Geometric Interpretation of Derivatives in Higher Dimensions

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6 February,2025

Overview

Introduction

- **Overview:**
  - Extends the concept of derivatives from one-dimensional calculus to functions of several variables.
  - Provides a framework for understanding how functions change along multiple directions.
- **Key Ideas:**
  - **Linear Approximation:** The derivative at a point is the best linear approximation of the function near that point.
- **Why It Matters:**
  - **Visualization:** Offers intuitive geometric insights into the behavior of multivariable functions.
  - **Applications:** Fundamental in optimization, solving differential equations and modeling real-world phenomena.

Geometric Meaning of the derivative of a vector Function I

The geometric meaning of the derivative of a vector function is closely related to the concept of the rate of change and the tangent vector.

- *Rate of change of position*:

Let $r : \mathbb{R} \to \mathbb{R}^n$, where r is differentiable meaning $r(t) = (r_1(t), r_2(2), \cdots, r_n(t))$ and all $r_i(t)$ such that $r_i : \mathbb{R} \to \mathbb{R}$ are differentiable functions, and $r'(t) = (r_1'(t), r_2'(2), \cdots, r_n'(t))$ describes how the position of the vector-valued function $r(t)$ changes with respect to the parameter t.

- *Tangent Vector*:

$r'(t)$ at a specific point on the curve indicates the direction of the tangent line to the curve at that point.

Geometric Meaning of the derivative of a vector Function II

- *Visualisation in 3D and 2D Spaces*:

If a particle moving along a curve, $r'(t)$ at any point shows how the particle would continue moving if it maintains its current velocity and direction

Consider the unit circle defined by

$$r(t) = (\cos t, \sin t).$$

Its derivative is

$$r'(t) = (-\sin t, \cos t),$$

Which represents the tangent vector at each point.
At $t = \frac{\pi}{4}$:

$$r\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$$

Geometric Meaning of the derivative of a vector Function III

$$r'\left(\frac{\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

The tangent line is then given by

$$L(s) = r\left(\frac{\pi}{4}\right) + s\, r'\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) + s\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad s \in \mathbb{R}.$$

Geometric Meaning of the derivative of a vector Function IV

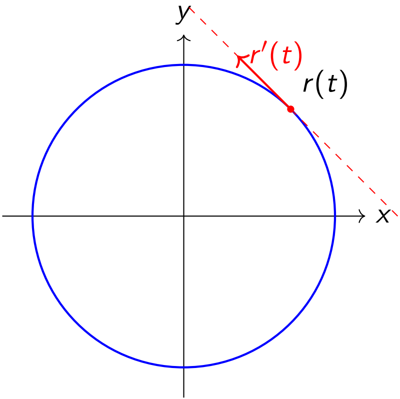

Figure: A unit circle with its tangent vector and the tangent line at $t = \pi/4$.

Understanding the Gradient of a Function

What is the Gradient?

The **gradient** of a function is a vector that tells us two things:

- **Direction** – Where the function increases the fastest.
- **Steepness** – How fast the function increases in that direction.

Mathematically, for a function $f(x, y)$, the gradient is written as:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

This means:

- $\frac{\partial f}{\partial x}$ tells how much f changes as we move in the x-direction.
- $\frac{\partial f}{\partial y}$ tells how much f changes as we move in the y-direction.

Example: Climbing a Hill

Imagine you are hiking on a hill where the height at any point is given by $f(x, y)$.

- The **gradient** tells you the steepest path to climb up.
- The bigger the gradient's length, the steeper the climb.

Mathematical Example: If $f(x, y) = x^2 + y^2$, then:

$$\nabla f = (2x, 2y)$$

- At $(1, 1)$, the gradient is $(2, 2)$, meaning the function increases fastest in the $(1, 1)$ direction.
- At $(0, 0)$, the gradient is $(0, 0)$, meaning it's a flat point (no steepest direction).

Geometric Meaning

- The **gradient vector** always points in the direction where the function increases the fastest.
- The **steepness** (magnitude of the gradient) tells how quickly the function increases.
- The gradient is **perpendicular** to the level curves (contour lines).

Physical Meaning (Real-World Examples)

The gradient is useful in many areas of science:

- **Heat Flow:** If $f(x, y)$ represents temperature, then ∇f shows the direction where heat increases fastest.
- **Electric Fields:** The gradient of voltage tells us how electric potential changes in space.
- **Water Flow:** Water flows **downhill**, opposite to the gradient of elevation.

Gradient of a Scalar-Valued Function Figure I

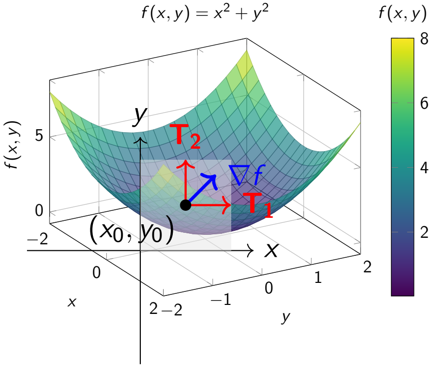

Figure: Illustration of the function $f(x, y) = x^2 + y^2$, its tangent plane at $(1, 1)$, and the gradient vector perpendicular to the plane.

The End