Matrices and Vectors

$$\underset{m\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

m rows
n columns

Special cases

column vector $\underset{m\times 1}{\underline{a}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$

row vector $\underset{1\times n}{\underline{b}} = [b_1 \; b_2 \; \cdots \; b_n]$

Product of Two Matrices

$$C = A\,B$$

$$\underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1l} \\ c_{21} & c_{22} & \cdots & c_{2l} \\ & & & \\ c_{n1} & c_{n2} & \cdots & c_{nl} \end{bmatrix}}_{C_{n\times l}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}}_{A_{n\times m}} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1l} \\ b_{21} & b_{22} & \cdots & b_{2l} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{ml} \end{bmatrix}}_{B_{m\times l}}$$

$$c_{ij} = \sum_{k=1}^{m} a_{ik}\, b_{kj}$$

Example

$$\begin{bmatrix} 2 & -1 & 6 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 11 & -18 \\ 19 & 9 \end{bmatrix}$$

Transpose of Matrix

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad A^T_{n\times m} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \Rightarrow \underline{a}^T = [a_1 \; a_2 \; \cdots \; a_n]$$

Dot Product of Two Vectors (a.k.a. inner product)

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\underline{a} \cdot \underline{b} = \underline{a}^T \underline{b} = [a_1 \; a_2 \; \cdots \; a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \sum_{k=1}^{n} a_k b_k$$

Determinant

- only defined for square matrix, e.g. $A_{n\times n}$
- Notation: $\det(A)$ or $|A|$

Case $n = 2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

Case $n=3$

$$A = \begin{bmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{bmatrix}$$

$$\det(A) = a_{11} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{co-factor of } a_{11}} + a_{12} \underbrace{\left(- \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \right)}_{\text{co-factor of } a_{12}} + a_{13} \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{co-factor of } a_{13}}$$

Example

$$A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}, \quad |A| = 7 \cdot (-2) + 2(3) + 1(9)$$
$$= 1$$

Matrix of Co-Factors

$$A = \begin{bmatrix} \overset{+}{7} & \overset{-}{2} & \overset{+}{1} \\ \overset{-}{0} & \overset{+}{3} & \overset{-}{-1} \\ \overset{+}{-3} & \overset{-}{4} & \overset{+}{-2} \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}$$

co-factor matrix

Adjoint of Matrix

$$\text{adj}(A) = C^T = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

Inverse of Matrix

- well-defined only for square matrix A with $\det(A) \neq 0$.

- A^{-1} satisfies : $A^{-1}A = AA^{-1} = I$

$\uparrow$ Identity matrix

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

Result

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$$

Example

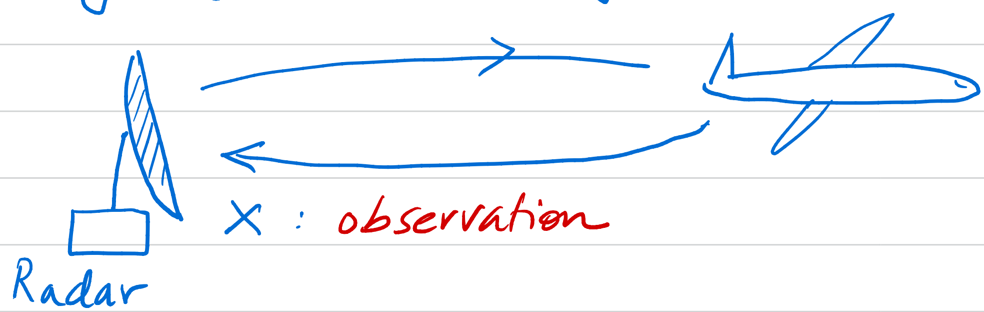
$$A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}$$

$$\det(A) = |A| = 1$$

$$\mathrm{adj}(A) = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix} = A^{-1} \quad (\text{since } |A| = 1)$$

Check : $\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Binary Hypothesis Testing (ECE 313)



X : observation

H_0 : plane is absent

H_1 : plane is present

Goal: Decide which hypothesis is true based on X

Prior information $P(H_0) = \pi_0$, $P(H_1) = \pi_1 = 1 - \pi_0$

Observation pdf/pmf $p(x|H_0)$, $p(x|H_1)$

Likelihood Ratio $\Lambda(x) = \dfrac{p(x|H_1)}{p(x|H_0)}$

Performance

Probability of False Alarm $P_{FA} = P(\text{decide } H_1 \mid H_0)$

Probability of Miss $P_M = P(\text{decide } H_0 \mid H_1)$

Overall probability of error $P_e = \pi_0 P_{FA} + \pi_1 P_M$

MAP Rule decide 1 if $\Lambda(x) \geq \dfrac{\pi_0}{\pi_1}$

decide 0 if $\Lambda(x) < \dfrac{\pi_0}{\pi_1}$

$\Lambda(x) \underset{0}{\overset{1}{\gtrless}} \dfrac{\pi_0}{\pi_1}$ MAP rule minimizes P_e.

Example $\quad \pi_0 = 0.5, \quad \pi_1 = 0.5.$

$$p(x|H_0) = \begin{cases} 0.1 & x=0 \\ 0.2 & x=1 \\ 0.3 & x=2 \\ 0.4 & x=3 \\ 0 & \text{otherwise} \end{cases} \qquad p(x|H_1) = \begin{cases} 0.25 & x=0 \\ 0.25 & x=1 \\ 0.2 & x=2 \\ 0.3 & x=3 \\ 0 & \text{otherwise} \end{cases}$$

(a) MAP Rule :

$$\Lambda(x) = \frac{p(x|H_1)}{p(x|H_0)} \underset{0}{\overset{1}{\gtrless}} \frac{\pi_0}{\pi_1} = 1$$

$\Lambda(x) > 1$ for $x = 0$, and $x = 1$

$\Lambda(x) < 1$ for $x = 2$, and $x = 3$

MAP rule decides $\begin{cases} 1 & \text{if } x = 0 \text{ or } x = 1 \\ 0 & \text{if } x = 2 \text{ or } x = 3 \end{cases}$

(b) Performance :

$$P_{FA} = P(\text{decide } H_1 | H_0) = p(0|H_0) + p(1|H_0)$$
$$= 0.1 + 0.2 = 0.3$$

$$P_M = P(\text{decide } H_0 | H_1) = p(2|H_1) + p(3|H_1)$$
$$= 0.2 + 0.3 = 0.5$$

$$P_e = \pi_0 P_{FA} + \pi_1 P_M$$
$$= 0.5(0.3 + 0.5) = 0.4$$