

4.4.2(c):

Prove that for any positive integer n,

$$\sum_{j=1}^n j^3 = \left(\frac{n(n+1)}{2}\right)^2$$

case 1: n = 1

$$\sum_{j=1}^1 j^3 = 1^3 = 1 \text{ and } \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1 \cdot 2}{2}\right)^2 = 1^2 = 1$$

case 2: n = k+1

$$\sum_{j=1}^n j^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{j=1}^{k+1} j^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

$$\sum_{j=1}^{k+1} j^3 = \sum_{j=1}^k j^3 + (k+1)^3$$

$$\sum_{j=1}^{k+1} j^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2 \left(\frac{k^2}{4}\right) \left(\frac{4k}{4}\right) \left(\frac{4}{4}\right) = (k+1)^2 \left(\frac{k^2+4k+4}{4}\right)$$

$$= (k+1)^2 \left(\frac{(k+2)^2}{4}\right) = \frac{(k+1)^2(k+2)^2}{4}$$

$$\sum_{j=1}^{k+1} j^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

4.5.1(e):

Prove that for any positive integer n, 2 evenly divides  $n^2 - 5n + 2$

Case1: n is even

$$n \equiv 0 \pmod{2}$$

$$1. \ n^2 \equiv 0^2 \equiv 0 \pmod{2}$$

$$2. \ 5n \equiv 5 \cdot 0 \equiv 0 \pmod{2}$$

$$3. \ n^2 - 5n + 2 \equiv 0 - 0 + 2 \equiv 2 \equiv 0 \pmod{2}$$

making  $n^2 - 5n + 2 = \text{even}$  when n is even

Case2: n is odd

$$n \equiv 1 \pmod{2}$$

$$1. \ n^2 \equiv 1^2 \equiv 1 \pmod{2}$$

$$2. \ 5n \equiv 5 \cdot 1 \equiv 5 \equiv 1 \pmod{2}$$

$$3. \ n^2 - 5n + 2 \equiv 1 - 1 + 2 \equiv 2 \equiv 0 \pmod{2}$$

making  $n^2 - 5n + 2$  odd when n is odd

Therefore, for any positive integer n,  $n^2 - 5n + 2$  is even, proving that 2 evenly divides  $n^2 - 5n + 2$  is divisible by 2 for any positive integer n.

4.4.3(c):

Prove that for  $n \geq 1$ ,

$$\sum_{j=1}^n \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Consider  $f(x) = \frac{1}{x^2}$

$$\int_1^{n+1} \frac{1}{x^2} dx < \sum_{j=1}^n \frac{1}{j^2} < 1 + \int_1^n \frac{1}{x^2} dx$$

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

$$\int_1^{n+1} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^{n+1} = -\frac{1}{n+1} + 1 = 1 - \frac{1}{n+1}$$

$$\int_1^n \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^n = -\frac{1}{n} + 1 = 1 - \frac{1}{n}$$

From the integrals, we now get

$$1 - \frac{1}{n+1} < \sum_{j=1}^n \frac{1}{j^2} < 1 + \left(1 - \frac{1}{n}\right) = 2 - \frac{1}{n}$$

thus for all  $n \geq 1$ :

$$\sum_{j=1}^n \frac{1}{j^2} < 2 - \frac{1}{n}$$