Discrete Math HW #8

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11. Let a, b, and c be integers such that $a \neq 0$. Prove that if $a|b$ and $a|c$, then $a|(sb + tc)$ for any integers s and t.

    Proof: Let $a, b, c \in \mathbb{Z}$ such that $a \neq 0$ and $s, t \in \mathbb{Z}$ be chosen arbitrarily. Assume $a|b$ and $a|c$ then by definition, $b = ak$ and $c = aj$ for some specific $k, j \in \mathbb{Z}$. If we multiply b by s we get $sb = sak$. Similarly, if we multiply c by t we get $tc = taj$. We want to prove that $a|(sb + tc)$; therefore,

$$\begin{aligned} sb + tc &= sak + taj \\ &= a(sk + tj) \end{aligned}$$

    Which is clearly divisible by a. Thus, $a|(sb + tc)$ is true. $\square$

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12. Let m and n be positive integers. Prove that $\gcd(m, m+n)|n$.

    By definition of GCD, the $\gcd(m, m+n) = d \iff d|m$ and $d|(m+n)$ and $\forall e$ if $e|m$ and $e|(m+n)$ then $e < d$. Therefore, let $d \in \mathbb{Z}$ represent the $\gcd(m, m+n)$. We know $d|m$ and $d|(m+n)$ so by definition $m = dk$ and $m + n = dj$ for some specific $j, k \in \mathbb{Z}$. Subtracting m from $m+n$ means that $n = dj - m$. Note: m will be substituted for dk later on.

$$\begin{aligned} n &= dj - m \\ &= dj - dk \\ &= d(j-k) \end{aligned}$$

    Recall that d, which clearly divides n represents the $\gcd(m, m+n)$. Therefore, $\gcd(m, m+n)|n$. $\square$

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13. Prove the generalized DeMorgan Law, or in other words that for any nonempty index set I

$$\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$$

Proof: We want to prove set equality, so we'll prove that $\overline{\bigcap_{i\in I} A_i} \subseteq \bigcup_{i\in I} \overline{A_i}$ and $\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcap_{i\in I} A_i}$, starting with the former.

Assume some element $x \in \overline{\bigcap_{i\in I} A_i}$, since $\overline{\bigcap_{i\in I} A_i}$ represents the complement of the intersection of $A_i \forall i \in I$, that means $x \notin A_i \forall i \in I$. x not being part of $A_i \forall i \in I$ means that $x \in \overline{A_i} \forall i \in I$ by definition of a set complement. Saying $x \in \overline{A_i} \forall i \in I$, which was proven, is essentially the same as saying $x \in \bigcup_{i\in I} \overline{A_i}$ because if $x \in A_i \forall i \in I$, then x will also be part of the union: $\bigcup_{i\in I} \overline{A_i}$ by definition of applying the union operation on sets. Since $x \in \overline{\bigcap_{i\in I} A_i}$ and $\bigcup_{i\in I} \overline{A_i}$, $\overline{\bigcap_{i\in I} A_i} \subseteq \bigcup_{i\in I} \overline{A_i}$

To prove that $\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcap_{i\in I} A_i}$, assume $x \in \bigcup_{i\in I} \overline{A_i}$. That implies that $x \in \overline{A_i} \forall i \in I$. x being part of $x \in \overline{A_i} \forall i \in I$ means that $x \notin A_i \forall i \in I$, which is equivalent to saying that $x \notin \bigcap_{i\in I} A_i$ since if $x \notin A_i \forall i \in I$ it won't be in the intersection of $A_i \forall i \in I$. $x \notin \bigcap_{i\in I} A_i \forall i \in I$ which was proven means that $x \in \overline{\bigcap_{i\in I} A_i} \forall i \in I$ by definition of complementing a set. Since $x \in \bigcup_{i\in I} \overline{A_i}$ and $x \in \overline{\bigcap_{i\in I} A_i} \forall i \in I$, $\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcap_{i\in I} A_i}$.

In conclusion, since $\overline{\bigcap_{i\in I} A_i} \subseteq \bigcup_{i\in I} \overline{A_i}$ and $\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcap_{i\in I} A_i}$, $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$ by definition of set equality. $\square$