Signal, Image, and Data Processing (236200)

Homework 1

Sean Man (206184798)
Dor Brekhman (000000000)

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Part I
Theory

1 Optimal Quantization for Minimum Expected Absolute Deviation

1.1

We have K representation points $\{r_i\}_1^K$, each between two of our $K+1$ decision points $\{d_i\}_0^K$. We want to optimize those $2K+1$ points in regard to the MAD criteria:

$$\begin{aligned}
&\min_{\phi_L=d_0<r_1<d_1<\ldots<d_{K-1}<r_K<d_K=\phi_H} E^1_{\varepsilon_Q} \\
=&\min_{\phi_L=d_0<r_1<d_1<\ldots<d_{K-1}<r_K<d_K=\phi_H} \sum_{i=1}^{K}\left(\int_{d_{i-1}}^{d_i}|x-r_i|p(x)dx\right)
\end{aligned}$$

1.2

Given our decision points $\{d_i\}_0^K$, we want to optimize our representation points $\{r_i\}_1^K$. Lets differentiate $E^1_{\varepsilon_Q}$ with regard to r_i and equal to 0:

$$\begin{aligned}
\frac{\partial E^1_{\varepsilon_Q}}{\partial r_i}(x) &= \frac{\partial\left(\sum_{j=1}^{K}\left(\int_{d_{j-1}}^{d_j}|x-r_j|p(x)dx\right)\right)}{\partial r_i}(x) \\
&= \frac{\partial\left(\int_{d_{i-1}}^{d_i}|x-r_i|p(x)dx\right)}{\partial r_i}(x) \\
&= -\int_{d_{i-1}}^{d_i} sign(x-r_i)p(x)dx \\
&= \int_{d_{i-1}}^{r_i} p(x)dx - \int_{r_i}^{d_i} p(x)dx = 0 \\
&\iff \int_{d_{i-1}}^{r_i} p(x)dx = \int_{r_i}^{d_i} p(x)dx
\end{aligned}$$

I.e, the area under $p(x)$ in the interval $[d_{i-1}, r_i]$ needs to be equal to the area under $p(x)$ in the interval $[r_i, d_i]$. This is exactly the definition of the *median* value over the distributation of X.

1.3

Given our representation points $\{r_i\}_1^K$, we want to optimize our decision points $\{d_i\}_0^K$. Lets differentiate $E^1_{\varepsilon_Q}$ with regard to d_i and equal to 0:

$$
\begin{aligned}
\frac{\partial E^1_{\varepsilon_Q}}{\partial d_i}(x) &= \frac{\partial\left(\sum_{j=1}^K\left(\int_{d_{j-1}}^{d_j}|x-r_j|p(x)dx\right)\right)}{\partial d_i}(x) \\
&= \frac{\partial\left(\int_{d_{i-1}}^{d_i}|x-r_i|p(x)dx+\int_{d_i}^{d_{i+1}}|x-r_{i+1}|p(x)dx\right)}{\partial d_i}(x) \\
&= |d_i-r_i|p(d_i)-|d_i-r_{i+1}|p(d_i) \\
&= p(d_i)(|d_i-r_i|-|d_i-r_{i+1}|)=0 \\
&\iff |d_i-r_i|=|d_i-r_{i+1}| \\
&\xLeftrightarrow{r_i<d_i<r_{i+1}} d_i-r_i=r_{i+1}-d_i \\
&\iff d_i=\frac{r_{i+1}+r_i}{2}
\end{aligned}
$$

I.e, our optimal decision point d_i is the middle-point of the interval $[r_{i+1}, r_i]$.

1.4

Algorithm 1: Max-Lloyd for MAD criteria

set some values to $\{r_i\}_1^K$ and $\{d_i\}_0^K$ s.t. $\phi_L = d_0 < r_1 < d_1 < \ldots < d_{k-1} < r_k < d_k = \phi_H$;
while *stop condition not met (e.g, error rate improvement is larger than a given threshold)* **do**
  update $\{r_i\}_1^K$: $r_i = median\ of\ p(x)\ in\ [d_{i-1}, d_i]$;
  update $\{d_i\}_0^K$: $d_i = \frac{r_{i+1}+r_i}{2}$;
end

2 L1 optimization: doing the proof right

2.1

$$
\begin{aligned}
\mathcal{L}(f, \hat{f}_{F^*}) &= \int_I |f(x) - \hat{f}_{F^*}(x)| dx \\
&= \sum^N \int_{I_k} |f(x) - F^*(x)| dx \\
&= \sum^N \left(\int_{x:f(x)<F_i^*} f(x) - F_i^* dx \;-\; \int_{x:f(x)>F_i^*} f(x) - F_i^* dx \;+\; \int_{x:f(x)=F_i^*} \underbrace{f(x) - F_i^*}_{=0} dx \right) \\
&= \sum^N \left(\int_{x:f(x)<F_i^*} f(x) dx \;-\; F_i^* \underbrace{\int_{x:f(x)<F_i^*} 1 dx}_{=0} \;-\; \int_{x:f(x)>F_i^*} f(x) dx \;-\; F_i^* \underbrace{\int_{x:f(x)<F_i^*} 1 dx}_{=0} \right) \\
&= \sum^N \left(\int_{x:f(x)<F_i^*} f(x) dx \;-\; \int_{x:f(x)>F_i^*} f(x) dx \right)
\end{aligned}
$$

2.2

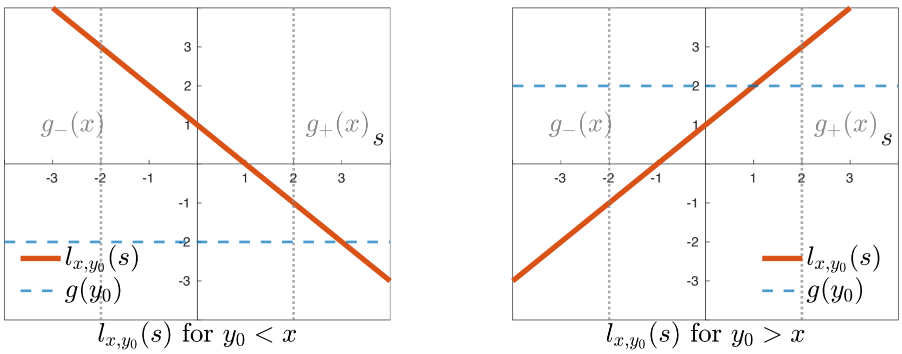

Figure 1: This is the caption text

3 Signal Discretization using a Piecewise-Linear Approximation

3.1

$$\begin{aligned}
\int_{\Delta_i} (t-t_i)^k dt &= \frac{1}{k+1}(t-t_i)^{k+1}\Big|_{\frac{i-1}{N}}^{\frac{i}{N}} \\
&= \frac{1}{k+1}((\frac{i}{N}-t_i)^{k+1} - (\frac{i-1}{N}-t_i)^{k+1}) \\
&= \frac{1}{k+1}((\frac{i}{N}-\frac{i+(i-1)}{2N})^{k+1} - (\frac{i-1}{N}-\frac{i+(i-1)}{2N})^{k+1}) \\
&= \frac{1}{k+1}((\frac{1}{2N})^{k+1} - (\frac{-1}{2N})^{k+1})
\end{aligned}$$

for odd k:

$$\begin{aligned}
\int_{\Delta_i} (t-t_i)^k dt &= \frac{1}{k+1}((\frac{1}{2N})^{k+1} - (\frac{-1}{2N})^{k+1}) \\
&= \frac{1}{k+1}(\frac{1}{(2N)^{k+1}} - \frac{1}{(2N)^{k+1}}) = 0
\end{aligned} \tag{1}$$

for even k:

$$\begin{aligned}
\int_{\Delta_i} (t-t_i)^k dt &= \frac{1}{k+1}\left(\left(\frac{1}{2N}\right)^{k+1} - \left(\frac{-1}{2N}\right)^{k+1}\right) \\
&= \frac{1}{k+1}\left(\frac{2}{(2N)^{k+1}}\right) \\
&= \frac{1}{k+1} * \frac{1}{2^k} * \left(\frac{1}{N}\right)^{k+1} = \frac{|\Delta_i|^{k+1}}{2^k(k+1)}
\end{aligned} \tag{2}$$

3.2

Lets formulates the MSE error of Piecewise-Linear approximation:

$$\begin{aligned}
\Phi_{MSE}^{PL}(\{a_i\}_1^N, \{c_i\}_1^N) &= \int_0^1 \left(\phi(t)-\hat{\phi}(t)\right)^2 dt \\
&= \sum_{i=1}^N \left(\int_{\Delta_i} \left(\phi(t)-\hat{\phi}(t)\right)^2 dt\right) \\
&= \sum_{i=1}^N \left(\int_{\Delta_i} \left(\phi^2(t) - 2\phi(t)\hat{\phi}(t) + \hat{\phi}^2(t)\right) dt\right) \\
&= \sum_{i=1}^N \left(\int_{\Delta_i} \phi^2(t)dt\right) - 2\sum_{i=1}^N \left(\int_{\Delta_i} \phi(t)\hat{\phi}(t)dt\right) + \sum_{i=1}^N \left(\int_{\Delta_i} \hat{\phi}^2(t)dt\right) \\
&= \sum_{i=1}^N \left(\int_{\Delta_i} \phi^2(t)dt\right) \underbrace{-2\sum_{i=1}^N \left(\int_{\Delta_i} \phi(t)\left(a_i(t-t_i)+c_i\right)dt\right)}_{(*)} + \underbrace{\sum_{i=1}^N \left(\int_{\Delta_i} \left(a_i(t-t_i)+c_i\right)^2 dt\right)}_{(**)}
\end{aligned}$$

Developing the terms separately:

$$
\begin{aligned}
(*) &= -2\sum_{i=1}^{N}\left(\int_{\Delta_i}\phi(t)\left(a_i(t-t_i)+c_i\right)dt\right)\\
&= -2\sum_{i=1}^{N}a_i\left(\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt\right)-2\sum_{i=1}^{N}c_i\left(\int_{\Delta_i}\phi(t)dt\right)
\end{aligned}
$$

$$
\begin{aligned}
(**) &= \sum_{i=1}^{N}\left(\int_{\Delta_i}\left(a_i(t-t_i)+c_i\right)^2 dt\right)\\
&= \sum_{i=1}^{N}a_i^2\left(\int_{\Delta_i}\left(t-t_i\right)^2 dt\right)+2\sum_{i=1}^{N}a_ic_i\left(\int_{\Delta_i}\left(t-t_i\right)dt\right)+\sum_{i=1}^{N}c_i^2|\Delta_i|\\
\text{(using eq 1,2)} &= \sum_{i=1}^{N}a_i^2\frac{|\Delta_i|^3}{12}+2\sum_{i=1}^{N}a_ic_i\,(0)+\sum_{i=1}^{N}c_i^2|\Delta_i|
\end{aligned}
$$

To find our optimal coefficient $\{a_i\}_1^N$, $\{c_i\}_1^N$ we will differentiate with regard to a_i and c_i each of the terms (only $(*)$ and $(**)$ contribute to the derivatives):

$$
\begin{aligned}
\frac{\partial(*)}{\partial a_i} &= \frac{\partial\left(-2\sum_{i=1}^{N}a_i\left(\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt\right)-2\sum_{i=1}^{N}c_i\left(\int_{\Delta_i}\phi(t)dt\right)\right)}{\partial a_i}\\
&= -2\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(*)}{\partial c_i} &= \frac{\partial\left(-2\sum_{i=1}^{N}a_i\left(\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt\right)-2\sum_{i=1}^{N}c_i\left(\int_{\Delta_i}\phi(t)dt\right)\right)}{\partial c_i}\\
&= -2\int_{\Delta_i}\phi(t)dt
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(**)}{\partial a_i} &= \frac{\partial\left(\sum_{i=1}^{N}a_i^2\frac{|\Delta_i|^3}{12}+\sum_{i=1}^{N}c_i^2|\Delta_i|\right)}{\partial a_i}\\
&= 2\frac{|\Delta_i|^3}{12}a_i
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(**)}{\partial a_i} &= \frac{\partial\left(\sum_{i=1}^{N}a_i^2\frac{|\Delta_i|^3}{12}+\sum_{i=1}^{N}c_i^2|\Delta_i|\right)}{\partial a_i}\\
&= 2|\Delta_i|c_i
\end{aligned}
$$

Summing the terms and equating to 0 we get the optimal coefficients:

$$
\begin{aligned}
\frac{\partial\Phi_{MSE}^{PL}}{\partial a_i} &= 2\frac{|\Delta_i|^3}{12}a_i-2\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt=0\\
\implies a_i^* &= \frac{12}{|\Delta_i|^3}\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt
\end{aligned}
$$

$$\frac{\partial \Phi_{MSE}^{PL}}{\partial c_i} = 2|\Delta_i|c_i - 2\int_{\Delta_i} \phi(t)dt = 0$$
$$\implies c_i^* = \frac{1}{|\Delta_i|}\int_{\Delta_i} \phi(t)dt$$

3.3

To formulate the minimal MSE we will place the optimal coefficients in the error function:

$$\begin{aligned}
\Phi_{MSE}^{PL}(\{a_i^*\}_1^N, \{c_i^*\}_1^N) &= \sum_{i=1}^{N}\left(\int_{\Delta_i}\phi^2(t)dt\right) - 2\sum_{i=1}^{N} a_i\left(\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt\right) - 2\sum_{i=1}^{N} c_i\left(\int_{\Delta_i}\phi(t)dt\right) \\
&\quad + \sum_{i=1}^{N} a_i^2\frac{|\Delta_i|^3}{12} + \sum_{i=1}^{N} c_i^2|\Delta_i| \\
&= \sum_{i=1}^{N}\left(\int_{\Delta_i}\phi^2(t)dt\right) - 2\sum_{i=1}^{N}\frac{12}{|\Delta_i|^3}\left(\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt\right)^2 - 2\sum_{i=1}^{N}\frac{1}{|\Delta_i|}\left(\int_{\Delta_i}\phi(t)dt\right)^2 \\
&\quad + \sum_{i=1}^{N}\frac{12}{|\Delta_i|^3}\left(\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt\right)^2 + \sum_{i=1}^{N}\frac{1}{|\Delta_i|}\left(\int_{\Delta_i}\phi(t)dt\right)^2 \\
&= \sum_{i=1}^{N}\left(\int_{\Delta_i}\phi^2(t)dt\right) - \sum_{i=1}^{N}\frac{12}{|\Delta_i|^3}\left(\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt\right)^2 - \sum_{i=1}^{N}\frac{1}{|\Delta_i|}\left(\int_{\Delta_i}\phi(t)dt\right)^2
\end{aligned}$$

3.4

To compare the Piecewise-Linear approximation to the Piecewise-Constant approximation we will compare them interval-by-interval as they are devided the same way.

We will show that in each interval, the Piecewise-Linear approximation has lower MSE than the Piecewise-Constant approximation:

The Piecewise-Linear MSE:

$$\Phi_{MSE_i}^{PL*} = \int_{\Delta_i}\phi^2(t)dt - \frac{12}{|\Delta_i|^3}\left(\int_{\Delta_i}\phi(t)\left(t-t_i\right)dt\right)^2 - \frac{1}{|\Delta_i|}\left(\int_{\Delta_i}\phi(t)dt\right)^2$$

The Piecewise-Constant MSE:

$$\begin{aligned}
\Phi_{MSE_i}^{PC*} &= \int_{\Delta_i}\phi^2(t)dt - |\Delta_i|\left(\frac{1}{|\Delta_i|}\int_{\Delta_i}\phi(t)dt\right)^2 \\
&= \int_{\Delta_i}\phi^2(t)dt - \frac{1}{|\Delta_i|}\left(\int_{\Delta_i}\phi(t)dt\right)^2
\end{aligned}$$

Hence:

$$\Phi_{MSE_i}^{PL*} \overset{?}{\leq} \Phi_{MSE_i}^{PC*}$$

$$\int_{\Delta_i} \phi^2(t)dt - \frac{12}{|\Delta_i|^3}\left(\int_{\Delta_i} \phi(t)\,(t-t_i)\,dt\right)^2 - \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} \phi(t)dt\right)^2 \overset{?}{\leq} \int_{\Delta_i} \phi^2(t)dt - \frac{1}{|\Delta_i|}\left(\int_{\Delta_i} \phi(t)dt\right)^2$$

$$-\frac{12}{|\Delta_i|^3}\left(\int_{\Delta_i} \phi(t)\,(t-t_i)\,dt\right)^2 \overset{?}{\leq} 0$$

As $|\Delta_i|$ is a strictly positive value, the above inequality holds:

$$\Phi_{MSE_i}^{PL*} \leq \Phi_{MSE_i}^{PC*}$$

Part II
Implementation

4 Quantization

5 Subsampling and Reconstruction