① ②

Workshop.

Introductions.

Pete Dowsett: facilitator

2 days, Saturday & Sunday.

1 pm – 7 pm, 6 hours each day.

Breaks, short 10 minutes every hour

1 long break ~5pm 30 minutes.

Logistics: Zoom call

Ask questions, clarifications

Mute, interruptions okay

Raise hand, comment in chat

↳ Pete can watch these things.

Recording on Patreon

Posted ASAP

Workshop Give-away Tweets.

Textbook Big Freq.

Structure of Workshop.

Programming / Software.

Circuit Theory / Electronics.

Math notes

Few proofs / theory

More practical Demonstrations.

Foundational knowledge → simple → complex → general

Goal: $\boxed{\text{HW}} \rightarrow \boxed{\text{SW}}$ Background Context.

Analog → Digital

Electricity → Code

Many different ways, research & industry.

Workshop is on one category.

1). Blackbox: inner aspects of HW unknown.
how does input → output. (Acoustica) Kemper

2). Whitebox: Components are modeled. (VA)
"Physically informed" Virtual Analog.

3) Graybox: Spectrum (range of detail)

The Art of VA Filter Design.

③

Approaches for component modeling

1) Not in workshop: Wave Digital Filter
   Different Paradigm
   Kurt Werner, DAFX
2) Traditional Circuit Analysis Techniques
   Nodal Analysis → Modified Nodal Analysis
   State-Space → Discrete Kirchhoff (DK)
   Presonus "branded"

Type of effect (system) determine modeling technique.

EQ, Reverb   Black-box   Sine sweep
Linear, Time-Invariant (LTI)
Non-linear: Distortion   Volterra-Series
Hammerstein-Weiner Model

[Schematic] → [Math] → [Prototype] → [Plug-in]

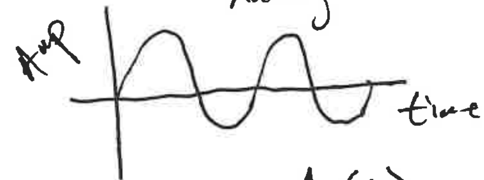
Prototype: Matlab, Python
Plug-in: JUCE/C++

Start with Digital as reference.

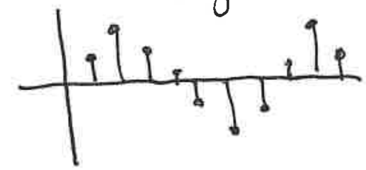
④.

Audio signals.

Analog



Digital



Continuous $A(t)$
Acoustic Sound Pressure
Electricity.

Discrete
$X[n]$
n = sample number
0, 1, 2, 3, ...

Signal Gain Change

in → [Amp] → out   in → ▷ → out

in $x[n]$ → $\otimes$ (g) → out $y[n]$

Matlab: audioread.
loop
sinewave synthesis.

DC sweep [-1:.001:1]
Characteristic Curve

Know the end goal, ~~then~~ introduce electronics

Circuit: a system of interconnected components that carry electricity.

Electricity: electrons that move
↳ negative charge, jump from one atom to next.

Current: Flow of electrons.
↳ units amps/ampere (A)

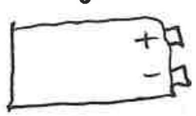
Voltage: force that causes current to flow.
unit Volt (V)

Analogy. water flowing, pipes
pump pushes water.

Relationship between voltage & current.

Circuit: we need a place to start.
What is the input.

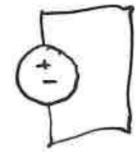
Sources: voltage & current

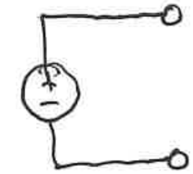
Battery  ← Terminals.

Not a complete circuit on its own.
no current flowing, supplying voltage.

Symbol V 

Use wire to connect paths for current to flow.

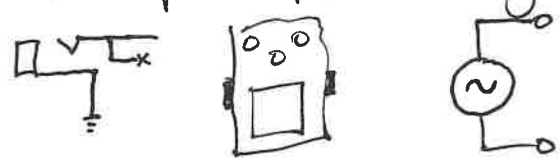
V  ↓i

 Open circuit

Voltage is identical when measured across wires.

Short circuit: current is the same at points in wire not separated by component.

Other types of Voltage Sources
microphone, electric guitar.

 AC vs. DC.

Battery DC. direct current (no frequency)

AC - alternating current (has frequency Hz)

Other times in schematic.

Vin o———

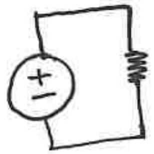
Current Sources

0.5A  There has to be exactly this current at this point in circuit.

Useful circuits need other components.

Resistors: restrict current flow.
units Ohm's Ω (omega).

Symbol

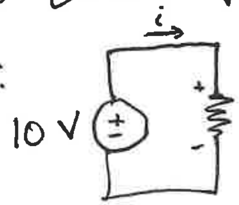


First full circuit.

Relationship between Volts, Amps, Ohms.

Ohm's Law $V = I \cdot R$ $\quad v = i \cdot R$

Examples:



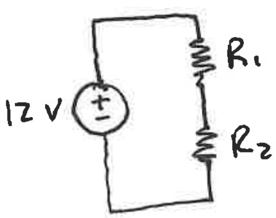
$R = 5\Omega$ What is current?

$10 = i \cdot R \quad 10 = i \cdot 5 \quad i = \frac{10}{5} = 2 \quad i = \frac{V}{R}$

Note: this amount of current is at all points in this circuit.

Voltage can be measured across source or resistor same value.

Combined Resistance (series)



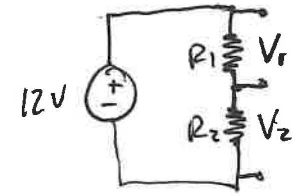
$R = R_1 + R_2$ equivalent resistance

$R_1 = 1\ \Omega$
$R_2 = 2\ \Omega$

What is current?

$$i = \frac{12}{R_1 + R_2} = \frac{12}{3} = 4\ A$$

What is voltage across individual resistors?



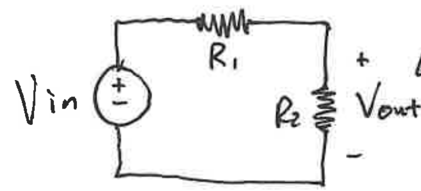
Current is same all the way around.

$V_1 = i \cdot R_1 \quad i = 4$
$V_2 = i \cdot R_2$

$V_1 = 4 \cdot 1 = 4$ volts. $\quad V_2 = 4 \cdot 2 = 8$ volts.

Input 12 volts divided between R_1 & R_2
"Voltage divider" circuit.

Transfer Function Form.



Define output of circuit

Interested in writing Vout as a function of Vin.

$i = \frac{V_{in}}{R_1 + R_2}$ Start by finding current through all elements.

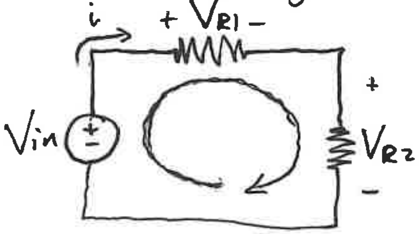
$V_{out} = i R_2$ Plug in current to find voltage across R_2 only.

$$V_{out} = \frac{V_{in} R_2}{R_1 + R_2} \qquad \frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2} = G$$

$y[n] = G \cdot x[n]$

$R_1 = R_2$	$R_1 > R_2$	$R_1 < R_2$
$G = 0.5$	$0 < G < 0.5$	$0.5 < G < 1$

Other analyses of circuit.



Kirchhoff's Voltage Law (KVL)

Mesh Loop.

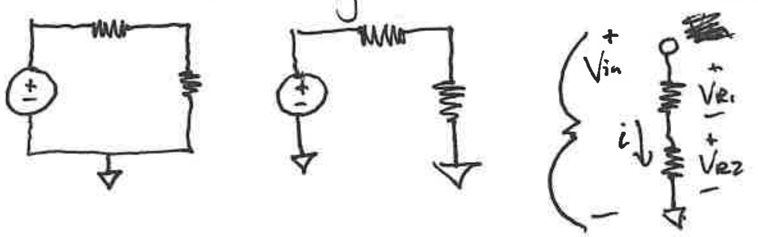
Sum of voltages around a closed loop of circuit must sum to zero.

Clockwise arrow

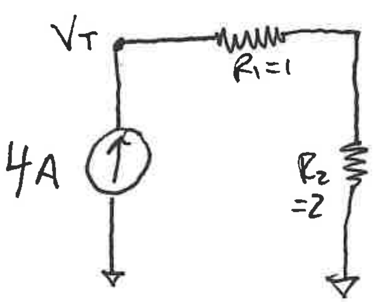
negative-to-positive = positive-to-negative.

$$V_{in} = V_{R1} + V_{R2}$$

New concept: "ground" reference node 0 V



Current Source.



$$V_{R1} = 4 \cdot 1 = 4$$

$$V_{R2} = 4 \cdot 2 = 8$$

$$V_T = V_{R1} + V_{R2} = 12$$

Common Resistor Values.

Rare to have 1, 2, 4, 8 Ω in audio.

Found in guitar speaker cabinets.

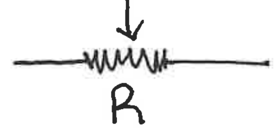
More often 1000 Ω, 100,000 Ω

units 1k = 1000   1M = 1000000 = 1000k

100k = 100000 $1M = 1 \times 10^6 = 1e^6$

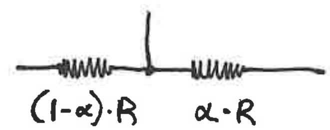
$1 \times 10^3 = 1e3$

Variable Resistor: Potentiometer



Symbol

How to think about it:

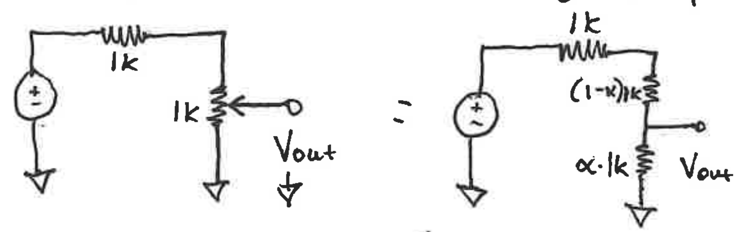


$0 < \alpha < 1$

2 separate resistors

Values sum to R

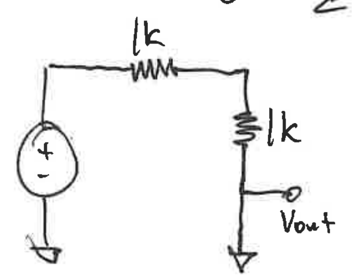
Examples:



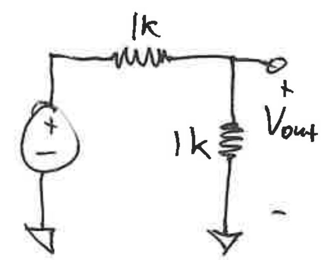
$\alpha = 0$ $V_{out} = 0$

$\alpha = 1$ $\frac{V_{out}}{V_{in}} = \left(\frac{1k}{1k+1k}\right)$

$\alpha = 0$



$\alpha = 1$



$$\frac{V_{out}}{V_{in}} = \left(\frac{1k}{1k+1k}\right) = \frac{1}{2}$$

$$G = \frac{1}{2}$$

Other Potentiometer Examples.

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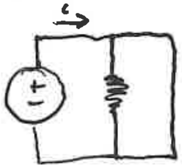
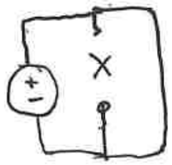
Side note: What happens when we "short" resistor?

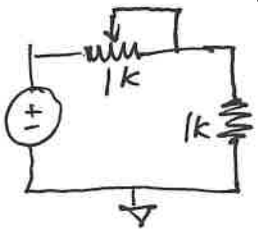


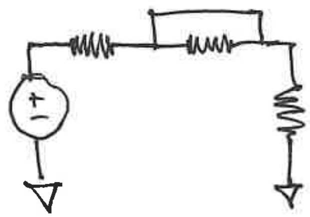
Wire connecting around resistor.
All the current chooses path of no resistance. No current flows across resistor.

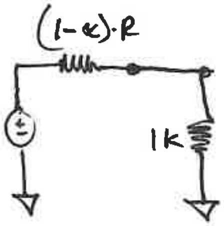
Bypassed Resistor



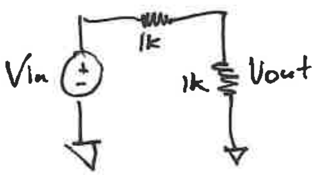
Used with pot





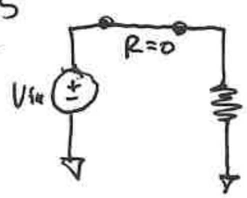


$\alpha = 0$



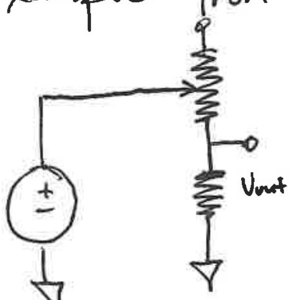
$\frac{V_{out}}{V_{in}} = \frac{1k}{2k} = 0.5$

$\alpha = 1$

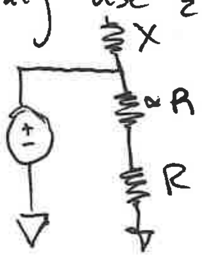


$G = 1$

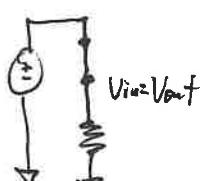
Example from clipper

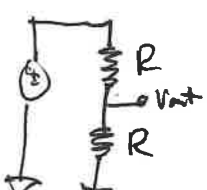


Only use $\frac{1}{2}$ of pot



$\alpha = 0$





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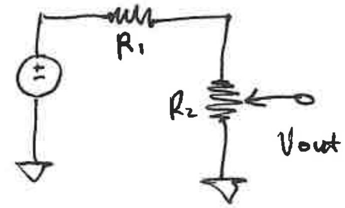
Matlab Implementations.

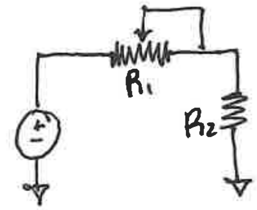


$R_1 = 1k \quad R_2 = 1k$

$P = \alpha \cdot R_2$

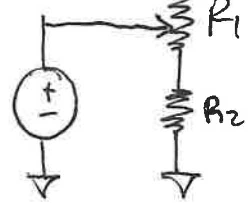
$V_{out} = G \cdot V_{in}$

$G = \left(\frac{P}{R_1 + P}\right)$



$R_1 = 1k \quad R_2 = 1k$

$P = (1-\alpha) \cdot R_1 \quad G = \left(\frac{R_2}{P + R_2}\right)$



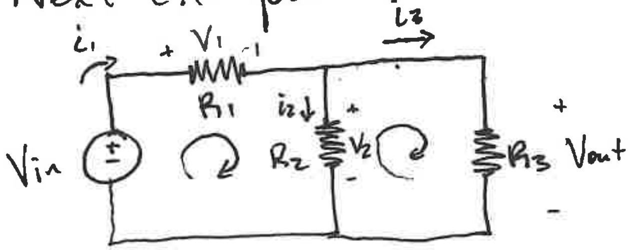
$P = (1-\alpha) R_1$

$G = \left(\frac{R_2}{P + R_2}\right)$

These last two end up being the same result.

The main thing we want is a variable (α) that can be controlled by a plug-in knob.

Next example



Goal: Write V_{out} as function of V_{in}.

Things we know: KVL

$V_{in} = V_1 + V_2 \quad V_2 = V_{out} \quad V_{in} = V_1 + V_{out}$

Ohm's Law

$V_1 = i_1 \cdot R_1$ What about current on other Rs?

New Rule: Kirchhoff's Current Law (KCL)

Water in pipes.

Current flow into node = current flowing out of node.

$$i_1 = i_2 + i_3$$

Go back to Ohm's Law

$V_2 = i_2 \cdot R_2 \qquad V_{out} = i_3 \cdot R_3$

$i_2 = \frac{V_2}{R_2} = \frac{V_{out}}{R_2} \qquad i_3 = \frac{V_{out}}{R_3}$

We know several relationships between elements of circuit. Let's combine them together to reach our goal.

This is what circuit analysis is all about.

Many different ways to do this.

$$V_{in} = V_1 + V_{out}$$

$$V_{in} = i_1 \cdot R_1 + V_{out}$$

$$V_{in} = (i_2 + i_3) \cdot R_1 + V_{out}$$

$$V_{in} = \left(\frac{V_{out}}{R_2} + \frac{V_{out}}{R_3}\right) \cdot R_1 + V_{out}$$

$$V_{in} = \frac{R_1}{R_2} V_{out} + \frac{R_1}{R_3} V_{out} + V_{out}$$

$$V_{in} = \left(\frac{R_1}{R_2} + \frac{R_1}{R_3} + 1\right) \cdot V_{out}$$

Option 1: $V_{in} \cdot G = V_{out}$

$$G = \frac{1}{\left(\frac{R_1}{R_2} + \frac{R_1}{R_3} + 1\right)}$$

Option 2 Expand. Want $R_2 \cdot R_3$ in denominator

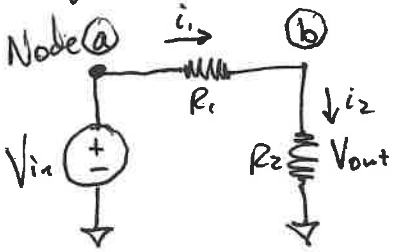
$$\left(\frac{R_1}{R_2} \cdot \frac{R_3}{R_3} + \frac{R_1}{R_3} \frac{R_2}{R_2} + \frac{R_2 R_3}{R_2 R_3}\right)$$

$$\left(\frac{R_1 R_3 + R_1 R_2 + R_2 R_3}{R_2 R_3}\right)$$

$$V_{in} \cdot \left(\frac{R_2 \cdot R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}\right) = V_{out}$$

Nodal Analysis: different way to define relationships in circuit.

We will use thing moving forward.



Pick nodes. What's a node? Location between components

$V_a = V_{in} \qquad V_b = V_{out}$

Currents into ~~a~~ node = currents out of node

$i_1 = i_2$

Normally we'd write

$i_1 = \frac{V_{R_1}}{R_1}$ also use $V_{in} = V_{R_1} + V_{out}$

Notice $V_{R1} = V_{in} - V_{out}$

We will now write voltages as the <u>drop</u> from node "a" to node "b"

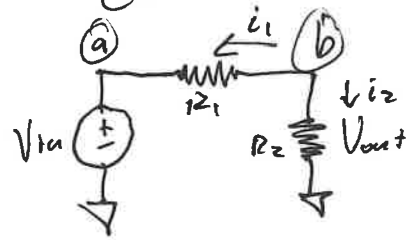
$$i_1 = \frac{V_a - V_b}{R_1} = \frac{V_{in} - V_{out}}{R_1}$$

$$i_2 = \frac{V_b - 0}{R_2} = \frac{V_b}{R_2} = \frac{V_{out}}{R_2}$$ When ~~tied~~ taken to ground

Node B:

KCL $i_1 = i_2$ $\quad \frac{V_{in} - V_{out}}{R_1} = \frac{V_{out}}{R_2} \quad \frac{V_{in}}{R_1} = V_{out}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ldots$

Note: what happens if we label current directions differently?



KCL $0 = i_1 + i_2$

both currents flowing out

$$i_2 = \frac{V_{out}}{R_2}$$

$$i_1 = \frac{V_b - V_a}{R_1} = \frac{V_{out} - V_{in}}{R_1}$$

$$0 = \frac{V_{out}}{R_1} - \frac{V_{in}}{R_1} + \frac{V_{out}}{R_2}$$

$$\frac{V_{in}}{R_1} = V_{out}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_{out}\left(\frac{R_2 + R_1}{R_1 R_2}\right)$$

$$\frac{V_{in}}{R_1}\left(\frac{R_1 R_2}{R_1 + R_2}\right) = V_{out}$$

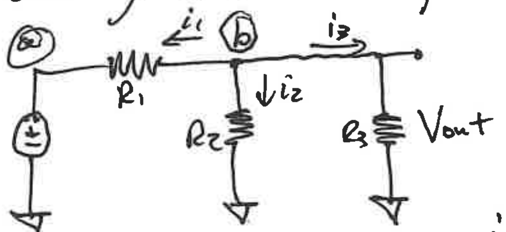
$$V_{in}\left(\frac{R_2}{R_1 + R_2}\right) = V_{out}$$

$$G = \frac{R_2}{R_1 + R_2}$$ same result.

Take-away: with resistors main thing

$\frac{\text{start} - \text{finish}}{\text{resistor}}$ pay attention KCL side of equation.

Redo example with parallel resistors using nodal analysis.



Use Node b
Write KCL

$$i_1 = \frac{V_{out} - V_{in}}{R_1} \quad i_2 = \frac{V_{out}}{R_2} \quad i_3 = \frac{V_{out}}{R_3}$$

$$0 = i_1 + i_2 + i_3$$

$$0 = \frac{V_{out} - V_{in}}{R_1} + \frac{V_{out}}{R_2} + \frac{V_{out}}{R_3}$$

$$\frac{V_{in}}{R_1} = \frac{V_{out}}{R_1} + \frac{V_{out}}{R_2} + \frac{V_{out}}{R_3}$$

$$\frac{V_{in}}{R_1} = V_{out}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\frac{V_{in}}{R_1} = V_{out}\left(\frac{R_2R_3}{R_1R_2R_3} + \frac{R_1R_3}{R_1R_2R_3} + \frac{R_1R_2}{R_1R_2R_3}\right)$$

$$\frac{V_{in}}{R_1} = V_{out}\left(\frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3}\right)$$

$$V_{in}\underbrace{\left(\frac{R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}\right)}_{G} = V_{out}$$

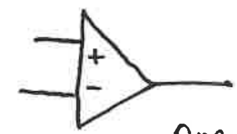
Up to this point, we have seen "passive" circuits

Resistors decrease amp, never increase $0 < G < 1$

Active circuits, amplifiers $G > 1$

New Component: operational amplifier
op-amp.

Symbol



Two input terminals

One output terminal

"Active Device"
Need to supply power to it.

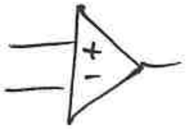
Inverting input
Non-inverting input

There is a circuit inside the op-amp
non-trivial, we will ignore it
Instead, we will focus on conceptual behavior how it performs input → output.
"Black box" Also assume "ideal" behavior
↳ okay usually.

We can make our circuit models even more realistic, if we model circuit inside op-amp.

Paper reference:

Ideal Op-amp behavior.

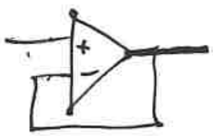


Take the difference between +/- terminals and amplify it.

In this configuration, amplifies more than is practically useful.

Almost never see this "open-loop" op-amp.

Instead: Feedback.

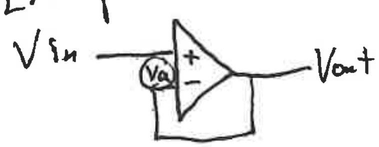


If there is any difference between +/- it will be fed back to input and removed instantaneously.

Therefore, assumption #1

positive and negative terminals have exact same voltage. Label as <u>single node</u>.

Example



V_a: node at input

$V_a = V_{in}$

$V_{out} = V_a$

$V_{out} = V_{in}$ Unity gain

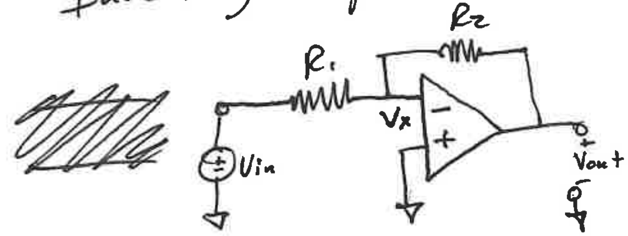
What is the use of that?

Assumption #2 internal resistance between +/- is infinite (open circuit) no current flows into op-amp.

Can be used to buffer sections of circuit.

Using resistors with op-amp.

Inverting amplifier.



Label V_x node

Note: $V_x = 0$ positive terminal

No current flows into op-amp, all current must flow through $R_1 \rightarrow R_2$

Label i_1, i_2 $\quad i_1 = i_2$ $\quad$ Nodal Analysis.

$$i_1 = \frac{V_{in} - V_x}{R_1} \qquad i_2 = \frac{V_x - V_{out}}{R_2}$$

$$\frac{V_{in} - 0}{R_1} = \frac{0 - V_{out}}{R_2} \qquad \frac{V_{in}}{R_1} = \frac{-V_{out}}{R_2}$$

$$V_{in} \cdot \frac{R_2}{R_1} = -V_{out} \quad \text{or} \quad V_{out} = \frac{-R_2}{R_1} \cdot V_{in}$$

inverting amplifier

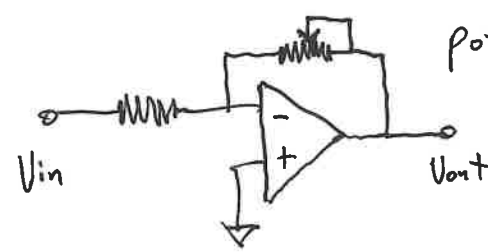
$$G = \frac{-R_2}{R_1}$$

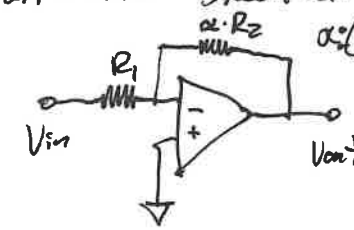
if $R_2 > R_1$ increase in amplitude

$R_2 = R_1$ unity gain $\boxed{\phi}$

$R_2 < R_1$ decrease in amplitude.

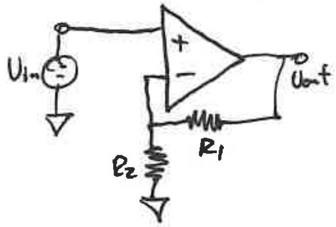
potentiometer shorted.

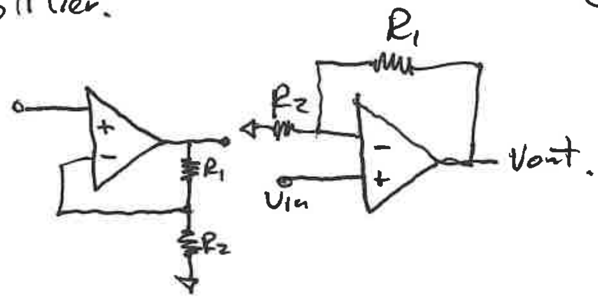




$\alpha:(0-1)$

Non-inverting Amplifier.





Label v_x node. $V_x = V_{in}$

Current i flows through R_1, R_2

$$\frac{V_{out} - V_{in}}{R_1} = \frac{V_{in} - 0}{R_2}$$

$$\frac{V_{out}}{R_1} - \frac{V_{in}}{R_1} = \frac{V_{in}}{R_2}$$

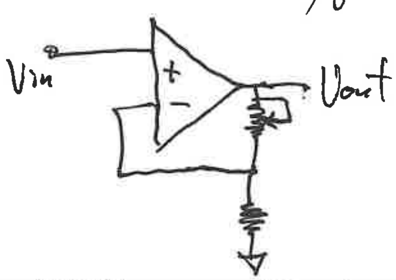
$$\frac{V_{out}}{R_1} = \frac{V_{in}}{R_1} + \frac{V_{in}}{R_2} = V_{in}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$V_{out} = \frac{R_1}{R_1} V_{in} + \frac{R_1}{R_2} V_{in} = V_{in}\left(1 + \frac{R_1}{R_2}\right)$$

$$V_{out} = V_{in}\left(\frac{R_2 + R_1}{R_2}\right)$$

$$G = 1 + \frac{R_1}{R_2}$$

unity gain + something.



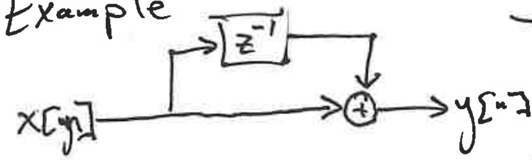
$$G = 1 + \frac{\alpha \cdot R_1}{R_2} \qquad \alpha\ [0:1]$$

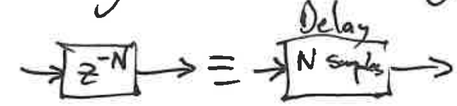
New Topic: Filtering

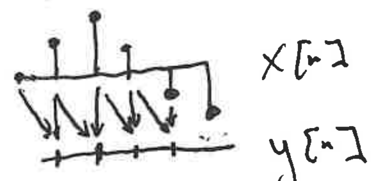
start with digital, move to analog.

Digital Filters created using short time delay.

Example



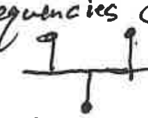




Moving Average Filter

*0.5 average of adjacent samples.

High Frequencies cancelled out.



Write output as a function of input.

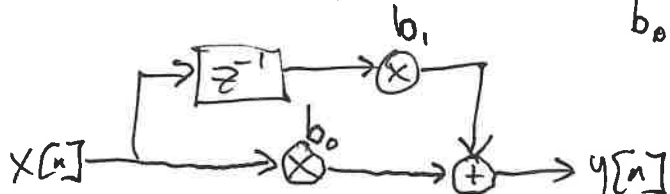
$$y[n] = x[n] + x[n-1]$$

1-sample delay.

Equation is called a <u>difference equation</u>

Usually we want to control the amplitude on each path.

$b_0\ b_1$ filter coefficients.



MATLAB

$b = [b_0, b_1]$

freqz(b)

1st-order system

Impulse Response 

Finite Impulse Response.

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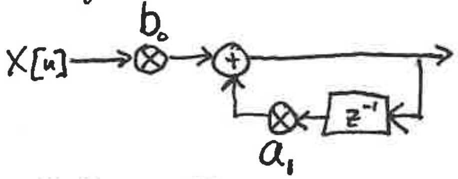
Digital Filters with feedback.

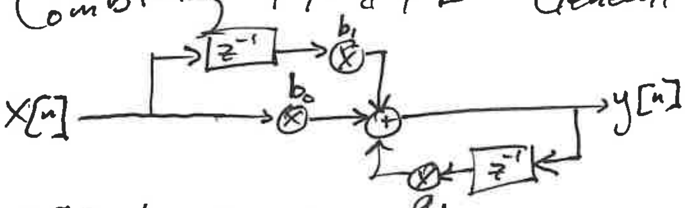


$$y[n] = b_0 x[n] + a_1 y[n-1]$$

Infinite Impulse Response (IIR)
↳ Analog Filters.

MATLAB Loop Example y_1 - state variable.

Combining FF & FB General Form



$$y[n] = b_0 x[n] + b_1 x[n-1] + a_1 y[n-1]$$

Transfer Function Form: $\frac{Y}{X}$ $\frac{\text{OUT}}{\text{IN}}$ = $\boxed{\text{Effect}}$
used for going analog → digital
We need a way to separate delay from X, Y
Z-transform (topic outside of workshop, use notation)

$$y[n] = b_0 x[n] + b_1 x[n-1] + a_1 y[n-1]$$

$x[n] \to X[z]$ $\quad$ $y[n] \to Y[z]$

$x[n-1] \to X[z] \cdot z^{-1}$ $\quad$ $y[n-1] = Y[z] z^{-1}$

$$Y[z] = b_0 X[z] + b_1 X[z] z^{-1} + a_1 Y[z] z^{-1}$$

move $Y[z]$ to LHS

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$$Y[z] - a_1 Y[z] z^{-1} = b_0 X[z] + b_1 X[z] z^{-1}$$

Factor out Y, X

$$Y[z](1 - a_1 z^{-1}) = X[z] \cdot (b_0 + b_1 z^{-1})$$

$$\frac{Y[z]}{X[z]} = \frac{(b_0 + b_1 z^{-1})}{(1 - a_1 z^{-1})} \qquad \begin{aligned} b &= [b_0, b_1] \\ a &= [1, -a_1] \end{aligned} \quad \text{freqz}$$

Example: convert higher-order TF to difference EQ.

$$\frac{Y[z]}{X[z]} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

second order biquad.

$$Y[z](1 + a_1 z^{-1} + a_2 z^{-2}) = X[z](b_0 + b_1 z^{-1} + b_2 z^{-2})$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2]$$

Side note: 2^{nd}-order biquad filter can be used to make LPF, HPF, Shelf, BPF, notch

Audio EQ Cookbook

f, amp, Q → b_0 b_1 b_2 a_1 a_2

Analog Filters. (25)

Symbols

New component: Capacitor 

Two metal plates close together

Think of static electricity.

Charge builds up on plate, eventually jumps (flows)

This component also impedes the flow of electricity but in a different way than resistor.

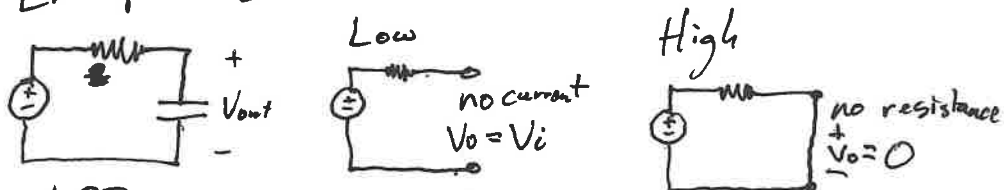
"Frequency Dependent"

Low Frequencies: open circuit or infinite resistor

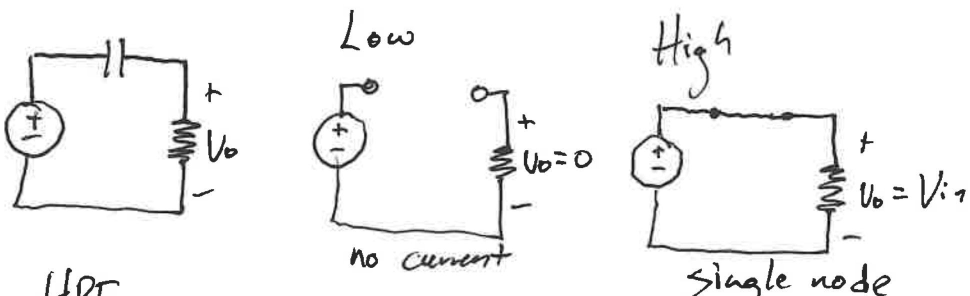
High Frequencies: short circuit or no resistor

In the middle: some resistance.

Examples: Basic Filters.



LPF



HPF

Now we have concept, let's study equation.

(26) Ohm's Law for resistance $V = I \cdot R$

Resistance is simple type of complex impedance

Written $V = I \cdot Z$ where Z = general impedance

for resistor $Z = R$

for capacitor slightly different relationship.

Equation involves calculus. We don't actually need to use calculus to use this relationship.

It is important to understand a couple of concepts from calculus.

$$\frac{dv}{dt} = i \cdot E \qquad \frac{dv}{dt} = i \cdot \frac{1}{C} \qquad \dot{v} = i\frac{1}{C}$$

elastance $E = \frac{1}{C}$ capacitance

Notation: time implied. differential equation

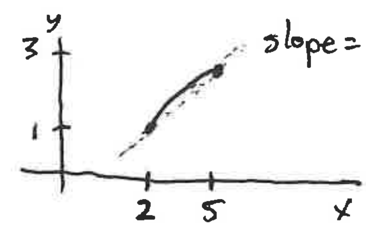
$\frac{dv}{dt}$: derivative of voltage over time

rate of change in voltage

Slope of voltage

For a function/signal we can determine its slope at some point in time

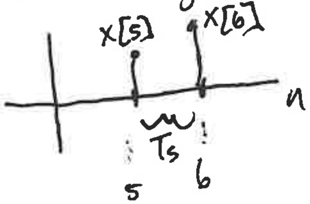
Simplified example without calculus notation



$$\text{slope} = \frac{\Delta y}{\Delta x} \qquad \frac{3-1}{5-2} = \frac{2}{3}$$

This is how we calculate derivative for digital signal

Audio signal $x[n]$

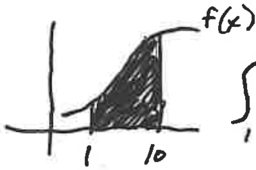


$$\text{slope} = \frac{x[6]-x[5]}{T_s}$$

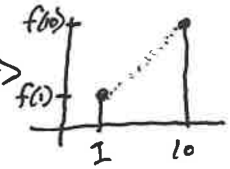
Another important calculus concept: integral

Notation $\int_1^{10} x(t)\,dt \Longrightarrow \sum_{1}^{10} x[t]$

Sum underneath curve of function



$$\int_1^{10} f(x)\,dx \Longrightarrow$$



Approximation using trapezoid.

$$(10-1)\cdot\left(\frac{f(1)+f(10)}{2}\right)$$

inverts
Integration complements differentiation

$$\int \frac{dv}{dt}\cdot dt = V \qquad \text{Therefore: } \int \frac{dv}{dt}\,dt = \int i\cdot\frac{1}{C}\,dt$$

$$V = \int i\cdot\frac{1}{C}\,dt$$

Voltage equals sum of current over time

Voltage accumulates current over time

If all this calculus is new to you, don't worry.

We can avoid a lot of it, if we use a tranform.

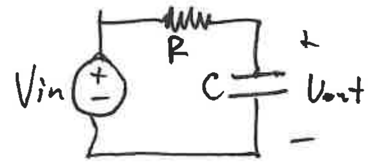
Simplied Notation, math steps easier.

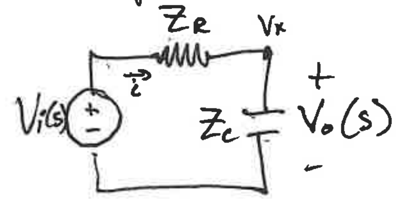
Laplace Transform (similar to Z-transform)

$$V(t) = \frac{1}{C}\int i(t)\,dt \Longrightarrow V(s) = \frac{1}{sC}\cdot I(s)$$

$$V = \frac{I}{sC} \qquad \frac{V}{I} = \frac{1}{sC} = \underbrace{Z}_{\text{impedance}} \text{ for a capacitor}$$

Apply to circuit example:





Voltage ~~through~~ across resistor $V = I\cdot Z_R$

$V = I\cdot R$

voltage across capacitor $V = I\cdot Z_c$

$V = I\cdot\frac{1}{sC}$

Solve circuit using "Z"s & Ohm's Law

substitute into transfer function.

$$\frac{V_i(s)-V_o(s)}{Z_R} = \frac{V_o(s)-0}{Z_c} \qquad \frac{V_i(s)}{Z_R} = \frac{V_o(s)}{Z_c} + \frac{V_o(s)}{Z_R}$$

$$\frac{V_i(s)}{Z_R} = V_o(s)\left(\frac{1}{Z_c}+\frac{1}{Z_R}\right) \qquad \frac{V_i(s)}{Z_R} = V_o(s)\left(\frac{Z_R+Z_c}{Z_c * Z_R}\right)$$

$$\frac{V_i(s)}{Z_R}\cdot\left(\frac{Z_c Z_R}{Z_R+Z_c}\right) = V_o(s) \qquad \frac{V_o(s)}{V_i(s)} = \frac{Z_c}{Z_R+Z_c} \text{ TF form}$$

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Substitute into transfer function.

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_c}{Z_R + Z_c} \qquad Z_c = \frac{1}{sC} \qquad Z_R = R$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$ Multiply all terms by sC $$\frac{\frac{1}{sC}\cdot sC}{R\cdot sC + \frac{1}{sC}\cdot sC}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{sRC+1}$$ Standard form for LPF

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$ Alternative.

This form allows us to do several things

(1) sDomainFilters $s = j\omega$ look at frequency spectrum

(2) ~~r~~ rcFilter $\overset{\text{angular freq}}{\omega_c = \frac{1}{Rc}}\ \frac{\text{rad}}{\text{sec}}$ $f_c = 2\pi \frac{1}{Rc}$ Hz

Now we know the transfer function of analog filter

Next we convert ~~a~~ analog to digital.

Use bilinear transform map continuous to discrete

"Trapezoidal Rule" $z = e^{sT}$ T - sampling period

approximation $s = \frac{2}{T}\frac{(z-1)}{(z+1)}$ let $k = \frac{2}{T}$ sometimes different.

$s \rightarrow k\frac{(z-1)}{(k+1)}$ anywhere we have "s", plug in this to get Z-domain TF

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$$H[z] = \frac{\omega_c}{\frac{k(z-1)}{(z+1)} + \omega_c} = \frac{\omega_c}{\frac{k(z-1) + \omega_c(z+1)}{(z+1)}}$$

$$H[z] = \frac{\omega_c(z+1)}{k(z-1) + \omega_c(z+1)} = \frac{\omega_c z + \omega_c}{kz - k + \omega_c z + \omega_c}$$

$$H[z] = \frac{\omega_c z + \omega_c}{(\omega_c + k)z + (\omega_c - k)}$$ Multiply all terms by z^{-1}

$$H[z] = \frac{\omega_c + \omega_c z^{-1}}{(\omega_c + k) + (\omega_c - k)z^{-1}}$$ $b_0 = \omega_c \quad b_1 = \omega_c$ $a_0 = \omega_c + k \quad a_1 = \omega_c - k$

Want $a_0 = 1$ divide all terms by a_0

$$b_0 = \frac{\omega_c}{\omega_c + k} \quad b_1 = \frac{\omega_c}{\omega_c + k} \quad a_0 = 1 \quad a_1 = \frac{\omega_c - k}{\omega_c + k}$$

$$y[n] = b_0 x[n] + b_1 x[n-1] - a_1 y[n-1]$$

~~Could~~ $k = \frac{2}{T_s} = 2\cdot F_s$ we get frequency warping.

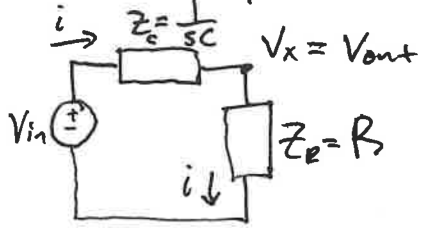
We can pick one frequency in the spectrum and make sure amp $|H(s)| = |H[z]|$

if we ~~let~~ let $k = \frac{2\pi f}{\tan\left(\frac{\pi f}{F_s}\right)}$

Typically we pick most important frequency to be $\underline{f_c}$

$$k = \frac{2\pi f_c}{\tan\left(\frac{\pi f_c}{F_s}\right)} = \frac{2\pi \frac{1}{2\pi RC}}{\tan\left(\frac{\pi \frac{1}{2\pi RC}}{F_s}\right)} = \frac{\frac{1}{RC}}{\tan\left(\frac{\frac{1}{2RC}}{F_s}\right)}$$

High-pass Filter Example



$$\frac{V_{in} - V_{out}}{Z_C} = \frac{V_{out}}{Z_R} \qquad \frac{V_{in}}{Z_C} = \frac{V_{out}}{Z_R} + \frac{V_{out}}{Z_C}$$

$$\frac{V_{in}}{Z_C} = V_{out}\left(\frac{1}{Z_R} + \frac{1}{Z_C}\right) \qquad \frac{V_{in}}{Z_C} = V_{out}\left(\frac{Z_C + Z_R}{Z_R \cdot Z_C}\right)$$

$$\frac{V_{in}}{Z_C}\left(\frac{Z_R \cdot Z_C}{Z_R + Z_C}\right) = V_{out} \qquad V_{in}\left(\frac{Z_R}{Z_R + Z_C}\right) = V_{out}$$

$$\frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_R + Z_C}$$ Substitute $$\frac{R}{R + \frac{1}{sC}}$$

Multiply all terms by sC

$$\frac{V_{out}}{V_{in}} = \frac{sCR}{sCR + 1}$$

Divide all terms by RC

$$\frac{V_{out}}{V_{in}} = \frac{s}{s + \frac{1}{RC}}$$

Plug in $s \rightarrow \frac{k(z-1)}{z+1}$

31 32 Turn $H(s) \rightarrow H[z]$ $\quad s \rightarrow \frac{k(z-1)}{z+1}$

$$H[z] = \frac{\frac{k(z-1)}{z+1}}{\frac{k(z-1)}{z+1} + \omega_c} = \frac{\frac{k(z-1)}{z+1}}{\frac{k(z-1) + \omega_c(z+1)}{z+1}}$$

$$H[z] = \frac{k(z-1)}{k(z-1) + \omega_c(z+1)} = \frac{kz - k}{kz - k + \omega_c z + \omega_c}$$

$$H[z] = \frac{kz - k}{(k + \omega_c)z + (\omega_c - k)} \cdot \quad \text{Multiply by } \frac{z^{-1}}{z^{-1}}$$

$$H[z] = \frac{k - kz^{-1}}{(\omega_c + k) + (\omega_c - k)z^{-1}} \qquad \begin{matrix} b_0 = k & b_1 = -k \\ a_0 = \omega_c + k & a_1 = \omega_c - k \end{matrix}$$

we want $a_0 = 1$ so divide by $\omega_c + k$

$$b_0 = \frac{k}{\omega_c + k} \quad b_1 = \frac{-k}{\omega_c + k} \quad a_0 = 1 \quad a_1 = \frac{\omega_c - k}{\omega_c + k}$$

$$y[n] = b_0 x[n] + b_1 x[n-1] - a_1 y[n-1]$$

We can apply this approach to more complicated circuits involving R & C's.

Examples: Baxandall Bennett AES

Bassman Tonestack Yeh Smith Dafx

Schematic → Laplace Transform → Bilinear Transform → Inverse Z
↳ Difference Eq

When circuit is complicated, that's a lot of steps to do by hand. Tools like Mathematica Wolfram Alpha can help. Solution still messy.

Alternative: Discretize Schematic from start.
substitute components with discrete approximation
specifically capacitor (memory storage)
using trapezoidal rule.

Discrete Schematic → Difference Equation.

Equation for capacitor: apply trapezoidal rule

$$\frac{dv(t)}{dt} = i(t)\frac{1}{C}$$

Current sample ↓ Previous sample ↓

$\frac{dv(t)}{dt}$ change in voltage → $V[n] - V[n-1]$

$$i(t)\frac{1}{C} \to T_s\left(\frac{i[n]+i[n-1]}{2}\right)\cdot\frac{1}{C}$$

$$V[n] - V[n-1] = \frac{T_s}{2C}\left(i[n]+i[n-1]\right)$$

relationship between current & voltage based entirely on discrete time

$$i[n] + i[n-1] = \frac{2C}{T_s}\left(V[n] - V[n-1]\right)$$

When we apply KCL to circuit we want to know current $i[n] =$ ~~~

$$i[n] = \frac{2C}{T_s}V[n] - \underbrace{\frac{2C}{T_s}V[n-1] - i[n-1]}_{\text{stuff from past}}$$

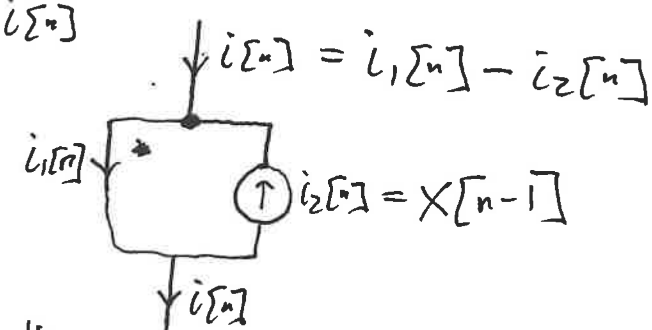
current voltage

Define a "state" for stuff from past

$$X[n-1] = \frac{2C}{T_s}V[n-1] + i[n-1]$$

$$i[n] = \frac{2C}{T_s}V[n] - X[n-1]$$

side note: from KCL if we have a current $i[n]$



notice $\frac{2C}{T_s}$ is not time varying.

nice form $i_1[n] = \frac{V[n]}{R}$ let $R = \frac{T_s}{2C}$

$$i[n] = \frac{V[n]}{R} - X[n-1]$$

How to handle state $X[n-1]$?

Initialize state $X[0] = 0$

How does $X[n-1]$ change over time?

We defined

$$X[n-1] = \frac{V[n-1]}{R} + i[n-1]$$

Therefore

$$X[n] = \frac{V[n]}{R} + i[n]$$

We already found $i[n]$

$$i[n] = \frac{V[n]}{R} - X[n-1] \quad \text{Plug in}$$

$$X[n] = \frac{V[n]}{R} + \frac{V[n]}{R} - X[n-1]$$

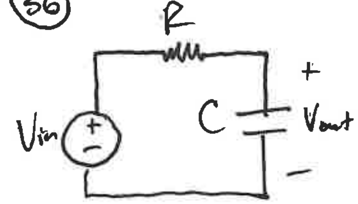
$$X[n] = \frac{2}{R} V[n] - X[n-1]$$

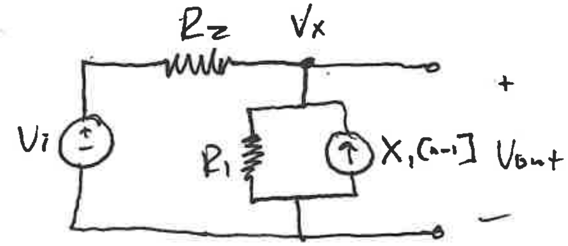
Every sample we will calculate $X[n]$ and use for subsequent sample as $X[n-1]$

Now let's apply this to some circuits to see how it works.

35

36





Node Vx

current in = current out

$$\frac{V_i - V_o}{R_2} = \frac{V_o}{R_1} - X_1[n-1]$$

$$\frac{V_i}{R_2} + X_1[n-1] = \frac{V_o}{R_1} + \frac{V_o}{R_2}$$

$$\frac{V_i}{R_2} + X_1[n-1] = V_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = V_o\left(\frac{R_1 + R_2}{R_1 R_2}\right)$$

$$V_o = V_i\left(\frac{R_1}{R_1 + R_2}\right) + \left(\frac{R_1 R_2}{R_1 + R_2}\right) X_1[n-1]$$

$$X_1[n] = \frac{2}{R_1} \cdot V_o - X_1[n-1]$$

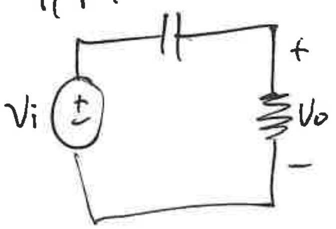
③ DKrcFilter

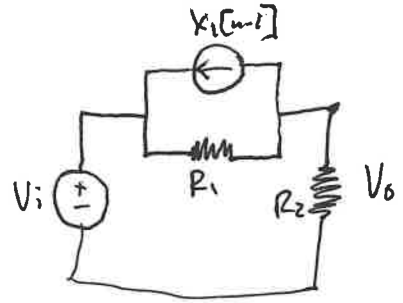
we calculate this first for each sample so we can use it to update state.

In just a few steps we converted this circuit to a difference equation that can easily be implemented as code.

More examples build up to Tube Screamer.

HPF





$$\frac{V_i - V_o}{R_1} - X_1[n-1] = \frac{V_o}{R_2}$$

$$\frac{V_i}{R_1} - X_1[n-1] = \frac{V_o}{R_1} + \frac{V_o}{R_2} = V_o\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\frac{V_i}{R_1} - X_1[n-1] = V_o\left(\frac{R_1 + R_2}{R_1 R_2}\right)$$

$$V_i\left(\frac{R_2}{R_1 + R_2}\right) - \left(\frac{R_1 R_2}{R_1 + R_2}\right) X_1[n-1] = V_o$$

$$b_0 = \frac{R_2}{R_1 + R_2} \qquad b_1 = \frac{-R_1 R_2}{R_1 + R_2}$$

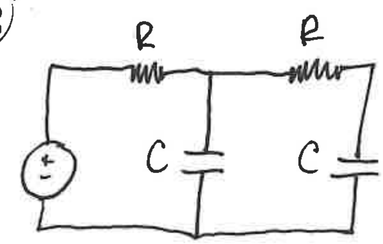
$$X_1[n] = \frac{2}{R_1}(V_i - V_o) - X_1[n-1]$$

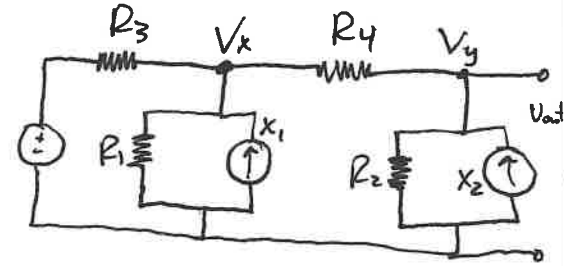
Both are known for current sample before updating state.

Next Example: 2nd-Order RC Filter

2 states, parallel components

(37) (38)





Node Vx

Current in Current out

$$\frac{V_i - V_x}{R_3} = \frac{V_x}{R_1} - X_1[n-1] + \frac{V_x - V_{out}}{R_4}$$

We will need to replace V_x in equation use node V_y and then come back.

Node V_y:

$$\frac{V_x - V_o}{R_4} = \frac{V_o}{R_2} - X_2 \qquad \frac{V_x}{R_4} = \frac{V_o}{R_2} + \frac{V_o}{R_4} - X_2$$

$$V_x = V_o\left(\frac{R_4}{R_2} + 1\right) - R_4 X_2 = V_o\left(\frac{R_4 + R_2}{R_2}\right) - R_4 X_2$$

Solve Node Vx

$$\frac{V_i}{R_3} + \frac{V_{out}}{R_4} + X_1 = V_x \underbrace{\left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4}\right)}_{G_x}$$

$$\underset{\text{replace}}{V_x} = \frac{V_i}{R_3 G_x} + \frac{V_{out}}{R_4 G_x} + \frac{X_1}{G_x}$$

$$V_o\left(\frac{R_4 + R_2}{R_2}\right) - R_4 X_2 = \frac{V_i}{R_3 G_x} + \frac{V_o}{R_4 G_x} + \frac{X_1}{G_x}$$

solve for V_o diff. EQ

$$\underbrace{V_o\left[\frac{R_4 + R_2}{R_2} - \frac{1}{R_4 G_x}\right]}_{a_0} = \underset{b_0}{\frac{V_i}{R_3 G_x}} + \underset{b_1}{\frac{X_1}{G_x}} + \underset{b_2}{X_2 \cdot R_4}$$

State update equations

$$X_2[n] = \frac{2}{R_2} V_o - X_2[n-1]$$

$$X_1[n] = \frac{2}{R_1} V_x - X_1[n-1]$$

We will need to find V_x in terms of in/out & states

$$V_x = V_o\left(\frac{R_4+R_2}{R_2}\right) - R_4 X_2$$

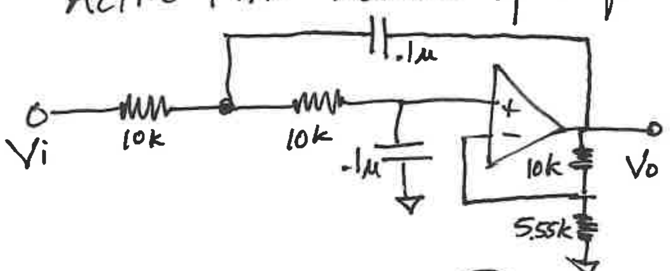
we already found this, other option has more computations.

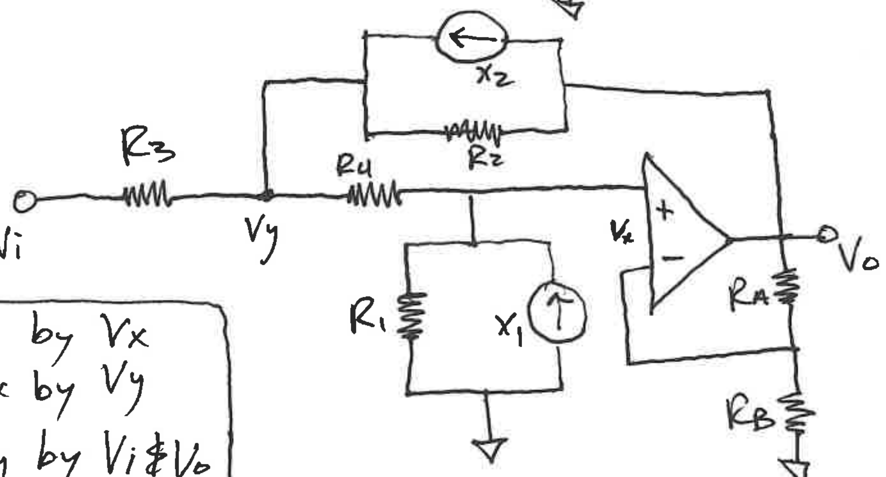
Make sure to calculate V_x <u>before</u> X_2

(4) DKrc2ndOrder

Sallen-Key 2nd Order Filter (w/ resonance Q)

Active Filter includes op-amp





V_o by V_x
V_x by V_y
V_y by V_i & V_o

(39) (40) start with current from Vout → ground.

$$\frac{V_{out} - V_x}{R_A} = \frac{V_x}{R_B} \qquad \frac{V_{out}}{R_A} = V_x\left(\frac{1}{R_A} + \frac{1}{R_B}\right) = V_x\left(\frac{R_A+R_B}{R_A \cdot R_B}\right)$$

$$V_x = V_{out}\left(\frac{R_B}{R_A+R_B}\right)$$ save for substitution

Node v_x

$$\frac{V_y - V_x}{R_4} = \frac{V_x}{R_1} - X_1 \qquad \frac{V_y}{R_4} = V_x\left(\frac{1}{R_1} + \frac{1}{R_4}\right) - X_1$$

$$V_y = V_x \cdot R_4\left(\frac{1}{R_1} + \frac{1}{R_4}\right) - R_4 X_1 = V_x\left(\frac{R_4}{R_1} + 1\right) - R_4 X_1$$

plug in →

$$V_y = V_o\left(\frac{R_B}{R_A+R_B}\right)\left(\frac{R_4}{R_1}+1\right) - R_4 X_1$$

Node v_y

$$\frac{V_i - V_y}{R_3} = \frac{V_y - V_x}{R_4} + \frac{V_y - V_o}{R_2} - X_2$$ isolate V_y

$$\frac{V_i}{R_3} + \frac{V_x}{R_4} + \frac{V_o}{R_2} + X_2 = V_y\Big(\underbrace{\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}}_{G_y}\Big)$$

plug in

$$\frac{V_i}{R_3} + \frac{V_x}{R_4} + \frac{V_o}{R_2} + X_2 = V_o \cdot G_y \cdot \left(\frac{R_B}{R_A+R_B}\right)\left(\frac{R_4}{R_1}+1\right) - R_4 G_y \cdot X_1$$

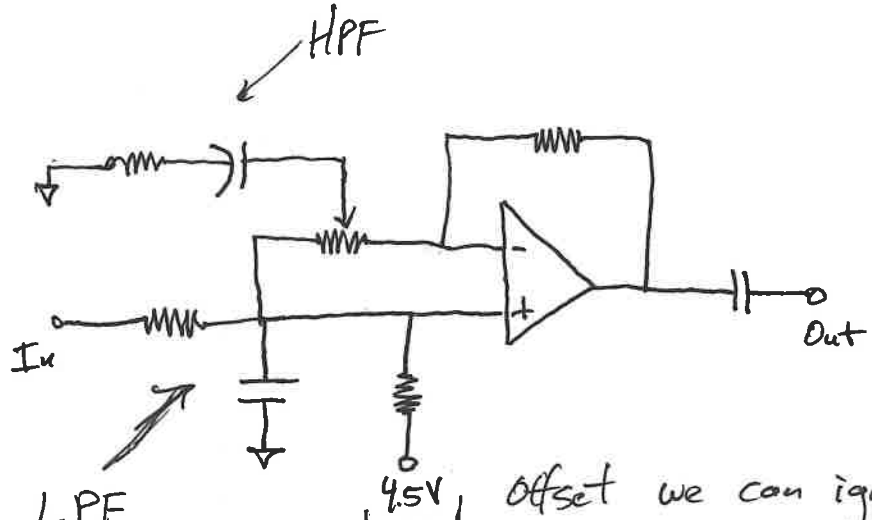
plug in

$$\frac{V_i}{R_3} + \frac{V_o}{R_4}\left(\frac{R_B}{R_A+R_B}\right) + \frac{V_o}{R_2} + X_2 = V_o G_y\left(\frac{R_B}{R_A+R_B}\right)\left(\frac{R_4}{R_1}+1\right) - R_4 G_y X_1$$

isolate V_o

$$V_o\underbrace{\left(G_y\left(\frac{R_B}{R_A+R_B}\right)\left(\frac{R_4}{R_1}+1\right) - \frac{1}{R_2} - \frac{R_B}{R_4(R_A+R_B)}\right)}_{a_0} = \underbrace{\frac{V_i}{R_3}}_{b_0} + \underbrace{R_4 G_y X_1}_{b_1} + \underbrace{X_2}_{b_2}$$

$$X_1[n] = \frac{2}{R_1} V_x - X_1[n-1] \qquad X_2[n] = \frac{2}{R_2}(V_y - V_o) - X_2[n-1]$$

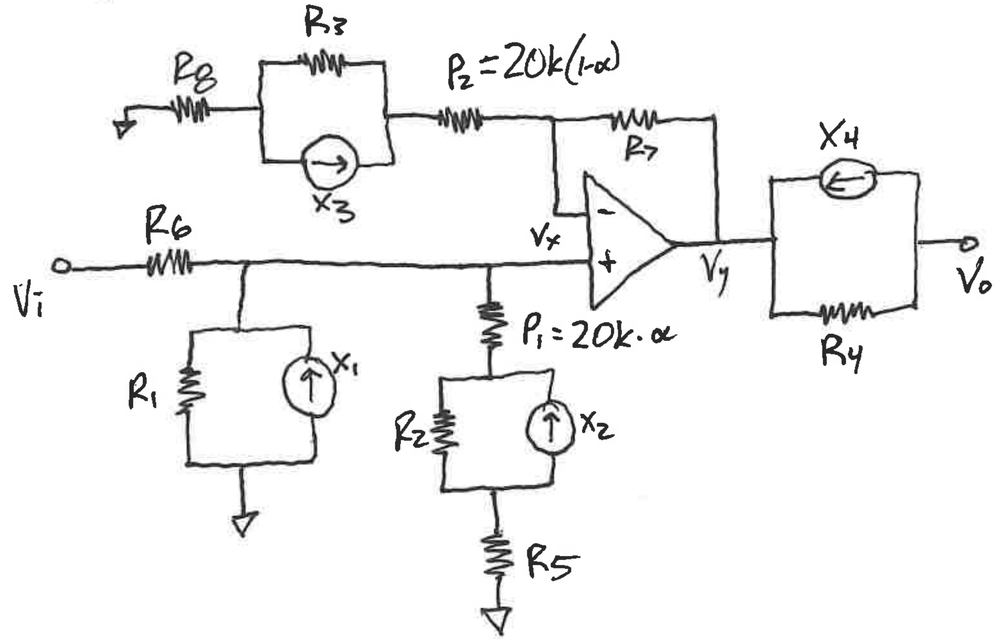


Offset we can ignore because op-amp 0-9V

Potentiometer: Tone control single knob

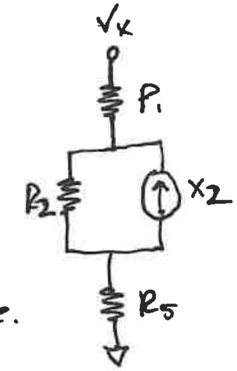
Resistor between "−" & "+" terminals

Assumption no current flow through/between terminals.



(41) (42) Start with sub-circuit

Want current "i" as function of Vx & X_1.



$$V_x = V_{P1} + V_{R1} + V_{R5}$$

same current "i" through each stage.

$$i = \frac{V_{P1}}{P_1} \quad i = \frac{V_{R5}}{R_5} \quad i = \frac{V_{R2}}{R_2} - X_2$$

Find each voltage as a function of V_{P1}

$$\frac{V_{R5}}{R_5} = \frac{V_{P1}}{P_1} \quad V_{R5} = \frac{R_5}{P_1} \cdot V_{P1}$$

$$\frac{V_{R2}}{R_2} - X_1 = \frac{V_{P1}}{P_1} \quad \frac{V_{R2}}{R_2} = \frac{V_{P1}}{P_1} + X_1 \quad V_{R2} = \frac{R_2 V_{P1}}{P_1} + R_2 X_2$$

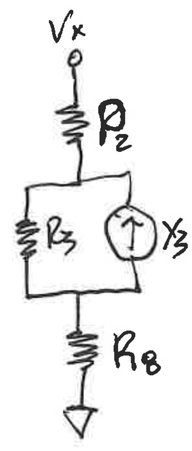
Plug in above

$$V_x = V_{P1} + \frac{R_2}{P_1} V_{P1} + R_2 X_2 + \frac{R_5}{P_1} V_{P1}$$

$$V_x - R_2 X_2 = V_{P1} \underbrace{\left(1 + \frac{R_2}{P_1} + \frac{R_5}{P_1}\right)}_{\boxed{G_2}}$$

$$V_{P1} = \frac{V_x - R_2 X_2}{G_2} \quad \text{Therefore} \quad i = \frac{V_{P1}}{P_1} = \frac{V_x - R_2 X_2}{G_2 \cdot P_2}$$

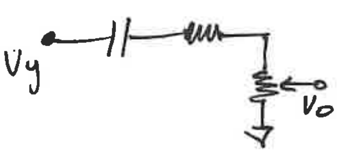
use this for current on this path.

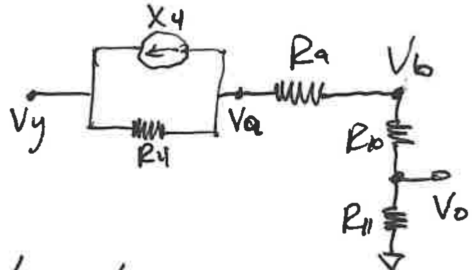


Repeat same process

$$i = \frac{V_x - R_3 X_3}{G_3 \cdot P_2} \qquad \boxed{G_3} = \left(1 + \frac{R_3}{P_2} + \frac{R_8}{P_2}\right)$$

Output Branch





Current going through each component is equal.
Write everything as a function of Vo for difference equation.

$$\frac{V_b - V_o}{R_{10}} = \frac{V_o}{R_{11}} \qquad \frac{V_b}{R_{10}} = V_o\left(\frac{1}{R_{10}}+\frac{1}{R_{11}}\right) \qquad V_b = V_o\left(1+\frac{R_{10}}{R_{11}}\right)$$

$$\frac{V_a - V_b}{R_9} = \frac{V_o}{R_{11}} \qquad V_a - V_b = \frac{R_9}{R_{11}}V_o \qquad V_a - \frac{R_9}{R_{11}}V_o = V_b$$

$$V_a - \frac{R_9}{R_{11}}V_o = V_o\left(1+\frac{R_{10}}{R_{11}}\right) \qquad V_a = V_o\left(1+\frac{R_{10}}{R_{11}}+\frac{R_9}{R_{11}}\right)$$

$$\frac{V_y - V_a}{R_4} - X_4 = \frac{V_o}{R_{11}} \qquad \frac{V_y - V_a}{R_4} = \frac{V_o}{R_{11}} + X_4$$

$$V_y - V_a = \frac{R_4}{R_{11}}V_o + R_4X_4 \qquad V_y - \frac{R_4}{R_{11}}V_o - R_4X_4 = V_a$$

$$V_y - \frac{R_4}{R_{11}}V_o - R_4X_4 = V_o\left(1+\frac{R_{10}}{R_{11}}+\frac{R_9}{R_{11}}\right)$$

$$V_y = V_o\underbrace{\left(1+\frac{R_{10}}{R_{11}}+\frac{R_9}{R_{11}}+\frac{R_4}{R_{11}}\right)}_{G_o} + R_4X_4$$

Save for later.

Now we need to connect branches together.

Node V_x "−" Terminal current flows $V_y \rightarrow V_x$

$$\frac{V_y - V_x}{R_7} = \frac{V_x}{G_3P_2} - \frac{R_3}{G_3P_2}X_3 \quad \text{(From previous page)}$$

Isolate V_y to plug in

$$V_y - V_x = \frac{R_7}{G_3P_2}V_x - \frac{R_7R_3}{G_3P_2}X_3$$

$$V_y = V_x + \frac{R_7}{G_3P_2}V_x - \frac{R_7R_3}{G_3P_2}X_3$$

plug in V_y

$$V_o(G_o) + R_4X_4 = V_x\left(1+\frac{R_7}{G_3P_2}\right) - \frac{R_3R_7}{G_3P_2}X_3$$

Isolate V_x

$$V_o\cdot G_o + R_4X_4 + \frac{R_3R_7}{G_3P_2}X_3 = V_x\underbrace{\left(1+\frac{R_7}{G_3P_2}\right)}_{G_x}$$

$$V_x = \frac{G_o}{G_x}\cdot V_o + \frac{R_3R_7}{G_xG_3P_2} + \frac{R_4}{G_x}X_4$$

save

now we need to connect input → output "+" terminal to "−"

Node V_x "+" terminal

$$\frac{V_i - V_x}{R_6} = \frac{V_x}{R_1} - X_1 + \frac{V_x}{G_2 P_1} - \frac{R_2 X_2}{G_2 P_1}$$ (from previous page)

Isolate V_x

$$\frac{V_i}{R_6} + X_1 + \frac{R_2 X_2}{G_2 P_1} = V_x \underbrace{\left(\frac{1}{R_1} + \frac{1}{G_2 P_1} + \frac{1}{R_6}\right)}_{G_Z}$$

$$V_x = \frac{V_i}{R_6 G_Z} + \frac{X_1}{G_Z} + \frac{R_2 X_2}{G_2 G_Z P_1}$$ set equal to other V_x

$$\frac{V_i}{R_6 G_Z} + \frac{X_1}{G_Z} + \frac{R_2 X_2}{G_2 G_Z P_1} = V_o \frac{G_o}{G_x} + \frac{R_3 R_7}{G_x G_3 P_2} X_3 + \frac{R_4}{G_x} X_4$$

solve for V_o for difference equation

$$\frac{V_i}{R_6 G_Z} + \frac{X_1}{G_Z} + \frac{R_2}{G_2 G_Z P_1} X_2 - \frac{R_3 R_7}{G_x G_3 P_2} X_3 - \frac{R_4}{G_x} X_4 = V_o \frac{G_o}{G_x}$$

$$V_o = \frac{G_x}{G_o R_6 G_Z} V_i + \frac{G_x}{G_o G_Z} X_1 + \frac{G_x R_2}{G_o G_2 G_Z P_1} X_2 - \frac{R_3 R_7}{G_o G_3 P_2} X_3 - \frac{R_4}{G_o} X_4$$

$$b_0 = \frac{G_x}{G_o R_6 G_Z} \qquad b_1 = \frac{G_x}{G_o G_Z}$$

$$b_2 = \frac{G_x R_2}{G_o G_2 G_Z P_1} \qquad b_3 = \frac{-R_3 R_7}{G_o G_3 P_2} \qquad b_4 = \frac{-R_4}{G_o}$$

State update equations

$$\boxed{X_1[n]} = \frac{2}{R_1} \cdot V_x - X_1[n-1]$$

↑need to calculate.

$$X_2[n] = \frac{2}{R_2} V_{R2} - X_2[n-1]$$ more complicated.

Let's find V_{R2} as function of V_x

Use $V_x = V_{P1} + V_{R2} + V_{R5}$

Write all terms with V_{R2}, use currents equal

$$\frac{V_{P1}}{P_1} = \frac{V_{R2}}{R_2} - X_2 \qquad V_{P1} = \frac{P_1 V_{R2}}{R_2} - P_1 X_2$$

$$\frac{V_{R5}}{R_5} = \frac{V_{R2}}{R_2} - X_2 \qquad V_{R5} = \frac{R_5}{R_2} V_{R2} - X_2 R_5$$

now plug in

$$V_x = \frac{P_1 V_{R2}}{R_2} - P_1 X_2 + V_{R2} + \frac{R_5}{R_2} V_{R2} - X_2 R_5$$

Solve for V_{R2}

$$V_x + P_1 X_2 + R_5 X_2 = V_{R2} \underbrace{\left(\frac{P_1}{R_2} + 1 + \frac{R_5}{R_2}\right)}_{G_R}$$

$$V_{R2} = \frac{V_x}{G_R} + \left(\frac{P_1 + R_5}{G_R}\right) X_2$$

Plug into original state equation

$$\boxed{X_2[n]} = \frac{2}{R_2}\left(\frac{V_x}{G_R} + \left(\frac{P_1 + R_5}{G_R}\right) X_2[n-1]\right) - X_2[n-1]$$

We can repeat exact same process for X_3. Here are the results. 47 48

$$\boxed{X_3[n]} = \frac{2}{R_3}\left(\frac{V_x}{G_s} + \frac{(R_2+R_8)}{G_s}X_3[n-1]\right) - X_3[n-1]$$

Where $\boxed{G_s} = 1 + \frac{R_2}{R_3} + \frac{R_8}{R_3}$

Output Branch

$$X_4[n] = \frac{2}{R_4} V_{R4} - X_4[n-1]$$

$$V_{R4} = V_y - V_a$$

$$X_4[n] = \frac{2}{R_4}(V_y - V_a) - X_4[n-1]$$

$$V_y = V_o\left(1 + \frac{R_{10}}{R_{11}} + \frac{R_9}{R_{11}} + \frac{R_4}{R_{11}}\right) + R_4 X_4$$

$$V_a = V_o\left(1 + \frac{R_{10}}{R_{11}} + \frac{R_9}{R_{11}}\right)$$

$$V_y - V_a = V_o \frac{R_4}{R_{11}} + R_4 X_4$$

$$X_4[n] = \frac{2}{R_4}\left(V_o \frac{R_4}{R_{11}} + R_4 X_4\right) - X_4[n-1]$$

$$X_4[n] = 2 \cdot \frac{V_o}{R_{11}} + 2 X_4[n-1] - X_4[n-1]$$

$$\boxed{X_4[n]} = 2\frac{V_o}{R_{11}} + X_4[n-1]$$