2017-02-16

1 Stability of Nonlinear Systems

Class of system:

$$\dot{a}(t) = f(a(t)) + g(a(t), u(t)) \tag{1}$$

Nonlinear model and the controller:

$$\begin{aligned}\dot{\hat{a}}(t) &= \hat{f}(\hat{a}(t)) + \hat{g}(\hat{a}(t), u(t)) \\ u(t) &= k(\hat{a}(t))\end{aligned} \tag{2}$$

Then,

$$\begin{aligned}\dot{a}(t) &= f(a(t)) + g(a(t), k(\hat{a}(t))) = f(a(t)) + m(a(t), \hat{a}(t)) \\ \dot{\hat{a}}(t) &= \hat{f}(\hat{a}(t)) + \hat{g}(\hat{a}(t), k(\hat{a}(t))) = \hat{f}(\hat{a}(t)) + \hat{m}(\hat{a}(t), \hat{a}(t))\end{aligned} \tag{3}$$

Assume the uncertainty between the model and the plant is additive:

$$\begin{aligned}\hat{f}(\zeta) &= f(\zeta) + \delta_f(\zeta) \\ \hat{m}(\zeta, \zeta) &= m(\zeta, \zeta) + \delta_m(\zeta)\end{aligned} \tag{4}$$

Then the error dynamics follows:

$$\begin{aligned}\dot{e}(t) &= f(a(t)) - \hat{f}(\hat{a}(t)) + m(a(t), \hat{a}(t)) - \hat{m}(\hat{a}(t), \hat{a}(t)) \\ &= f(a(t)) - f(\hat{a}(t)) - \delta_f(\hat{a}(t)) + m(a(t), \hat{a}(t)) - m(\hat{a}(t), \hat{a}(t)) - \delta_m(\hat{a}(t))\end{aligned} \tag{5}$$

Assume the following Lipschitz conditions:

$$\begin{aligned}\|f(x) - f(y)\| &\leq K_f \|x - y\| \\ \|m(x, s) - m(y, s)\| &\leq K_m(s) \|x - y\| \\ \|\delta_f(x) - \delta_f(y)\| &\leq K_{\delta_f} \|x - y\| \\ \|\delta_m(x) - \delta_m(y)\| &\leq K_{\delta_m} \|x - y\|\end{aligned} \tag{6}$$

For $K_m(s)$, define $K_{m,\max} \doteq \max_{s \in B_s}(K_m(s))$ for B_s a ball centered in the origin. Then

$$\begin{aligned}\|\dot{e}(t)\| &\leq K_f \|e(t)\| + K_{\delta_f} \|\hat{a}(t)\| + K_{m,\max} \|e(t)\| + K_{\delta_m} \|\hat{a}(t)\| \\ &= (K_f + K_{m,\max}) \|e(t)\| + (K_{\delta_f} + K_{\delta_m}) \|\hat{a}(t)\|\end{aligned} \tag{7}$$

Also assume that the model-based controller exponentially stabilizes the origin of the model, then with $\hat{a}(t_k) \in B_s$, $\hat{a}(t) \in B_s$, for $t \in [t_k, t_k + h)$, and

$$\|\hat{a}(t)\| \leq \alpha \|\hat{a}(t_k)\| e^{-\beta(t-t_k)} \tag{8}$$

where $\alpha, \beta > 0$. Then

$$\begin{aligned}
e(t) &= e(t_k) + \int_{t_k}^{t} [f(a(s)) - f(\hat{a}(s)) - \delta_f(\hat{a}(s)) + m(a(s), \hat{a}(s)) - m(\hat{a}(s), \hat{a}(s)) - \delta_m(\hat{a}(s))] ds \\
&= \int_{t_k}^{t} [f(a(s)) - f(\hat{a}(s)) - \delta_f(\hat{a}(s)) + m(a(s), \hat{a}(s)) - m(\hat{a}(s), \hat{a}(s)) - \delta_m(\hat{a}(s))] ds \\
\|e(t)\| &\leq \int_{t_k}^{t} [(K_f + K_{m,\max})\|e(s)\| + \alpha(K_{\delta_f} + K_{\delta_m})\|\hat{a}(t_k)\| e^{-\beta(s-t_k)}] ds \\
&= (K_f + K_{m,\max}) \int_{t_k}^{t} \|e(s)\| ds + \alpha(K_{\delta_f} + K_{\delta_m})\|\hat{a}(t_k)\| \int_{t_k}^{t} e^{-\beta(s-t_k)}] ds \\
&= \frac{\alpha(K_{\delta_f} + K_{\delta_m})\|\hat{a}(t_k)\|}{-\beta} e^{-\beta(s-t_k)} \Big|_{t_k}^{t} + (K_f + K_{m,\max}) \int_{t_k}^{t} \|e(s)\| ds \\
&= \frac{\alpha(K_{\delta_f} + K_{\delta_m})\|\hat{a}(t_k)\|}{\beta} (1 - e^{-\beta(t-t_k)}) + (K_f + K_{m,\max}) \int_{t_k}^{t} \|e(s)\| ds
\end{aligned} \tag{9}$$

From the Gronwall-Bellman Inequality, if a continuous real-valued function $y(t)$ satisfies:

$$y(t) \leq \lambda(t) + \int_{a}^{t} \mu(s) y(s) ds \tag{10}$$

with $\lambda(t)$ and $\mu(t)$ continuous real-valued functions and $\mu(t)$ non-negative for $t \in [a, b]$, then

$$y(t) \leq \lambda(t) + \int_{a}^{t} \lambda(s)\mu(s) e^{\int_s^t \mu(t) d\tau} ds \tag{11}$$

In our case,

$$\begin{aligned}
y(t) &= \|e(t)\| \\
\lambda(t) &= \frac{\alpha(K_{\delta_f} + K_{\delta_m})\|\hat{a}(t_k)\|}{\beta} (1 - e^{-\beta(t-t_k)}) \\
\mu(t) &= K_f + K_{m,\max}
\end{aligned} \tag{12}$$

Therefore,

$$
\begin{aligned}
\|e(t)\| \leq{}& \frac{\alpha(K_{\delta_f}+K_{\delta_m})\|\hat{a}(t_k)\|}{\beta}(1-e^{-\beta(t-t_k)}) \\
&+ (K_f+K_{m,\max})\int_{t_k}^{t}\frac{\alpha(K_{\delta_f}+K_{\delta_m})\|\hat{a}(t_k)\|}{\beta}(1-e^{-\beta(s-t_k)})e^{\int_s^t(K_f+K_{m,\max})d\tau}ds \\
={}& \frac{\alpha(K_{\delta_f}+K_{\delta_m})\|\hat{a}(t_k)\|}{\beta}(1-e^{-\beta(t-t_k)}) \\
&+ \frac{\alpha(K_f+K_{m,\max})(K_{\delta_f}+K_{\delta_m})\|\hat{a}(t_k)\|}{\beta}\int_{t_k}^{t}(1-e^{-\beta(s-t_k)})e^{(K_f+K_{m,\max})(t-s)}ds
\end{aligned}
\tag{13}
$$

To simplify the notations, define:

$$
\begin{aligned}
K_\delta &\doteq K_{\delta_f}+K_{\delta_m} \\
K_{fm} &\doteq K_f+K_{m,\max}
\end{aligned}
\tag{14}
$$

Then,

$$
\begin{aligned}
\|e(t)\| \leq{}& \frac{\alpha K_\delta\|\hat{a}(t_k)\|}{\beta}(1-e^{-\beta(t-t_k)}) + \frac{\alpha K_\delta K_{fm}\|\hat{a}(t_k)\|}{\beta}\int_{t_k}^{t}(1-e^{-\beta(s-t_k)})e^{K_{fm}(t-s)}ds \\
={}& \frac{\alpha K_\delta\|\hat{a}(t_k)\|}{\beta}\left[(1-e^{-\beta(t-t_k)}) + K_{fm}\int_{t_k}^{t}(1-e^{-\beta(s-t_k)})e^{K_{fm}(t-s)}ds\right] \\
={}& \frac{\alpha K_\delta\|\hat{a}(t_k)\|}{\beta}\left[(1-e^{-\beta(t-t_k)}) + K_{fm}\int_{t_k}^{t}(e^{K_{fm}(t-s)}-e^{K_{fm}t-K_{fm}s-\beta s+\beta t_k})ds\right] \\
={}& \frac{\alpha K_\delta\|\hat{a}(t_k)\|}{\beta}\left[(1-e^{-\beta(t-t_k)}) - e^{K_{fm}(t-s)}\Big|_{t_k}^{t} + \frac{K_{fm}}{K_{fm}+\beta}e^{K_{fm}t-K_{fm}s-\beta s+\beta t_k}\Big|_{t_k}^{t}\right] \\
={}& \frac{\alpha K_\delta\|\hat{a}(t_k)\|}{\beta}\left[1-e^{-\beta(t-t_k)}+e^{K_{fm}(t-t_k)}-1+\frac{K_{fm}}{K_{fm}+\beta}(e^{-\beta(t-t_k)}-e^{K_{fm}(t-t_k)})\right] \\
={}& \frac{\alpha K_\delta\|\hat{a}(t_k)\|}{\beta}\left[e^{K_{fm}(t-t_k)}-e^{-\beta(t-t_k)}-\frac{K_{fm}}{K_{fm}+\beta}\left(e^{K_{fm}(t-t_k)}-e^{-\beta(t-t_k)}\right)\right] \\
={}& \frac{\alpha K_\delta\|\hat{a}(t_k)\|}{\beta}\left(1-\frac{K_{fm}}{K_{fm}+\beta}\right)[e^{K_{fm}(t-t_k)}-e^{-\beta(t-t_k)}] \\
={}& \frac{\alpha K_\delta\|\hat{a}(t_k)\|}{K_{fm}+\beta}[e^{K_{fm}(t-t_k)}-e^{-\beta(t-t_k)}]
\end{aligned}
\tag{15}
$$

Then, combining Eq.8 and Eq.15 yields

$$
\begin{aligned}
\|a(t)\| &\leq \|\hat{a}(t)\| + \|e(t)\| \\
&= \alpha\|\hat{a}(t_k)\|e^{-\beta(t-t_k)} + \frac{\alpha K_\delta \|\hat{a}(t_k)\|}{K_{fm}+\beta}[e^{K_{fm}(t-t_k)} - e^{-\beta(t-t_k)}] \\
&= \alpha\|\hat{a}(t_k)\| \left[e^{-\beta(t-t_k)} + \frac{K_\delta}{K_{fm}+\beta}[e^{K_{fm}(t-t_k)} - e^{-\beta(t-t_k)}]\right] \\
\|a(t_{k+1})\| &\leq \alpha\|\hat{a}(t_k)\| \left[e^{-\beta h} + \frac{K_\delta}{K_{fm}+\beta}[e^{K_{fm}h} - e^{-\beta h}]\right]
\end{aligned}
\tag{16}
$$

To ensure stability, we need to require

$$
\|a(t_{k+1})\| < \|a(t_k)\| = \|\hat{a}(t_k)\| \tag{17}
$$

Alternatively, we enfore the following stability condition

$$
\begin{aligned}
\alpha\|\hat{a}(t_k)\| \left[e^{-\beta h} + \frac{K_\delta}{K_{fm}+\beta}[e^{K_{fm}h} - e^{-\beta h}]\right] &< \|\hat{a}(t_k)\| \\
\alpha \left[e^{-\beta h} + \frac{K_\delta}{K_{fm}+\beta}[e^{K_{fm}h} - e^{-\beta h}]\right] &< 1 \\
1 - \alpha \left[e^{-\beta h} + \frac{K_\delta}{K_{fm}+\beta}[e^{K_{fm}h} - e^{-\beta h}]\right] &> 0
\end{aligned}
\tag{18}
$$

h need to be sufficiently small such that the stability condition in Eq.18 holds.