Singular Value Decomposition (SVD)

Generalization of eigen-decomposition that applies to non-square (non-symmetric) matrices:

$$A_{m\times n} = U_{m\times m} \; S_{m\times n} \; V^{\top}_{n\times n}$$

$m<n$ "fat"

$$= \begin{bmatrix} \boldsymbol{u}_1 \; \boldsymbol{u}_2 \dots \boldsymbol{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \dots & 0 \\ 0 & \dots & 0 & \sigma_m & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^{\top} \\ \boldsymbol{v}_2^{\top} \\ \vdots \\ \vdots \\ \boldsymbol{v}_n^{\top} \end{bmatrix}$$

$m>n$ "tall"

$$= \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \ddots & 0 \\ \vdots & \ddots & \dots & 0 \\ 0 & \dots & 0 & \sigma_n \\ 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^{\top} \\ \boldsymbol{v}_2^{\top} \\ \vdots \\ \boldsymbol{v}_n^{\top} \end{bmatrix}$$

S is $m\times n$ diagonal matrix with leading diagonal equal to $(\sigma_1, \sigma_2, \dots, \sigma_{\min\{m,n\}})$,

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\min\{m,n\}} \geq 0$$

U, V are orthogonal: $UU^T = I_{m\times m}$, $VV^T = I_{n\times n}$

U, V come from eigen-decompositions of AA^T, A^TA

$$AA^T = UDU^T \qquad A^TA = V\tilde{D}V^T$$

$m\times m$ symmetric $n\times n$ symmetric

If $m < n$,

$$D = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_m^2 \end{bmatrix}, \qquad \tilde{D} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \sigma_2^2 & \ddots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \sigma_m^2 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & 0 & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix},$$

If $m > n$, D and $\tilde{D}$ swap roles along with m and n

Example Find SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$ $\quad m = 2$, $n = 3$

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$$\det(AA^T - \lambda I) = \begin{vmatrix} (17-\lambda) & 8 \\ 8 & (17-\lambda) \end{vmatrix} = \lambda^2 - 34\lambda + 225$$

$$= 0 \Rightarrow (\lambda - 25)(\lambda - 9) = 0 \Rightarrow \lambda_1 = 25, \lambda_2 = 9$$

$$\Rightarrow \sigma_1 = \sqrt{25} = 5, \quad \sigma_2 = \sqrt{9} = 3$$

To find U, need eigenvectors of AA^T

$$\begin{bmatrix} (17-25) & 8 \\ 8 & (17-25) \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \underline{0} \Rightarrow u_{11} = u_{12}$$

Set $\underline{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similarly, can show $\underline{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

To find V, need eigenvectors of A^TA.

$$A^TA = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

Check: A^TA has eigenvalues $25, 9, 0$

$\underline{v}_1$ satisfies $(A^TA - 25I)\underline{v}_1 = \underline{0}$

$$(A^TA - 25I)\underline{v}_1 = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -12v_{11} + 12v_{12} + 2v_{13} &= 0 \\ 2v_{11} - 2v_{12} - 17v_{13} &= 0 \end{aligned}\right\} \Rightarrow v_{13} = 0$$

$$\Rightarrow -v_{11} + v_{12} = 0 \Rightarrow v_{11} = v_{12}.$$

Set $\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ Similarly, $\underline{v}_2 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, $\underline{v}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

SVD of A:

$$\begin{array}{cc} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} & \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \\ \underline{u}_1 \quad \underline{u}_2 & \sigma_1 = 5,\ \sigma_2 = 3 \end{array} \begin{array}{c} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \underline{v}_1^T \\ \leftarrow \underline{v}_2^T \\ \leftarrow \underline{v}_3^T \end{array}$$

Note that $\underline{v}_3$ is zeroed out in RHS.

Compact Form (Economy) SVD:

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{bmatrix}$$

SVD can also be written as:

$$A = \sum_{i=1}^{\min\{m,n\}} \sigma_i \, \underline{u}_i \, \underline{v}_i^T$$

For example,

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = 5 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{bmatrix}$$

Low-Rank Approximation (LRA)

Since $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min\{m,n\}} \geq 0$, we can get a low-rank approximation of A using $k < \min\{m,n\}$ terms in SVD expansion

$$\hat{A} = \sum_{i=1}^{k} \sigma_i \, \underline{u}_i \, \underline{v}_i^T, \qquad k < \min\{m,n\}.$$

LRA's are useful in many ways.

LRA for Image Compression

Image is a matrix of pixel values $A_{m \times n}$

Example "Larry the Cat" $m = 630$, $n = 420$

- Uncompressed image: $m \cdot n = 264{,}600$ values
- Rank-k SVD: Send k $\underline{u}_i$'s, $\underline{v}_i$'s, σ_i's reconstruct $\hat{A}$ at receiver

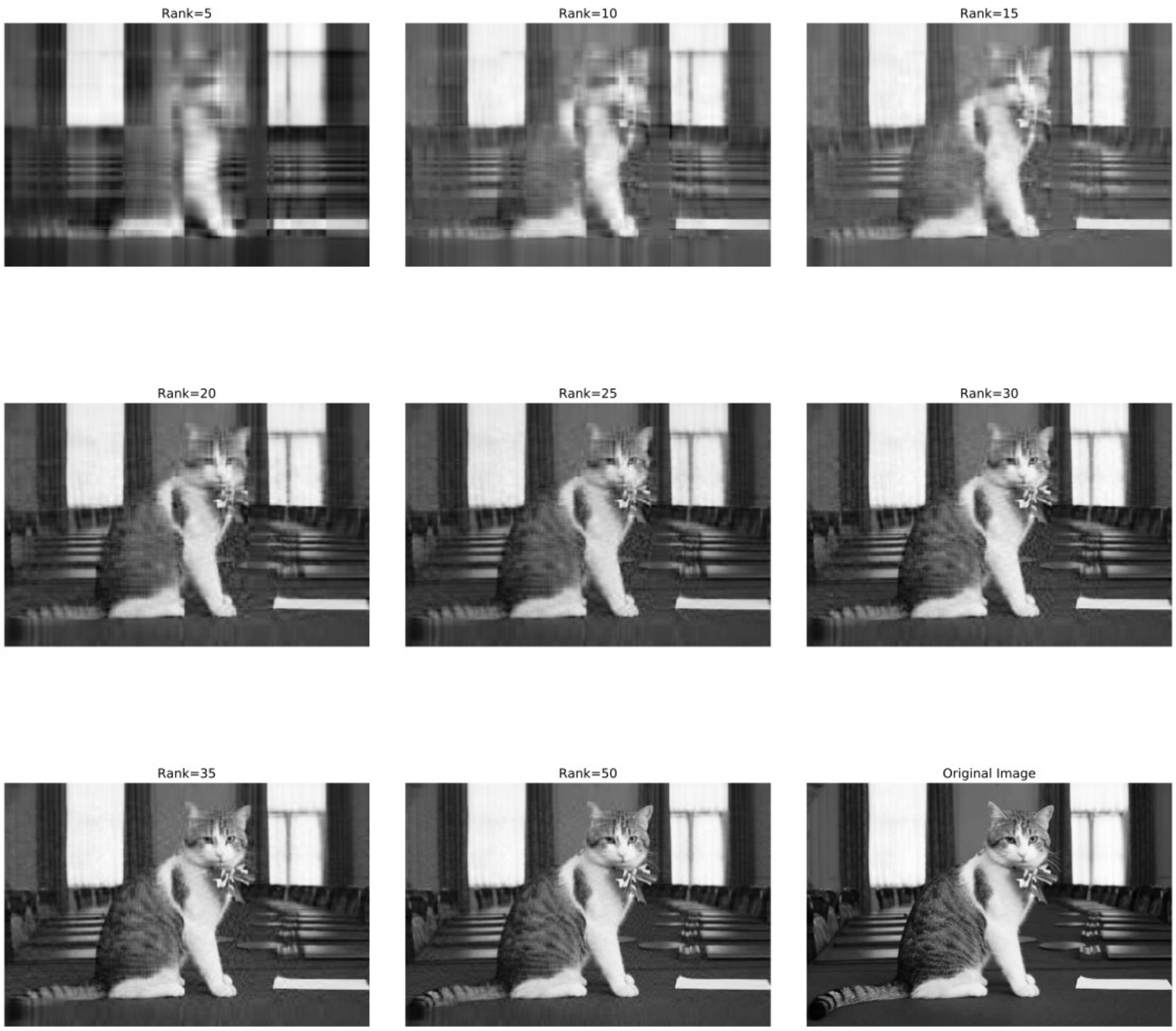
Values used in compression

$$= \underbrace{630k}_{\text{dim of } \underline{u}_i\text{'s}} + \underbrace{420k}_{\text{dim. of } \underline{v}_i\text{'s}} + \underbrace{1 \cdot k}_{\text{dim. of } \sigma_i\text{'s}} = 1051k \text{ values}$$

Uncompressed : 264,600 values

Compressed : 1051 k values

k	5	10	20	50
# Values	5255	10,510	21,020	52,250
% Compression	2%	4%	8%	20%



Direct LRA leads to global distortion

Solution : Apply LRA to blocks of image (JPEG)