite string composed of propositional variables and logical symbols according to certain rules. Atomic propositions are propositions, and we can further create propositions from simpler propositions and logical symbols: for example, for the logical connective $\wedge$, we write first the symbol '(', then the first proposition, the symbol '$\wedge$', the second proposition, and finally the symbol ')'.

**Definition 2.1.2** (Proposition)**.** A *proposition* (*propositional formula*) in the language $\mathbb{P}$ is an element of the set $\mathrm{PF}_{\mathbb{P}}$ defined as follows: $\mathrm{PF}_{\mathbb{P}}$ is the smallest set satisfying[3]

- for every atomic proposition $p \in \mathbb{P}$, $p \in \mathrm{PF}_{\mathbb{P}}$,
- for every proposition $\varphi \in \mathrm{PF}_{\mathbb{P}}$, $(\neg\varphi)$ is also an element of $\mathrm{PF}_{\mathbb{P}}$,
- for every $\varphi, \psi \in \mathrm{PF}_{\mathbb{P}}$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, and $(\varphi \leftrightarrow \psi)$ are also elements of $\mathrm{PF}_{\mathbb{P}}$.

Propositions are usually denoted by Greek letters $\varphi, \psi, \chi$ ($\varphi$ from the word "formula"). To avoid listing all four binary logical connectives, we sometimes use a placeholder symbol $\square$. Thus, the third point of the definition could be expressed as:

- for every $\varphi, \psi \in \mathrm{PF}_{\mathbb{P}}$ and $\square \in \{\wedge, \vee, \to, \leftrightarrow\}$, $(\varphi \square \psi)$ is also an element of $\mathrm{PF}_{\mathbb{P}}$.

A *subproposition* (*subformula*) is a substring that is itself a proposition. Note that all propositions are necessarily finite strings, created by applying a finite number of steps from the definition to their subpropositions.

*Example* 2.1.3. The proposition $\varphi = ((p \vee (\neg q)) \leftrightarrow (r \to (p \wedge q)))$ has the following subpropositions: $p, q, (\neg q), (p \vee (\neg q)), r, (p \wedge q), (r \to (p \wedge q)), \varphi$.

A proposition in the language $\mathbb{P}$ does not have to contain all atomic propositions from $\mathbb{P}$ (and it cannot if $\mathbb{P}$ is an infinite set). Therefore, it will be useful to denote by $\mathrm{Var}(\varphi)$ the set of atomic propositions occurring in $\varphi$.[4] In our example, $\mathrm{Var}(\varphi) = \{p, q, r\}$.

We introduce abbreviations for two special propositions: $\top = (p \vee (\neg p))$ (*tautology*) and $\bot = (p \wedge (\neg p))$ (*contradiction*), where $p \in \mathbb{P}$ is fixed (e.g., the first atomic proposition from $\mathbb{P}$). Thus, the proposition $\top$ is always true, and the proposition $\bot$ is always false.

When writing propositions, we may omit some parentheses for better readability. For example, the proposition $\varphi$ from Example 2.1.3 can be represented by the string $p \vee \neg q \leftrightarrow (r \to p \wedge q)$. We omit outer parentheses and use priority of operators: $\neg$ has the highest priority, followed by $\wedge, \vee$, and finally $\to, \leftrightarrow$ have the lowest priority. Furthermore, the notation $p \wedge q \wedge r \wedge s$ means the proposition $(p \wedge (q \wedge (r \wedge s)))$, and similarly for $\vee$.[5][6]

[3] This kind of definition is called *inductive*. It can also be naturally expressed using a *formal grammar*, see the course NTIN071 Automata and Grammars.

[4] If we do not specify the language of a proposition (and if it is not clear from the context), we mean that it is in the language $\mathrm{Var}(\varphi)$.

[5] Due to the associativity of $\wedge, \vee$, the placement of parentheses does not matter.

[6] Sometimes finer priorities are introduced, $\wedge$ often has higher priority than $\vee$, and $\to$ has higher priority than $\leftrightarrow$. Also, sometimes $p \to q \to r$ is written instead of $(p \to (r \to q))$, although $\to$ is not associative and so here the placement of parentheses does matter. We prefer to avoid both of those conventions.

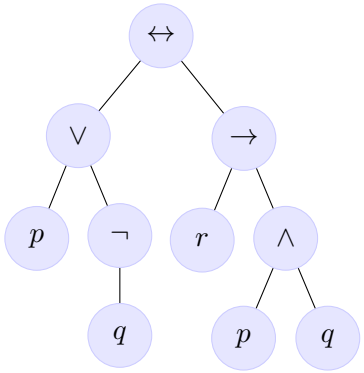



Figure 2.1: Tree of the proposition $\varphi = ((p \vee (\neg q)) \leftrightarrow (r \rightarrow (p \wedge q)))$

### 2.1.3 Tree of a Proposition

In the definition of a proposition, we chose *infix* notation (with parentheses) purely for human readability. Nothing would prevent us from using *prefix* ("Polish") notation, i.e., defining propositions as follows:

- every atomic proposition is a proposition, and
- if $\varphi, \psi$ are propositions, then $\neg\varphi$, $\wedge\varphi\psi$, $\vee\varphi\psi$, $\rightarrow\varphi\psi$, and $\leftrightarrow\varphi\psi$ are also propositions.

The proposition $\varphi = ((p \vee (\neg q)) \leftrightarrow (r \rightarrow (p \wedge q)))$ would then be written as $\varphi = \leftrightarrow\vee p\neg q\rightarrow r\wedge pq$. We could also use *postfix* notation and write $\varphi = pq\neg\vee rpq\wedge\rightarrow\leftrightarrow$. The essential information about a proposition is actually contained in its tree structure, which captures how it is composed of simpler propositions, similar to the tree of an arithmetic expression.

*Example* 2.1.4. The tree of the proposition $\varphi = ((p \vee (\neg q)) \leftrightarrow (r \rightarrow (p \wedge q)))$ is illustrated in Figure 2.1. Notice also how the subpropositions of $\varphi$ correspond to subtrees. The proposition $\varphi$ is obtained by traversing the tree from the root, and at each node:

- if the label is an atomic proposition, write it out,
- if the label is a negation: write '(¬', recursively call the child, write ')',
- otherwise (for binary logical connectives): write '(', call the left child, write the label, call the right child, write ')'.[7]

Now we will define the tree of a proposition formally, *by induction on the structure of the proposition*:[8]

**Definition 2.1.5** (Tree of a Proposition)**.** The *tree of a proposition* $\varphi$, denoted $\mathrm{Tree}(\varphi)$, is a rooted ordered tree, defined inductively as follows:

[7] Prefix and postfix notations would be obtained similarly, but we do not write parentheses, and the label is written immediately upon entering or just before leaving the node.

[8] Once we have the tree of a proposition defined, we can understand induction on the structure of the proposition as induction on the depth of the tree. For now, understand it as induction on the number of steps in Definition 2.1.2 by which the proposition was created. Alternatively, induction on the length of the proposition or the number of logical connectives would work as well.

- If $\varphi$ is an atomic proposition $p$, Tree($\varphi$) contains a single node, and its label is $p$.
- If $\varphi$ is of the form $(\neg\varphi')$, Tree($\varphi$) has a root labeled $\neg$, and its single child is the root of Tree($\varphi'$).
- If $\varphi$ is of the form $(\varphi' \square \varphi'')$ for $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, Tree($\varphi$) has a root labeled $\square$ with two children: the left child is the root of the tree Tree($\varphi'$), and the right child is the root of Tree($\varphi''$).

*Exercise* 2.1. Prove that every proposition has a uniquely determined proposition tree, and vice versa.

### 2.1.4 Theory

In practical applications, we do not express the desired properties with a single proposition — it would have to be very long and complex and difficult to work with — but with many simpler propositions.

**Definition 2.1.6** (Theory)**.** A *theory* in the language $\mathbb{P}$ is any set of propositions in $\mathbb{P}$, that is, any subset $T \subseteq \mathrm{PF}_{\mathbb{P}}$. The individual propositions $\varphi \in T$ are also called *axioms*.

*Finite* theories could be replaced by a single proposition: the conjunction of all their axioms. But that would not be practical. Moreover, we also allow infinite theories (a trivial example is the theory $T = \mathrm{PF}_{\mathbb{P}}$), and the empty theory $T = \emptyset$.[9]

## 2.2 Semantics of Propositional Logic

In our logic, the semantics are given by one of two possible values: *True* or *False*. (In other logical systems, semantics can be more interesting.)

### 2.2.1 Truth Value

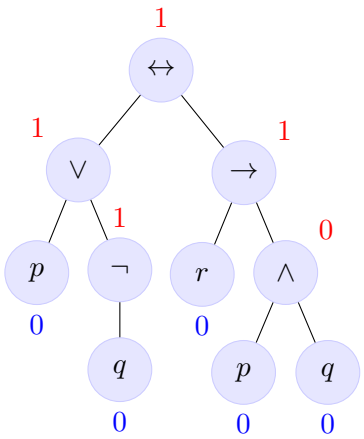
Propositions can be assigned one of two possible truth values: *True* (1) or *False* (0). Atomic propositions represent simple, indivisible statements (hence the name '*atomic*'); their truth value must be assigned to correspond to what we want to model (that is why we call them *propositional variables*). Once we *assign* truth values to the atomic propositions, the truth value of any compound proposition is uniquely determined and can be easily calculated according to the tree of the proposition:

*Example* 2.2.1. Let us calculate the truth value of the proposition $\varphi = ((p \vee (\neg q)) \leftrightarrow (r \rightarrow (p \wedge q)))$ for the assignments (a) $p = 0$, $q = 0$, $r = 0$ and (b) $p = 1$, $q = 0$, $r = 1$. We proceed from the leaves towards the root, similarly to evaluating an arithmetic expression. The proposition $\varphi$ is *true* under assignment (a) and *false* under assignment (b). See Figure 2.2.

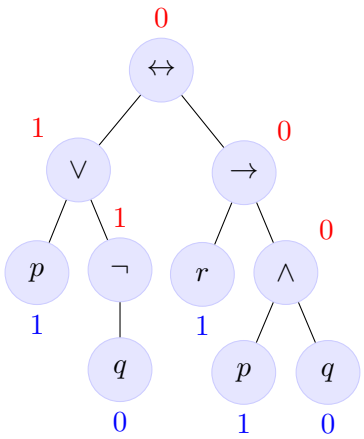
Logical connectives in the inner nodes are evaluated according to their *truth tables*, see Table 2.1.[10]

[9] Infinite theories are useful, for example, for describing the development of a system over (discrete) time steps $t = 0, 1, 2, \ldots$. The empty theory is not useful for anything, but it would be awkward to formulate statements about logic if theories had to be non-empty.

[10] Let us recall once again that disjunction is not exclusive, i.e., $p \vee q$ is true even if both $p$ and $q$ are true, and that implication is purely logical, i.e., $p \rightarrow q$ is true whenever $p$ is false.




(a) $p = 0$, $q = 0$, $r = 0$




(b) $p = 1$, $q = 0$, $r = 1$

Figure 2.2: Truth value of a proposition

| $p$ | $q$ | $\neg p$ | $p \wedge q$ | $p \vee q$ | $p \to q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 |

Table 2.1: Truth tables of logical connectives.

### 2.2.2 Propositions and Boolean Functions

To formalize the truth value of a proposition, we first look at the relationship between propositions and Boolean functions.

A *Boolean function* is a function $f\colon \{0,1\}^n \to \{0,1\}$, meaning that the input is an $n$-tuple of zeros and ones, and the output is 0 or 1. Each logical connective represents a Boolean function. In the case of negation, it is a unary function $f_\neg(x) = 1 - x$, while the other logical connectives correspond to binary functions described in Table 2.2.

**Definition 2.2.2** (Truth Function)**.** The *truth function* of a proposition $\varphi$ in a finite language $\mathbb{P}$ is the function $f_{\varphi,\mathbb{P}}\colon \{0,1\}^{|\mathbb{P}|} \to \{0,1\}$ defined inductively:

- If $\varphi$ is the $i$-th atomic proposition from $\mathbb{P}$, then $f_{\varphi,\mathbb{P}}(x_0, \ldots, x_{n-1}) = x_i$,
- If $\varphi = (\neg\varphi')$, then

$$f_{\varphi,\mathbb{P}}(x_0, \ldots, x_{n-1}) = f_\neg(f_{\varphi',\mathbb{P}}(x_0, \ldots, x_{n-1})),$$

$f_\wedge(x,y)$:

| | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$f_\vee(x,y)$:

| | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

$f_\to(x,y)$:

| | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 1 |

$f_\leftrightarrow(x,y)$:

| | 0 | 1 |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |

Table 2.2: Boolean functions of logical connectives

- If $(\varphi' \square \varphi'')$ where $\square \in \{\wedge, \vee, \to, \leftrightarrow\}$, then

$$f_{\varphi,\mathbb{P}}(x_0, \ldots, x_{n-1}) = f_{\square}(f_{\varphi',\mathbb{P}}(x_0, \ldots, x_{n-1}), f_{\varphi'',\mathbb{P}}(x_0, \ldots, x_{n-1})).$$

*Example* 2.2.3. Let us calculate the truth function of the proposition $\varphi = ((p \vee (\neg q)) \leftrightarrow (r \to (p \wedge q)))$ in the language $\mathbb{P}' = \{p, q, r, s\}$:

$$f_{\varphi,\mathbb{P}'}(x_0, x_1, x_2, x_3) = f_{\leftrightarrow}(f_{\vee}(x_0, f_{\neg}(x_1)), f_{\to}(x_2, f_{\wedge}(x_0, x_1)))$$

Truth value of the proposition $\varphi$ for the truth assignment $p = 1$, $q = 0$, $r = 1$, $s = 1$ is calculated as follows (compare with Figure 2.2(b)):

$$\begin{aligned} f_{\varphi,\mathbb{P}'}(1,0,1,1) &= f_{\leftrightarrow}(f_{\vee}(1, f_{\neg}(0)), f_{\to}(1, f_{\wedge}(1,0))) \\ &= f_{\leftrightarrow}(f_{\vee}(1,1), f_{\to}(1,0)) \\ &= f_{\leftrightarrow}(1,0) \\ &= 0 \end{aligned}$$

**Observation 2.2.4.** *The truth function of a proposition $\varphi$ over $\mathbb{P}$ depends only on the variables corresponding to the atomic propositions in* $\mathrm{Var}(\varphi) \subseteq \mathbb{P}$.

Thus, even if we have a proposition $\varphi$ in an *infinite* language $\mathbb{P}$, we can restrict ourselves to the language $\mathrm{Var}(\varphi)$ (which is finite) and consider the truth function over this language.

### 2.2.3 Models

A given truth assignment of propositional variables is a representation of the 'real world' (system) in our chosen 'formal world,' hence it is also called a *model*.

**Definition 2.2.5** (Model of a Language)**.** A *model* of a language $\mathbb{P}$ is any truth assignment $v\colon \mathbb{P} \to \{0,1\}$. The *set of (all) models of a language* $\mathbb{P}$ is denoted by $\mathrm{M}_{\mathbb{P}}$:

$$\mathrm{M}_{\mathbb{P}} = \{v \mid v\colon \mathbb{P} \to \{0,1\}\} = \{0,1\}^{\mathbb{P}}$$

Models will be denoted by letters $v, u, w$, etc. ($v$ from the word 'valuation'). A model of a language is therefore a function, formally a set of pairs (input, output). For example, for the language $\mathbb{P} = \{p, q, r\}$ and the truth assignment where $p$ is true, $q$ false, and $r$ true, we have the model

$$v = \{(p,1), (q,0), (r,1)\}.$$

For simplicity, we will write it as $v = (1,0,1)$. For the language $\mathbb{P} = \{p, q, r\}$, we have $2^3 = 8$ models:

$$\mathrm{M}_{\mathbb{P}} = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

*Remark* 2.2.6. Formally speaking, we identify the set $\{0,1\}^{\mathbb{P}}$ with the set $\{0,1\}^{|\mathbb{P}|}$ using the ordering $\iota$ of the language $\mathbb{P}$ (see Remark 2.1.1). Specifically, instead of the element $v = \{(p,1), (q,0), (r,1)\} \in \{0,1\}^{\mathbb{P}}$, we write $(1,0,1) = (v \circ \iota)(0,1,2) = (v(\iota(0)), v(\iota(1)), v(\iota(2))) \in \{0,1\}^{|\mathbb{P}|}$ (where we allow the functions $v, \iota$ to act 'component-wise').[11] If this seems confusing, imagine the model $v$ as a set of atomic propositions that are true, i.e., $\{p, r\} \subseteq \mathbb{P}$, our notation $v = (1,0,1)$ is then the characteristic vector of this set. This identification will be used henceforth without further notice.

[11] Alternatively, we could require (at least for countable languages) that the language *be* $\mathbb{P} = \{0, 1, 2, \ldots\}$ and use symbols $p_0, p_1, p, q, r$ only for readability.

### 2.2.4 Validity

We are now ready to define the key concept of logic, *validity* of a proposition in a given model. Informally, a proposition is valid in a model (i.e., under a specific truth assignment of atomic propositions) if its truth value, as calculated in Example 2.2.1, equals 1. In the formal definition, we will use the truth function of the proposition (Definition 2.2.2).[12]

**Definition 2.2.7** (Validity of a Proposition in a Model, Model of a Proposition)**.** Given a proposition $\varphi$ in a language $\mathbb{P}$ and a model $v \in \mathrm{M}_{\mathbb{P}}$, if $f_{\varphi,\mathbb{P}}(v) = 1$, we say that the proposition $\varphi$ is *valid* in the model $v$, $v$ is a *model* of $\varphi$, and we write $v \models \varphi$. The set of all models of the proposition $\varphi$ is denoted by $\mathrm{M}_{\mathbb{P}}(\varphi)$.

Models of a language that are not models of $\varphi$ will sometimes be called *non-models* of $\varphi$. They form the complement of the set of models of $\varphi$. Using the standard notation for function inverse, we can write:

$$\mathrm{M}_{\mathbb{P}}(\varphi) = \{v \in \mathrm{M}_{\mathbb{P}} \mid v \models \varphi\} = f^{-1}_{\varphi,\mathbb{P}}[1]$$
$$\overline{\mathrm{M}_{\mathbb{P}}(\varphi)} = \mathrm{M}_{\mathbb{P}} \setminus \mathrm{M}_{\mathbb{P}}(\varphi) = \{v \in \mathrm{M}_{\mathbb{P}} \mid v \not\models \varphi\} = f^{-1}_{\varphi,\mathbb{P}}[0]$$

If the language is clear from the context, we can simply write $\mathrm{M}(\varphi)$. We must be really sure, though: for example, in the language $\mathbb{P} = \{p, q\}$ we have

$$\mathrm{M}_{\{p,q\}}(p \to q) = \{(0,0), (0,1), (1,1)\},$$

while in the language $\mathbb{P}' = \{p, q, r\}$ we would have

$$\mathrm{M}_{\mathbb{P}'}(p \to q) = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,1,0), (1,1,1)\}.$$

**Definition 2.2.8** (Validity of a Theory, Model of a Theory)**.** Let $T$ be a theory in a language $\mathbb{P}$. The theory $T$ is *valid* in a model $v$ if every axiom $\varphi \in T$ is valid in $v$. In this case, we also say that $v$ is a *model* of $T$, and we write $v \models T$. The set of all models of the theory $T$ in a language $\mathbb{P}$ is denoted by $\mathrm{M}_{\mathbb{P}}(T)$.

When dealing with a finite theory, or adding a finite number of new axioms to a theory, we will use the following simplified notation:

- $\mathrm{M}_{\mathbb{P}}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ instead of $\mathrm{M}_{\mathbb{P}}(\{\varphi_1, \varphi_2, \ldots, \varphi_n\})$,
- $\mathrm{M}_{\mathbb{P}}(T, \varphi)$ instead of $\mathrm{M}_{\mathbb{P}}(T \cup \{\varphi\})$.

Note that $\mathrm{M}_{\mathbb{P}}(T, \varphi) = \mathrm{M}_{\mathbb{P}}(T) \cap \mathrm{M}_{\mathbb{P}}(\varphi)$, $\mathrm{M}_{\mathbb{P}}(T) = \bigcap_{\varphi \in T} \mathrm{M}_{\mathbb{P}}(\varphi)$, and for a finite theory (similarly for countable theories), we have

$$\mathrm{M}_{\mathbb{P}}(\varphi_1) \supseteq \mathrm{M}_{\mathbb{P}}(\varphi_1, \varphi_2) \supseteq \mathrm{M}_{\mathbb{P}}(\varphi_1, \varphi_2, \varphi_3) \supseteq \cdots \supseteq \mathrm{M}_{\mathbb{P}}(\varphi_1, \varphi_2, \ldots, \varphi_n).$$

We can use this when finding models by brute force.

*Example* 2.2.9. We can find models of the theory $T = \{p \vee q \vee r, q \to r, \neg r\}$ (in the language $\mathbb{P} = \{p, q, r\}$) as follows. First, we find the models of the proposition $\neg r$:

$$\mathrm{M}_{\mathbb{P}}(r) = \{(x, y, 0) \mid x, y \in \{0, 1\}\} = \{(0,0,0), (0,1,0), (1,0,0), (1,1,0)\},$$

then we determine in which of these models the proposition $q \to r$ is valid:

[12] For *validity*, we use the symbol $\models$, which we read as 'satisfies' or 'models', in LaTeX as \models.

- $(0,0,0) \models q \to r$,
- $(0,1,0) \not\models q \to r$,
- $(1,0,0) \models q \to r$,
- $(1,1,0) \not\models q \to r$,

Thus $\mathrm{M}_{\mathbb{P}}(r, q \to r) = \{(0,0,0),(1,0,0)\}$. The proposition $p \vee q \vee r$ is valid only in the second of these models, so we get

$$\mathrm{M}_{\mathbb{P}}(r, q \to r, p \vee q \vee r) = \mathrm{M}_{\mathbb{P}}(T) = \{(1,0,0)\}.$$

This procedure is more efficient than determining the sets of models of the individual axioms and taking their intersection. (But much less efficient than using a formal proof system, such as the tableau method, which we will see later.)

### 2.2.5 Additional semantic notions

Following the notion of validity, we will use several other notions. For some properties, several different names are in use, depending on the context in which the property is discussed.

**Definition 2.2.10** (Semantic notions)**.** We say that a proposition $\varphi$ (in a language $\mathbb{P}$) is

- *true*, *tautology*, *valid (in logic/logically)*, and we write $\models \varphi$, if it is valid in every model (of the language $\mathbb{P}$), $\mathrm{M}_{\mathbb{P}}(\varphi) = \mathrm{M}_{\mathbb{P}}$,
- *false*, *contradictory*, if it has no model, $\mathrm{M}_{\mathbb{P}}(\varphi) = \emptyset$,[13]
- *independent*, if it is valid in some model, and it is not valid in some other model, i.e., it is neither true nor false, $\emptyset \subsetneq \mathrm{M}_{\mathbb{P}}(\varphi) \subsetneq \mathrm{M}_{\mathbb{P}}$,
- *satisfiable*, if it has some model, i.e., it is not false, $\mathrm{M}_{\mathbb{P}}(\varphi) \neq \emptyset$.

Furthermore, we say that propositions $\varphi, \psi$ (in the same language $\mathbb{P}$) are *(logically) equivalent*, we write $\varphi \sim \psi$, if they have the same models, i.e.,

$$\varphi \sim \psi \text{ if and only if } \mathrm{M}_{\mathbb{P}}(\varphi) = \mathrm{M}_{\mathbb{P}}(\psi).$$

*Example* 2.2.11. For example, the following hold:

- propositions $\top$, $p \vee q \leftrightarrow q \vee p$ are true,
- propositions $\bot$, $(p \vee q) \wedge (p \vee \neg q) \wedge \neg p$ are false,
- propositions $p, p \wedge q$ are independent, and also satisfiable, and
- the following propositions are equivalent:
  - $p \sim p \vee p \sim p \vee p \vee p$,
  - $p \to q \sim \neg p \vee q$,

[13] Note that being *false* is not the same as not being *true*!

– $\neg p \to (p \to q) \sim \top$.

The notions from Definition 2.2.10 can also be relativized with respect to a given theory. This means that we restrict the individual definitions to the models of this theory:

**Definition 2.2.12** (Semantic Notions Relative to a Theory)**.** Let $T$ be a theory in a language $\mathbb{P}$. We say that a proposition $\varphi$ in the language $\mathbb{P}$ is

- *true in* $T$, a *consequence of* $T$, *valid in* $T$, and we write $T \models \varphi$, if $\varphi$ is valid in every model of the theory $T$, i.e., $\mathrm{M}_{\mathbb{P}}(T) \subseteq \mathrm{M}_{\mathbb{P}}(\varphi)$,
- *false in* $T$, *contradictory in* $T$, if it is not valid in any model of $T$, i.e., $\mathrm{M}_{\mathbb{P}}(\varphi) \cap \mathrm{M}_{\mathbb{P}}(T) = \mathrm{M}_{\mathbb{P}}(T, \varphi) = \emptyset$.
- *independent in* $T$, if it is valid in some model of $T$, and not valid in some other model of $T$, i.e., it is neither true in $T$ nor false in $T$, $\emptyset \subsetneq \mathrm{M}_{\mathbb{P}}(T, \varphi) \subsetneq \mathrm{M}_{\mathbb{P}}(T)$,
- *satisfiable in* $T$, *consistent with* $T$, if it is valid in some model of $T$, i.e., it is not false in $T$, $\mathrm{M}_{\mathbb{P}}(T, \varphi) \neq \emptyset$.

And we say that propositions $\varphi, \psi$ (in the same language $\mathbb{P}$) are *equivalent in* $T$, $T$*-equivalent*, we write $\varphi \sim_T \psi$ if they hold in the same models of $T$, i.e.,

$$\varphi \sim_T \psi \text{ if and only if } \mathrm{M}_{\mathbb{P}}(T, \varphi) = \mathrm{M}_{\mathbb{P}}(T, \psi).$$

Note that for the empty theory $T = \emptyset$, we have $\mathrm{M}_{\mathbb{P}}(T) = \mathrm{M}_{\mathbb{P}}$ and the above concepts for $T$ coincide with the original ones. Again, we illustrate the concepts with several examples:

*Example* 2.2.13. Let $T = \{p \vee q, \neg r\}$. The following hold:

- propositions $q \vee p$, $\neg p \vee \neg q \vee \neg r$ are true in $T$,
- the proposition $(\neg p \wedge \neg q) \vee r$ is false in $T$,
- propositions $p \leftrightarrow q, p \wedge q$ are independent in $T$, and also satisfiable, and
- $p$ and $p \vee r$ are $T$-equivalent, $p \sim_T p \vee r$ (but $p \not\sim p \vee r$).

### 2.2.6 Universality of Logical Connectives

In the language of propositional logic, we use the following logical connectives: $\neg, \wedge, \vee, \to, \leftrightarrow$. This is not the only possible choice; to build a fully-fledged logic, we could make do, for example, only with negation and implication,[14] or negation, conjunction, and disjunction.[15] And as we will see below, we could use other logical connectives as well. Our choice is the golden middle path between expressiveness on the one hand and succintness of syntactic definitions and proofs on the other.

What do we mean by saying that logic is fully-fledged? We say that a set of logical connectives $S$ is *universal*[16] if any Boolean function $f$ can be expressed as the truth function $f_{\varphi,\mathbb{P}}$ of some proposition $\varphi$ built from the logical connectives in $S$ (where $|\mathbb{P}| = n$ if $f$ is

[14] Negation is needed to describe the state of a system, and implication to describe behavior over time.
[15] These are sufficient to build logical circuits.
[16] Some people would say *[functionally] complete*.

an $n$-ary function). Equivalently, for any finite language $\mathbb{P}$ (say $n$-element) and any set of models $M \subseteq \mathrm{M}_{\mathbb{P}}$, there must exist a proposition $\varphi$ such that $\mathrm{M}_{\mathbb{P}}(\varphi) = M$. (The equivalence of these two statements follows from the fact that if we have a Boolean function $f$ and choose $M = f^{-1}[1]$, then $f_{\varphi,\mathbb{P}} = f$ if and only if $\mathrm{M}_{\mathbb{P}}(\varphi) = M$.)

**Proposition 2.2.14.** *The sets of logical connectives* $\{\neg, \wedge, \vee\}$ *and* $\{\neg, \to\}$ *are universal.*

*Proof.* Let us have a function $f\colon \{0,1\}^n \to \{0,1\}$, or equivalently a set of models $M = f^{-1}[1] \subseteq \{0,1\}^n$. Our language will be $\mathbb{P} = \{p_1, \ldots, p_n\}$. If the set $M$ contained only one model, say $v = (1,0,1,0)$, we could represent it with the proposition $\varphi_v = p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4$, which says 'I must be the model $v$.' For a general model $v$, we would write the proposition $\varphi_v$ as follows:

$$\varphi_v = p_1^{v_1} \wedge p_2^{v_2} \wedge \cdots \wedge p_n^{v_n} = \bigwedge_{i=1}^{n} p_i^{v(p_i)} = \bigwedge_{p \in \mathbb{P}} p^{v(p)},$$

where we introduce the following useful notation: $p^{v(p)}$ is the proposition $p$ if $v(p) = 1$, and the proposition $\neg p$ if $v(p) = 0$.

If the set $M$ contains more models, we say 'I must be at least one of the models from $M$':

$$\varphi_M = \bigvee_{v \in M} \varphi_v = \bigvee_{v \in M} \bigwedge_{p \in \mathbb{P}} p^{v(p)}$$

Clearly, $\mathrm{M}_{\mathbb{P}}(\varphi_M) = M$, or equivalently $f_{\varphi_M,\mathbb{P}} = f$. (If $M = \emptyset$, then by definition $\bigvee_{v \in M} \varphi_v = \bot$.)[17]

The universality of $\{\neg, \to\}$ follows from the universality of $\{\neg, \wedge, \vee\}$ and the fact that conjunction and disjunction can be expressed using negation and implication: $p \wedge q \sim \neg(p \to \neg q)$ and $p \vee q \sim \neg p \to q$. □

*Remark* 2.2.15. Note that in the construction of the proposition $\varphi_M$, it is crucial that the set $M$ is finite (it has at most $2^n$ elements). If it were infinite, the symbol '$\bigvee_{v \in M}$' would mean 'disjunction' of infinitely many propositions, and thus the result would not be a finite string, i.e., '$\varphi_M$' would not be a proposition. (If we have a countably infinite language $\mathbb{P}'$, then not every subset $M \subseteq \mathrm{M}_{\mathbb{P}'}$ can be represented by a proposition—there are uncountably many such subsets, while propositions are only countably many.)

What other logical connectives could we use? Nullary Boolean functions,[18] i.e. constants 0, 1, could be introduced as symbols TRUE and FALSE; we will suffice with propositions $\top, \bot$. There are four unary Boolean functions ($4 = 2^{2^1}$), but negation is the only 'interesting' one: the others are $f(x) = x$, $f(x) = 0$, and $f(x) = 1$. There are more interesting binary logical connectives that occur naturally, for example:

- NAND or *Sheffer's stroke*, sometimes denoted as $p \uparrow q$, where $p \uparrow q \sim \neg(p \wedge q)$,
- NOR or *Peirce's arrow*, sometimes denoted as $p \downarrow q$, where $p \downarrow q \sim \neg(p \vee q)$,
- XOR, or *exclusive-OR*, sometimes denoted as $\oplus$, where $p \oplus q \sim (p \vee q) \wedge \neg(p \wedge q)$, i.e. the sum of truth values modulo 2.

[17] Similar to how the sum of an empty set of summands equals 0.

[18] In formalizing mathematics or computer science, a function of arity 0 means it has no inputs, so the output cannot depend on the input and is constant. Formally, these are functions $f\colon \emptyset \to \{0,1\}$. If this is confusing, assume that functions must have an arity of at least 1, and instead of 'nullary function', say 'constant.'

*Exercise* 2.2. Express $(p \oplus q) \oplus r$ using $\{\neg, \wedge, \vee\}$.

*Exercise* 2.3. Show that {NAND} and {NOR} are universal.

*Exercise* 2.4. Consider the ternary logical connective IFTE, where $IFTE(p, q, r)$ is satisfied if and only if 'if $p$ then $q$ else $r$'. Determine the truth table of this logical connective (i.e., the function $f_{\text{IFTE}}$) and show that $\{\text{TRUE}, \text{FALSE}, \text{IFTE}\}$ is universal.

## 2.3 Normal Forms

Let us recall that propositions are equivalent if they have the same set of models. For each proposition, there exist infinitely many equivalent propositions; it is often useful to express a proposition in a 'nice' (useful) 'shape', i.e., to find an equivalent proposition of that 'shape'. This concept of 'shape' in mathematics is called a *normal form*. We will introduce two most common normal forms: *conjunctive normal form (CNF)* and *disjunctive normal form (DNF)*.

The following terminology and notation are needed:

- A *literal* $\ell$ is either a propositional variable $p$ or the negation of a propositional variable $\neg p$. For a propositional variable $p$, denote $p^0 = \neg p$ and $p^1 = p$. If $\ell$ is a literal, then $\bar{\ell}$ denotes the *opposite literal* to $\ell$. If $\ell = p$ (a *positive literal*), then $\bar{\ell} = \neg p$; if $\ell = \neg p$ (a *negative literal*), then $\bar{\ell} = p$.
- A *clause* is a disjunction of literals $C = \ell_1 \vee \ell_2 \vee \cdots \vee \ell_n$. A *unit clause* is a single literal $(n = 1)$ and the *empty clause* $(n = 0)$ is interpreted as $\bot$.
- A proposition is in *conjunctive normal form (CNF)* if it is a conjunction of clauses. The *empty CNF proposition* is $\top$.
- An *elementary conjunction* is a conjunction of literals $E = \ell_1 \wedge \ell_2 \wedge \cdots \wedge \ell_n$. A *unit elementary conjunction* is a single literal $(n = 1)$. The *empty elementary conjunction* $(n = 0)$ is $\top$.
- A proposition is in *disjunctive normal form (DNF)* if it is a disjunction of elementary conjunctions. The *empty DNF proposition* is $\bot$.

*Example* 2.3.1. The proposition $p \vee q \vee \neg r$ is in CNF (it is a single clause) as well as in DNF (it is a disjunction of unit elementary conjunctions). The proposition $(p \vee q) \wedge (p \vee \neg q) \wedge \neg p$ is in CNF, while the proposition $\neg p \vee (p \wedge q)$ is in DNF.

*Example* 2.3.2. The proposition $\varphi_v$ from the proof of Proposition 2.2.14 is in CNF (it is a conjunction of unit clauses, i.e., literals) and also in DNF (it is a single elementary conjunction). The proposition $\varphi_M$ is in DNF.

**Observation 2.3.3.** *Note that a proposition in CNF is a tautology if and only if each of its clauses contains a pair of opposite literals. Similarly, a proposition in DNF is satisfiable if and only if not every elementary conjunction contains a pair of opposite literals.*

### 2.3.1 On Duality

Note that if we interchange the values for truth and falsehood in propositional logic, i.e., 0 and 1, the truth table for negation remains the same, while conjunction becomes disjunction, and vice versa. This concept is called *duality*; we will see many examples of this in logic.

We have $\neg(p \wedge q) \sim (\neg p \vee \neg q)$, and from *duality*, we also know that $\neg(\neg p \vee \neg q) \sim (\neg\neg p \wedge \neg\neg q)$, from which we can easily deduce $\neg(p \vee q) \sim (\neg p \wedge \neg q)$.[19] More generally, $n$-ary Boolean functions $f, g$ are *dual* to each other if $f(\neg x) = \neg g(x)$. If we have a proposition $\varphi$ built from $\{\neg, \wedge, \vee\}$ and we interchange $\wedge$ and $\vee$, and negate the propositional variables (i.e., interchange literals with their opposite literals), we obtain a proposition $\psi \sim \neg\varphi$ (i.e., the models of $\varphi$ are the non-models of $\psi$ and vice versa), and the functions $f_{\varphi,\mathbb{P}}$ and $f_{\psi,\mathbb{P}}$ are dual to each other.

The notion of DNF is dual to the notion of CNF; 'is a tautology' is dual to 'is not satisfiable', thus the previous observation can be understood as an example of duality. For every statement in propositional logic, we obtain a *dual* statement 'for free', resulting from the interchange of $\wedge$ and $\vee$, truth and falsehood.

### 2.3.2 Conversion to Normal Forms

We have already encountered the disjunctive normal form in the proof of Proposition 2.2.14. The key part of the proof can be formulated as follows: 'If the language is finite, any set of models can be *axiomatized* by a proposition in DNF'. From duality, we also obtain axiomatization in CNF since the complement of a set of models is also a set of models:

**Proposition 2.3.4.** *Given a finite language $\mathbb{P}$ and any set of models $M \subseteq \mathrm{M}_{\mathbb{P}}$, there exists a proposition $\varphi_{\mathrm{DNF}}$ in DNF and a proposition $\varphi_{\mathrm{CNF}}$ in CNF such that $M = \mathrm{M}_{\mathbb{P}}(\varphi_{\mathrm{DNF}}) = \mathrm{M}_{\mathbb{P}}(\varphi_{\mathrm{CNF}})$. Specifically:*

$$\varphi_{\mathrm{DNF}} = \bigvee_{v \in M} \bigwedge_{p \in \mathbb{P}} p^{v(p)}$$

$$\varphi_{\mathrm{CNF}} = \bigwedge_{v \in \overline{M}} \bigvee_{p \in \mathbb{P}} \overline{p^{v(p)}} = \bigwedge_{v \notin M} \bigvee_{p \in \mathbb{P}} p^{1-v(p)}$$

*Proof.* For the proposition $\varphi_{\mathrm{DNF}}$, see the proof of Proposition 2.2.14, where each elementary conjunction describes one model. The proposition $\varphi_{\mathrm{CNF}}$ is dual to the proposition $\varphi'_{\mathrm{DNF}}$ constructed for the complement $M' = \overline{M}$. Alternatively, we can prove it directly: the models of the clause $C_v = \bigvee_{p \in \mathbb{P}} p^{1-v(p)}$ are all models except $v$, $\mathrm{M}_C = \mathrm{M}_{\mathbb{P}} \setminus \{v\}$, so each clause in the conjunction excludes one non-model. □

Proposition 2.3.4 provides a method for converting a proposition to disjunctive or conjunctive normal form:

*Example* 2.3.5. Consider the proposition $\varphi = p \leftrightarrow (q \vee \neg r)$. First, we find the set of models: $M = \mathrm{M}(\varphi) = \{(0,0,1), (1,0,0), (1,1,0), (1,1,1)\}$. Then, we find the propositions $\varphi_{\mathrm{DNF}}$ and $\varphi_{\mathrm{CNF}}$ according to Proposition 2.3.4. Those have the same models as $\varphi$ and are therefore equivalent with it.

We find the proposition $\varphi_{\mathrm{DNF}}$ by constructing an elementary conjunction for each model that enforces precisely that model:

$$\varphi_{\mathrm{DNF}} = (\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$$

For constructing $\varphi_{\mathrm{CNF}}$, we need the *non-models* of $\varphi$, $\overline{M} = \{(0,0,0), (0,1,0), (0,1,1), (1,0,1)\}$. Each clause excludes one non-model:

$$\varphi_{\mathrm{CNF}} = (p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$$

[19] Since $p, q$ are propositional variables, they can be assigned both values 0 and 1, thus we can interchange them with their opposite literals.

**Corollary 2.3.6.** *Every proposition (in any, even infinite, language $\mathbb{P}$) is equivalent to some proposition in CNF and some proposition in DNF.*

*Proof.* Even if the language $\mathbb{P}$ is infinite, the proposition $\varphi$ contains only finitely many propositional variables, so we can use Proposition 2.3.4 for the language $\mathbb{P}' = \mathrm{Var}(\varphi)$ and the set of models $M = \mathrm{M}_{\mathbb{P}'}(\varphi)$. Since $M = \mathrm{M}_{\mathbb{P}'}(\varphi_{\mathrm{DNF}}) = \mathrm{M}_{\mathbb{P}'}(\varphi_{\mathrm{CNF}})$, we have $\varphi \sim \varphi_{\mathrm{DNF}} \sim \varphi_{\mathrm{CNF}}$. □

*Exercise* 2.5. Describe how one can easily generate models from a DNF proposition and non-models from a CNF proposition.

*Remark* 2.3.7. When can a *theory* be axiomatized by a proposition in DNF or CNF? Consider the language $\mathbb{P}' = \mathrm{Var}(T)$ (i.e., all propositional variables occurring in the axioms of $T$). If $T$ in the language $\mathbb{P}'$ has finitely many models (i.e., $\mathrm{M}_{\mathbb{P}'}(T)$ is finite), we can construct a proposition in DNF, and if it has finitely many *non-models*, we can construct a proposition in CNF. Generally, however, not every theory can be axiomatized by a *single* proposition in CNF or DNF. We can always convert individual axioms to CNF (or DNF), and we can also axiomatize the theory using (potentially infinitely many) clauses.

This method of conversion to CNF or DNF requires knowledge of the set of models of the proposition, so it is quite inefficient. Additionally, the resulting normal form can be very long. We will now show another method.

### Conversion Using Equivalent Transformations

We use the following observation: If we replace some subproposition $\psi$ of a proposition $\varphi$ with an equivalent proposition $\psi'$, the resulting proposition $\varphi'$ will also be equivalent to $\varphi$. We will first demonstrate the method on an example:

*Example* 2.3.8. Again, we will convert the proposition $\varphi = p \leftrightarrow (q \vee \neg r)$. First, we eliminate the equivalence, expressing it as a conjunction of two implications. In the next step, we remove the implications using the rule $\varphi \to \psi \sim \neg\varphi \vee \psi$:

$$\begin{aligned} p \leftrightarrow (q \vee \neg r) &\sim (p \to (q \vee \neg r)) \wedge ((q \vee \neg r) \to p) \\ &\sim (\neg p \vee q \vee \neg r) \wedge (\neg(q \vee \neg r) \vee p) \end{aligned}$$

Now, imagine the tree of the proposition; in the next step, we want to push the negations as low as possible in the tree, just above the leaves: we use $\neg(q \vee \neg r) \sim \neg q \wedge \neg\neg r$ and eliminate the double negation $\neg\neg r \sim r$. We obtain the proposition

$$(\neg p \vee q \vee \neg r) \wedge ((\neg q \wedge r) \vee p)$$

At this point, we leave the literals untouched and apply the distributivity of $\wedge$ over $\vee$, or vice versa, depending on whether we want DNF or CNF. For conversion to CNF, we use the transformation $(\neg q \wedge r) \vee p \sim (\neg q \vee p) \wedge (r \vee p)$, which pushes the $\vee$ symbol lower in the tree. (Draw the tree!) This gives us a proposition in CNF; for better readability, we sort the literals in the clauses:

$$(\neg p \vee q \vee \neg r) \wedge (p \vee \neg q) \wedge (p \vee r)$$

For conversion to DNF, we proceed similarly by repeatedly applying distributivity. Here, we start from the CNF form and combine each literal from the first clause with each literal from the second and each literal from the third clause. Note that the same literal does not need to

be repeated twice in the elementary conjunction, and if the elementary conjunction contains a pair of opposite literals, it is contradictory and can be omitted in the DNF. We can also omit an elementary conjunction $E$ if we have another elementary conjunction $E'$ such that $E'$ contains all the literals contained in $E$, e.g., $E = (p \wedge \neg r)$ and $E' = (p \wedge q \wedge \neg r)$. (Think about why, and formulate the dual simplification when converting to CNF.) The resulting proposition in DNF is:

$$(\neg p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r) \vee (p \wedge \neg r)$$

We now list all equivalent transformations needed for the method. The proof that every proposition can be converted to DNF and CNF can be easily carried out by induction on the structure of the proposition (depth of the proposition tree).

- Implications and equivalences:

  $$\varphi \to \psi \sim \neg\varphi \vee \psi$$
  $$\varphi \leftrightarrow \psi \sim (\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi)$$

- Negations:

  $$\neg(\varphi \wedge \psi) \sim \neg\varphi \vee \neg\psi$$
  $$\neg(\varphi \vee \psi) \sim \neg\varphi \wedge \neg\psi$$
  $$\neg\neg\varphi \sim \varphi$$

- Conjunctions (conversion to DNF):

  $$\varphi \wedge (\psi \vee \chi) \sim (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$
  $$(\varphi \vee \psi) \wedge \chi \sim (\varphi \wedge \chi) \vee (\psi \wedge \chi)$$

- Disjunctions (conversion to CNF):

  $$\varphi \vee (\psi \wedge \chi) \sim (\varphi \vee \psi) \wedge (\varphi \vee \chi)$$
  $$(\varphi \wedge \psi) \vee \chi \sim (\varphi \vee \chi) \wedge (\psi \vee \chi)$$

As we will see in the next chapter, CNF is much more important in practice than DNF (even though they are dual concepts). For describing a real system, it is more natural to express it as a conjunction of many simpler properties than as a single very long disjunction. There are many other forms of representing Boolean functions. Similar to data structures, we choose the appropriate form of representation depending on the operations we need to perform on the function.[20]

## 2.4 Properties and Consequences of Theories

Let us delve deeper into properties of theories. Similar to propositions, we say that two theories $T, T'$ in the language $\mathbb{P}$ are *equivalent* if they have the same set of models:

$$T \sim T' \text{ if and only if } \mathrm{M}_{\mathbb{P}}(T) = \mathrm{M}_{\mathbb{P}}(T')$$

These are theories expressing the same properties of models, just formulated (*axiomatized*) differently. In logic, we are mainly interested in properties of theories that are independent of the specific *axiomatization*.

*Example* 2.4.1. For example, the theory $T = \{p \to q, p \leftrightarrow r\}$ is equivalent to the theory $T' = \{(\neg p \vee q) \wedge (\neg p \vee r) \wedge (p \vee \neg r)\}$.

**Definition 2.4.2** (Properties of Theories)**.** We say that a theory $T$ in the language $\mathbb{P}$ is

- *inconsistent* (*unsatisfiable*), if it includes $\bot$ (a contradiction), equivalently, if it has no model, equivalently, if it includes all propositions,

[20] See, for example, the course NAIL031 Representations of Boolean Functions.

- *consistent* (*satisfiable*) if it is not inconsistent, i.e., it has some model,
- *complete* if it is not inconsistent and every proposition in it is either true or false (i.e., it has no independent propositions), equivalently, if it has exactly one model.

Let us verify that the equivalence of properties in the definition holds. Notice that in an inconsistent theory, all propositions are indeed valid! A proposition is valid in $T$ if it is valid in every model of $T$, but there are none. Conversely, if the theory has at least one model, $\bot = p \wedge \neg p$ cannot be valid in this model.

If a theory is complete, it cannot have two different models $v \neq v'$. The proposition $\varphi_v = \bigwedge_{p \in \mathbb{P}} p^{v(p)}$ (which we encountered in the proof of Proposition 2.2.14) would be independent in $T$ because it is valid in the model $v$ but not in $v'$. Conversely, if $T$ has a single model $v$, then every proposition is either valid in $v$ and thus in $T$, or it is not valid in $v$ and therefore false in $T$.

*Example* 2.4.3. An example of an inconsistent theory is $T_1 = \{p, p \to q, \neg q\}$. The theory $T_2 = \{p \vee q, r\}$ is consistent but not complete, for instance, the proposition $p \wedge q$ is neither true (it does not hold in the model $(1, 0, 1)$) nor false (it holds in the model $(1, 1, 1)$). The theory $T_2 \cup \{\neg p\}$ is complete, with its only model being $(0, 1, 1)$.

### 2.4.1 Consequences of Theories

Recall that a consequence of a theory $T$ is any proposition that is true in $T$ (i.e., valid in every model of $T$), and let us denote the *set of all consequences* of a theory $T$ in the language $\mathbb{P}$ as

$$\mathrm{Csq}_{\mathbb{P}}(T) = \{\varphi \in \mathrm{PF}_{\mathbb{P}} \mid T \models \varphi\}$$

If the theory $T$ is in the language $\mathbb{P}$, we can write:

$$\mathrm{Csq}_{\mathbb{P}}(T) = \{\varphi \in \mathrm{PF}_{\mathbb{P}} \mid \mathrm{M}_{\mathbb{P}}(T) \subseteq \mathrm{M}_{\mathbb{P}}(\varphi)\}$$

(However, it is also meaningful to talk about consequences of a theory that are in some smaller language, which is a subset of the language of $T$).

Let us demonstrate some simple properties of consequences:

**Proposition 2.4.4.** *Let $T, T'$ be theories and $\varphi, \varphi_1, \ldots, \varphi_n$ be propositions in a language $\mathbb{P}$. Then the following holds:*

*(i)* $T \subseteq \mathrm{Csq}_{\mathbb{P}}(T)$,

*(ii)* $\mathrm{Csq}_{\mathbb{P}}(T) = \mathrm{Csq}_{\mathbb{P}}(\mathrm{Csq}_{\mathbb{P}}(T))$,

*(iii) if* $T \subseteq T'$, *then* $\mathrm{Csq}_{\mathbb{P}}(T) \subseteq \mathrm{Csq}_{\mathbb{P}}(T')$,

*(iv)* $\varphi \in \mathrm{Csq}_{\mathbb{P}}(\{\varphi_1, \ldots, \varphi_n\})$ *if and only if the proposition* $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi$ *is a tautology.*

*Proof.* The proof is straightforward, using the fact that $\varphi$ is a consequence of $T$ if and only if $\mathrm{M}_{\mathbb{P}}(T) \subseteq \mathrm{M}_{\mathbb{P}}(\varphi)$, and recognizing the following relationships:

- $\mathrm{M}(\mathrm{Csq}(T)) = \mathrm{M}(T)$,
- if $T \subseteq T'$, then $\mathrm{M}(T) \supseteq \mathrm{M}(T')$,[21]

[21] The more properties we prescribe, the fewer objects will satisfy them all.

- $\psi \to \varphi$ is a tautology if and only if $\mathrm{M}(\psi) \subseteq \mathrm{M}(\varphi)$,
- $\mathrm{M}(\varphi_1 \wedge \cdots \wedge \varphi_n) = \mathrm{M}(\varphi_1, \ldots, \varphi_n)$.

□

*Exercise* 2.6. Give a detailed proof of Proposition 2.4.4.

### 2.4.2 Extensions of Theories

Informally speaking, an *extension* of a theory $T$ refers to any theory $T'$ that satisfies everything that is true in $T$ (and more, apart from the trivial case). If theory $T$ models some system, it can be extended in two ways: by adding additional requirements about the system (which we will call a *simple extension*) or by expanding the system with some new parts. If in the second case there are no additional requirements on the original part of the system, i.e., exactly the same hold about the original part as before, we say that the extension is *conservative*.

*Example* 2.4.5. Let us return to the initial example of graph coloring, Example 1.1.2. The theory $T_3$ (complete colorings satisfying the edge condition) is a simple extension of the theory $T_1$ (partial colorings of vertices with no regard for edges). The theory $T_3'$ from Section 1.2.1 (adding a new vertex to the graph) is a conservative, non-simple extension of $T_3$. And it is an extension of $T_1$ that is neither simple nor conservative.

Let us now finally present the formal definitions:

**Definition 2.4.6** (Extension of a Theory)**.** Let $T$ be a theory in the language $\mathbb{P}$.

- An *extension* of the theory $T$ is any theory $T'$ in a language $\mathbb{P}' \supseteq \mathbb{P}$ that satisfies $\mathrm{Csq}_{\mathbb{P}}(T) \subseteq \mathrm{Csq}_{\mathbb{P}'}(T')$,
- it is a *simple extension* if $\mathbb{P}' = \mathbb{P}$,
- it is a *conservative extension* if $\mathrm{Csq}_{\mathbb{P}}(T) = \mathrm{Csq}_{\mathbb{P}}(T') = \mathrm{Csq}_{\mathbb{P}'}(T') \cap \mathrm{PF}_{\mathbb{P}}$.

Thus, an extension means that it satisfies all the consequences of the original theory. An extension is simple if we do not add any new propositional variables to the language, and it is conservative if it does not change the validity of statements expressible in the original language, meaning that any new consequence must contain some of the newly added propositional variables.

What do these concepts mean *semantically*, in terms of models? Let us first formulate a general observation, which we will then illustrate with an example:

**Observation 2.4.7.** *Let $T$ be a theory in the language $\mathbb{P}$, and $T'$ a theory in a language $\mathbb{P}'$ containing the language $\mathbb{P}$. Then the following holds:*

- *$T'$ is a simple extension of $T$ if and only if $\mathbb{P}' = \mathbb{P}$ and $\mathrm{M}_{\mathbb{P}}(T') \subseteq \mathrm{M}_{\mathbb{P}}(T)$,*
- *$T'$ is an extension of $T$ if and only if $\mathrm{M}_{\mathbb{P}'}(T') \subseteq \mathrm{M}_{\mathbb{P}'}(T)$. We thus consider models of theory $T$ in the extended language $\mathbb{P}'$.*[22] *In other words, the* restriction[23] *of any*

[22] Note that we cannot write $\mathrm{M}_{\mathbb{P}}(T')$ because models of $T'$ must be truth assignments of the larger language $\mathbb{P}'$, and values only for the propositional variables in $\mathbb{P}$ are not sufficient to determine the truth value of axioms of $T'$. And we cannot write $\mathrm{M}_{\mathbb{P}'}(T') \subseteq \mathrm{M}_{\mathbb{P}}(T)$ either, as these are sets of vectors of different dimensions.

[23] *Restriction* means forgetting the values for the new propositional variables, i.e., deleting the corresponding coordinates in the vector representation of the model.

*model* $v \in \mathrm{M}_{\mathbb{P}'}(T')$ *to the original language* $\mathbb{P}$ *must be a model of* $T$*. We could write* $v\restriction_{\mathbb{P}} \in \mathrm{M}_{\mathbb{P}}(T)$ *or:*

$$\{v\restriction_{\mathbb{P}} \mid v \in \mathrm{M}_{\mathbb{P}'}(T')\} \subseteq \mathrm{M}_{\mathbb{P}}(T)$$

- $T'$ *is a conservative extension of* $T$ *if it is an extension and, moreover, every model of* $T$ *(in the language* $\mathbb{P}$*) can be* expanded[24] *(in some way, not necessarily uniquely) to a model of* $T'$ *(in the language* $\mathbb{P}'$*), in other words,* every *model of* $T$ *(in the language* $\mathbb{P}$*) is obtained by restricting* some *model of* $T'$ *to the language* $\mathbb{P}$*. We could write:*

$$\{v\restriction_{\mathbb{P}} \mid v \in \mathrm{M}_{\mathbb{P}'}(T')\} = \mathrm{M}_{\mathbb{P}}(T)$$

- $T'$ *is an extension of* $T$ *and* $T$ *is an extension of* $T'$*, if and only if* $\mathbb{P}' = \mathbb{P}$ *and* $\mathrm{M}_{\mathbb{P}}(T') = \mathrm{M}_{\mathbb{P}}(T)$*, in other words,* $T' \sim T$*.*

- *Complete simple extensions of* $T$ *correspond to models of* $T$*, uniquely up to equivalence.*

*Example* 2.4.8. Consider the theory $T = \{p \to q\}$ in the language $\mathbb{P} = \{p, q\}$. The theory $T_1 = \{p \wedge q\}$ in the language $\mathbb{P}$ is a simple extension of $T$, we have $\mathrm{M}_{\mathbb{P}}(T_1) = \{(1,1)\} \subseteq \{(0,0),(0,1),(1,1)\} = \mathrm{M}_{\mathbb{P}}(T)$. It is a complete theory, and other complete simple extensions of theory $T$ are, for example, $T_2 = \{\neg p, q\}$ and $T_3 = \{\neg p, \neg q\}$. Each complete simple extension of theory $T$ is equivalent to $T_1$, $T_2$, or $T_3$.

Now consider the theory $T' = \{p \leftrightarrow (q \wedge r)\}$ in the language $\mathbb{P}' = \{p, q, r\}$. It is an extension of $T$ because $\mathbb{P} = \{p, q\} \subseteq \{p, q, r\} = \mathbb{P}'$ and:

$$\begin{aligned}\mathrm{M}_{\mathbb{P}'}(T') &= \{(0,0,0),(0,0,1),(0,1,0),(1,1,1)\} \\ &\subseteq \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,1,0),(1,1,1)\} = \mathrm{M}_{\mathbb{P}'}(T)\end{aligned}$$

In other words, by restricting the models of $T'$ to the language $\mathbb{P}$, we get $\{(0,0),(0,1),(1,1)\}$, which is a subset of $\mathrm{M}_{\mathbb{P}}(T)$.

Because $\{(0,0),(0,1),(1,1)\} = \mathrm{M}_{\mathbb{P}}(T)$, in other words, every model $v \in \mathrm{M}_{\mathbb{P}}(T)$ can be expanded to a model $v' \in \mathrm{M}_{\mathbb{P}'}(T')$ (e.g., $(0,1)$ can be expanded by defining $v'(r) = 0$ to the model $(0,1,0)$), $T'$ is even a conservative extension of $T$. This means that every proposition in the language $\mathbb{P}$ is valid in $T$ if and only if it is valid in $T'$. But the proposition $p \to r$ (which is in the language $\mathbb{P}'$, but not in the language $\mathbb{P}$) is a new consequence: it is valid in $T'$ but not in $T$ (see the model $(1,1,0)$).

The theory $T'' = \{\neg p \vee q, \neg q \vee r, \neg r \vee p\}$ in the language $\mathbb{P}'$ is an extension of $T$ but not a conservative extension because $p \leftrightarrow q$ is valid in $T''$, but not in $T$. Or because the model $(0,1)$ of the theory $T$ cannot be extended to a model of $T''$: neither $(0,1,0)$ nor $(0,1,1)$ satisfy the axioms of $T''$.

The theory $T$ is a (simple) extension of the theory $\{\neg p \vee q\}$ in the language $\mathbb{P}$, and vice versa: $T \sim \{\neg p \vee q\}$. It is also, like any theory, a simple conservative extension of itself.

*Exercise* 2.7. Show (in detail) that if a theory $T$ has a complete conservative extension, then it must itself be complete.

[24] By adding values for the new propositional variables, i.e., adding the corresponding coordinates in the vector representation.

## 2.5 Algebra of Propositions

In logic, we are usually[25] interested in propositions (or theories) *up to equivalence.*[26] The correct answer to the question 'How many different propositions are there in the language $\mathbb{P} = \{p, q, r\}$?' is 'Infinitely many'. However, we are probably interested in propositions *up to equivalence* (in other words, *mutually inequivalent*). There are as many of these as there are different subsets of models of the language, which is $2^{|\,\mathrm{M}_\mathbb{P}\,|} = 2^8 = 256$. Indeed, if two propositions have the same set of models, they are equivalent by definition. And for each set of models, we can find a corresponding proposition, e.g., in DNF (see 2.3.4). Let us consider slightly more complex reasoning:

*Example* 2.5.1. Let $T$ be a theory in the language $\mathbb{P} = \{p, q, r\}$ having exactly five models. How many propositions over $\mathbb{P}$ (up to equivalence) are independent in the theory $T$? Let $|\mathbb{P}| = n = 3$ and $|\,\mathrm{M}_\mathbb{P}(T)| = k = 5$.

We count the sets $M = \mathrm{M}_\mathbb{P}(\varphi)$ and require that $\emptyset \neq M \cap \mathrm{M}_\mathbb{P}(T) \neq \mathrm{M}_\mathbb{P}(T)$. Thus, we have a total of $2^k - 2 = 30$ possibilities for what the set $M \cap \mathrm{M}_\mathbb{P}(T)$ may be. And for each model of the language that is not a model of $T$ (there are $2^n - k = 3$ of these), we can choose arbitrarily whether or not it will be in $M$. Altogether, we get $(2^k - 2) \cdot 2^{2^n - k} = 30 \cdot 2^{8-5} = 240$ possible sets $M$. Hence that is the number of propositions independent in $T$, up to equivalence.

Let us explore the matter on a more abstract level. Formally, we consider the set of equivalence classes of $\sim$ on the set of all propositions $\mathrm{PF}_\mathbb{P}$, which we denote as ${}^{\mathrm{PF}_\mathbb{P}}/{\sim}$. The elements of this set are sets of equivalent propositions, e.g., $[p \to q]_\sim = \{p \to q, \neg p \vee q, \neg(p \wedge \neg q), \neg p \vee q \vee q, \dots\}$. And we have a mapping $h : {}^{\mathrm{PF}_\mathbb{P}}/{\sim} \to \mathcal{P}(\mathrm{M}_\mathbb{P})$ (where $\mathcal{P}(X)$ is the set of all subsets of $X$) defined by:

$$h([\varphi]_\sim) = \mathrm{M}(\varphi)$$

That is, to an equivalence class of propositions we assign the set of models of any member of that class. It is easy to verify that this mapping is well-defined (the definition does not depend on the choice of proposition from the class), injective, and if the language $\mathbb{P}$ is finite, then $h$ is even bijective. (Do verify this!)

On the set ${}^{\mathrm{PF}_\mathbb{P}}/{\sim}$, we can define operations $\neg, \wedge, \vee$ by:

$$\begin{aligned} \neg[\varphi]_\sim &= [\neg\varphi]_\sim \\ [\varphi]_\sim \wedge [\psi]_\sim &= [\varphi \wedge \psi]_\sim \\ [\varphi]_\sim \vee [\psi]_\sim &= [\varphi \vee \psi]_\sim \end{aligned}$$

In words, we select the representative or representatives and perform the operation with them, e.g., the 'conjunction' of the classes $[p \to q]_\sim$ and $[q \vee \neg r]_\sim$ is:

$$[p \to q]_\sim \wedge [q \vee \neg r]_\sim = [(p \to q) \wedge (q \vee \neg r)]_\sim$$

By adding *constants* $\bot = [\bot]_\sim$ and $\top = [\top]_\sim$, we obtain the *(mathematical) structure*[27]

$$\mathbf{AV}_\mathbb{P} = \langle {}^{\mathrm{PF}_\mathbb{P}}/{\sim}; \neg, \wedge, \vee, \bot, \top \rangle$$

[25] Unless, for example, we have a syntactic-transformation-based method such as conversion to CNF.

[26] We can view them as a kind of abstract 'properties' of models, regardless of their specific description.

[27] A structure is a non-empty set together with relations, operations, and constants. For example, a (directed) graph, a group, a field, a vector space. Structures will play an important role in predicate logic.

which we call the *algebra of propositions* of the language $\mathbb{P}$. It is an example of a so-called *Boolean algebra*. This means that its operations 'behave' like the operations $^-$, $\cap$, $\cup$ on the set of all subsets $\mathcal{P}(X)$ of some non-empty set $X$, and the constants correspond to $\emptyset$, $X$ (such a Boolean algebra is called a *power set algebra*).[28]

The mapping $h : {}^{\mathrm{PF}_\mathbb{P}}/{\sim} \to \mathcal{P}(\mathrm{M}_\mathbb{P})$ is thus an injective mapping from the algebra of propositions $\mathbf{AV}_\mathbb{P}$ to the power set algebra

$$\mathcal{P}(\mathrm{M}_\mathbb{P}) = \langle \mathcal{P}(\mathrm{M}_\mathbb{P}); {}^-, \cap, \cup, \emptyset, \mathrm{M}_\mathbb{P} \rangle$$

and if the language is finite, it is a bijection. This mapping 'preserves' the operations and constants, i.e., it holds that $h(\bot) = \emptyset$, $h(\top) = \mathrm{M}_\mathbb{P}$, and

$$\begin{aligned} h(\neg[\varphi]_\sim) &= \overline{h([\varphi]_\sim)} = \overline{\mathrm{M}(\varphi)} = \mathrm{M}_\mathbb{P} \setminus \mathrm{M}(\varphi) \\ h([\varphi]_\sim \wedge [\psi]_\sim) &= h([\varphi]_\sim) \cap h([\psi]_\sim) = \mathrm{M}(\varphi) \cap \mathrm{M}(\psi) \\ h([\varphi]_\sim \vee [\psi]_\sim) &= h([\varphi]_\sim) \cup h([\psi]_\sim) = \mathrm{M}(\varphi) \cup \mathrm{M}(\psi) \end{aligned}$$

Such a mapping is called a *homomorphism* of Boolean algebras, and if it is a bijection, it is an *isomorphism*.

*Remark* 2.5.2. We can apply these relationships when searching for models: for the proposition $\varphi \to (\neg\psi \wedge \chi)$, it holds (using the fact that $\mathrm{M}(\varphi \to \varphi') = \mathrm{M}(\neg\varphi \vee \varphi')$):

$$\mathrm{M}(\varphi \to (\neg\psi \wedge \chi)) = \overline{\mathrm{M}(\varphi)} \cup (\overline{\mathrm{M}(\psi)} \cap \mathrm{M}(\chi))$$

We can also relativize all of the above reasoning with respect to a given theory $T$ in the language $\mathbb{P}$ by replacing the equivalence $\sim$ with $T$-equivalence $\sim_T$ and the set of models of the language $\mathrm{M}_\mathbb{P}$ with the set of models of the theory $\mathrm{M}_\mathbb{P}(T)$. We get:

$$\begin{aligned} h(\bot) &= \emptyset, \\ h(\top) &= \mathrm{M}(T) \\ h(\neg[\varphi]_{\sim_T}) &= \mathrm{M}(T) \setminus \mathrm{M}(T, \varphi) \\ h([\varphi]_{\sim_T} \wedge [\psi]_{\sim_T}) &= \mathrm{M}(T, \varphi) \cap \mathrm{M}(T, \psi) \\ h([\varphi]_{\sim_T} \vee [\psi]_{\sim_T}) &= \mathrm{M}(T, \varphi) \cup \mathrm{M}(T, \psi) \end{aligned}$$

The resulting *algebra of propositions with respect to the theory* $T$ is denoted as $\mathbf{AV}_\mathbb{P}(T)$. The algebra of propositions of the language is thus the same as the algebra of propositions with respect to the empty theory. For technical reasons, we need $\mathrm{M}(T)$ to be non-empty, i.e., $T$ must be consistent. Let's summarize our considerations:

**Corollary 2.5.3.** *If $T$ is a consistent theory over a* finite *language $\mathbb{P}$, then the algebra of propositions $\mathbf{AV}_\mathbb{P}(T)$ is isomorphic to the power set algebra $\mathcal{P}(\mathrm{M}_\mathbb{P}(\mathbf{T}))$ through the mapping $h([\varphi]_{\sim_T}) = \mathrm{M}(T, \varphi)$.*

Thus, we know that negation, conjunction, and disjunction correspond to complement, intersection, and union of sets of models, and that if we want to find the number of propositions up to equivalence or $T$-equivalence, we just need to determine the number of corresponding sets of models. Let us summarize a few such calculations in the form of a proposition, leaving its proof as an exercise.

[28] That is, they satisfy certain algebraic laws, such as the distributive law of $\wedge$ over $\vee$. Boolean algebras will be formally defined later, but let us mention another important example: the set of all $n$-bit vectors with operations $\sim$, &, | (coordinate-wise) and with constants $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$.

**Proposition 2.5.4.** *Let $\mathbb{P}$ be a language of size $n$ and $T$ a consistent theory in $\mathbb{P}$ with exactly $k$ models. Then in $\mathbb{P}$ there are,* up to equivalence*:*

- $2^{2^n}$ *propositions (or theories),*
- $2^{2^n-k}$ *propositions true (or false) in $T$,*
- $2^{2^n} - 2 \cdot 2^{2^n-k}$ *propositions independent in $T$,*
- $2^k$ *simple extensions of the theory $T$ (of which 1 is inconsistent),*
- $k$ *complete simple extensions of $T$.*

*Furthermore,* up to $T$-equivalence*, there are:*

- $2^k$ *propositions,*
- 1 *proposition true in $T$,* 1 *false in $T$,*
- $2^k - 2$ *propositions independent in $T$.*

*Exercise* 2.8. Choose a suitable theory $T$ and use it as an example to demonstrate that Proposition 2.5.4 holds.

*Exercise* 2.9. Prove Proposition 2.5.4 in detail. (Draw a Venn diagram.)

*Exercise* 2.10. Prove in detail that the mapping $h$ from Corollary 2.5.3 is well-defined, injective, and if the language is finite, also surjective.

# Chapter 3

# The Boolean Satisfiability Problem

*The boolean satisfiability problem*, also known as the *SAT problem*, is the following computational task: The input is a proposition $\varphi$ in CNF (in some reasonable encoding[1]), and the goal is to decide whether $\varphi$ is *satisfiable*.[2]

As we demonstrated in the previous chapter, we can convert every proposition, or even every propositional theory in a finite language, into a CNF formula. The SAT problem is thus, in a sense, universal; it answers the question of whether there exists a model.

The well-known Cook-Levin theorem states that the SAT problem is *NP-complete*, meaning it is in the NP class (if an oracle provides the right truth assignment, we can easily verify that all clauses are satisfied) and that every problem in the NP class can be reduced to it in polynomial time (specifically, the computation of a Turing machine can be described using a CNF formula).[3]

However, practical SAT solvers can handle instances containing many (up to tens of millions) propositional variables and clauses. In this chapter, we will first demonstrate the practical application of a SAT solver to a 'real-life' problem, then show two fragments of the SAT problem, namely *2-SAT* and *Horn-SAT*, for which there are polynomial algorithms, and finally, we will also present the DPLL algorithm, which is the basis of (almost) all SAT solvers. (Later, in Chapter 4, we will also see the connection with the *resolution method*.)

## 3.1 SAT Solvers

The first SAT solvers were developed in the 1960s. They are almost always based on the DPLL (Davis–Putnam–Logemann–Loveland) algorithm, which we will introduce in Section 3.4, or some of its improvements. After 2000, there has been a somewhat surprising, dramatic development of technologies for SAT solvers, leading to a rapid increase in their utility in various areas of applied computer science.

Modern SAT solvers use a range of technologies for efficiently solving typical instances from various application domains, strategies, and heuristics for exploring the solution space (including, for examle, the use of machine learning and neural networks), and other enhancements. These modern tools typically have several tens of thousands of lines of code. The

[1] For example, the DIMACS-CNF format.

[2] Beware, in some literature SAT means satisfiability of an *arbitrary* proposition, the restriction of inputs to CNF is done only for the problem $k$-SAT (see below).

[3] See the course NTIN090 Introduction to Complexity and Computability.

availability of efficient SAT solvers has significantly impacted developments in areas such as software verification, program analysis, optimization, and artificial intelligence. The best SAT solvers regularly compete in the SAT competition.

To try out SAT solving, we will use the solver `Glucose`. It accepts input in the simple DIMACS CNF format. Let's demonstrate the usage on the following puzzle called *boardomino*:

*Example* 3.1.1 (Boardomino). Can a chessboard with two opposite corners cut out be perfectly covered with domino tiles?

How to formalize this problem? Let us choose propositional variables $h_{i,j}, v_{i,j}$ $(1 \leq i, j \leq n)$, where $h_{i,j}$ means "the left half of a horizontally oriented domino lies at position $(i, j)$" and similarly $v_{i,j}$ for the top half of a vertically oriented domino. Here $n = 8$, but we can try it for other (even) dimensions of the chessboard. Now we axiomatize all required properties:

- the upper left and lower right corners are missing: $\neg h_{11}, \neg v_{11}, \neg h_{n,n-1}, \neg v_{n-1,n}$
- dominos do not extend beyond the chessboard (to the right or down): $\neg h_{i,n}, \neg v_{n,i}$ for $1 \leq i \leq n$
- each square is covered by at least one domino (the first row and column separately):

$$\begin{aligned} h_{i,j-1} \vee h_{i,j} \vee v_{i-1,j} \vee v_{i,j} &\text{ for } 1 < i, j \leq n \\ h_{1,j-1} \vee h_{1,j} \vee v_{1,j} &\text{ for } 1 < j \leq n \\ h_{i,1} \vee v_{i-1,1} \vee v_{i,1} &\text{ for } 1 < i \leq n \end{aligned}$$

- each square is covered by at most one domino (the first row and column separately):

$$\begin{aligned} (\neg h_{i,j-1} \vee \neg h_{i,j}) \wedge (\neg h_{i,j-1} \vee \neg v_{i-1,j}) \wedge (\neg h_{i,j-1} \vee \neg v_{i,j}) \wedge \\ (\neg h_{i,j} \vee \neg v_{i-1,j}) \wedge (\neg h_{i,j} \vee \neg v_{i,j}) \wedge (\neg v_{i-1,j} \vee \neg v_{i,j}) &\text{ for } 1 < i, j \leq n \\ (\neg h_{1,j-1} \vee \neg h_{1,j}) \wedge (\neg h_{1,j-1} \vee \neg v_{1,j}) \wedge (\neg h_{1,j} \vee \neg v_{1,j}) &\text{ for } 1 < j \leq n \\ (\neg h_{i,1} \vee \neg v_{i-1,1}) \wedge (\neg h_{i,1} \vee \neg v_{i,1}) \wedge (\neg v_{i-1,1} \vee \neg v_{i,1}) &\text{ for } 1 < i \leq n \end{aligned}$$

The resulting theory is already in CNF, and we can easily write it in the DIMACS CNF format and solve it using the Glucose solver. In practice, we might program this conversion or use one of the many high-level languages from the *constraint programming* area that allow translation to SAT.

We will see that such instances of the SAT problem will be difficult for the solver, and for relatively small dimensions of the chessboard, we will not get a solution. As mathematicians, we can easily see that there is no solution: Each domino covers one white and one black square, but we removed two white squares, so necessarily two black squares will remain. However, this view is not available in the CNF encoding. It is possible to find partial truth assignments of almost all variables without violating any condition. The solver will have to search almost the entire solution space before proving unsatisfiability.[4] The key insight into SAT solving is that such difficult instances almost never occur in practice.

[4]Similar properties also hold for the encoding of the *pigeonhole principle* into SAT.

## 3.2 2-SAT and Implication Graph

A proposition $\varphi$ is in $k$-CNF if it is in CNF and each clause has at most $k$ literals. The $k$-SAT problem asks whether a given $k$-CNF proposition is satisfiable. For $k \geq 3$, $k$-SAT remains NP-complete; any CNF proposition can be encoded into a 3-CNF proposition:

*Exercise* 3.1. Show that for any proposition $\varphi$ in CNF, there exists an *equisatisfiable* proposition $\varphi'$ in 3-CNF (i.e., $\varphi$ is satisfiable if and only if $\varphi'$ is satisfiable), which can be constructed in linear time.

For the 2-SAT problem, however, there is a polynomial (linear, even) algorithm, which we will now introduce. The algorithm utilizes the notion of *implication graph*. We will demonstrate the procedure with an example:

*Example* 3.2.1. Consider the following 2-CNF proposition $\varphi$:

$$(\neg p_1 \vee p_2) \wedge (\neg p_2 \vee \neg p_3) \wedge (p_1 \vee p_3) \wedge (p_3 \vee \neg p_4) \wedge (\neg p_1 \vee p_5) \wedge (p_2 \vee p_5) \wedge p_1 \wedge \neg p_4$$

### Implication Graph

The implication graph of a 2-CNF proposition $\varphi$ is based on the idea that a 2-clause $\ell_1 \vee \ell_2$ (where $\ell_1$ and $\ell_2$ are literals) can be viewed as a pair of implications: $\overline{\ell_1} \to \ell_2$ and $\overline{\ell_2} \to \ell_1$.[5] For example, from the clause $\neg p_1 \vee p_2$, we get the implications $p_1 \to p_2$ and $\neg p_2 \to \neg p_1$. Therefore, if $p_1$ holds in some model, $p_2$ must hold as well, and if $p_2$ does not hold, then $p_1$ must not hold either. A unit clause $\ell$ can also be expressed as an implication, namely $\overline{\ell} \to \ell$, e.g., from $p_1$ we get $\neg p_1 \to p_1$.

The implication graph $\mathcal{G}_\varphi$ is thus a directed graph, whose vertices are all the literals (variables from $\mathrm{Var}(\varphi)$ and their negations), and edges are given by the implications described above:

- $V(\mathcal{G}_\varphi) = \{p, \neg p \mid p \in \mathrm{Var}(\varphi)\}$,
- $E(\mathcal{G}_\varphi) = \{(\overline{\ell_1}, \ell_2), (\overline{\ell_2}, \ell_1) \mid \ell_1 \vee \ell_2 \text{ is a clause in } \varphi\} \cup \{(\overline{\ell}, \ell) \mid \ell \text{ is a unit clause in } \varphi\}$

In our example, the set of vertices is

$$V(\mathcal{G}_\varphi) = \{p_1, p_2, p_3, p_4, p_5, \neg p_1, \neg p_2, \neg p_3, \neg p_4, \neg p_5\}$$

and the edges are:

$$E(\mathcal{G}_\varphi) = \{(p_1, p_2), (\neg p_2, \neg p_1), (p_2, \neg p_3), (p_3, \neg p_2), (\neg p_1, p_3), (\neg p_3, p_1), (\neg p_3, \neg p_4), \\ (p_4, p_3), (p_1, p_5), (\neg p_5, \neg p_1), (\neg p_2, p_5), (\neg p_5, p_2), (\neg p_1, p_1), (p_4, \neg p_4)\}$$

The resulting graph is shown in Figure 3.1.

### 3.2.1 Strongly Connected Components

We now need to find the strongly connected components[6] of this graph. In our example, we get the following components: $C_1 = \{p_4\}$, $C_2 = \{\neg p_5\}$, $C_3 = \{\neg p_1, \neg p_2, p_3\}$, $\overline{C_3} = \{p_1, p_2, \neg p_3\}$, $\overline{C_2} = \{p_5\}$, $\overline{C_1} = \{\neg p_4\}$.

[5] In the previous chapter, we expressed $p_1 \to p_2$ as $\neg p_1 \vee p_2$; here, we are performing the reverse procedure.

[6] *Strong connectivity* means that there is a directed path from $u$ to $v$ and from $v$ to $u$, i.e., every two vertices in one component lie in a directed cycle. Conversely, every directed cycle lies within some component.

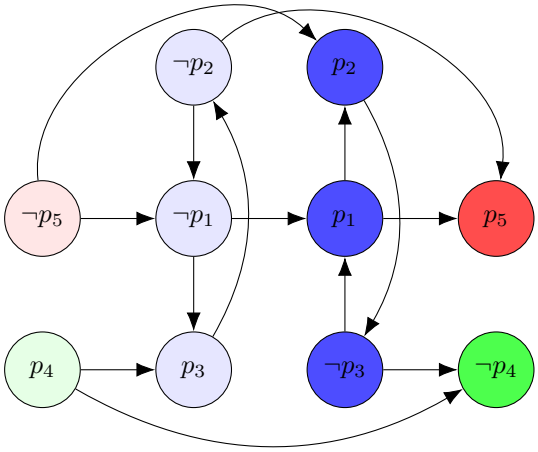



Figure 3.1: Implication graph $\mathcal{G}_\varphi$. Strongly connected components are distinguished by colors.

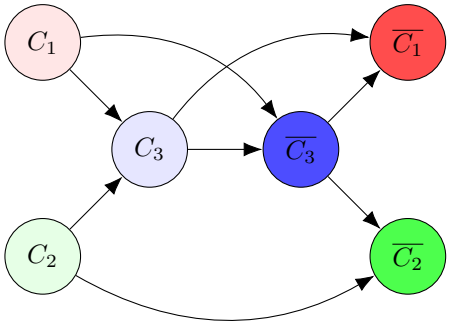



Figure 3.2: Implication graph $\mathcal{G}_\varphi$. Graph of strongly connected components $\mathcal{G}^*_\varphi$.

All literals in one component must be assigned the same value. Therefore, if we find a pair of opposite literals in one component, it means that the proposition is unsatisfiable. Otherwise, we can always find a satisfying assignment, as we will prove in Proposition 3.2.2. We need to ensure that no component assigned 1 has an edge leading to a component assigned 0. If we contract the components (and remove loops), the resulting graph $\mathcal{G}^*_\varphi$ is acyclic (every cycle was within some component), see Figure 3.2. Thus we can draw it in a *topological ordering* (i.e., an ordering on a line where edges only go to the right), see Figure 3.3 below.

When searching for a satisfying assignment (if we are not content with the information that the proposition is satisfiable), we proceed by taking the leftmost unassigned component, assigning it 0, assigning the opposite component 1, and repeating this process until no unassigned components remain. For example, the topological ordering in Figure 3.3 corresponds to the model $v = (1, 1, 0, 0, 1)$.

Finally, we summarize our reasoning in the following proposition:

**Proposition 3.2.2.** *A proposition $\varphi$ is satisfiable if and only if no strongly connected component in $\mathcal{G}_\varphi$ contains a pair of opposite literals $\ell, \overline{\ell}$.*

*Proof.* Every model, i.e., valid truth assignment, must assign all literals in the same compo-

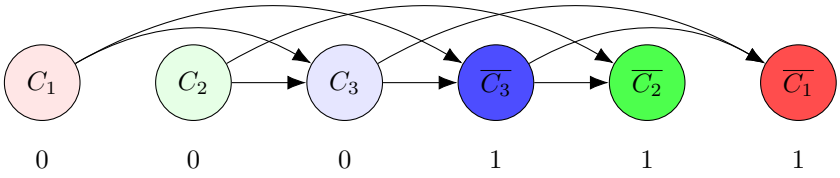



Figure 3.3: Implication graph $\mathcal{G}_\varphi$. Topological ordering of the graph $\mathcal{G}_\varphi^*$ and satisfying assignment of the components.

nent the same truth value. (Otherwise, there would necessarily be an implication $\ell_1 \to \ell_2$ where $\ell_1$ is valid in the model but $\ell_2$ is not.) Therefore, there cannot be opposite literals in one component.

Conversely, assume that no component contains a pair of opposite literals, and let us show that there exists a model. Let $\mathcal{G}_\varphi^*$ be the graph obtained from $\mathcal{G}_\varphi$ by contracting the strongly connected components. This graph is acyclic; choose a topological ordering. We construct a model by choosing the first unassigned component in our topological ordering, assigning all literals in it the value 0, and assigning the opposite literals the value 1. We continue this process until all components are assigned.

Why does the proposition $\varphi$ hold in the model thus obtained? If it did not, some clause would not hold. A unit clause $\ell$ must hold because there is an edge $\overline{\ell} \to \ell$ in the graph $\mathcal{G}_\varphi$. The same edge is also in the graph of components, so $\overline{\ell}$ precedes the component containing $\ell$ in the topological ordering. In constructing the model, we must have assigned $\overline{\ell} = 0$, so $\ell = 1$. Similarly, a 2-clause $\ell_1 \vee \ell_2$ must also hold: there are edges $\overline{\ell_1} \to \ell_2$ and $\overline{\ell_2} \to \ell_1$. If we assigned $\ell_1$ first, we must have assigned $\overline{\ell_1} = 0$ due to the edge $\overline{\ell_1} \to \ell_2$, so $\ell_1$ holds. Similarly, if we assigned $\ell_2$ first, $\overline{\ell_2} = 0$ and $\ell_2 = 1$. □

**Corollary 3.2.3.** *The 2-SAT problem is solvable in linear time. We can also construct a model in linear time, if one exists.*

*Proof.* The strongly connected components can be found in $\mathcal{O}(|V| + |E|)$ time, and the topological ordering can also be constructed in $\mathcal{O}(|V| + |E|)$ time. □

*Exercise* 3.2. Find an unsatisfiable 2-CNF proposition, construct its implication graph, and verify that there is a pair of opposite literals in the same strongly connected component.

*Exercise* 3.3. Find all topological orderings of the graph $\mathcal{G}_\varphi^*$ from the example above and the corresponding models. Think about why we obtain exactly all models of the proposition $\varphi$ in this way.

*Exercise* 3.4. Explain how to find the components and topological ordering in $\mathcal{O}(|V| + |E|)$ time.

## 3.3 Horn-SAT and Unit Propagation

Next, we will introduce another fragment of SAT that can be solved in polynomial time, the so-called *Horn-SAT*, the *Horn satisfiability problem*. A propositional formula is *Horn*

(*in Horn form*)[7] if it is a conjunction of *Horn clauses*, i.e., clauses containing *at most one *positive* literal*. The significance of Horn clauses is evident from their equivalent expression in the form of an implication:

$$\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n \vee q \;\sim\; (p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$$

Horn formulas are thus well suited to model systems where the satisfaction of certain conditions guarantees the satisfaction of another condition. Note that a unit clause $\ell$ is also Horn. In the context of logic programming, it is called a *fact* if the literal is positive, and a *goal* if it is negative.[8] Horn formulas with at least one positive and at least one negative literal are *rules*.

*Example* 3.3.1. An example of a proposition that is in CNF but not Horn is, for instance, $(p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_3)$. As an example to illustrate the algorithm, we will use the following Horn formula:

$$\varphi = (\neg p_1 \vee p_2) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_2 \vee \neg p_3) \wedge (\neg p_5 \vee \neg p_4) \wedge p_4$$

The polynomial algorithm for solving the Horn-SAT problem is based on the simple idea of *unit propagation*: If our proposition contains a *unit* clause, we know how the propositional variable in this clause must be valued. And this knowledge can be *propagated*—exploited to simplify the proposition.

Our proposition $\varphi$ contains the unit clause $p_4$. We know, therefore, that every model $v \in \mathrm{M}(\varphi)$ must satisfy $v(p_4) = 1$. This means that in any model of the proposition $\varphi$:

- Every clause containing the positive literal $p_4$ is satisfied, so we can remove it from the proposition,
- The negative literal $\neg p_4$ cannot be satisfied, so we can remove it from all clauses that contain it.

This step is called *unit propagation*. The result is the following simplified proposition, denoted as $\varphi^{p_4}$ (in general $\varphi^{\ell}$ if we have a unit clause $\ell$):

$$\varphi^{p_4} = (\neg p_1 \vee p_2) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_2 \vee \neg p_3) \wedge \neg p_5$$

**Observation 3.3.2.** *Note that $\varphi^{\ell}$ no longer contains the literal $\ell$ nor $\overline{\ell}$, and it is obvious that the models of $\varphi$ are exactly the models of $\{\varphi^{\ell}, \ell\}$, i.e., the models of $\varphi^{\ell}$ in the original language $\mathbb{P}$, in which $\ell$ is valid.*

By unit propagation, we obtained a new unit clause $\neg p_5$ in the proposition $\varphi^{p_4}$, so we can continue by setting $v(p_5) = 0$ and performing another unit propagation:

$$(\varphi^{p_4})^{\neg p_5} = (\neg p_1 \vee p_2) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_2 \vee \neg p_3)$$

The resulting proposition no longer contains a unit clause. This means that each clause contains at least two literals, and at most one of them can be positive! (Here we need the 'Horn property' of the proposition.) Because each clause contains a negative literal, it is sufficient to evaluate all remaining variables as 0, and the proposition will be satisfied: $v(p_1) = v(p_2) = v(p_3) = 0$. Thus, we get the model $v = (0, 0, 0, 1, 0)$.

[7] Mathematician Alfred Horn discovered the significance of this form of logical formulas (thus laying the foundation for logic programming) in 1951.

[8] Because we are proving by contradiction, more on this in Chapter 4 and Section **??**.

*Example* 3.3.3. What would happen if the proposition was not satisfiable? Let us look at the proposition

$$\psi = p \wedge (\neg p \vee q) \wedge (\neg q \vee r) \wedge \neg r$$

and perform unit propagation as in the previous example: we have $v(p) = 1$ and $\psi^p = q \wedge (\neg q \vee r) \wedge \neg r$, further $v(q) = 1$ and $(\psi^p)^q = r \wedge \neg r$. This proposition is unsatisfiable because it contains a pair of opposite unit clauses. [9]

Let us summarize the algorithm for solving the Horn-SAT problem:

**Algorithm** (Horn-SAT)**. input**: a proposition $\varphi$ in Horn form, **output**: a model of $\varphi$ or information that $\varphi$ is unsatisfiable

1. If $\varphi$ contains a pair of opposite unit clauses $\ell, \overline{\ell}$, it is unsatisfiable.
2. If $\varphi$ does not contain any unit clause, it is satisfiable, evaluate remaining variables to 0.
3. If $\varphi$ contains a unit clause $\ell$, evaluate the literal $\ell$ to 1, perform unit propagation, replace $\varphi$ with the proposition $\varphi^\ell$, and return to the beginning.

**Proposition 3.3.4.** *The algorithm is correct.*

*Proof.* Correctness follows from the Observation and the preceding discussion. □

**Corollary 3.3.5.** *Horn-SAT can be solved in linear time.*

*Proof.* In each step, it is sufficient to traverse the proposition once, and unit propagation always shortens the proposition. This easily gives a quadratic upper bound, but with a suitable implementation, we can achieve linear time with respect to the length of $\varphi$. □

*Exercise* 3.5. Propose an implementation of the algorithm for Horn-SAT in linear time.

*Exercise* 3.6. Propose a modification of the Horn-SAT algorithm that finds all models.

## 3.4 The DPLL Algorithm for Solving the SAT Problem

To conclude the chapter on the satisfiability problem, we will introduce the most widely used algorithm, by far, for solving the general SAT problem: the DPLL algorithm.[10] Although it has exponential complexity in the worst case, it performs very efficiently in practice.

The algorithm uses unit propagation along with the following observation: We say that a literal $\ell$ has *pure occurrence* in $\varphi$ if it occurs in $\varphi$ but the opposite literal $\overline{\ell}$ does not occur in $\varphi$. If we have a literal with pure occurrence, we can set its value to 1, and thus satisfy (and remove) all clauses containing it. If the proposition cannot be simplified neither by unit propagation nor by pure literal, we branch the computation by assigning both possible values to a selected propositional variable.

**Algorithm** (DPLL)**. input**: a proposition $\varphi$ in CNF, **output**: a model of $\varphi$ or information that $\varphi$ is unsatisfiable

[9] In other words, in the next step, we would perform unit propagation on $r$, remove the unit clause $r$, and from the remaining unit clause $\neg r$, we would remove the literal $\neg r$, resulting in an *empty clause*, which is unsatisfiable.

[10] Named after its creators, Davis-Putnam-Logemann-Loveland, it was invented in 1961.

1. While $\varphi$ contains a unit clause $\ell$, set the literal $\ell$ to 1, perform unit propagation, and replace $\varphi$ with the proposition $\varphi^\ell$.
2. While there is a literal $\ell$ with pure occurrence in $\varphi$, set $\ell$ to 1, and remove clauses containing $\ell$.
3. If $\varphi$ contains no clauses, it is satisfiable.
4. If $\varphi$ contains an empty clause, it is unsatisfiable.
5. Otherwise, choose an unassigned propositional variable $p$, and recursively call the algorithm on $\varphi \wedge p$ and on $\varphi \wedge \neg p$.

The algorithm runs in exponential time: the number of branching points in the computation cannot exceed the number of variables. It can be shown that in the worst case, exponential time is indeed needed. The correctness of the algorithm is not difficult to verify.

**Proposition 3.4.1.** *The DPLL algorithm solves the SAT problem.*

*Example* 3.4.2. We will demonstrate the algorithm on the following example:

$$(\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee \neg s) \wedge (p \vee \neg r \vee \neg s) \wedge (q \vee \neg r \vee s) \wedge (p \vee s) \wedge (p \vee \neg s) \wedge (q \vee s)$$

The proposition does not contain any unit clauses. The literal $\neg r$ has pure occurrence, so we set $v(r) = 0$ and remove the clauses containing $\neg r$:

$$(\neg p \vee \neg q \vee \neg s) \wedge (p \vee s) \wedge (p \vee \neg s) \wedge (q \vee s)$$

No other literal has pure occurrence. We therefore recursively run the algorithm:

(p=1) Add the unit clause $p$:

$$(\neg p \vee \neg q \vee \neg s) \wedge (p \vee s) \wedge (p \vee \neg s) \wedge (q \vee s) \wedge p$$

Set $v(p) = 1$ and perform unit propagation: $(\neg q \vee \neg s) \wedge (q \vee s)$. Now we branch on the variable $q$:

(q=1) $(\neg q \vee \neg s) \wedge (q \vee s) \wedge q$. After setting $v(q) = 1$ and unit propagation, we get $\neg s$, and after setting $v(s) = 0$ and unit propagation, we get a proposition containing no clauses, which is thus satisfiable by the model $(1, 1, 0, 0)$. We already have the answer to the satisfiability problem, so we do not need to complete the other branches of the computation. For illustration, we will do so.

(q=0) $(\neg q \vee \neg s) \wedge (q \vee s) \wedge \neg q$. By unit propagation with $v(q) = 0$, we get $s$, and after setting $v(s) = 1$ and unit propagation, we get an empty set of clauses. We obtain the model $(1, 0, 0, 1)$.

(p=0) Add the unit clause $\neg p$:

$$(\neg p \vee \neg q \vee \neg s) \wedge (p \vee s) \wedge (p \vee \neg s) \wedge (q \vee s) \wedge \neg p$$

After performing unit propagation on $\neg p$, we have $s \wedge \neg s \wedge (q \vee s)$. After unit propagation on $s$, we have $\square \wedge q$, where $\square$ is an empty clause. The proposition is thus unsatisfiable, and we do not get any models in this branch.

We found that the original proposition is satisfiable. We found 2 models: $(1, 1, 0, 0)$ and $(1, 0, 0, 1)$. However, there may be other models, the valuation $v(r) = 0$ for the literal $\neg r$ with pure occurrence may not be necessary to satisfy all clauses; this step does not preserve the set of models, only satisfiability.

What's next? The basic DPLL algorithm, which systematically explores the solution space, was enhanced and extended in various ways at the end of 1990s. Let us mention the algorithm called *Conflict-driven clause learning (CDCL)*. It is based on the idea that we can *learn* a new clause from the failure of a branch in the search tree, which prevents future repetitions of the same failure ("conflict"). Additionally, we can backtrack in the tree by more than one level at once (called *back-jumping*) to the point where we started evaluating variables in this new clause. Thus we prevent repeatedly discovering the "same" conflict. You can learn much more about SAT solvers in the course NAIL094 Decision procedures and SAT/SMT solvers.

# Chapter 4

# The Method of Analytic Tableaux

In this chapter, we introduce the *Method of Analytic Tableaux*. It is a syntactic procedure that can be used to determine whether a given proposition is true in a given theory without having to deal with semantics (e.g., searching for all models, which is impractical). We will prove its *soundness* ('it gives correct answers') and *completeness* ('it always works'). Moreover, we will use it to prove the so-called *Compactness Theorem* ('properties of an infinite object can be established by proving them only for all its finite parts').

## 4.1 Formal Proof Systems

A *formal proof system* formalizes 'proof' (e.g., in mathematics) as a precisely (algorithmically) given syntactic procedure. A *proof* of the fact that a proposition $\varphi$ holds in a theory $T$ (i.e., $T \models \varphi$) is a finite syntactic object derived from the axioms of $T$ and the proposition $\varphi$. If a proof exists, it can be found 'algorithmically'.[1] Moreover, one must be able to algorithmically (and reasonably efficiently) verify that a given object is indeed a valid proof.

If a proof exists, we say that $\varphi$ is *provable* [in the given proof system] from $T$, and we write $T \vdash \varphi$. We require the following two properties from a proof system:

- *soundness*: if a proposition is provable from a theory, then it is true in that theory ($T \vdash \varphi \Rightarrow T \models \varphi$)
- *completeness*: if a proposition is true in a theory, then it is provable from that theory ($T \models \varphi \Rightarrow T \vdash \varphi$)

(While soundness is always required, an effective proof system can be practical even if it is not complete, especially if it is complete for some interesting class of propositions or theories.)

In this chapter, besides the *tableaux method*, we will also introduce the *Hilbert calculus*, and in the next chapter, we will present another proof system, the so-called *resolution method*.

## 4.2 Introduction to the Tableaux Method

For the rest of this chapter, we will always assume that we have a *countable* language $\mathbb{P}$. This implies that any theory over $\mathbb{P}$ is also countable. We will first focus on the case where $T = \emptyset$, i.e., we are proving that a proposition $\varphi$ is *logically* valid (it is a *tautology*).

[1] Here we must be cautious in the case of an infinite theory $T$: how is it given? The algorithm must have effective access to all axioms.

A *tableau* is a labeled tree representing the search for a counterexample, i.e., a model in which $\varphi$ does not hold. The labels on the nodes, which we will call *entries*, consist of the symbol T or F ('True'/'False') followed by a proposition $\psi$ and represent the assumption (requirement) that the proposition $\psi$ is or is not valid in a model, respectively. At the root of the tableau, we place the entry $\mathrm{F}\varphi$, i.e., we are looking for a model in which $\varphi$ *is not valid*. We will then develop the tableau using rules for *reducing* the entries. These rules ensure the following invariant:

> Any model that *agrees* the entry at the root (i.e., in which $\varphi$ is not valid) must *agree* with some branch of the tableau (i.e., satisfy all the requirements expressed by the entries on that branch).

If a branch contains entries of the form $\mathrm{T}\psi$ and $\mathrm{F}\psi$ for the same $\psi$, we say that the branch *fails* (is *contradictory*) and we know that no model can agree with it. Thus, if all branches fail, we can conclude that there is no model in which $\varphi$ is invalid, and thus we have a *proof* that $\varphi$ is a tautology. (Note that this is a *proof by contradiction*.)

If a branch does not fail and is *finished*, i.e., all entries are reduced, we know that $\varphi$ is not a tautology, and we will be able to construct a specific model from this branch in which $\varphi$ is not valid.

*Example* 4.2.1. Let us illustrate the entire procedure with two examples, see Figure 4.2.1.

(a) First, let us construct a tableau proof of the proposition $\varphi = ((p \to q) \to p) \to p$. We start with the root entry $\mathrm{F}\varphi$. This entry is of the form $\mathrm{F}\varphi_1 \to \varphi_2$ ('implication does not hold'), so if any model agrees with it, it must satisfy $\mathrm{T}(p \to q) \to p$ and $\mathrm{F}p$. We thus append these two entries. (Technically, we append the *atomic tableau* for this case, see Table 4.1, but we omit the root of this atomic tableau to avoid repeating the same entry.) This reduces the entry at the root.

We continue with the entry $\mathrm{T}(p \to q) \to p$, which is of the form 'implication holds'. We split into two branches: the model agrees with $\mathrm{F}(p \to q)$ or with $\mathrm{T}p$ (or both). The right branch *fails* (is *contradictory*) because it contains the entries $\mathrm{T}p$ and $\mathrm{F}p$, so no model agrees with it; we mark it with the symbol $\otimes$. In the left branch, we further reduce the entry $\mathrm{F}p \to q$ and also get a contradictory branch. All branches are contradictory, so no counterexample exists, and we have a proof of the proposition $\varphi$. We write $\vdash \varphi$.

(b) Now let us construct a tableau with the entry $\mathrm{F}(\neg q \vee p) \to p$ in the root. We are trying to find a counterexample: a model in which $(\neg q \vee p) \to p$ does not hold. We first used the atomic tableau for 'implication does not hold' and then reduced the entry $\mathrm{T}\neg q \vee p$ by adding the atomic tableau for 'disjunction holds'. The right branch failed. In the left branch, we further reduced $\mathrm{T}\neg q$ to $\mathrm{F}q$ (the atomic tableau for 'negation holds'), obtaining a finished branch since all entries have been reduced. This finished branch is non-contradictory (we mark it with the symbol ✓). This means that a counterexample exists: we have the entries $\mathrm{F}p$ and $\mathrm{F}q$, which correspond to the model $(0, 0)$, where $(\neg q \vee p) \to p$ does not hold.

In the next section, we will formalize the entire procedure and explain how to proceed when we want to prove propositions not in logic, but rather in some theory $T$ (spoiler alert: we append the entries $\mathrm{T}\alpha$ for the axioms $\alpha \in T$ during the construction). We will also see an example with an infinite theory where a *finished* branch sometimes needs to be infinite.

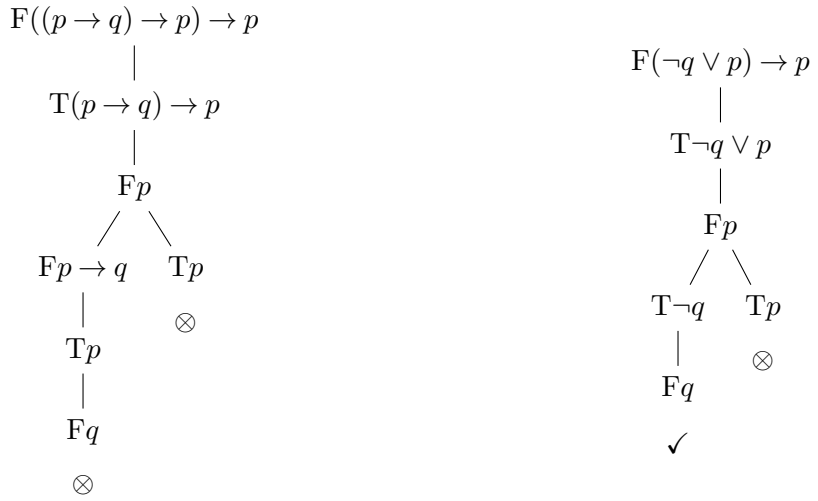



Figure 4.1: Examples of tableaux. (a) Tableau proof of the proposition $((p \to q) \to p) \to p$. (b) Tableau for the proposition $(\neg q \vee p) \to p$. The left branch gives a counterexample, the model $(0, 0)$ where the proposition is not valid.

In the rest of this section, we will introduce all the *atomic tableaux* needed for construction, and formalize the concept of a *tree*.

### 4.2.1 Atomic Tableaux

Atomic tableaux represent the rules by which we reduce the entries. For each logical connective and each of the two signs T/ F, we have one atomic tableau, shown in Table 4.1.

| | ¬ | ∧ | ∨ | → | ↔ |
|---|---|---|---|---|---|
| True | $\mathrm{T}\neg\varphi$<br>$\mathrm{F}\varphi$ | $\mathrm{T}\varphi \wedge \psi$<br>$\mathrm{T}\varphi$<br>$\mathrm{T}\psi$ | $\mathrm{T}\varphi \vee \psi$<br>$\mathrm{T}\varphi \quad \mathrm{T}\psi$ | $\mathrm{T}\varphi \to \psi$<br>$\mathrm{F}\varphi \quad \mathrm{T}\psi$ | $\mathrm{T}\varphi \leftrightarrow \psi$<br>$\mathrm{T}\varphi \quad \mathrm{F}\varphi$<br>$\mathrm{T}\psi \quad \mathrm{F}\psi$ |
| False | $\mathrm{F}\neg\varphi$<br>$\mathrm{T}\varphi$ | $\mathrm{F}\varphi \wedge \psi$<br>$\mathrm{F}\varphi \quad \mathrm{F}\psi$ | $\mathrm{F}\varphi \vee \psi$<br>$\mathrm{F}\varphi$<br>$\mathrm{F}\psi$ | $\mathrm{F}\varphi \to \psi$<br>$\mathrm{T}\varphi$<br>$\mathrm{F}\psi$ | $\mathrm{F}\varphi \leftrightarrow \psi$<br>$\mathrm{T}\varphi \quad \mathrm{F}\varphi$<br>$\mathrm{F}\psi \quad \mathrm{T}\psi$ |

Table 4.1: Atomic tableaux

The tableaux from Example 4.2.1 are constructed by sequentially adding atomic tableaux, see Figure 4.2.1. The roots of the atomic tableaux are marked in blue; we will adopt the convention of not drawing them.

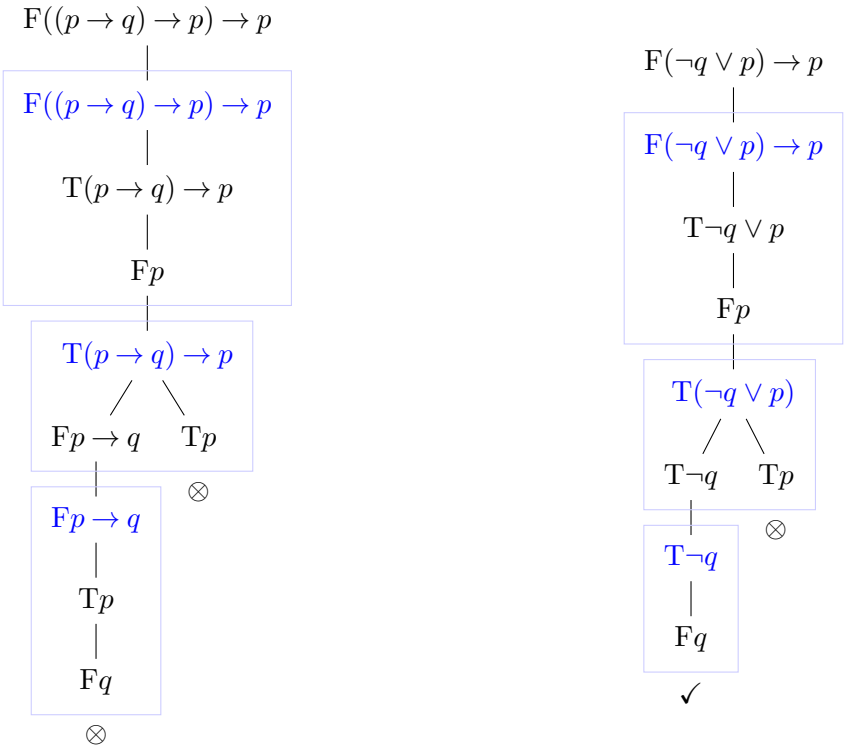



Figure 4.2: Construction of tableaux from Example 4.2.1.

*Exercise* 4.1. Try to construct a tableau with the entry $\mathrm{F}((\neg p \wedge \neg q) \vee p) \to (\neg p \wedge \neg q)$ at the root and also a tableau with the entry $\mathrm{T}(p \to q) \leftrightarrow (p \wedge \neg q)$. When constructing, use only atomic tableaux (check if your construction matches the definition of a tableau from the following section). Think about what these tableaux say about the propositions at their roots.

*Exercise* 4.2. Verify that all atomic tableaux satisfy the invariant: if a model agrees with the entry at the root, it agrees with some branch.

*Exercise* 4.3. Propose atomic tableaux for the logical connectives NAND, NOR, XOR, IFTE.

### 4.2.2 On Trees

Before we proceed to the formal definition and proofs, let us specify what we mean by a 'tree'. In graph theory, a tree would mean a connected graph without cycles, but our trees are rooted, ordered (i.e., with left-right order among the children of each node), and labeled. They can, and often will, be infinite. Formally:

**Definition 4.2.2** (Tree)**.**

- A *tree* is a non-empty set $T$ with a partial order $<_T$ that has a (single) minimal element (*root*) and in which the set of ancestors of any node is *well-ordered*.[2]
- A *branch* of a tree $T$ is a maximal[3] linearly ordered subset of $T$.

[2] That is, every non-empty subset has a least element. (This prevents infinite descending chains of ancestors.)
[3] That is, it cannot be extended by adding more nodes from the tree.

- An *ordered tree* is a tree $T$ along with a linear order $<_L$ on the set of children of each node. We call the order of children *left-right* while the order $<_T$ is *tree order*.
- A *labeled tree* is a tree along with a labeling function label: $V(T) \to$ Labels.

We will use standard terminology about trees, for example, we will talk about the *n-th level of a tree* or the *depth* of a tree (which is infinite, if and only if we have an infinite branch). In one of the theorems that we will prove below, we will need the following famous result, which is a consequence of the axiom of choice.

**Lemma 4.2.3** (König's Lemma)**.** *An infinite, finitely branching tree has an infinite branch.*

(A tree is *finitely branching* if each node has finitely many children.)

## 4.3 Tableau Proof

We now present the formal definition of a tableau. In the definition, we also include a theory $T$: its axioms can be added during construction with the sign T ("true"). Recall that an *entry* is an expression T$\varphi$ or F$\varphi$, where $\varphi$ is some proposition.

**Definition 4.3.1** (Tableau)**.** A *finite tableau from a theory* $T$ is an ordered, labeled tree constructed by applying finitely many of the following rules:

- A single-node tree labeled with any entry is a tableau from the theory $T$.
- For any entry $E$ on any branch $B$, we can append the atomic tableau for the entry $E$ at the end of the branch $B$.
- We can append the entry T$\alpha$ for any axiom $\alpha \in T$ at the end of any branch.

A *tableau from the theory* $T$ can be either finite or *infinite*: in the latter case, it is constructed in countably many steps. Formally, it can be expressed as the union $\tau = \bigcup_{i \geq 0} \tau_i$, where $\tau_i$ are finite tableaux from $T$, $\tau_0$ is a single-node tableau, and $\tau_{i+1}$ is obtained from $\tau_i$ in one step.[4]

A tableau *for an entry* $E$ is a tableau that has the entry $E$ at its root.

Recall the convention that we do not write the root of an atomic tableau (since the node labeled by the entry $E$ is already in the tableau). The definition does not specify the order in which to perform the steps; however, later we will describe a concrete construction procedure (algorithm) that we will call *systematic tableau*.

To obtain a proof system, it remains to define the concept of a *tableau proof* (and related terms). Recall once again that it is a proof by contradiction, i.e., we assume the proposition is not valid in $T$, and find a contradiction (a contradictory tableau):

**Definition 4.3.2** (Tableau Proof)**.** A *tableau proof* of the proposition $\varphi$ from the theory $T$ is a *contradictory* tableau from the theory $T$ with the entry F$\varphi$ at the root. If it exists, then $\varphi$ is *(tableau) provable* from $T$, written as $T \vdash \varphi$. (Also, we define *tableau refutation* as a contradictory tableau with the entry T$\varphi$ at the root. If it exists, then $\varphi$ is *(tableau) refutable* from $T$, i.e., $T \vdash \neg\varphi$ holds.)

- A tableau is *contradictory* if every branch is contradictory.

[4] We take the union because, in each step, we add new nodes to the tableau, so $\tau_i$ is a subtree of $\tau_{i+1}$.

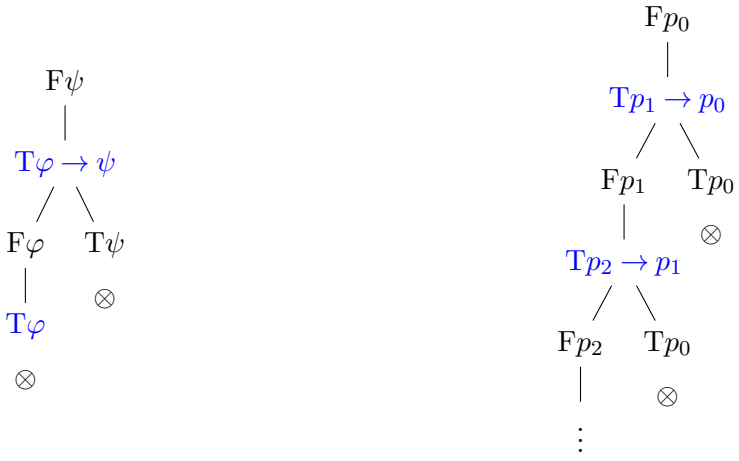



Figure 4.3: Tableaux from Example 4.3.3. Entries coming from axioms are marked in blue.

- A branch is *contradictory* if it contains entries $\mathrm{T}\psi$ and $\mathrm{F}\psi$ for some proposition $\psi$; otherwise, it is *non-contradictory*.
- A tableau is *finished* if every branch is finished.
- A branch is *finished* if
  - it is contradictory, or
  - every entry on this branch is *reduced* and it contains the entry $\mathrm{T}\alpha$ for every axiom $\alpha \in T$.
- An entry $E$ is *reduced* on a branch $B$ passing through this entry if
  - it is of the form $\mathrm{T}p$ or $\mathrm{F}p$ for some propositional variable $p \in \mathbb{P}$, or
  - it appears on $B$ as the root of an atomic tableau[5] (i.e., typically, during the construction of the tableau, it has already been developed on $B$).

*Example* 4.3.3. Let us look at two examples. The tableaux are shown in Figure 4.3.

(a) Tableau proof of the proposition $\psi$ from the theory $T = \{\varphi, \varphi \to \psi\}$, i.e., $T \vdash \psi$ (where $\varphi, \psi$ are some fixed propositions). This fact is called the *Deduction Theorem*.

(b) Finished tableau for the proposition $p_0$ from the theory $T = \{p_{n+1} \to p_n \mid n \in \mathbb{N}\}$. The leftmost branch is non-contradictory and finished. It contains the entries $\mathrm{T}p_{i+1} \to p_i$ and $\mathrm{F}p_i$ for all $i \in \mathbb{N}$. Thus, it agrees with the model $v = (0, 0, \dots)$, i.e., $v : \mathbb{P} \to \{0, 1\}$ where $v(p_i) = 0$ for all $i$.

*Exercise* 4.4. Let us return to the tableaux from Exercise 4.1. Are they tableau proofs or refutations (from the theory $T = \emptyset$)? Which entries on which branches are reduced? Which branches are contradictory, and which are finished?

[5] Although, by convention, we do not write this root.

## 4.4 Finiteness and Systematicity of Proofs

In this section, we will prove that if a tableau proof exists, there is always also a *finite* tableau proof. Additionally, we will present an algorithm by which a tableau proof can always be found; however, we will need the Soundness Theorem and the Completeness Theorem from the next section to prove this fact. For now, we will show that this algorithm will always allow us to construct a finished tableau.

Note that when reducing an entry, we only add entries containing shorter propositions to the tableau. Therefore, if we have a finite theory and do not make unnecessary steps (e.g., repeatedly adding the same axiom or the same atomic tableau), it is easy to construct a finished tableau that will be finite.

If the theory $T$ is infinite, we must be more careful. We could be constructing the tableau forever, without ever reducing a particular entry or using one of the axioms. Therefore, we define a specific algorithm for constructing a tableau, and the resulting tableau will be called a *systematic tableau*. The idea of the construction is simple: we alternate between reducing an entry (simultaneously on all non-contradictory branches passing through it) and using an axiom. We traverse the entries by levels, and within a level in left-right order. We go through the axioms of the theory one by one in a fixed ordering.

**Definition 4.4.1.** Let $R$ be an entry and $T = \{\alpha_1, \alpha_2, \dots\}$ a theory (finite or infinite[6]). A *systematic tableau* from the theory $T$ for the entry $R$ is the tableau $\tau = \bigcup_{i \geq 0} \tau_i$, where $\tau_0$ is a single-node tableau labeled by $R$, and for each $i \geq 0$:

- Let $E$ be the leftmost entry at the smallest level that is not reduced on some non-contradictory branch passing through $E$. We first define the tableau $\tau_i'$ as the tableau obtained from $\tau_i$ by appending the atomic tableau for $E$ to each non-contradictory branch passing through $E$. (If such an entry does not exist, then $\tau_i' = \tau_i$.)
- Subsequently, $\tau_{i+1}$ is the tableau obtained from $\tau_i'$ by appending $\mathrm{T}\alpha_{i+1}$ to each non-contradictory branch of $\tau_i'$. That is while $i < |T|$, otherwise (if $T$ is finite and we have already used all the axioms) skip this step and define $\tau_{i+1} = \tau_i'$.

**Lemma 4.4.2.** *The systematic tableau is finished.*

*Proof.* We will show that each branch is finished. Contradictory branches are finished. Non-contradictory branches contain entries $\mathrm{T}\alpha_i$ (which we added in the $i$-th step), and every entry on them is reduced. Indeed, if $E$ were not reduced on a non-contradictory branch $B$, it would be processed in some step since there are only finitely many entries above $E$ and to the left of $E$. (We use the obvious fact that any prefix of a non-contradictory branch is also a non-contradictory branch, so during the construction, $B$ is never contradictory.) □

Now, let us return to the question of finiteness of proofs:

**Theorem 4.4.3** (Finiteness of Contradiction)**.** *If $\tau = \bigcup_{i \geq 0} \tau_i$ is a contradictory tableau, then there exists $n \in \mathbb{N}$ such that $\tau_n$ is a finite contradictory tableau.*

*Proof.* Consider the set $S$ of all nodes of the tree $\tau$ that do not contain a contradiction above them (in the tree order), i.e., a pair of entries $\mathrm{T}\psi$, $\mathrm{F}\psi$.

[6]Recall that $T$ is countable since the language is (in the entire chapter) countable.

If the set $S$ were infinite, by König's lemma applied to the subtree of $\tau$ on the set $S$, we would have an infinite, non-contradictory branch in $S$. However, this would mean that we have a non-contradictory branch in $\tau$, which contradicts the fact that $\tau$ is contradictory. (In detail: The branch on $S$ would be a sub-branch of some branch $B$ in $\tau$, which is contradictory, i.e., it contains some (specific) contradictory pair of entries, which exists already in some finite prefix of $B$.)

Therefore, the set $S$ is finite. This means that there exists $d \in \mathbb{N}$ such that the entire $S$ lies at a depth of at most $d$. Thus, every node at level $d+1$ has a contradiction above it. Choose $n$ such that $\tau_n$ already contains all the nodes of $\tau$ from the first $d+1$ levels: each branch of $\tau_n$ is therefore contradictory. □

**Corollary 4.4.4.** *If we never extend contradictory branches during the construction of a tableau (e.g., for systematic tableau), then a contradictory tableau is finite.*

*Proof.* We use Theorem 4.4.3, and we have $\tau = \tau_n$ since we do not further extend the tableau once it is contradictory. □

**Corollary 4.4.5** (Finiteness of Proofs)**.** *If $T \vdash \varphi$, then there exists a* finite *tableau proof of $\varphi$ from $T$.*

*Proof.* It easily follows from Corollary 4.4.4: during the construction of $\tau$, we ignore steps that would extend a contradictory branch. □

We also state the following corollary. We will prove it in the next section.

**Corollary 4.4.6** (Systematicity of Proofs)**.** *If $T \vdash \varphi$, then the systematic tableau is a (finite) tableau proof of $\varphi$ from $T$.*

To prove this, we will need two facts: if $\varphi$ is provable from $T$, then it holds in $T$ (Soundness Theorem), i.e., no counterexample can exist. And if the systematic tableau had a non-contradictory branch, it would mean that a counterexample exists (which is the key to the Completeness Theorem).

## 4.5 Soundness and Completeness

In this section, we will prove that the tableau method is a *sound* and *complete* proof system, i.e., that $T \vdash \varphi$ holds if and only if $T \models \varphi$.

### 4.5.1 Soundness Theorem

We say that a model $v$ *agrees* with an entry $E$ if $E = \mathrm{T}\varphi$ and $v \models \varphi$, or $E = \mathrm{F}\varphi$ and $v \not\models \varphi$. Furthermore, $v$ agrees with a branch $B$ if it agrees with every entry on that branch.

As already mentioned, the design of atomic tableaux ensures that if a model agrees with the entry at the root of the tableau, it agrees with some branch. It is not difficult to show the following lemma by induction on the construction of the tableau:

**Lemma 4.5.1.** *If a model of a theory $T$ agrees with the root entry of a tableau from $T$, then it agrees with some branch.*

*Proof.* Consider a tableau $\tau = \bigcup_{i \geq 0} \tau_i$ from the theory $T$ and a model $v \in \mathrm{M}(T)$ that agrees with the root of $\tau$, i.e., with the (single-element) branch $B_0$ in the (single-element) tableau $\tau_0$.

By induction on $i$ (steps in the construction of the tableau), we find a sequence $B_0 \subseteq B_1 \subseteq \ldots$ such that $B_i$ is a branch in the tableau $\tau_i$ agreeing with the model $v$, and $B_{i+1}$ extends $B_i$. The branch of the tableau $\tau$ we are looking for is then $B = \bigcup_{i \geq 0} B_i$.

- If $\tau_{i+1}$ is obtained from $\tau_i$ without extending the branch $B_i$, we define $B_{i+1} = B_i$.
- If $\tau_{i+1}$ is obtained from $\tau_i$ by adding the entry $\mathrm{T}\alpha$ (for some axiom $\alpha \in T$) to the end of the branch $B_i$, we define $B_{i+1}$ as this extended branch. Since $v$ is a model of $T$, it satisfies the axiom $\alpha$, so it agrees with the new entry $\mathrm{T}\alpha$.
- Let $\tau_{i+1}$ be obtained from $\tau_i$ by adding the atomic tableau for some entry $E$ to the end of the branch $B_i$. Since the model $v$ agrees with the entry $E$ (which is already on the branch $B_i$), it agrees with the root of the appended atomic tableau and thus with some branch of the atomic tableau. (This property is easily verified for all atomic tableaux.) We define $B_{i+1}$ as the extension of $B_i$ by this branch of the atomic tableau.[7]

□

We are now ready to prove the Soundness Theorem. In short, if both a proof and a counterexample existed, the counterexample would have to agree with some branch of the proof, but all branches are contradictory.

**Theorem 4.5.2** (Soundness)**.** *If a proposition $\varphi$ is tableau provable from a theory $T$, then $\varphi$ is true in $T$, i.e., $T \vdash \varphi \Rightarrow T \models \varphi$.*

*Proof.* The proof is by contradiction. Assume that $\varphi$ is not true in $T$, i.e., there is a counterexample: a model $v \in \mathrm{M}(T)$ in which $\varphi$ does not hold.

Since $\varphi$ is provable from $T$, there exists a tableau proof of $\varphi$ from $T$, which is a contradictory tableau from $T$ with root entry $\mathrm{F}\varphi$. The model $v$ agrees with the entry $\mathrm{F}\varphi$, so by Lemma 4.5.1, it agrees with some branch $B$. However, all branches are contradictory, including $B$. Therefore, $B$ contains the entries $\mathrm{T}\psi$ and $\mathrm{F}\psi$ (for some proposition $\psi$), and the model $v$ agrees with these entries. Thus, we have $v \models \psi$ and simultaneously $v \not\models \psi$, which is a contradiction. □

### 4.5.2 Completeness Theorem

We will show that if proving fails, i.e., if we obtain a *non-contradictory* branch in a *finished* tableau from the theory $T$ with root entry $\mathrm{F}\varphi$, then this branch provides a counterexample: a model of the theory $T$ that agrees with the entry $\mathrm{F}\varphi$ at the root of the tableau, i.e., it does not satisfy $\varphi$. There may be multiple such models; we will define a specific one:

**Definition 4.5.3** (Canonical Model)**.** If $B$ is a non-contradictory branch of a finished tableau, then the *canonical model* for $B$ is a model defined by the rule (for $p \in \mathbb{P}$):

$$v(p) = \begin{cases} 1 \text{ if the entry } \mathrm{T}p \text{ appears on } B, \\ 0 \text{ otherwise.} \end{cases}$$

[7] Or by any such branch: the model $v$ may agree with more than one branch of the atomic tableau.

**Lemma 4.5.4.** *The canonical model for a (non-contradictory finished) branch $B$ agrees with $B$.*

*Proof.* We will show that the canonical model $v$ agrees with all entries $E$ on the branch $B$, by induction on the structure of the proposition in the entry.[8] First, the base case of the induction:

- If $E = \mathrm{T}p$ for some atomic proposition $p \in \mathbb{P}$, we have $v(p) = 1$ by definition; $v$ agrees with $E$.
- If $E = \mathrm{F}p$, then the entry $\mathrm{T}p$ cannot appear on the branch $B$, otherwise $B$ would be contradictory. By definition, we have $v(p) = 0$, and $v$ again agrees with $E$.

Now, the induction step. We will cover two cases; the other cases are proved similarly.

- Let $E = \mathrm{T}\varphi \wedge \psi$. Since $B$ is a finished branch, the entry $E$ is reduced on it. This means that the entries $\mathrm{T}\varphi$ and $\mathrm{T}\psi$ appear on $B$. By the induction hypothesis, the model $v$ agrees with these entries, so $v \models \varphi$ and $v \models \psi$. Thus, $v \models \varphi \wedge \psi$ holds, and $v$ agrees with $E$.
- Let $E = \mathrm{F}\varphi \wedge \psi$. Since $E$ is reduced on $B$, the entry $\mathrm{F}\varphi$ or the entry $\mathrm{F}\psi$ appears on $B$. Therefore, $v \not\models \varphi$ or $v \not\models \psi$, from which it follows that $v \not\models \varphi \wedge \psi$, and $v$ agrees with $E$.

□

**Theorem 4.5.5** (Completeness)**.** *If a proposition $\varphi$ is true in a theory $T$, then it is tableau provable from $T$, i.e., $T \models \varphi \Rightarrow T \vdash \varphi$.*

*Proof.* We will show that any *finished* (thus, for example, *systematic*) tableau from $T$ with root entry $\mathrm{F}\varphi$ must be contradictory. The proof is by contradiction: If such a tableau were not contradictory, it would contain a non-contradictory (finished) branch $B$. Consider the canonical model $v$ for this branch. Since $B$ is finished, it contains the entry $\mathrm{T}\alpha$ for all axioms $\alpha \in T$. By Lemma 4.5.4, the model $v$ agrees with all entries on $B$, thus satisfying all the axioms and we have $v \models T$. However, since $v$ also agrees with the root entry $\mathrm{F}\varphi$, we have $v \not\models \varphi$, which means $T \not\models \varphi$, a contradiction. Thus, the tableau must have been contradictory, i.e., a tableau proof of $\varphi$ from $T$. □

*Proof of Corollary 4.4.6.* From the previous proof, we also obtain the 'systematicity of proofs,' i.e., that a proof can always be found by constructing a systematic tableau: If $T \models \varphi$, then the systematic tableau for the entry $\mathrm{F}\varphi$ must be contradictory, and thus it is a tableau proof of $\varphi$ from $T$. □

*Exercise* 4.5. Verify the remaining cases in the proof of Lemma 4.5.4.

*Exercise* 4.6. Describe what *all* models agreeing with a given non-contradictory finished branch look like.

*Exercise* 4.7. Suggest a procedure to use the tableau method to find all models of a given theory $T$.

[8] Recall that this means induction on the depth of the proposition tree.

## 4.6 Consequences of Soundness and Completeness

The Soundness and Completeness Theorems together state that *provability* is the same as *validity*. This allows us to formulate syntactic analogues of semantic concepts and properties.

The analogue of *consequences* are *theorems* of the theory $T$:

$$\mathrm{Thm}_{\mathbb{P}}(T) = \{\varphi \in \mathrm{PF}_{\mathbb{P}} \mid T \vdash \varphi\}$$

**Corollary 4.6.1** (Provability = Validity)**.** *For any theory $T$ and propositions $\varphi, \psi$, the following hold:*

- *$T \vdash \varphi$ if and only if $T \models \varphi$*
- $\mathrm{Thm}_{\mathbb{P}}(T) = \mathrm{Csq}_{\mathbb{P}}(T)$

*Proof.* Follows immediately from the Soundness and Completeness Theorems. □

In all definitions and theorems, we can therefore replace the notion of '*validity*' with the notion of '*provability*' (i.e., the symbol '$\models$' with the symbol '$\vdash$') and the notion of '*consequence*' with the notion of '*theorem*'. For example:

- A theory is *contradictory* if it proves a contradiction (i.e., $T \vdash \bot$).
- A theory is *complete* if for every proposition $\varphi$, either $T \vdash \varphi$ or $T \vdash \neg\varphi$ (but not both, otherwise it would be contradictory).

Let us state one more easy consequence:

**Theorem 4.6.2** (Deduction Theorem)**.** *For a theory $T$ and propositions $\varphi, \psi$, we have that: $T, \varphi \vdash \psi$ if and only if $T \vdash \varphi \to \psi$.*

*Proof.* It suffices to prove that $T, \varphi \models \psi \Leftrightarrow T \models \varphi \to \psi$, which is straightforward. □

*Exercise* 4.8. Prove the Deduction Theorem directly by a transformation of tableau proofs.

## 4.7 Compactness Theorem

An important consequence of the Soundness and Completeness Theorems is the so-called *Compactness Theorem*.[9] This principle allows for converting statements about infinite objects or processes into statements about (all) their finite parts.

**Theorem 4.7.1** (Compactness Theorem)**.** *A theory has a model if and only if every finite subset of it has a model.*

*Proof.* Every model of a theory $T$ is evidently a model of each of its subsets. For the other implication, we will prove its contrapositive: Assume that $T$ has no model, i.e., it is contradictory, and find a finite part $T' \subseteq T$ that is also contradictory.

Since $T$ is contradictory, we have $T \vdash \bot$ (here we need the Completeness Theorem). According to Corollary 4.4.5, there exists a *finite* tableau proof $\tau$ of the contradiction $\bot$

[9] The word *compactness* comes from compact (i.e., bounded and closed) sets in Euclidean spaces, where one can select a convergent subsequence from every sequence. You can imagine a sequence of increasing finite parts 'converging' to the infinite whole.

from $T$. The construction of this proof involves only finitely many steps, so we used only finitely many axioms from $T$. If we define $T' = \{\alpha \in T \mid \mathrm{T}\alpha \text{ is an entry in the tableau } \tau\}$, then $\tau$ is also a tableau proof of the contradiction from the theory $T'$. Therefore, $T'$ is a finite contradictory part of $T$. □

### 4.7.1 Applications of Compactness

The following simple application of the Compactness Theorem can be seen as a template followed by many other, more complex applications of this theorem.

**Corollary 4.7.2.** *A countably infinite graph is bipartite if and only if every finite subgraph of it is bipartite.*

*Proof.* Every subgraph of a bipartite graph is evidently also bipartite. We will prove the other implication. A graph is bipartite if and only if it is 2-colorable. Let the colors be 0 and 1.

We will construct a propositional theory $T$ in the language $\mathbb{P} = \{p_v \mid v \in V(G)\}$, where the value of the propositional variable $p_v$ represents the color of the vertex $v$.

$$T = \{p_u \leftrightarrow \neg p_v \mid \{u, v\} \in E(G)\}$$

Clearly, $G$ is bipartite if and only if $T$ has a model. According to the Compactness Theorem, it suffices to show that every finite part of $T$ has a model. Consider a finite $T' \subseteq T$. Let $G'$ be the subgraph of $G$ induced by the set of vertices mentioned in the theory $T'$, i.e., $V(G') = \{v \in V(G) \mid p_v \in \mathrm{Var}(T')\}$. Since $T'$ is finite, $G'$ is also finite, and by assumption, it is 2-colorable. Any 2-coloring of $V(G')$ determines a model of the theory $T'$. □

The essence of this technique is to describe the desired property of an infinite object using an (infinite) propositional theory. Additionally, note how we construct a finite subobject having the given property from a finite part of the theory (in our case, a finite subgraph that is bipartite).

*Exercise* 4.9. Generalize Corollary 4.7.2 for more colors, i.e., show that a countably infinite graph is $k$-colorable if and only if every finite subgraph of it is $k$-colorable. (See Section 1.1.7.)

*Exercise* 4.10. Show that every partial order on a countable set can be extended to a linear order.

*Exercise* 4.11. State and prove the 'countably infinite' analog of Hall's theorem.

## 4.8 Hilbert Calculus

At the end of the chapter on the tableau method, we will for comparison describe another proof system, the so-called *Hilbert deductive system* or *Hilbert calculus*. This is the oldest proof system, modeled after mathematical proofs. As we will see from the example, proving in it is quite laborious, so it is more suitable for theoretical purposes. It is also a sound and complete proof system. (We will show soundness, but leave completeness without proof.)

The Hilbert calculus uses only two basic logical connectives: negation and implication. (Recall that other logical connectives can be derived from these.) The system consists of logical axioms given by the following *schemata* and a single *inference rule*, the so-called *modus ponens*:

**Definition 4.8.1** (Axiom Schemata in Hilbert Calculus)**.** For any propositions $\varphi, \psi, \chi$, the following propositions are logical axioms:

(i) $\varphi \to (\psi \to \varphi)$

(ii) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(iii) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

Note that all logical axioms are indeed tautologies. It should be noted that other systems of logical axioms can be chosen; there are many, see the article List of Hilbert systems on Wikipedia.

**Definition 4.8.2** (Modus Ponens)**.** The *modus ponens* inference rule states that if we have already proved the proposition $\varphi$ and also the proposition $\varphi \to \psi$, we can infer the proposition $\psi$. We write it as follows:

$$\frac{\varphi, \varphi \to \psi}{\psi}$$

Note that modus ponens is *sound*, i.e., if $T \models \varphi$ and $T \models \varphi \to \psi$ hold in some theory, we also have $T \models \psi$.

We are now ready to define a *proof*. It will be a finite sequence of propositions, in which each newly written proposition is either an axiom (logical or from the theory in which we are proving), or can be inferred from some previous ones using modus ponens:

**Definition 4.8.3** (Hilbert Proof)**.** A *Hilbert proof* of a proposition $\varphi$ from a theory $T$ is a *finite* sequence of propositions $\varphi_0, \ldots, \varphi_n = \varphi$, in which for each $i \leq n$, the following holds:

- $\varphi_i$ is a logical axiom, or
- $\varphi_i$ is an axiom of the theory ($\varphi_i \in T$), or
- $\varphi_i$ can be derived from some previous propositions $\varphi_j, \varphi_k$ (where $j, k < i$) using modus ponens.

If a Hilbert proof exists, we say that $\varphi$ is *(Hilbert) provable*, and we write $T \vdash_H \varphi$.

We will illustrate the concept of a Hilbert proof with a simple example:

*Example* 4.8.4. Let us show that for the theory $T = \{\neg\varphi\}$ and for any proposition $\psi$, we have $T \vdash_H \varphi \to \psi$. The Hilbert proof is the following sequence of propositions:

| | |
|---|---|
| *1.* $\neg\varphi$ | *axiom of the theory* |
| *2.* $\neg\varphi \to (\neg\psi \to \neg\varphi)$ | *logical axiom by (i)* |
| *3.* $\neg\psi \to \neg\varphi$ | *modus ponens on 1 and 2* |
| *4.* $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$ | *logical axiom by (iii)* |
| *5.* $\varphi \to \psi$ | *modus ponens on 3 and 4* |

As mentioned earlier, the Hilbert calculus is a sound and complete proof system.

**Theorem 4.8.5** (Soundness of Hilbert Calculus)**.** *For any theory $T$ and proposition $\varphi$:*

$$T \vdash_H \varphi \Rightarrow T \models \varphi$$

*Proof.* By induction on the index $i$, we will show that every proposition $\varphi_i$ in the proof (thus also $\varphi_n = \varphi$) holds in $T$.

If $\varphi_i$ is a logical axiom, $T \models \varphi_i$ holds because logical axioms are tautologies. If $\varphi_i \in T$, then clearly $T \models \varphi_i$ also holds. If $\varphi_i$ is obtained using modus ponens from $\varphi_j$ and $\varphi_k = \varphi_j \to \varphi_i$ (for some $j, k < i$), we know from the induction hypothesis that $T \models \varphi_j$ and $T \models \varphi_j \to \varphi_i$. Then, by the soundness of modus ponens, $T \models \varphi_i$ also holds. □

For completeness, we will state the Completeness theorem but leave it without proof.

**Theorem 4.8.6** (Completeness of Hilbert Calculus)**.** *For any theory $T$ and proposition $\varphi$:*

$$T \models \varphi \Rightarrow T \vdash_H \varphi$$

# Chapter 5

# Resolution Method

In this chapter, we introduce another proof system more suitable for practical applications, called the *resolution method*. This method is the foundation of, among others, *logical programming*, *automated theorem proving*, or *software verification*. In this chapter, we limit ourselves to the resolution method in propositional logic, but later, in Chapter **??**, we will introduce the concept of *unification*, which allows us to search for resolution proofs in predicate logic.

The resolution method works with propositions in *conjunctive normal form (CNF)*. Recall that every proposition can be converted to CNF. This conversion is, in the worst case, of exponential time copmlexity (there are even propositions whose shortest CNF equivalent is exponentially longer), but in practice, this is not an issue.

Similar to the tableau method, the resolution method is based on proof by contradiction, i.e., we add the *negation* of the proposition we want to prove to the theory in which we are proving (both converted to CNF), and show that this leads to a contradiction.

To find the contradiction, the resolution method uses a single inference rule called the *resolution rule*. This is a special case of the *cut rule*, which states: *"From the propositions* $\varphi \vee \psi$ *and* $\neg\varphi \vee \chi$*, we can infer the proposition* $\psi \vee \chi$*,"* written as:

$$\frac{\varphi \vee \psi, \neg\varphi \vee \chi}{\psi \vee \chi}$$

In the *resolution rule*, which we will demonstrate shortly, $\varphi$ is a *literal*, and $\psi, \chi$ are *clauses*.

*Exercise* 5.1. Show that the cut rule is *sound*. (What does it mean?)

## 5.1 Set Representation

First, we introduce a more compact notation for CNF propositions, called *set notation*. For it would be impractical to represent CNF propositions as strings including brackets and logical symbols.

- Recall that a *literal* $\ell$ is a propositional variable or its negation, and that $\bar{\ell}$ denotes the *opposite literal* to $\ell$.
- A *clause* $C$ is a finite set of literals. The *empty clause*, which is never satisfied,[1] is denoted by $\square$.

[1] It represents the disjunction of an empty set of literals, meaning none of the disjuncts are satisfied.

- A *(CNF) formula* $S$ is a (finite, or even infinite) set of clauses. The *empty formula* $\emptyset$ is always satisfied.[2]

*Remark* 5.1.1. Note that a *CNF formula* can also be an *infinite* set of clauses. If we convert an infinite propositional theory to CNF, we take (in set representation) all infinitely many clauses as elements of a single formula (set). In practical applications, the formula is (almost always) finite.

In set notation, models correspond to sets of literals that contain exactly one of the literals $p, \neg p$ for each propositional variable $p$:

- A *(partial) assignment* $\mathcal{V}$ is any set of literals that is *consistent*, i.e., it does not contain both a literal and its negation.
- An assignment is *complete* if it contains either the positive or negative literal for each propositional variable.
- An assignment $\mathcal{V}$ *satisfies* a formula $S$, written $\mathcal{V} \models S$, if $\mathcal{V}$ contains some literal from each clause in $S$, i.e.,
$$\mathcal{V} \cap C \neq \emptyset \text{ for each } C \in S.$$

*Example* 5.1.2. The proposition $\varphi = (\neg p_1 \vee p_2) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_3 \vee \neg p_4) \wedge (\neg p_4 \vee \neg p_5) \wedge p_4$ is written in set notation as:
$$S = \{\{\neg p_1, p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_3, \neg p_4\}, \{\neg p_4, \neg p_5\}, \{p_4\}\}$$
The assignment $\mathcal{V} = \{\neg p_1, \neg p_3, p_4, \neg p_5\}$ satisfies $S$, written $\mathcal{V} \models S$. It is not complete, but we can extend it with any literal for $p_2$: both $\mathcal{V} \cup \{p_2\} \models S$ and $\mathcal{V} \cup \{\neg p_2\} \models S$ hold. These two complete assignments correspond to the models $(0, 1, 0, 1, 0)$ and $(0, 0, 0, 1, 0)$.

## 5.2 Resolution Proof

First, we define one inference step in the resolution proof, the *resolution rule*, which we apply to a pair of clauses; the result is a clause called the *resolvent*, which is a logical consequence of the original pair of clauses:

**Definition 5.2.1** (Resolution Rule)**.** Given clauses $C_1$ and $C_2$ and a literal $\ell$ such that $\ell \in C_1$ and $\bar{\ell} \in C_2$, the *resolvent* of the clauses $C_1$ and $C_2$ *over the literal* $\ell$ is the clause
$$C = (C_1 \setminus \{\ell\}) \cup (C_2 \setminus \{\bar{\ell}\}).$$

So, we remove the literal $\ell$ from the first clause and the literal $\bar{\ell}$ from the second clause (both had to be there!) and take the union of all remaining literals to be the resulting resolvent. Using the symbol $\dot\cup$ for disjoint union, we could also write:
$$C_1' \cup C_2' \text{ is the resolvent of the clauses } C_1' \mathbin{\dot\cup} \{\ell\} \text{ and } C_2' \mathbin{\dot\cup} \{\bar{\ell}\}$$

*Example* 5.2.2. From the clauses $C_1 = \{\neg q, r\}$ and $C_2 = \{\neg p, \neg q, \neg r\}$, we can derive the resolvent $\{\neg p, \neg q\}$ over the literal $r$. From the clauses $\{p, q\}$ and $\{\neg p, \neg q\}$, we can derive $\{p, \neg p\}$ over the literal $q$ or $\{q, \neg q\}$ over the literal $p$ (both of which are tautologies).[3]

[2] It represents the conjunction of an empty set of clauses, meaning all clauses in $S$ are satisfied.

[3] We cannot, however, derive $\square$ 'by resolving over $p$ and $q$ simultaneously' (a common mistake). Note that $\{\{p, q\}, \{\neg p, \neg q\}\}$ is not unsatisfiable, e.g., $(1, 0)$ is a model.

**Observation 5.2.3** (Soundness of the Resolution Rule)**.** *The resolution rule is sound, i.e., for any assignment $\mathcal{V}$, the following holds:*

$$\textit{If } \mathcal{V} \models C_1 \textit{ and } \mathcal{V} \models C_2, \textit{ then } \mathcal{V} \models C.$$

A resolution proof is defined similarly to a Hilbert proof as a finite sequence of clauses, ensuring the validity of each clause in the sequence: at each step, we can either write an 'axiom' (a clause from $S$) or a resolvent of some two previously written clauses.

**Definition 5.2.4** (Resolution Proof)**.** A *resolution proof* of a clause $C$ from a CNF formula $S$ is a *finite* sequence of clauses $C_0, C_1, \ldots, C_n = C$ such that for each $i$, either $C_i \in S$ or $C_i$ is the resolvent of some $C_j, C_k$ where $j < i$ and $k < i$.

If a resolution proof exists, we say that $C$ is *resolution provable* from $S$, written $S \vdash_R C$. A *(resolution) refutation* of the CNF formula $S$ is a resolution proof of $\square$ from $S$, in which case $S$ is *(resolution) refutable*.

*Example* 5.2.5. The CNF formula $S = \{\{p, \neg q, r\}, \{p, \neg r\}, \{\neg p, r\}, \{\neg p, \neg r\}, \{q, r\}\}$ is (resolution) refutable, one possible refutation is:

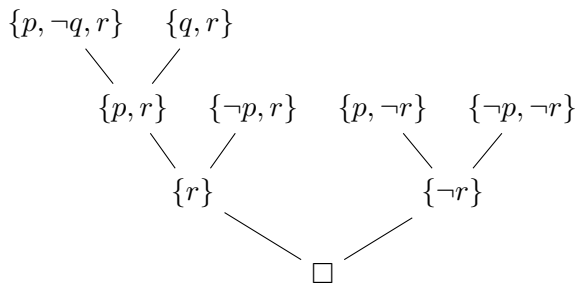
$$\{p, \neg q, r\}, \{q, r\}, \{p, r\}, \{\neg p, r\}, \{r\}, \{p, \neg r\}, \{\neg p, \neg r\}, \{\neg r\}, \square$$

A resolution proof has a natural tree structure: the leaves are labeled by axioms and the inner nodes represent individual resolution steps.

**Definition 5.2.6** (Resolution Tree)**.** A resolution tree of a clause $C$ from a CNF formula $S$ is a *finite* binary tree with nodes labeled by clauses, where:

- The root is labeled $C$,
- The leaves are labeled by clauses from $S$,
- Each inner node is labeled by a resolvent of its child nodes.

*Example* 5.2.7. The resolution tree of the empty clause $\square$ from the CNF formula $S$ from Example 5.2.5 is:




It is easy to prove the following observation, by induction on the depth of the tree and the length of the resolution proof:

**Observation 5.2.8.** *A clause $C$ has a resolution tree from a CNF formula $S$ if and only if $S \vdash_R C$.*

Each resolution proof corresponds to a unique resolution tree. Conversely, from a single resolution tree, we can derive multiple resolution proofs: they are given by any traversal of the tree nodes where an inner node is visited only after both of its children have been visited.

We also introduce another concept, called the *resolution closure*, which contains all clauses that can be 'learned' by resolution from a given formula. It is a useful theoretical perspective on resolution; in applications, constructing the entire resolution closure would be impractical.

**Definition 5.2.9** (Resolution Closure)**.** The *resolution closure* $\mathcal{R}(S)$ of the formula $S$ is defined inductively as the smallest set of clauses satisfying:

- $C \in \mathcal{R}(S)$ for all $C \in S$,
- If $C_1, C_2 \in \mathcal{R}(S)$ and $C$ is the resolvent of $C_1, C_2$, then $C \in \mathcal{R}(S)$.

*Example* 5.2.10. Let us compute the resolution closure of the formula $S = \{\{p, \neg q, r\}, \{p, \neg r\}, \{\neg p, r\}, \{\neg p, \neg r$
The clauses from $S$ are in blue; additional clauses are obtained by resolving them in pairs (first with the first, second with the first, second with the second, etc., over all possible literals):

$$\begin{aligned}\mathcal{R}(S) = \{&\{p, \neg q, r\}, \{p, \neg r\}, \{\neg p, r\}, \{\neg p, \neg r\}, \{q, r\},\\ &\{p, \neg q\}, \{\neg q, r\}, \{r, \neg r\}, \{p, \neg p\}, \{r, \neg s\}, \{p, r\}, \{p, q\}, \{r\}, \{p\}\}\end{aligned}$$

## 5.3 Soundness and Completeness of the Resolution Method

The resolution method is also both sound and complete.

### 5.3.1 Soundness of Resolution

Soundness can be easily proved by induction on the length of the resolution proof.

**Theorem 5.3.1** (Soundness of Resolution)**.** *If a formula $S$ is resolution refutable, then $S$ is unsatisfiable.*

*Proof.* Suppose $S \vdash_R \square$ and consider a resolution proof $C_0, C_1, \ldots, C_n = \square$. Suppose, for a contradiction, that $S$ is satisfiable, i.e., $\mathcal{V} \models S$ for some assignment $\mathcal{V}$. By induction on $i$, we prove that $\mathcal{V} \models C_i$. For $i = 0$, this holds because $C_0 \in S$. For $i > 0$, there are two cases:

- $C_i \in S$, in which case $\mathcal{V} \models C_i$ follows from the assumption that $\mathcal{V} \models S$,
- $C_i$ is the resolvent of $C_j, C_k$, where $j, k < i$: by the induction hypothesis, $\mathcal{V} \models C_j$ and $\mathcal{V} \models C_k$. $\mathcal{V} \models C_i$ follows from the soundness of the resolution rule.

(Alternatively, we could proceed by induction on the depth of the resolution tree.) $\square$

### 5.3.2 Substitution Tree

In the completeness proof, we need to construct a resolution tree, whose construction is based on the so-called *substitution tree*. By *substituting* a literal into a formula, we mean simplifying the formula under the assumption that the given literal is true. Substitution was already encountered in Section 3.3 during *unit propagation*: we remove clauses containing this literal and remove the opposite literal from the remaining clauses.

**Definition 5.3.2** (Literal Substitution)**.** Given a formula $S$ and a literal $\ell$, the *substitution* of $\ell$ into $S$ is the formula:

$$S^\ell = \{C \setminus \{\bar{\ell}\} \mid \ell \notin C \in S\}$$

**Observation 5.3.3.** *Here we summarize several simple facts about substitution:*

- $S^\ell$ *is the result of* unit propagation *applied to* $S \cup \{\{\ell\}\}$.
- $S^\ell$ *does not contain the literal* $\ell$ *or its opposite* $\bar{\ell}$ *(it does not contain the propositional variable from* $\ell$ *at all).*
- *If* $S$ *did not contain the literal* $\ell$ *or its opposite* $\bar{\ell}$*, then* $S^\ell = S$.
- *If* $S$ *contained the unit clause* $\{\bar{\ell}\}$*, then* $\square \in S^\ell$*, meaning* $S^\ell$ *is unsatisfiable.*

The key property of substitution is expressed by the following lemma:

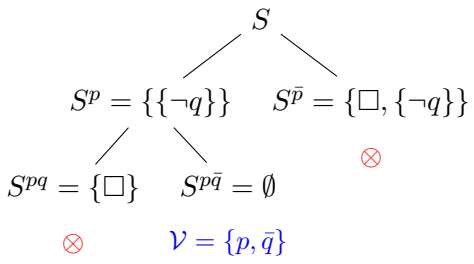
**Lemma 5.3.4.** *$S$ is satisfiable if and only if $S^\ell$ or $S^{\bar{\ell}}$ is satisfiable.*

*Proof.* Let $\mathcal{V} \models S$, it cannot contain both $\ell$ and $\bar{\ell}$ (it must be consistent); without loss of generality, assume $\bar{\ell} \notin \mathcal{V}$, and show that $\mathcal{V} \models S^\ell$. Consider any clause in $S^\ell$. It is of the form $C \setminus \{\bar{\ell}\}$ for a clause $C \in S$ (not containing the literal $\ell$). We know that $\mathcal{V} \models C$, but since $\mathcal{V}$ does not contain $\bar{\ell}$, the assignment $\mathcal{V}$ must satisfy some other literal of $C$, so $\mathcal{V} \models C \setminus \{\bar{\ell}\}$.

Conversely, suppose there exists an assignment $\mathcal{V}$ satisfying $S^\ell$ (again, without loss of generality). Since $\bar{\ell}$ (or $\ell$) does not appear in $S^\ell$, it holds that $\mathcal{V} \setminus \{\bar{\ell}\} \models S^\ell$. The assignment $\mathcal{V}' = (\mathcal{V} \setminus \{\bar{\ell}\}) \cup \{\ell\}$ then satisfies each clause $C \in S$: if $\ell \in C$, then $\ell \in C \cap \mathcal{V}'$ and $C \cap \mathcal{V}' \neq \emptyset$, otherwise $C \cap \mathcal{V}' = (C \setminus \{\bar{\ell}\}) \cap \mathcal{V}' \neq \emptyset$ because $\mathcal{V} \setminus \{\bar{\ell}\} \models C \setminus \{\bar{\ell}\} \in S^\ell$. We have verified that $\mathcal{V}' \models S$, so $S$ is satisfiable. □

Whether a given *finite* formula is satisfiable can thus be determined recursively (using the *divide and conquer* method) by substituting both possible literals for (some, say the first) propositional variable appearing in the formula and branching the computation. Essentially, this is a similar principle to the DPLL algorithm (see Section 3.4). The resulting tree is called a *substitution tree*.

*Example* 5.3.5. Let's illustrate this concept with an example by constructing a substitution tree for the formula $S = \{\{p\}, \{\neg q\}, \{\neg p, \neg q\}\}$:




Once a branch contains the empty clause □, it is unsatisfiable, and we need not continue in that branch. In the leaves, there are either unsatisfiable theories or empty theories: in the latter case, the sequence of substitutions gives us a satisfying assignment.

From the construction, it is evident how to proceed in the case of a finite formula. However, the substitution tree makes sense, and the following corollary holds, even for infinite formulas:

**Corollary 5.3.6.** *A formula $S$ (over a countable language) is unsatisfiable if and only if every branch of the substitution tree contains the empty clause $\square$.*

*Proof.* For a finite formula $S$, this follows from the above discussion; it can be easily proved by induction on the size of $\mathrm{Var}(S)$:

- If $|\mathrm{Var}(S)| = 0$, we have $S = \emptyset$ or $S = \{\square\}$, in both cases, the substitution tree is one node, and the statement holds.
- In the inductive step, we choose any literal $\ell \in \mathrm{Var}(S)$ and apply Lemma 5.3.4.

If $S$ is infinite and satisfiable, it has a satisfying assignment, which 'matches' the corresponding (infinite) branch in the substitution tree. If $S$ is infinite and unsatisfiable, then by the Compactness Theorem, there is a finite part $S' \subseteq S$ that is also unsatisfiable. Substitution for all variables from $\mathrm{Var}(S')$ will lead to $\square$ in each branch, which will happen after finitely many steps. $\square$

### 5.3.3 Completeness of Resolution

**Theorem 5.3.7** (Completeness of Resolution)**.** *If $S$ is unsatisfiable, it is resolution refutable (i.e., $S \vdash_R \square$).*

*Proof.* If $S$ is infinite, it has a finite unsatisfiable part $S'$ by the Compactness Theorem. A resolution refutation of $S'$ is also a resolution refutation of $S$. Suppose $S$ is finite.

The proof is by induction on the number of variables in $S$. If $|\mathrm{Var}(S)| = 0$, the only possible unsatisfiable formula without variables is $S = \{\square\}$, and we have a one-step proof $S \vdash_R \square$. Otherwise, choose $p \in \mathrm{Var}(S)$. By Lemma 5.3.4, both $S^p$ and $S^{\bar{p}}$ are unsatisfiable. They have one fewer variable, so by the induction hypothesis, there are resolution trees $T$ for $S^p \vdash_R \square$ and $T'$ for $S^{\bar{p}} \vdash_R \square$.

We show how to construct the resolution tree $\widehat{T}$ for $S \vdash_R \neg p$. Similarly, we construct $\widehat{T'}$ for $S \vdash_R p$, and then easily construct the resolution tree for $S \vdash_R \square$: we attach the roots of the trees $\widehat{T}$ and $\widehat{T'}$ as the left and right children of the root $\square$ (i.e., in the last step of the resolution proof, we obtain $\square$ by resolving $\{\neg p\}$ and $\{p\}$).

It remains to show the construction of the tree $\widehat{T}$: the set of nodes and the order remain the same; we only change some clauses in the nodes by adding the literal $\neg p$. At each leaf of the tree $T$ is some clause $C \in S^p$, and either $C \in S$ or it is not, but $C \cup \{\neg p\} \in S$. In the first case, we leave the label unchanged. In the second case, we add the literal $\neg p$ to $C$ *and to all clauses above this leaf.* In the leaves, there are now only clauses from $S$, in the root, we have changed $\square$ to $\neg p$. And each inner node remains the resolvent of its children. $\square$

*Exercise* 5.2. The proof of the Completeness Theorem for resolution gives a method for recursively 'growing' a resolution refutation. Think about how to do this and apply it to an example of an unsatisfiable formula.

## 5.4 LI-Resolution and Horn-SAT

We begin with a different view of the resolution proof, called the *linear proof.*

### 5.4.1 Linear Proof

In addition to the resolution tree, a resolution proof can also be organized as a *linear proof*, where in each step we have one *central* clause, which we resolve with a *side* clause, which is either one of the previous central clauses or an axiom from $S$. The resolvent then becomes the new central clause.[4]

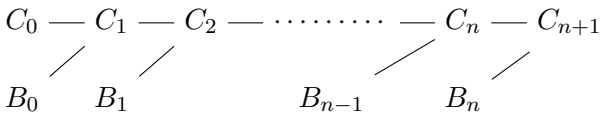
**Definition 5.4.1** (Linear Proof)**.** A *linear proof* (by resolution) of a clause $C$ from a formula $S$ is a finite sequence

$$\begin{bmatrix} C_0 \\ B_0 \end{bmatrix}, \begin{bmatrix} C_1 \\ B_1 \end{bmatrix}, \ldots, \begin{bmatrix} C_n \\ B_n \end{bmatrix}, C_{n+1}$$

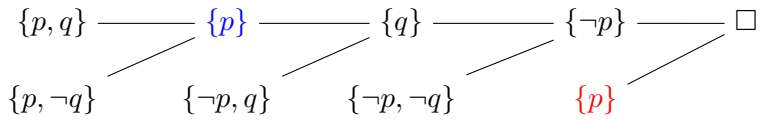
where $C_i$ are called *central* clauses, $C_0$ is the *initial* clause, $C_{n+1} = C$ is the *final* clause, $B_i$ are *side* clauses, and it holds that:

- $C_0 \in S$, for $i \leq n$, $C_{i+1}$ is the resolvent of $C_i$ and $B_i$,
- $B_0 \in S$, for $i \leq n$, $B_i \in S$ or $B_i = C_j$ for some $j < i$.

A *linear refutation* of $S$ is a linear proof of $\square$ from $S$. A linear proof can be illustrated as follows:




*Example* 5.4.2. Let's construct a linear refutation of the formula $S = \{\{p, q\}, \{p, \neg q\}, \{\neg p, q\}, \{\neg p, \neg q\}\}$ (i.e., a linear proof of $\square$ from $S$). A linear proof might look like this:




The last side clause $\{p\}$ (in red) is not from $S$, but is equal to the previous central clause (in blue).

*Exercise* 5.3. Convert the linear proof from Example 5.4.2 into a resolution tree.

*Remark* 5.4.3. $C$ has a linear proof from $S$ if and only if $S \vdash_R C$.

A resolution tree can easily be produced from a linear proof by induction on the length of the proof: the base case is obvious, and if there is a side clause $B_i$ that is not an axiom from $S$, then $B_i = C_j$ for some $j < i$ and we only need to attach the resolution tree for proving $C_j$ from $S$ instead of $B_i$. Notice that this also implies the *soundness* of linear resolution.

We will not present the proof of the reverse implication. It follows from the *completeness* of linear resolution, whose proof can be found in the textbook *A. Nerode, R. Shore: Logic for Applications* [3].

[4] While the construction of the resolution tree can be easily described recursively, the linear proof better corresponds to a procedural computation. It is only about finding a suitable side clause.

### 5.4.2 LI-Resolution

In a general linear proof, each subsequent side clause can either be an axiom from $S$ or one of the previous central clauses. If we forbid the latter option and require that all side clauses must be from $S$, we get the so-called *LI (linear-input) resolution*:

**Definition 5.4.4** (LI-Proof)**.** An *LI-proof* (by resolution) of a clause $C$ from a formula $S$ is a linear proof

$$\begin{bmatrix} C_0 \\ B_0 \end{bmatrix}, \begin{bmatrix} C_1 \\ B_1 \end{bmatrix}, \ldots, \begin{bmatrix} C_n \\ B_n \end{bmatrix}, C$$

in which each side clause $B_i$ is an axiom from $S$. If an LI-proof exists, we say that $C$ is *LI-provable* from $S$, and we write $S \vdash_{LI} C$. If $S \vdash_{LI} \square$, then $S$ is *LI-refutable*.

*Remark* 5.4.5. An LI-proof directly gives a resolution tree (all leaves are axioms) in a special form that we might call a 'hairy path'. Conversely, from a resolution tree in the form of a hairy path, we immediately obtain an LI-proof: the vertices on the path are central clauses, the hairs are side clauses.

While *linear resolution*[5] is just another view of the general resolution proof, *LI-resolution* introduces a significant restriction: we lose *completeness* (not every unsatisfiable formula has an LI-refutation). On the other hand, LI-proofs are simpler to construct.[6]

### 5.4.3 Completeness of LI-Resolution for Horn Formulas

As we will now show, LI-resolution is *complete for Horn formulas*. As we will see in the next section, it is the basis of Prolog interpreters, which work with Horn formulas. First, let us recall the terminology related to hornness and also programs, in set representation:

- A *Horn clause* is a clause containing at most one positive literal.
- A *Horn formula* is (finite or even infinite) a set of Horn clauses.
- A *fact* is a positive unit (Horn) clause, i.e., $\{p\}$, where $p$' is a propositional variable.
- A *rule* is a (Horn) clause with exactly one positive and at least one negative literal.
- Rules and facts are called *program clauses*.
- A *goal* is a non-empty (Horn) clause without a positive literal.[7]

We will find the following simple observation useful:

**Observation 5.4.6.** *If a Horn formula $S$ is unsatisfiable and $\square \notin S$, then it contains a fact and a goal.*

*Proof.* If it does not contain a fact, we can evaluate all variables to 0; if it does not contain a goal, we evaluate to 1. $\square$

[5] That is, a proof system based on finding *linear* proofs or refutations.
[6] In each step, we only have to choose from clauses in $S$, not from previously proven central clauses.
[7] Recall that we prove by *contradiction*, so the *goal* is the negation of what we want to prove.

Now we will state and prove the Completeness Theorem for LI-Resolution for Horn formulas. The proof also provides a guide on how to construct an LI-refutation, based on the process of unit propagation. This procedure is illustrated in the example below, which you can follow along with the reading of the proof.

**Theorem 5.4.7** (Completeness of LI-Resolution for Horn Formulas)**.** *If a Horn formula $T$ is satisfiable, and $T \cup \{G\}$ is unsatisfiable for a goal $G$, then $T \cup \{G\} \vdash_{LI} \square$, by an LI-refutation that starts with the goal $G$.*

*Proof.* Similarly to the Completeness Theorem for Resolution, we can assume by the Compactness Theorem that $T$ is finite. The proof (construction of the LI-refutation) will be done by induction on the number of variables in $T$.

By Observation 5.4.6, $T$ contains a fact $\{p\}$ for some propositional variable $p$. Since $T \cup \{G\}$ is unsatisfiable, according to Lemma 5.3.4, $(T \cup \{G\})^p = T^p \cup \{G^p\}$ is also unsatisfiable, where $G^p = G \setminus \{\neg p\}$.

If $G^p = \square$, then $G = \{\neg p\}$, $\square$ is the resolvent of $G$ and $\{p\} \in T$, and we have a one-step LI-refutation of $T \cup \{G\}$ (this is the base case of the induction).

Otherwise, we use the induction hypothesis. Note that the formula $T^p$ is satisfiable (by the same assignment as $T$, because it must contain $p$ due to the fact $\{p\}$, so it does not contain $\neg p$) and has fewer variables than $T$. Thus, by the induction hypothesis, there exists an LI-derivation of $\square$ from $T^p \cup \{G^p\}$ starting with $G^p = G \setminus \{\neg p\}$.

We construct the required LI-refutation of $T \cup \{G\}$ starting with $G$ (similarly to the proof of the Completeness Theorem for Resolution) by adding the literal $\neg p$ to all leaves that are not already in $T \cup \{G\}$ (i.e., they were created by removing $\neg p$), and to all vertices above them. This gives us $T \cup \{G\} \vdash_{LI} \neg p$, and finally we add the side clause $\{p\}$ and derive $\square$. $\square$

*Example* 5.4.8. Consider a (satisfiable, Horn) theory $T$, written in set representation as the formula $T = \{\{p, \neg r, \neg s\}, \{\neg q, r\}, \{q, \neg s\}, \{s\}\}$. Suppose we want to prove that $p \wedge q$ holds in the theory $T$.[8] In the resolution method, we consider the goal $G = \{\neg p, \neg q\}$ and show that $T \cup \{G\} \vdash_{LI} \square$.

Following the guide from the proof, we find a fact in the formula $T$, and perform unit propagation in both $T$ and the goal $G$. We repeat the process until the formula is empty:

- $T = \{\{p, \neg r, \neg s\}, \{\neg q, r\}, \{q, \neg s\}, \{s\}\}$, $G = \{\neg p, \neg q\}$
- $T^s = \{\{p, \neg r\}, \{\neg q, r\}, \{q\}\}$, $G^s = \{\neg p, \neg q\}$
- $T^{sq} = \{\{p, \neg r\}, \{r\}\}$, $G^{sq} = \{\neg p\}$
- $T^{sqr} = \{\{p\}\}$, $G^{sqr} = \{\neg p\}$
- $T^{sqrp} = \emptyset$, $G^{sqrp} = \square$

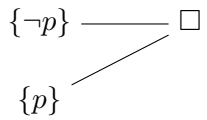
Now, we construct the resolution refutation in reverse:

- $T^{sqrp}, G^{sqrp} \vdash_{LI} \square$:

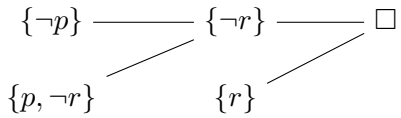
$\square$

[8] In Prolog, we would pose the 'query': `?-p,q.`

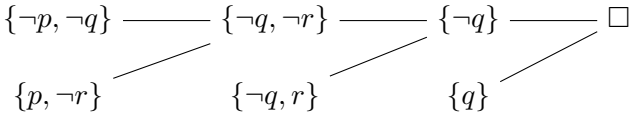
- $T^{sqr}, G^{sqr} \vdash_{LI} \square$:



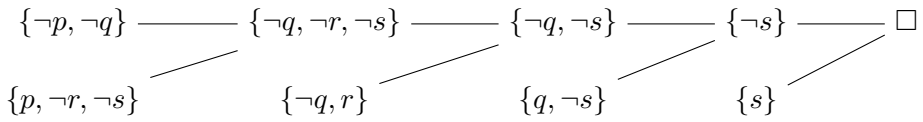

- $T^{sq}, G^{sq} \vdash_{LI} \square$:




- $T^{s}, G^{s} \vdash_{LI} \square$:




- $T, G \vdash_{LI} \square$




### 5.4.4 Prolog Program

Although the real power of Prolog comes from *unification* and resolution in predicate logic, we will show how Prolog uses the resolution method with a *propositional* program. Adaptation to predicates will be straightforward later.

A *program* in Prolog is a Horn formula containing only *program clauses*, i.e., *facts* or *rules*. A *query* is a conjunction of facts, the negation of the query is the *goal*.

*Example* 5.4.9. As an example of a Prolog program, we will use the theory (formula) $T$ and the query $p \wedge q$ from Example 5.4.8. For instance, the clause $\{p, \neg r, \neg s\}$, which is equivalent to $r \wedge s \to p$, is written in Prolog as: `p:-r,s.`

```
p:-r,s.
r:-q.
q:-s.
s.
```

And we pose the query to the program:

```
?-p,q.
```

**Corollary 5.4.10.** *Let $P$ be a program and $Q = p_1 \wedge \cdots \wedge p_n$, and denote $G = \{\neg p_1, \ldots, \neg p_n\}$ (i.e., $G \sim \neg Q$). The following conditions are equivalent:*

- $P \models Q$,
- $P \cup \{G\}$ *is unsatisfiable,*
- $P \cup \{G\} \vdash_{LI} \square$, *and there is an LI-refutation starting with the goal $G$.*

*Proof.* The equivalence of the first two conditions is by contradiction, and the equivalence of the second and third is by the Completeness Theorem for LI-Resolution for Horn formulas. (Note that the Program is always satisfiable.) $\square$

# Part II

# Predicate logic

# Part III

# Advanced topics

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