Discrete Math HW #5

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12. Let A and B be sets in a universe U. Prove or disprove: $A \cup B = A \cap B$ if and only if $A = B$.

    Proof: Since the prompt is a biconditional, we must prove both implications alone. We will do this following the direct proof method.

    Assume $A \cup B = A \cap B$, elaborating on both sides of the equation: $A \cup B$ means that $(x \in A) \vee (x \in B)$ for any element x in U. As for the other expression, $A \cap B$ means that $(x \in A) \wedge (x \in B)$ for any element x in U. Effectively this means that every element in the union of sets A and B is an element of A and B on their own respectively. Since we know that $\forall x(x \in A \wedge x \in B)$, by definition $A \subseteq B$ and $B \subseteq A$; therefore, $A = B$.

    Now Assume $A = B$, if that's true, that means $\forall x(x \in A \wedge x \in B)$. Since every element of A is also an element of B, the union of the two, $A \cup B$ is really the same just A and B on their own, since we don't count elements twice in sets. Similaraly, if we were to take the intersection of A and B, $A \cap B$, that set would equal just A or B on their own, since we don't count the same elements twice. Therefore, $A = B \implies A \cup B = A \cap B$

    Since $A = B \implies A \cup B = A \cap B$ and $A \cup B = A \cap B \implies A = B$, the biconditional $A \cup B = A \cap B \iff A = B$ is true. $\square$

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13. Let A, B and C be sets in a universe U. Prove or disprove that:

    (a) $A - B = A \cap \overline{B}$

    (b) $A - (B - C) = A \cap (\overline{B} \cup C)$

    (c) $A - (B - C) = (A - B) - C$

    (a) $A - B$ is the same as saying $\forall x \in U(x \in A \land x \notin B)$ otherwise known as A's relative complement. That's the same as saying $A \cap \overline{B}$ which means that $\forall x \in U(x \in A \land x \notin U - B)$, otherwise known as A's complement. The relative complement is the same as the intersection of the actual set with the regular complement because, $AU - B = A$, and if you substitute that to the definition of A's complement defined above it would be exactly identical to the definition of A's relative complement; hence, $A - B = A \cap \overline{B}$ is in fact true. □

    (b) $A - (B - C)$ is the same as saying $\forall x \in U(x \in A \land (x \in B \land x \notin C)$. Breaking it down further, $x \in A$ and x is not in the set of elements which are in B but not in C. As for, $A \cap (\overline{B} \cup C)$, that means that that $\forall x \in U(x \in A \land (x \notin B \lor x \in C))$. $(x \in B \land x \notin C)$ is logically equivalent to $B \land \overline{C}$. And $(x \notin B \lor x \in C)$ is logically equivalent to $\overline{B} \lor C$.

    $$\overline{B \land \overline{C}} = \overline{B} \lor C \qquad \textit{DeMorgan}$$

    Therefore, the membership requirement for the L.H.S. is logically equivalent to the membership requirement for the R.H.S. and the elements of both sets are equivalent, so $A - (B - C) = A \cap (\overline{B} \cup C)$ □

    (c) Let A = {1,2,3,4}, B = {3,4,5}, and C = {4,9}. $B - C$ = {3,5}. A - {3,5} = {1,2,4}. As for the R.H.S. expression, $A - B$ = {1,2} and {1,2} - C = {1,2}. $\{1,2,4\} \nsubseteq \{1,2\}$. Therefore, $A - (B - C) \neq (A - B) - C$ proven by counter-example. □

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14. Let U be a universal set. Let $A_i \subseteq U$ be a family of sets with $i \in I$ for some index set I.

    (a) Suppose $B \subseteq A_i$ for every $i \in I$. Prove that $B \subseteq \bigcap_{i \in I} A_i$.

    (b) Suppose $A_i \subseteq D$ for every $i \in I$. Prove that $\bigcup_{i \in I} A_i \subseteq D$.

    (a) If $B \subseteq A_i$ for every $i \in I$ that means that all the elements of B are in each A_i for all values of i. If you were to take the intersection of A_i and A_{i2} (assuming i_2 is the next index), both of which contain B individually, B would still be present in their intersection. The same can be said for the intersection of all A_i for every $i \in I$, since we're told the $B \subseteq A_i$. Therefore, $B \subseteq \bigcap_{i \in I} A_i$ is true because an intersection of sets containing the same set of elements will always contain that same set of elements. $\square$

    (b) Assume $A_i \subseteq D$ for every $i \in I$, that means that for any set generated, A, from the index set I, denoted A_i, will be present in a larger set D. Taking the union of two sets we assume are present in a larger set named D, will still be in the encapsulating set after they're joined, as they both were in there to begin with. Following that logic, $\bigcup_{i \in I} A_i \subseteq D$ will always be true since every possible A_i is already in D, so their union will still be in D as well. $\square$