Discrete Math HW #4

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11. Prove or disprove with a counterexample the following statement: If the sum of two positive prime numbers is prime, then one of the prime addends must be 2.

Proof: By definition, all prime numbers except for 2 are odd, since if they were even, they would be divisible by 2, and would not be prime. Therefore, assume that x and y are prime numbers > 2 which are odd. x and y could be expressed as $2k+1$ for some integer $k \in \mathbb{N}$ where $0 \notin \mathbb{N}$ (Since 1 is not prime). $2k$ is even because 2 multiplied by a natural number is always even. Since x and y are odd, $x+y$ could be rewritten as:

$$(2k_1 + 1) + (2k_2 + 1)$$

We've already established that $2k_1$ and $2k_2$ are even since they're a multiple of 2. Since addition is commutative, the above expression could be rewritten as:

$$(2k_1 + 2k_2) + 1 + 1$$

The sum of any two even numbers is even, in our case, $2(k_1 + k_2) = (2k_1 + 2k_2)$, so the sum of $k_1 + k_2$ multiplied by 2 is even for reasons stated above. Hence:

$$2(k_1 + k_2) + 1 + 1 = 2(k_1 + k_2) + 2$$

is even as addition between odd numbers will always results in an even sum. For that reason, the sum of two prime integers where one of the integers is not 2 cannot be prime. However, if one of the integers were 2, a prime sum would be producible, an example being: $2 + 3 = 5$ where both addends are prime as well as the sum. $\square$

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12. Let P be the set of Pythagorean triples; that is,

$$P = (a, b, c)|a, b, c \in \mathbb{Z} \text{ and } a^2 + b^2 = c^2$$

and let T be the set

$$T = (p, q, r)|p = x^2 - y^2, q = 2xy, \text{ and } r = x^2 + y^2 \text{ where } x, y \in \mathbb{Z}.$$

Prove that $T \subseteq P$.

Proof: For $T \subseteq P$ to be true, the implication $\forall(p, q, r) \in T \implies \forall(p, q, r) \in P$ must be true. For the implication to be true, $\forall(p, q, r) \in T$ must equal $\forall(a, b, c) \in P$. We're told the relationship between a, b, c is $a^2 + b^2 = c^2$, thus the relationship between p, q, r must also be $p^2 + q^2 = r^2$ for T to be a subset of P. Substituting the equations from the prompt starting with the left hand side:

$$\begin{aligned} p^2 + q^2 &= r^2 \\ (x^2 - y^2)^2 + (2xy)^2 &= \\ (x^4 - 2x^2y^2 + y^4) + 4x^2y^2 &= \\ x^4 + 2x^2y^2 + y^4 &= \end{aligned}$$

Followed by the right hand side:

$$\begin{aligned} (x^2 + y^2) &= r^2 \\ x^4 + 2x^2y^2 + y^4 &= \end{aligned}$$

As can be seen, $p^2 + q^2 = r^2$ is in fact true since the manipulated expressions on both the left and right side of the equation are identical. Therefore, the triplet (p, q, r), is also a pythagorean triplet proving that $\forall(p, q, r) \in T \implies \forall(p, q, r) \in P$ and that $T \subseteq P$ is in fact true. $\square$

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16. Show that given any rational number x, and any positive integer k, there exists an integer y such that $x^k y$ is an integer.

Proof: A rational number is one that can be expressed in terms of a fraction. Let x be some arbitrary number represented as $\frac{m}{n}$, where $m, n \in \mathbb{N}$ and are not further reducible. If you raise the fraction by a positive integer k you'd get $(\frac{m}{n})^k = \frac{m^k}{n^k}$. Since k is a positive integer, and m is an integer, m^k is equivalent to $(m)(m)...(m)$ k times, and m is still an integer as integers are closed under multiplication. The same can be said for n, substituting n for m in the previous sentence. Hence m and n are still integers. Given that, let y equal the denominator raised to k, which in our case is n^k, $x^k y = \frac{m^k(n^k)}{n^k} = m^k$. This proves that given a rational number, raised to a positive integer, multiplied by the denominator raised to the same positive integer – leaves you with just the numerator alone, which was already proven to be an integer, making the original statement true. $\square$