Discrete Math HW #10

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17. Let $f : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Q}$ be defined as $f(a,b) = a/b$.

    (a) Prove that f is a well-defined function.

    (b) Prove or disprove that f is injective.

    (c) Prove or disprove that f is surjective.

    Proof (a): We will prove that function $f : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Q}$ where $f(a,b) = a/b$ is a well-defined function. Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}^*$ be arbitrary integers. Then $\forall x, y \in \mathbb{Z} \times \mathbb{Z}^*, f(x,y) = x/y$ where x/y is a unique fraction within $\mathbb{Q}$, as a fraction between 2 non-zero integers would be in the set of rational numbers. Since all paired-values within the domain are mapped to a exactly one unique fraction within the codomain f can be said to be a well-defined function. $\square$

    Proof (b): We will prove that f is not injective by counterexample. Let $x_1 = 2, y_1 = 4$ and let $x_2 = 3, y_2 = 6$. Then it can be said that, $x_1, x_2, y_1, y_2 \in \mathbb{Z} \times \mathbb{Z}^*$. Following f's definition, $f(x_1, y_1) = x_1/y_1 = 2/4 = 1/2$. Similarly, $f(x_2, y_2) = x_2/y_2 = 3/6 = 1/2$. Since the pair $(x_1, y_1) \neq (x_2, y_2)$ but $f(x_1, y_1) = f(x_2, y_2)$ It can be said that the function f is not injective. $\square$

    Proof (c): We will prove that function f is surjective by proving that every element in the image of f is also in its codomain. Let x, y be arbitrary variables in f's domain. Then by definition of f, the image of f is the set containing $\forall x, y \in \mathbb{Z} \times \mathbb{Z}^* | f(a,b) = a/b$. Following the definition of the set of rational numbers (f's codomain), $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$. Since the domain for the image of f and $\mathbb{Q}$ are the same, and the range of the image of f is equivalent to its codomain $\mathbb{Q}$ since they share the same definition, it can be said that f is surjective. $\square$

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18. Let $f : A \to B$ and let $X \subseteq A$ and $U \subseteq B$. Prove that $f(X) \subseteq U$ if and only if $X \subseteq f^{-1}(U)$

Proof: In order to prove a biconditional statement, we will first prove that if $f(X) \subseteq U$ then $X \subseteq f^{-1}(U)$. Followed by proving that if $X \subseteq f^{-1}(U)$ then $f(X) \subseteq U$. Beginning with the former:

Assume $f(X) \subseteq U$, then $\forall x \in X, \exists u \in U$ such that $f(x) = u$ by the definition of an image of a function over a set. Since $f(x) = u$, then by definition of an inverse function, $\exists u \in U, \exists x \in X$ such that $f^{-1}(u) = x$. We can say that the preceding statement is true for elements of U which are equivalent to $f(x)$ for some $x \in X$. Since $\forall x \in X$, x is also in $f^{-1}(U)$, we can conclude that $X \subseteq f^{-1}(U)$

Assume $X \subseteq f^{-1}(U)$, then we can say that all elements of X are in $f^{-1}(U)$. Additionally, $\exists u \in U, f^{-1}(u) = x \iff u = f(x)$ by definition of an inverse function. Since $f^{-1}(u) = x \iff u = f(x)$, we can conclue that for every u in which the inverse function $f^{-1}(u) = x$, there exists an $f(x)$ which is equal to u. Since all element of X are in $f^{-1}(U)$, by the inverse function, $f^{-1}(u) = x$ for all values of u which are equivalent to $f(x)$. Since $\forall u \in U$ where $f(x) = u$ the inverse function is defined, we can say that $f(X) \subseteq U$, since $\forall f(x) \exists u \in U$ where $f(x) = u$.

Since $f(X) \subseteq U$ implies $X \subseteq f^{-1}(U)$ and $X \subseteq f^{-1}(U)$ implies $f(X) \subseteq U$ we can conclude that $f(X) \subseteq U$ if and only if $X \subseteq f^{-1}(U)$. $\square$

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19. Suppose $f : A \to B$ and $g : B \to C$ are functions. Prove that if both g and $g \circ f$ are one-to-one, then f is also one-to-one.

    Proof: Assume g and $g \circ f$ are injective functions, that means that $\forall x, y \in A$, if $(g \circ f)(x) = (g \circ f)(y)$ then $x = y$. Since we know that if $g(f(x)) = g(f(y)), x = y$ by the assumption that $g \circ f$ is injective, we can proceed to say that $f(x) = f(y)$ since g is also injective. Since we know that $x = y$ and that $f(x) = f(y)$ for all $x, y \in A$ we can say that function f is injective as well. $\square$