Discrete Math HW #10

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16. For the relation $V = (x, y) \mid xy > 0$ on $\mathbb{Z}$, prove or disprove that V is reflexive, irreflexive, symmetric, antisymmetric and transitive.

Proof: The relation V is not reflexive because if $x = 2$ and $y = 3$, $xy = 6$ and 6 is greater than 0. So the pair $(2, 3) \in V$ but since $2 \neq 3$, V is not reflexive.

The relation V is not irreflexive, because for the value pair $(1, 1)$, 1 times itself is greater than 0 and $1 \in \mathbb{Z}$ so $(1, 1) \in V$ but since $1 = 1$, the relation V is not irreflexive.

The relation V is symmetric, because $\forall x, y \in \mathbb{Z}, xy > 0 \implies yx > 0$ by the commutative property of multiplication.

The relation V is not antisymmetric, because $(2, 3) \in V$ and $(3, 2) \in V$ as previously proved, but $2 \neq 3$.

x, y can either both be negative, or both me positive, and still be in V because if their parity was different, their product would always be negative and not found in relation V. In the case where $x, y < 0$, if $xy > 0$ and $yz > 0$ then surely $xz > 0$. A similar argument can be used for when $x, y, z > 0$. Since xVy and $yVx \implies xVz \forall x, y, z \in \mathbb{Z}$, V is transitive. $\square$

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17. Prove Theorem 7.3.1. If $\sim$ is an equivalence relation on A, then $a \sim b \iff [a] = [b]$.

Proof: To prove that, $a \sim b \implies [a] = [b]$. Let $a, b \in A$, $a \sim b$, and $x \in [a]$. By definition of equivalence classes, $a \sim x$. Since we're told $\sim$ is an equivalence relation, that means its reflexive, symmetric, and transitive. By symmetry, $a \sim b \implies b \sim a$, and we assumed $a \sim x$. So by transitivity, $b \sim a$ and $a \sim x \implies b \sim x$ and therefore $x \in [b]$. So it can be said that $x \in [a] x \in [b]$. For the same reasons, $x \in [b] \implies x \in [a]$, as:

$$\begin{aligned} x \in [b] &\implies b \sim x \\ &\implies a \sim x \\ &\implies x \in [a] \end{aligned}$$

Since $x \in [b] \iff x \in [a]$, we can say $[a] = [b]$ derived from $a \sim b$.

To prove that $[a] = [b] \iff a \sim b$, let $x \in [a]$ by definition of the equivalence class, $a \sim x$ and assume $x \in [b]$, so $b \sim x$. Since $\sim$ is reflexive, $a = x = b$ and given $a \sim x$ from $x \in [a]$, we can substitute b for x such that $[a] = [b] \implies a \sim b$. $\square$

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18. Let $A = \{x \in \mathbb{N} \mid 2 \leq x \leq 100\}$. For p a prime number let $A_p = \{a \in A \mid p \text{ is the smallest prime that divides } a\}$. Note you proved these form a partition on A in Question 15.

    Define a relation $\sim$ on A, such that $a \sim b$ is $a \in A_p$ and $b \in A_p$ for some prime p. Prove that $\sim$ is an equivalence relation.

    Proof: By the fundamental theorem of arithmetic, any number in the given range can be described as the product of certain prime numbers if it isn't prime itself. Let $a, b \in A$, Let P_1 be the set of prime numbers that when multiplied yield a and let P_2 be the set of prime numbers that when multiplied yield b. The relation $\sim$ then can be described as saying that $a \sim b \implies min(p_1) = min(P_2)$, note that $min(P_1)$ and $min(P_2)$ must equal the prime subscript of the partition a, b are part of. Therefore, the relation $a \sim b \implies a \in A_p$ and $b \in A_p \forall a, b \in [2, 100]$.

    I don't however see how $\sim$ can be an equivalence relation, as $4 \sim 6$, $4, 6 \in [2, 100]$, and $4, 6 \in A_2$ but $4 \neq 6$ so $\sim$ can't be reflexive and hence not an equivalence relation. $\square$