Chapter 1

Model Theory

In this chapter, we will step away from the typical applications of logic in computer science[1] and look at a higher level of abstraction in the field of *mathematical* logic. *Model theory* seeks to describe the relationship between general properties of theories (in predicate logic) and properties of classes of their models. We will inevitably work with infinite theories and infinite structures. We will only look at a sample of some selected results that are accessible to us. We will not attempt to cover all the main areas of model theory, which is very rich and deep. In this chapter, we also include material about properties of models that did not fit elsewhere.

1.1 Elementary Equivalence

First, we will look at some properties related to the concept of *elementary equivalence*. Recall that L-structures $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent* ($\mathcal{A} \equiv \mathcal{B}$) if they satisfy the same L-sentences.

In model theory, we are often interested in what properties (sentences) hold in a given, specific structure:

Definition 1.1.1 (Theory of a Structure)**.** Given an L-structure $\mathcal{A}$, the *theory of the structure* $\mathcal{A}$, denoted $\mathrm{Th}(\mathcal{A})$, is the set of all L-sentences that are valid in $\mathcal{A}$:

$$\mathrm{Th}(\mathcal{A}) = \{\varphi \mid \varphi \text{ is an } L\text{-sentence and } \mathcal{A} \models \varphi\}$$

Example 1.1.2. As an important example, consider the *standard model of arithmetic*, the structure $\underline{\mathbb{N}} = \langle \mathbb{N}, S, +, \cdot, 0, \leq \rangle$. The theory $\mathrm{Th}(\underline{\mathbb{N}})$ is called the *arithmetic of natural numbers*. In the next chapter, we will show that it is *(algorithmically) undecidable*.[2]

We summarize several basic properties of the theory of a structure in the following observation:

Observation 1.1.3. *Let $\mathcal{A}$ be an L-structure and T be an L-theory. Then:*

(i) The theory $\mathrm{Th}(\mathcal{A})$ *is complete.*

[1] For example, using resolution to determine whether a given sentence φ holds in a given finite theory T.

[2] A theory T is *(algorithmically) decidable* if there is an algorithm that, for every input sentence φ, terminates and answers whether $T \models \varphi$.

(ii) If $\mathcal{A} \in \mathrm{M}_L(T)$, then $\mathrm{Th}(\mathcal{A})$ is a (complete) simple extension of the theory T.

(iii) If $\mathcal{A} \in \mathrm{M}_L(T)$ and T is complete, then $\mathrm{Th}(\mathcal{A})$ is equivalent to T, in which case $\mathrm{Th}(\mathcal{A}) = \mathrm{Csq}_L(T)$.

Using the notion of the *theory of a structure*, we can also express elementary equivalence. For L-structures $\mathcal{A}, \mathcal{B}$, it holds that:

$$\mathcal{A} \equiv \mathcal{B} \text{ if and only if } \mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B}).$$

Example 1.1.4. Consider the standard orders of the reals, rationals, and integers, i.e., the structures $\langle \mathbb{R}, \leq \rangle$, $\langle \mathbb{Q}, \leq \rangle$, and $\langle \mathbb{Z}, \leq \rangle$. As mentioned in Example **??**, it is not difficult to show that $\langle \mathbb{R}, \leq \rangle \equiv \langle \mathbb{Q}, \leq \rangle$ (using *density* of these orders). However, the structures $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{Z}, \leq \rangle$ are not elementarily equivalent: In $\langle \mathbb{Z}, \leq \rangle$, every element has an immediate successor, which is not true in $\langle \mathbb{Q}, \leq \rangle$. For the following sentence φ, we have $\varphi \in \mathrm{Th}(\langle \mathbb{Z}, \leq \rangle)$ but $\varphi \notin \mathrm{Th}(\langle \mathbb{Q}, \leq \rangle)$:

$$\varphi = (\forall x)(\exists y)(x \leq y \wedge \neg x = y \wedge (\forall z)(x \leq z \rightarrow z = x \vee y \leq z))$$

1.1.1 Complete Simple Extensions

Given a theory T, we want to know what its models look like. Recall that:

- A theory is *complete* if it has a single model up to elementary equivalence.[3]
- The models of a theory T, up to elementary equivalence, uniquely correspond to the complete simple extensions of T, up to equivalence.

The complete simple extensions of an L-theory T are thus (up to equivalence) of the form $\mathrm{Th}(\mathcal{A})$ for $\mathcal{A} \in \mathrm{M}_L(T)$, and (as mentioned above) $\mathcal{A} \equiv \mathcal{B}$ if and only if $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$. Instead of finding all models, it suffices to find all complete simple extensions.

Remark 1.1.5. One motivation for dealing with complete simple extensions is Proposition **??** from the next chapter, which states that if we can *effectively (algorithmically) describe* all complete simple extensions[4] of a *given* theory T,[5] then T is *(algorithmically) decidable*.

The ability to (effectively) describe all complete simple extensions is relatively rare and requires strong assumptions. However, it can be done for many important theories. Let us give one example: the *theory of dense linear order*.

Example: DeLO*

The theory of *dense linear order (DeLO*)* is an extension of the theory of order by the following axioms:

- the *linearity* axiom (sometimes called the *dichotomy* axiom):

$$x \leq y \vee y \leq x$$

[3] That is, all its models are elementarily equivalent.

[4] Imagine an algorithm that, for given inputs i, j, returns the j-th axiom of the i-th complete simple extension (in some fixed numbering); such an algorithm does not always exist!

[5] T can be infinite, but there must be an algorithm that generates axioms of T, and eventually outputs every one of them.

- the *density* axiom:

$$x \leq y \wedge \neg x = y \rightarrow (\exists z)(x \leq z \wedge z \leq y \wedge \neg z = x \wedge \neg z = y)$$

Sometimes, the *non-triviality* axiom $(\exists x)(\exists y)(\neg x = y)$ is added to exclude the one-element model. This theory is not complete, but we can describe all its complete simple extensions:

Proposition 1.1.6. *Let $\varphi = (\exists x)(\forall y)(x \leq y)$ and $\psi = (\exists x)(\forall y)(y \leq x)$, expressing the existence of a minimal and maximal element, respectively. The following four theories are exactly all (up to equivalence) complete simple extensions of the theory DeLO*:*

- $\text{DeLO} = \text{DeLO}^* \ \cup \{\neg\varphi, \neg\psi\}$
- $\text{DeLO}^+ = \text{DeLO}^* \ \cup \{\neg\varphi, \psi\}$
- $\text{DeLO}^- = \text{DeLO}^* \ \cup \{\varphi, \neg\psi\}$
- $\text{DeLO}^\pm = \text{DeLO}^* \ \cup \{\varphi, \psi\}$

It suffices to show that these four theories are complete. Then it is clear that no other complete simple extension of DeLO* can exist. As we will explain in Section 1.3, their completeness follows from the fact that they are *ω-categorical*, i.e., they have a unique countable model up to *isomorphism*. See Corollary 1.3.5.

1.1.2 Consequences of Löwenheim-Skolem Theorem

In Section **??**, we proved the Löwenheim-Skolem theorem, in particular its variant for languages without equality:

Theorem (Löwenheim-Skolem)**.** *If L is a countable language without equality, then every consistent L-theory has a countably infinite model.*

This theorem has the following simple corollary:

Corollary 1.1.7. *If L is a countable language without equality, then for every L-structure, there exists an elementarily equivalent countably infinite structure.*

Proof. Let $\mathcal{A}$ be an L-structure. The theory $\text{Th}(\mathcal{A})$ is consistent (it has $\mathcal{A}$ as its model), so by the Löwenheim-Skolem theorem, it has a countably infinite model $\mathcal{B} \models \text{Th}(\mathcal{A})$. This means that $\mathcal{B} \equiv \mathcal{A}$. $\square$

In a language without equality, we cannot express, for example, that 'the model has exactly 42 elements'.

In the proof of the Löwenheim-Skolem theorem, we obtained the constructed model as the canonical model for a non-contradictory branch of the tableau from T for the entry $\text{F}\bot$. The following version for languages with equality is proved in the same way, we just need to take the quotient by the relation $=^A$:

Theorem (Löwenheim-Skolem with Equality)**.** *If L is a countable language with equality, then every consistent L-theory has a countable model (i.e., finite or countably infinite).*

This version also has an easy corollary for specific structures:

Corollary 1.1.8. *If L is a countable language with equality, then for every* infinite *L-structure, there exists an elementarily equivalent countably infinite structure.*

Proof. Let $\mathcal{A}$ be an infinite L-structure. As in the proof of Corollary 1.1.7 (but using the Löwenheim-Skolem theorem with equality), we find a countable structure $\mathcal{B} \equiv \mathcal{A}$. Since for every $n \in \mathbb{N}$, $\mathcal{A}$ satisfies the sentence expressing 'there exist at least n elements' (which can be easily written using equality), these sentences hold in $\mathcal{B}$ as well, so $\mathcal{B}$ cannot be finite and must be countably infinite. □

We will use this corollary to show that there exists a countable field that is algebraically closed:

Countable Algebraically Closed Field

A field $\mathcal{A}$ is *algebraically closed* if every non-zero degree polynomial has a root in it. The field of real numbers $\mathbb{R}$ is not algebraically closed because $x^2 + 1$ has no root in $\mathbb{R}$, and similarly, the field of rationals $\mathbb{Q}$ is not (even $x^2 - 2$ has no root in $\mathbb{Q}$). The field of complex numbers $\mathbb{C}$ is algebraically closed, but it is uncountable.

Algebraic closure can be expressed using the following sentences ψ_n, for each $n > 0$:

$$(\forall x_{n-1}) \dots (\forall x_0)(\exists y)(y^n + x_{n-1} \cdot y^{n-1} + \cdots + x_1 \cdot y + x_0) = 0$$

where y^k is shorthand for the term $y \cdot y \cdot \cdots \cdot y$ (where $\cdot$ is applied $(k-1)$ times).

Corollary 1.1.9. *There exists a countable algebraically closed field.*

Proof. By Corollary 1.1.8, there exists a countably infinite structure $\mathcal{A}$ elementarily equivalent to the field $\mathbb{C}$. Since $\mathbb{C}$ is a field and satisfies the sentences ψ_n for all $n > 0$, $\mathcal{A}$ is also an algebraically closed field. □

1.2 Isomorphism of Structures

Let us take a closer look at the concept of *isomorphism of structures*, which generalizes the isomorphism of graphs, vector spaces, etc. Informally, structures are *isomorphic* if they differ only in the naming of specific elements.

Definition 1.2.1. Given structures $\mathcal{A}, \mathcal{B}$ in a language $L = \langle \mathcal{R}, \mathcal{F} \rangle$, an *isomorphism between $\mathcal{A}$ and $\mathcal{B}$* (or 'from $\mathcal{A}$ to $\mathcal{B}$') is a bijection $h \colon A \to B$ satisfying the following properties:

- For each (n-ary) function symbol $f \in \mathcal{F}$ and for all $a_i \in A$, it holds that:

  $$h(f^{\mathcal{A}}(a_1, \dots, a_n)) = f^{\mathcal{B}}(h(a_1), \dots, h(a_n))$$

  (In particular, if $c \in \mathcal{F}$ is a constant symbol, we have $h(c^{\mathcal{A}}) = c^{\mathcal{B}}$.)

- For each (n-ary) relation symbol $R \in \mathcal{R}$ and for all $a_i \in A$, it holds that:

  $$R^{\mathcal{A}}(a_1, \dots, a_n) \text{ if and only if } R^{\mathcal{B}}(h(a_1), \dots, h(a_n))$$

If an isomorphism exists, we say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* (or '$\mathcal{A}$ is *isomorphic to $\mathcal{B}$ via h*') and write $\mathcal{A} \simeq \mathcal{B}$ (or $\mathcal{A} \simeq_h \mathcal{B}$). An *automorphism* of $\mathcal{A}$ is an isomorphism from $\mathcal{A}$ to $\mathcal{A}$.

Note that the relation of 'being isomorphic' is an equivalence relation. Let us look at an example:

Example 1.2.2. If $|X| = n$, the power set algebra $\underline{\mathcal{P}(X)} = \langle \mathcal{P}(X), -, \cap, \cup, \emptyset, X\rangle$ is isomorphic to the Boolean algebra $\underline{2^n} = \langle \{0,1\}^n, -_n, \wedge_n, \vee_n, \overline{(0,\dots,0)}, (1,\dots,1)\rangle$ (where the operations are applied component-wise) via $h(A) = \chi_A$, where χ_A is the characteristic vector of the subset $A \subseteq X$.

Now we show that an isomorphism is a bijection that 'preserves semantics':

Proposition 1.2.3. *Let $\mathcal{A}, \mathcal{B}$ be structures in a language $L = \langle \mathcal{R}, \mathcal{F}\rangle$. A bijection $h\colon A \to B$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ if and only if the following hold:*

(i) For every L-term t and variable assignment $e\colon \mathrm{Var} \to A$:

$$h(t^{\mathcal{A}}[e]) = t^{\mathcal{B}}[e \circ h]$$

(ii) For every L-formula φ and variable assignment $e\colon \mathrm{Var} \to A$:

$$\mathcal{A} \models \varphi[e] \quad \textit{if and only if} \quad \mathcal{B} \models \varphi[e \circ h]$$

Proof. If h is an isomorphism, the properties are easily proved by induction on the structure of the term or formula. Conversely, if h is a bijection satisfying (i) and (ii), substituting $t = f(x_1, \dots, x_n)$ and $\varphi = R(x_1, \dots, x_n)$ gives the properties from the definition of isomorphism. $\square$

As an immediate consequence, we get the fact that isomorphic structures are elementarily equivalent:

Corollary 1.2.4. *If $\mathcal{A} \simeq \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$.*

Remark 1.2.5. The converse implication does not hold in general. For example, for ordered sets of rational and real numbers, $\langle \mathbb{Q}, \leq\rangle \equiv \langle \mathbb{R}, \leq\rangle$, but $\langle \mathbb{Q}, \leq\rangle \not\simeq \langle \mathbb{R}, \leq\rangle$ because $\mathbb{Q}$ is a countable set while $\mathbb{R}$ is not (so there is no bijection between them).

For finite models, however, isomorphism is the same as elementary equivalence if we have a language with equality, as we prove in the following proposition:

Proposition 1.2.6. *If L is a language with equality and $\mathcal{A}, \mathcal{B}$ are finite L-structures, then:*

$$\mathcal{A} \simeq \mathcal{B} \quad \textit{if and only if} \quad \mathcal{A} \equiv \mathcal{B}$$

Proof. We proved one implication in Corollary 1.2.4. Suppose $\mathcal{A} \equiv \mathcal{B}$ and show that there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Since the language includes equality, we can express by a sentence that 'there are exactly n elements'. This implies that $|A| = |B|$.

Let $\mathcal{A}'$ be the expansion of $\mathcal{A}$ by names of elements from A; it is a structure in the language $L' = L \cup \{c_a \mid a \in A\}$. We will show that $\mathcal{B}$ can be expanded to an L'-structure $\mathcal{B}$ such that $\mathcal{A}' \equiv \mathcal{B}'$. Then, as can be easily verified, the mapping $h(a) = c_a^{\mathcal{B}'}$ is an isomorphism from $\mathcal{A}'$ to $\mathcal{B}'$, and thus also an isomorphism of their L-reducts, and $\mathcal{A} \simeq \mathcal{B}$.

It suffices to show that for each $c_a^{\mathcal{A}'} = a \in A$, there exists an element $b \in B$ such that for expansions by the interpretation of the constant symbol c_a, it holds that $\langle \mathcal{A}, a\rangle \equiv \langle \mathcal{B}, b\rangle$. Let Ω be the set of formulas $\varphi(x)$ such that $\langle \mathcal{A}, a\rangle \models \varphi(x/c_a)$, i.e., $\mathcal{A} \models \varphi[e(x/a)]$. Since A

is a finite set, there are finitely many formulas $\varphi_1(x), \ldots, \varphi_m(x)$ such that for each formula $\varphi \in \Omega$, there exists i such that $\mathcal{A} \models \varphi \leftrightarrow \varphi_i$. Then $\mathcal{B} \models \varphi \leftrightarrow \varphi_i$ (this follows from $\mathcal{A} \equiv \mathcal{B}$, just take the general closure of this formula, which is a sentence).

Since in $\mathcal{A}$ the sentence $(\exists x) \bigwedge_{i=1}^{m} \varphi_i$ holds (it is satisfied by the element $a \in A$), and $\mathcal{B} \equiv \mathcal{A}$, we also have $\mathcal{B} \models (\exists x) \bigwedge_{i=1}^{m} \varphi_i$. In other words, there exists $b \in B$ such that $\mathcal{B} \models \bigwedge_{i=1}^{m} \varphi_i[e(x/b)]$. Thus, for each $\varphi \in \Omega$, $\mathcal{B} \models \varphi[e(x/b)]$, i.e., $\langle \mathcal{B}, b \rangle \models \varphi(x/c_a)$, which is what we wanted to prove. □

Corollary 1.2.7. *If a complete theory in a language with equality has a finite model, then all of its models are isomorphic.*

1.2.1 Definability and Automorphisms

Recall the concept of a definable set from Section **??**. We will show a useful property of definable sets: they are closed ('invariant') under automorphisms of the given structure.

It should surprise no one that under an automorphism, an isolated vertex of a given graph must map to an isolated vertex, a vertex of degree 4 to a vertex of the same degree, or a triple of vertices forming a triangle to a triangle. This can help us, for example, in finding automorphisms.

Proposition 1.2.8. *If $D \subseteq A^n$ is definable in the structure $\mathcal{A}$, then for every automorphism $h \in \mathrm{Aut}(\mathcal{A})$, it holds that $h[D] = D$ (where $h[D]$ denotes $\{h(\overline{a}) \mid \overline{a} \in D\}$).*

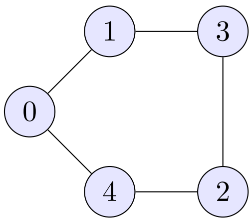
*If D is definable with parameters $\overline{b}$, the same holds for automorphisms that are identical on $\overline{b}$ (*fix $\overline{b}$*), i.e., such that $h(\overline{b}) = \overline{b}$ (meaning $h(b_i) = b_i$ for all i).*

Proof. We will show only the version with parameters. Let $D = \varphi^{\mathcal{A},\overline{b}}(\overline{x}, \overline{y})$. Then for every $\overline{a} \in A^n$, the following chain of equivalences holds:

$$\begin{aligned}
\overline{a} \in D \;&\Leftrightarrow\; \mathcal{A} \models \varphi[e(\overline{x}/\overline{a}, \overline{y}/\overline{b})] \\
&\Leftrightarrow\; \mathcal{A} \models \varphi[(e \circ h)(\overline{x}/\overline{a}, \overline{y}/\overline{b})] \\
&\Leftrightarrow\; \mathcal{A} \models \varphi[e(\overline{x}/h(\overline{a}), \overline{y}/h(\overline{b}))] \\
&\Leftrightarrow\; \mathcal{A} \models \varphi[e(\overline{x}/h(\overline{a}), \overline{y}/\overline{b})] \\
&\Leftrightarrow\; h(\overline{a}) \in D.
\end{aligned}$$

□

Example 1.2.9. Consider the following graph $\mathcal{G}$. Find all sets definable from $\mathcal{G}$ with parameter 0, i.e., the set $\mathrm{Df}^1(\mathcal{G}, \{0\})$.



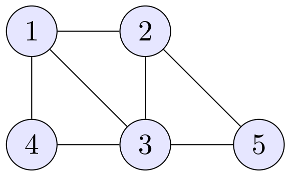
This graph has a single non-trivial automorphism preserving the vertex 0: $h(i) = (5-i) \bmod 5$. Its *orbits* are $\{0\}$, $\{1, 4\}$, and $\{2, 3\}$. These sets are definable:

- $\{0\}$ is defined by the formula $x = y$, i.e., $(x = y)^{\mathcal{G},\{0\}} = \{0\}$,
- $\{1,4\}$ can be defined by the formula $E(x, y)$, and
- $\{2,3\}$ by the formula $\neg E(x, y) \wedge \neg x = y$.

The set $\mathrm{Df}^1(\mathcal{G}, \{0\})$ is a subalgebra of the power set algebra $\mathcal{P}(V(\mathcal{G}))$, so it must be closed under complement, union, intersection, and contain $\emptyset$ and $V(\mathcal{G})$. The subalgebra generated by $\{\{0\}, \{1,4\}, \{2,3\}\}$, however, already contains all the subsets preserving the automorphism h. We get:

$$\mathrm{Df}^1(\mathcal{G}, \{0\}) = \{\emptyset, \{0\}, \{1,4\}, \{2,3\}, \{0,1,4\}, \{0,2,3\}, \{1,2,3,4\}, \{0,1,2,3,4\}\}$$

Exercise 1.1. Consider the following graph. Find all automorphisms. Determine which subsets are definable, provide defining formulas. Which binary relations are definable?



1.3 ω-categorical Theories

Now we will look at theories that have a single countably infinite model (up to isomorphism); these are called *ω-categorical* theories.[6]

Definition 1.3.1 (Isomorphism Spectrum, κ-categoricity)**.** The *isomorphism spectrum* of a theory T is the number $I(\kappa, T)$ of models of T of cardinality κ up to isomorphism, for each cardinality κ (including *transfinite* ones). A theory T is *κ-categorical* if $I(\kappa, T) = 1$.

From now on, we will be interested only in the case $\kappa = \omega$, that is, theories with a single countably infinite model (up to isomorphism). As an example, consider the theory of dense linear orders without endpoints:

Proposition 1.3.2. *The theory DeLO is ω-categorical.*

Proof. Consider two countably infinite models $\mathcal{A}, \mathcal{B}$, and enumerate their elements: $A = \{a_i \mid i \in \mathbb{N}\}$, $B = \{b_i \mid i \in \mathbb{N}\}$. By induction on n, we can, thanks to density, find a sequence $h_0 \subseteq h_1 \subseteq h_2 \subseteq \dots$ of injective (partial) functions from A to B such that $\{a_0, \dots, a_{n-1}\} \subseteq \operatorname{dom} h_n$, $\{b_0, \dots, b_{n-1}\} \subseteq \operatorname{rng} h_n$,[7] and *preserve the order*[8] Then $\mathcal{A} \simeq \mathcal{B}$ via $h = \bigcup_{n\in\mathbb{N}} h_n$. □

Corollary 1.3.3. *The isomorphism spectrum of the theory DeLO* is as follows:*

$$I(\kappa, DeLO^*) = \begin{cases} 0 & \text{for } \kappa \in \mathbb{N}, \\ 4 & \text{for } \kappa = \omega. \end{cases}$$

Countable models up to isomorphism are, for example:

$$\mathbb{Q} = \langle \mathbb{Q}, \leq \rangle \simeq \mathbb{Q} \restriction (0,1),\ \mathbb{Q} \restriction (0,1],\ \mathbb{Q} \restriction [0,1),\ \mathbb{Q} \restriction [0,1]$$

[6] The symbol ω is used for the smallest infinite *ordinal* number, in other words, the set of all natural numbers.

[7] Here, dom denotes the *domain* and rng denotes the *range* of a function.

[8] That is, if $a_i, a_j \in \operatorname{dom} h_n$, then $a_i \leq^{\mathcal{A}} a_j$ if and only if $h(a_i) \leq^{\mathcal{B}} h(a_j)$.

Proof. A dense order obviously cannot be finite. An isomorphism must map the smallest element to the smallest element and the largest to the largest. □

The concept of *ω-categoricity* can be understood as a weakening of the concept of *completeness*. The following useful criterion applies:

Theorem 1.3.4 (ω-Categorical Completeness Criterion)**.** *Let T be an ω-categorical theory in a countable language L. If*

- *L is without equality, or*
- *L is with equality and T has no finite models,*

then the theory T is complete.

Proof. For a language without equality, we know from Corollary 1.1.7 of the Löwenheim-Skolem theorem that every model is elementarily equivalent to some countably infinite model. However, that model is unique up to isomorphism, so all models are elementarily equivalent, which is the semantic definition of completeness.

For a language with equality, we similarly use Corollary 1.1.8 and obtain that all infinite models are elementarily equivalent. There could exist elementarily inequivalent finite models, but we have excluded that possibility. □

Corollary 1.3.5. *The theories* DeLO*,* DeLO$^+$*,* DeLO$^-$*, and* DeLO$^\pm$ *are complete. These are all (mutually inequivalent) complete simple extensions of the theory DeLO*.*

Remark 1.3.6*.* An analogous criterion applies for cardinalities κ greater than ω.

1.4 Axiomatizability

Finally, in this chapter, we will look at the circumstances under which a class of models or a theory can be 'described' (*axiomatized*). We will also be interested in when we can manage with finitely many axioms, and when it can be done by (possibly infinitely many) open axioms. Compare with Proposition **??** from propositional logic.

Definition 1.4.1 (Axiomatizability)**.** Let $K \subseteq \mathrm{M}_L$ be a class of structures in some language L. We say that K is

- *axiomatizable* if there exists an L-theory T such that $\mathrm{M}_L(T) = K$,
- *finitely axiomatizable* if it is axiomatizable by a finite theory, and
- *openly axiomatizable* if it is axiomatizable by an open theory.

We say that an L-theory T' is *finitely*, or *openly axiomatizable*, if this is true of the class $K = \mathrm{M}_L(T')$ of its models.

Example 1.4.2*.* Here are some examples:

- Graphs or partial orders are both finitely and openly axiomatizable.
- Fields are finitely, but not openly axiomatizable.

- Infinite groups are axiomatizable, but not finitely axiomatizable.
- Finite graphs are not axiomatizable.

We will explain why this is so below.

Let us start with a simple fact:

Observation 1.4.3. *If K is axiomatizable, it must be closed under elementary equivalence.*

From the compactness theorem, we easily obtain the following statement, which can be used to show the non-axiomatizability of, for example, finite graphs, finite groups, or finite fields.

Theorem 1.4.4. *If a theory has arbitrarily large finite models, then it also has an infinite model. In that case, the class of all its finite models is not axiomatizable.*

Proof. If the language is without equality, it suffices to take the canonical model for some non-contradictory branch in the tableau from T for the entry $\mathrm{F}\bot$ (T is consistent, as it has models, so the tableau is not contradictory).

Suppose we have a language with equality, and let T' be the following extension of the theory T to the language expanded by countably many new constant symbols c_i:

$$T' = T \cup \{\neg c_i = c_j \mid i \neq j \in \mathbb{N}\}$$

Every finite part of the theory T' has a model: let k be the largest such that the symbol c_k appears in this finite part of T'. Then it suffices to take any model of T with at least $(k+1)$ elements and interpret the constants $c_0, \ldots, c_k$ as distinct elements of this model.

By the compactness theorem, T' also has a model. That model is necessarily infinite. Its reduct to the original language (forgetting the constants $c_i^{\mathcal{A}}$) is an infinite model of T. □

Remark 1.4.5. The class of all *infinite* models of a theory is always axiomatizable if we have a language with equality: it suffices to add to the theory, for each $n \in \mathbb{N}$, an axiom expressing 'there are at least n elements'.

1.4.1 Finite Axiomatizability

We will show the following criterion for finite axiomatizability: both the class of structures K and its complement $\overline{K}$ must be axiomatizable.

Theorem 1.4.6 (Finite Axiomatizability)**.** *Let $K \subseteq \mathrm{M}_L$ be a class of structures and also consider its complement $\overline{K} = \mathrm{M}_L \setminus K$. Then K is finitely axiomatizable if and only if both K and $\overline{K}$ are axiomatizable.*

Proof. If K is finitely axiomatizable, then it is axiomatizable by finitely many sentences $\varphi_1, \ldots, \varphi_n$ (we can replace formulas with their general closures). To axiomatize $\overline{K}$, it suffices to take the sentence $\psi = \neg(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$. Clearly, $\mathrm{M}(\psi) = \overline{K}$.

Conversely, suppose T and S are theories such that $\mathrm{M}(T) = K$ and $\mathrm{M}(S) = \overline{K}$. Consider the theory $T \cup S$. This theory is inconsistent, as:

$$\mathrm{M}(T \cup S) = \mathrm{M}(T) \cap \mathrm{M}(S) = K \cap \overline{K} = \emptyset$$

By the compactness theorem,[9] there exist finite subtheories $T' \subseteq T$ and $S' \subseteq S$ such that:

$$\emptyset = \mathrm{M}(T' \cup S') = \mathrm{M}(T') \cap \mathrm{M}(S')$$

Now note that:

$$\mathrm{M}(T) \subseteq \mathrm{M}(T') \subseteq \overline{\mathrm{M}(S')} \subseteq \overline{\mathrm{M}(S)} = \mathrm{M}(T)$$

Thus, we have shown that $\mathrm{M}(T) = \mathrm{M}(T')$, that is, the theory T' is the desired finite axiomatization of K. □

As an application, we will prove that fields of characteristic 0 are not finitely axiomatizable.

Example: Fields of Characteristic 0

Let T be the theory of fields. The characteristic of a field is the smallest number of ones that must be added to get zero (in that case, the characteristic must be a prime number—prove it!), or if we never get zero by adding ones, we say that the characteristic is 0. More formally:

Definition 1.4.7 (Characteristic of a Field)**.** We say that a field $\mathcal{A} = \langle A, +, -, 0, \cdot, 1\rangle$ is of

- *characteristic* p if p is the smallest prime number such that $\mathcal{A} \models p1 = 0$, where $p1$ denotes the term $1 + 1 + \cdots + 1$ with p ones, or
- *characteristic 0* if it is not of characteristic p for any prime number p.

Let T be the theory of fields. Then the class of fields of characteristic p is finitely axiomatized by the theory $T \cup \{p1 = 0\}$. The class of fields of characteristic 0 is axiomatized by the following (infinite) theory:

$$T' = T \cup \{\neg\, p1 = 0 \mid p \text{ is a prime number}\}$$

However, a finite axiomatization does not exist.

Proposition 1.4.8. *The class K of fields of characteristic* 0 *is not finitely axiomatizable.*

Proof. Thanks to Theorem 1.4.6, it suffices to show that $\overline{K}$ (consisting of fields of non-zero characteristic and structures that are not fields) is not axiomatizable, which we will prove by contradiction. Suppose there exists a theory S such that $\mathrm{M}(S) = \overline{K}$. Then the theory $S' = S \cup T'$ has a model, as every finite part of it has a model: it suffices to take a field of prime characteristic greater than any p appearing in the axioms of T' of the form $\neg\, p1 = 0$. Let $\mathcal{A}$ be a model of S'. Then $\mathcal{A} \in \mathrm{M}(S) = \overline{K}$. At the same time, $\mathcal{A} \in \mathrm{M}(T') = K$, which is a contradiction. □

1.4.2 Open Axiomatizability

For open axiomatizability, there is a simple semantic criterion: the class of its models must be closed under taking substructures. In fact, the two properties are equivalent, but we will prove only one implication (the proof of the other is more difficult).

Theorem 1.4.9. *If a theory T is openly axiomatizable, then every substructure of a model of T is also a model of T.*

[9] See how useful it is!

Remark 1.4.10. The converse implication also holds: If every substructure of a model of T is also a model, then T is openly axiomatizable. We will omit the proof.

Proof. Suppose T' is an open axiomatization of T. Let $\mathcal{A} \models T'$ be a model and let $\mathcal{B} \subseteq \mathcal{A}$ be a substructure. For every formula $\varphi \in T'$, we have $\mathcal{B} \models \varphi$ (since φ is open). Thus $\mathcal{B} \models T'$. □

Example 1.4.11. Here are some examples:

- The theory DeLO is not openly axiomatizable; for example, no finite substructure of a model of DeLO can be dense.
- The theory of fields is not openly axiomatizable; the substructure $\mathbb{Z} \subseteq \mathbb{Q}$ of the field of rational numbers is not a field, as there is no multiplicative inverse of 2 in $\mathbb{Z}$.
- For a given $n \in \mathbb{N}$, the theories of at most n-element groups are openly axiomatizable (subgroups are certainly also at most n-element). As an open axiomatization, we can take the following extension of the (open) theory of groups T:

$$T \cup \{ \bigvee_{1 \leq i < j \leq n+1} x_i = x_j \}$$

Bibliography