

1T4: Some basic mathematics

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Sinusoidal functions (sinewaves)

$$x[n] = A \cos(\omega nT + \phi) = A \cos(2\pi f nT + \phi)$$

A : amplitude

ω : angular frequency in radians/seconds

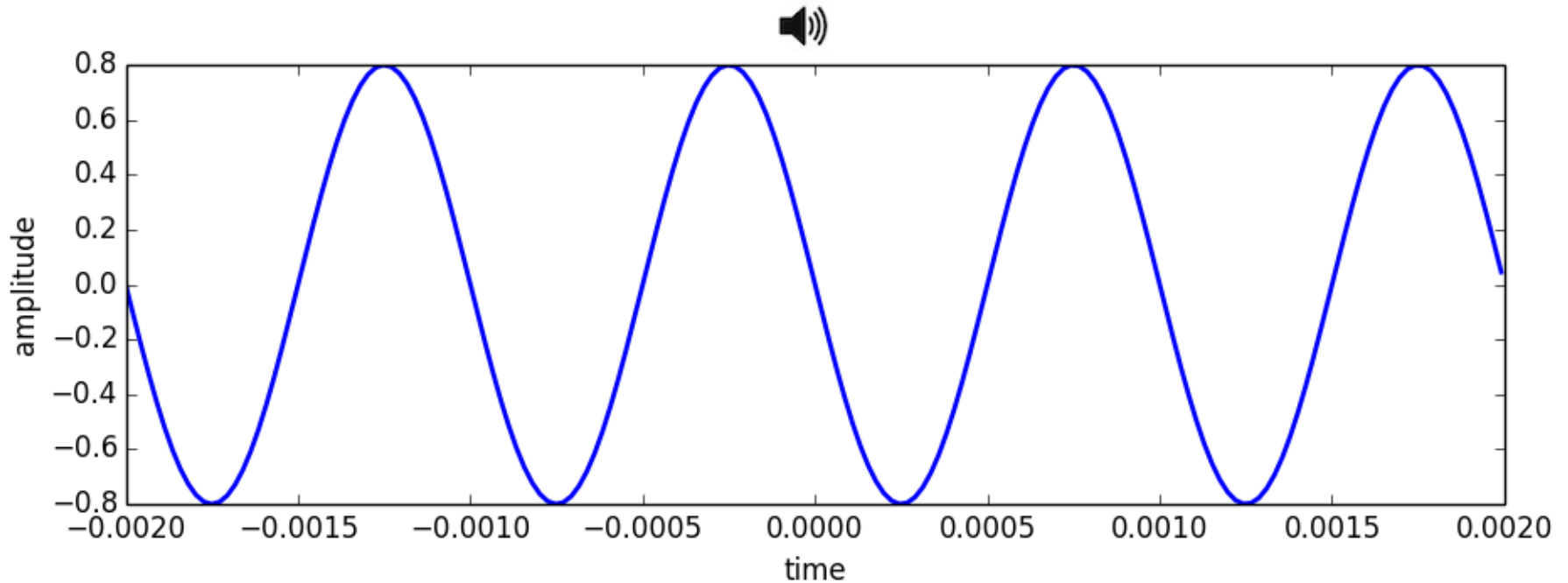
$f = \omega / 2\pi$: frequency in Hertz (cycles/seconds)

ϕ : initial phase in radians

n : time index

$T = 1 / f_s$: sampling period in seconds ($t = nT = n / f_s$)

Sinewave plot



```
A = .8  
f0 = 1000  
phi = np.pi/2  
fs = 44100  
t = np.arange(-.002, .002, 1.0/fs)  
x = A * np.cos(2*np.pi*f0*t+phi)
```

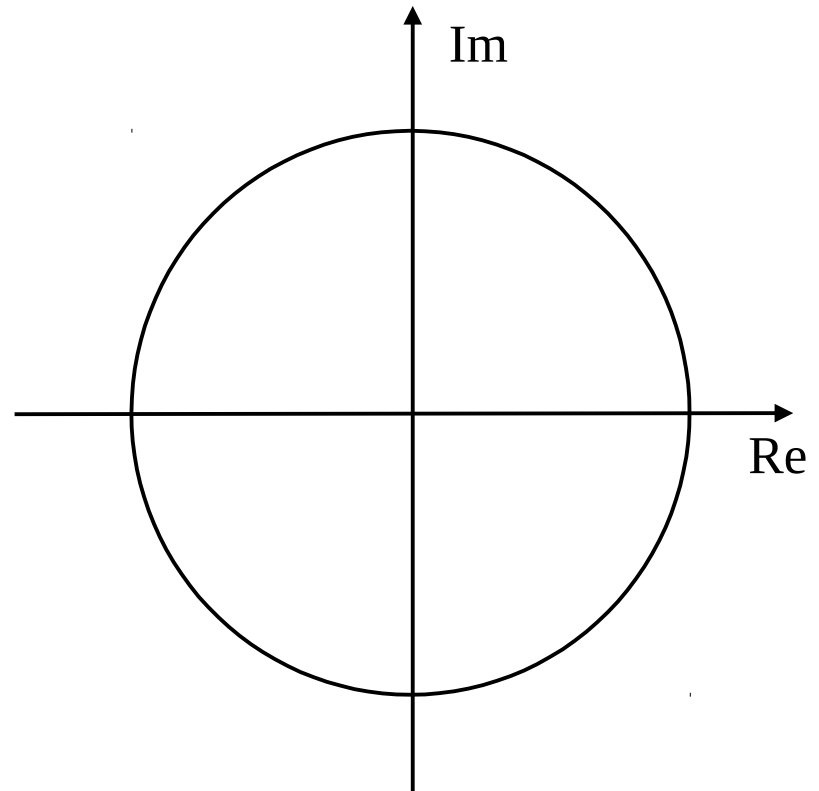
Complex numbers

$(a + jb)$ a, b : real numbers
 $j = \sqrt{-1}$: imaginary unit

Complex plane:

Re (real axis)

Im (imaginary axis)



Rectangular form:
 $(a + jb)$

Polar form:

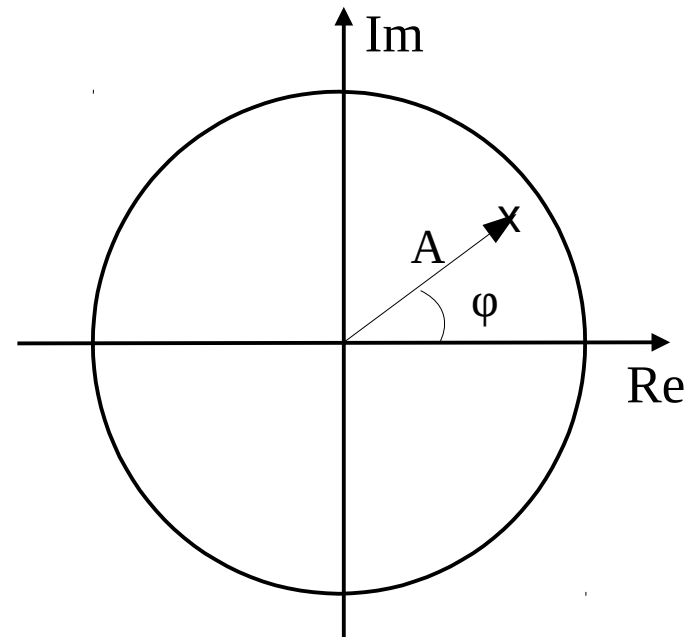
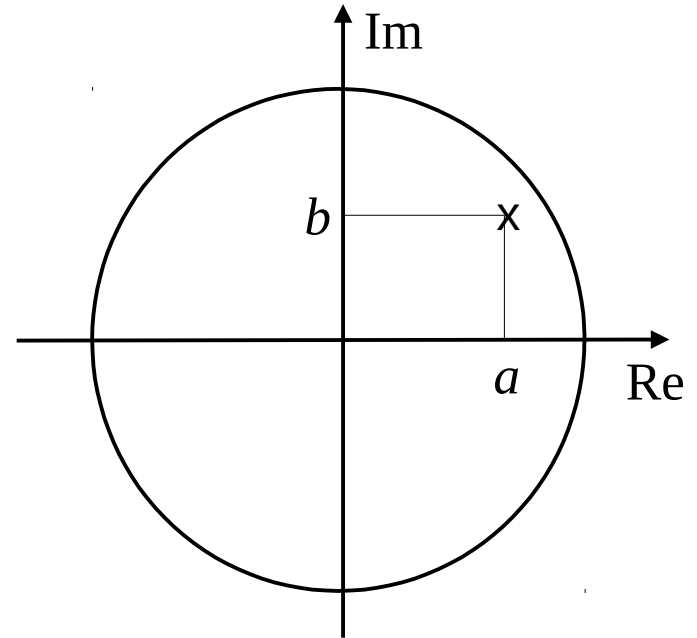
$$A = \sqrt{a^2 + b^2}$$

$$\phi = \text{atan2}\left(\frac{b}{a}\right)$$

where:

$$\text{if } (a > 0) \quad \text{atan2}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\text{else if } (a < 0) \quad \text{atan2}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{b}{a}\right) - \pi$$

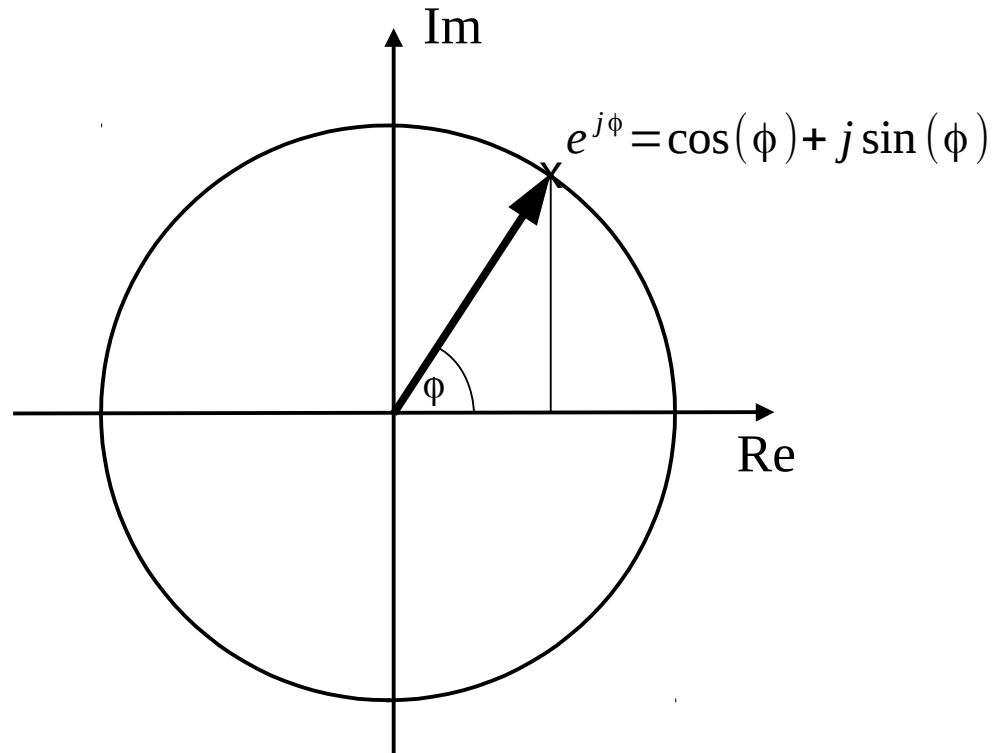


Euler's formula

$$e^{j\phi} = \cos \phi + j \sin \phi$$

$$\cos \phi = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\sin \phi = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

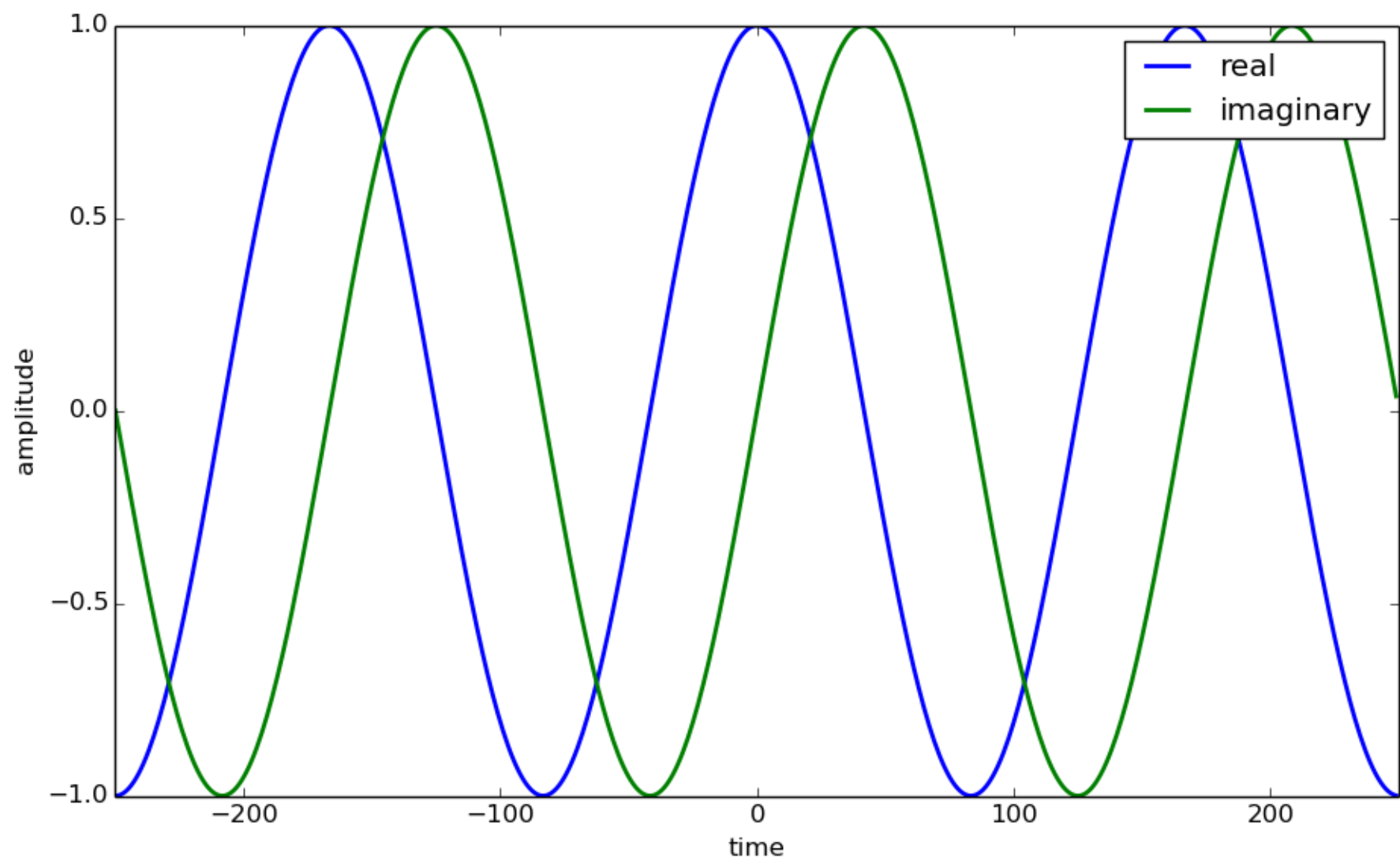


Complex sinewave

$$\begin{aligned}\bar{x}[n] &= A e^{j(\omega nT + \phi)} = A e^{j\phi} e^{j\omega nT} = X e^{j(\omega nT)} \\ &= A \cos(\omega nT + \phi) + j A \sin(\omega nT + \phi)\end{aligned}$$

Real sinewave:

$$\begin{aligned}x[n] &= A \cos(\omega nT + \phi) = A \left(\frac{e^{j(\omega nT + \phi)} + e^{-j(\omega nT + \phi)}}{2} \right) \\ &= \frac{1}{2} X e^{j(\omega nT)} + \frac{1}{2} X^* e^{-j(\omega nT)} = \frac{1}{2} \bar{x}[n] + \frac{1}{2} \bar{x}^*[n] \\ &= \Re \{ \bar{x}[n] \}\end{aligned}$$



Scalar (dot) product of sequences

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x[n] y^*[n]$$

Example:

$$x[n] = [0, j, 1]; y[n] = [1, j, j]$$

$$\langle x, y \rangle = 0 \times 1 + j \times (-j) + 1 \times (-j) = 0 + 1 + (-j) = 1 - j$$

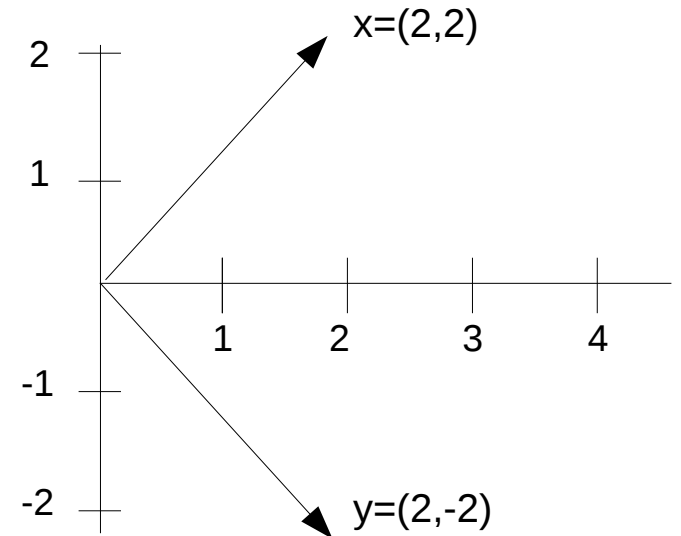
Orthogonality of sequences

$$x \perp y \Leftrightarrow \langle x, y \rangle = 0$$

Example:

$$x[n] = [2, 2]; y[n] = [2, -2]$$

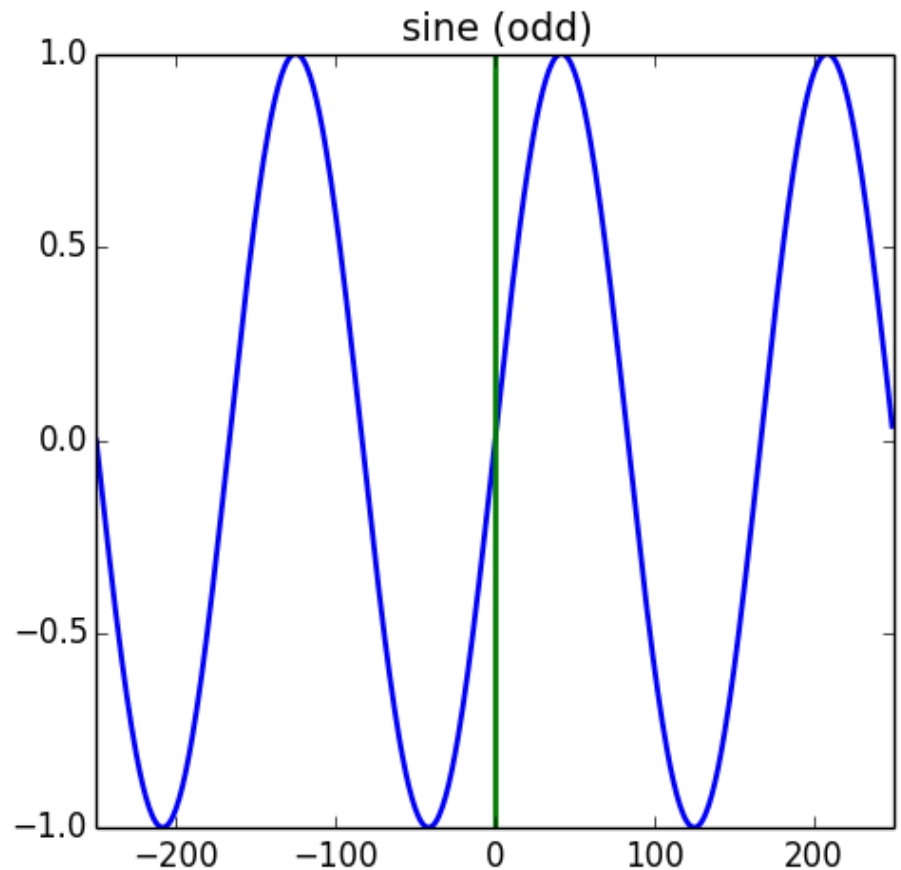
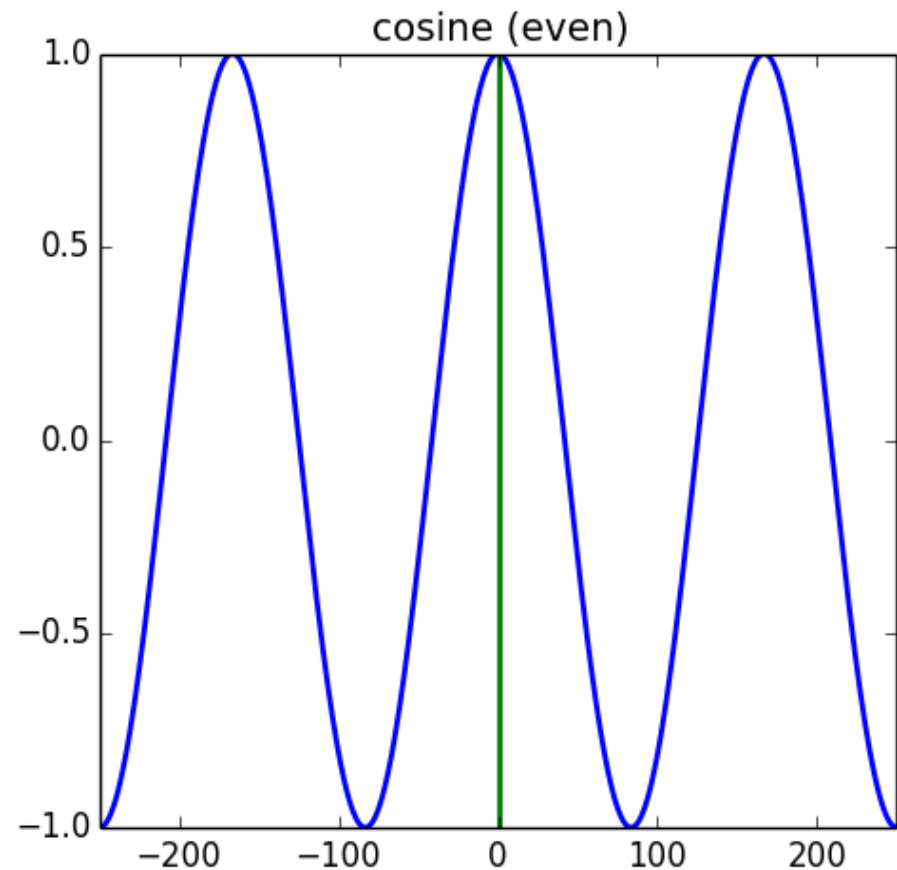
$$\langle x, y \rangle = 2 \times 2^* + 2 \times (-2)^* = 4 - 4 = 0$$



Even and odd functions

$f[n]$ is *even* if $f[-n] = f[n]$ [symmetric]

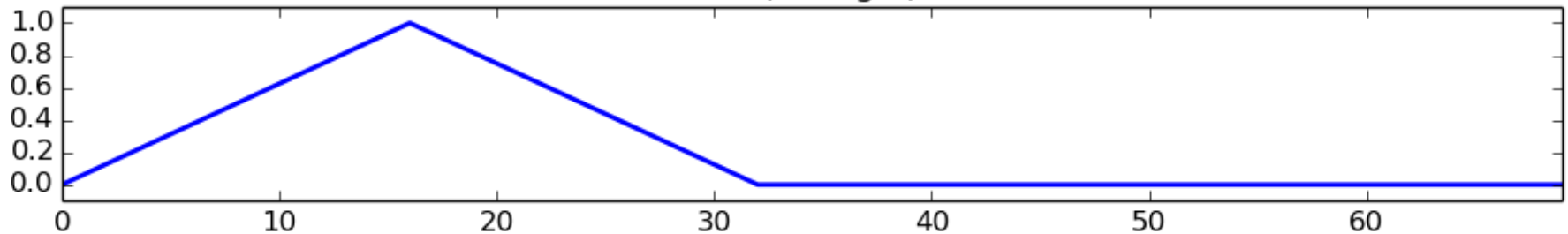
$f[n]$ is *odd* if $f[-n] = -f[n]$ [antisymmetric]



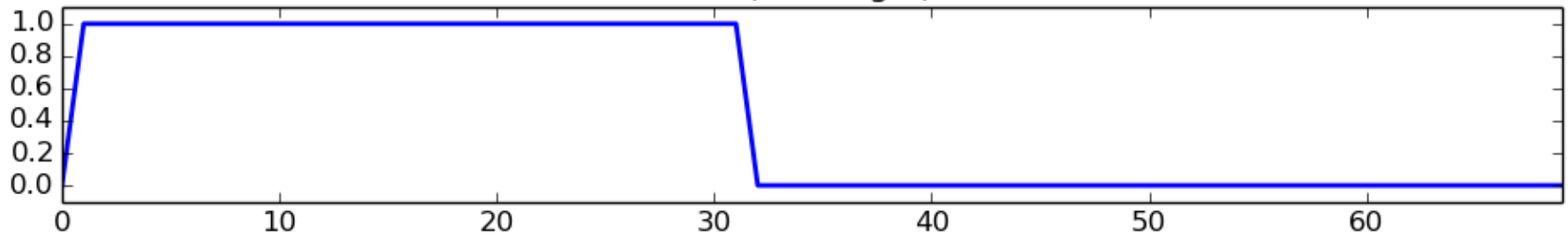
Convolution

$$y[n] = (x_1[n] * x_2[n])_n = \sum_{m=0}^{N-1} x_1[m] x_2[n-m]$$

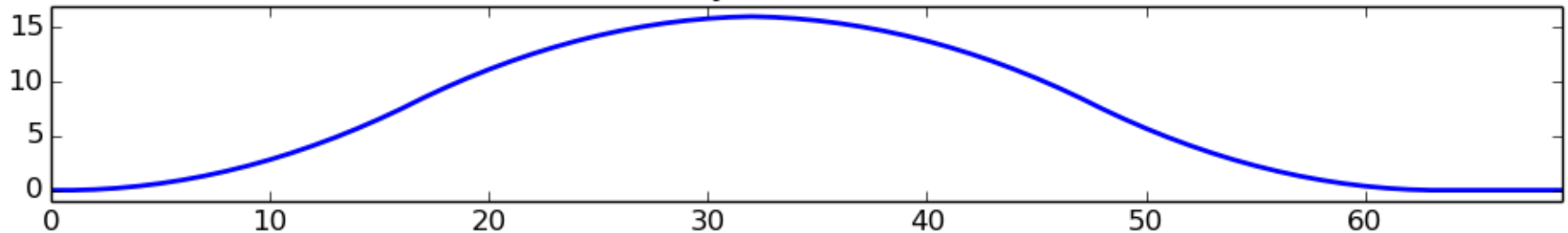
x1 (triangle)



x2 (rectangle)



y = x1 * x2



References and credits

- More information in:
 - <https://en.wikipedia.org/wiki/Sinusoid>
 - https://en.wikipedia.org/wiki/Complex_numbers
 - https://en.wikipedia.org/wiki/Euler_formula
 - http://en.wikipedia.org/wiki/Dot_product
 - <https://en.wikipedia.org/wiki/Convolution>
- Reference for the mathematics of the DFT by Julius O. Smith: <https://ccrma.stanford.edu/~jos/mdft/>
- Slides released under CC Attribution-Noncommercial-Share Alike license and code under Affero GPL license; available from <https://github.com/MTG/sms-tools>

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