2.Fubini定理

定理4.2.1

$$\text{设}(E_1,\Sigma_1,\mu_1)\text{和}(E_2,\Sigma_2,\mu_2)\text{是两个概率空间,}$$
$$f\text{是}(E_1\times E_2,\Sigma_1\times\Sigma_2)\text{上非负可测函数，则}$$
$$\iint\limits_{E_1\times E_2} f(x_1,x_2)d(\mu_1\times\mu_2)=\int_{E_1}\mu_1(dx_1)\int_{E_2}f(x_1,x_2)\mu_2(dx_2)=\int_{E_2}\mu_2(dx_2)\int_{E_1}f(x_1,x_2)\mu_1(dx_1)$$

证明：$f=1_B,B\in\Sigma_1\times\Sigma_2$时，

$$\begin{aligned}\iint\limits_{E_1\times E_2} f(x_1,x_2)d(\mu_1\times\mu_2)&=\iint\limits_{E_1\times E_2}1_B(x_1,x_2)d(\mu_1\times\mu_2)\\&=\int_{E_2}\mu_1(B(x_2))\mu_2(dx_2)(\text{乘积测度的定义})\\&=\int_{E_2}\mu_2(dx_2)\int_{E_1}1_{B(x_2)}(x_1)\mu_1(dx_1)\\&=\int_{E_2}\mu_2(dx_2)\int_{E_1}1_B(x_1,x_2)\mu_1(dx_1)\\&=\int_{E_2}\mu_2(dx_2)\int_{E_1}f(x_1,x_2)\mu_1(dx_1)\end{aligned}$$

当$f=\sum_{i=1}^n b_i1_{B_i}$时，

$$\begin{aligned}\iint\limits_{E_1\times E_2} f(x_1,x_2)d(\mu_1\times\mu_2)&=\sum_{i=1}^n b_i\iint\limits_{E_1\times E_2}1_{B_i}d(\mu_1\times\mu_2)\\&=\sum_{i=1}^n b_i\int_{E_2}\mu_2(dx_2)\int_{E_1}1_{B_i}(x_1,x_2)\mu_1(dx_1)\\&=\int_{E_2}\mu_2(dx_2)\int_{E_1}f(x_1,x_2)\mu_1(dx_1)\end{aligned}$$

当f为非负可测函数时，存在非负简单函数$f_n\uparrow f$，则

$$\begin{aligned}\iint\limits_{E_1\times E_2} f(x_1,x_2)d(\mu_1\times\mu_2)&=\lim_{n\to\infty}\iint\limits_{E_1\times E_2}f_n d(\mu_1\times\mu_2)\\&=\lim_{n\to\infty}\int_{E_2}\mu_2(dx_2)\int_{E_1}f_n(x_1,x_2)\mu_1(dx_1)\\&=\int_{E_2}\mu_2(dx_2)\lim_{n\to\infty}\int_{E_1}f_n(x_1,x_2)\mu_1(dx_1)\\&=\int_{E_2}\mu_2(dx_2)\int_{E_1}f(x_1,x_2)\mu_1(dx_1)\end{aligned}$$

由上一定理可以得到如下定理：

定理4.2.2

设 (E_1,Σ_1,μ_1)和(E_2,Σ_2,μ_2)是两个概率空间，

f是 $(E_1\times E_2,\Sigma_1\times\Sigma_2)$上实可测函数，若$\iint\limits_{E_1\times E_2} fd(\mu_1\times\mu_2)$存在，则

$$\iint\limits_{E_1\times E_2} f(x_1,x_2)d(\mu_1\times\mu_2)=\int_{E_1}\mu_1(dx_1)\int_{E_2}f(x_1,x_2)\mu_2(dx_2)=\int_{E_2}\mu_2(dx_2)\int_{E_1}f(x_1,x_2)\mu_1(dx_1)$$

证明：由上一定理可知

$$\iint\limits_{E_1\times E_2} f^{\pm}d(\mu_1\times\mu_2)=\int_{E_1}\mu_1(dx_1)\int_{E_2}f^{\pm}\mu_2(dx_2)$$

所以

$$\begin{aligned}\iint\limits_{E_1\times E_2} fd(\mu_1\times\mu_2)&=\iint\limits_{E_1\times E_2} f^{+}d(\mu_1\times\mu_2)-\iint\limits_{E_1\times E_2} f^{-}d(\mu_1\times\mu_2)\\&=\int_{E_1}\mu_1(dx_1)\int_{E_2}f^{+}\mu_2(dx_2)-\int_{E_1}\mu_1(dx_1)\int_{E_2}f^{-}\mu_2(dx_2)\\&=\int_{E_1}\mu_1(dx_1)\int_{E_2}(f^{+}-f^{-})\mu_2(dx_2)\\&=\int_{E_1}\mu_1(dx_1)\int_{E_2}f\mu_2(dx_2)\end{aligned}$$

例1

设X是$(\Omega,\mathcal{F},\mathbb{P})$上非负$r.v$列，则

$$\begin{aligned}\mathbb{E}[X]&=\int_0^{+\infty}\mathbb{P}(X>x)dx\\&=\int_0^{+\infty}(1-F(x))dx\\&=\int_0^{+\infty}\mathbb{P}(X\geq x)dx\end{aligned}$$

最后一个等号是因为$\mathbb{P}(X>x)$单调，最多有可数多个不连续点，从而

$$\int_0^{+\infty}\mathbb{P}(X>x)dx=\int_0^{+\infty}\mathbb{P}(X\geq x)dx$$

该例子的证明见一下题。

例2

X是非负随机变量，那么$\forall r>0$，

$$\mathbb{E}[X^r]=r\int_0^{+\infty}x^{r-1}\mathbb{P}(X>x)dx$$

证明：

$$
\begin{aligned}
\mathbb{E}[X^r] &= \int_{-\infty}^{+\infty} x^r d(F(x)) \\
&= \int_{-\infty}^{+\infty} x^r \mu_X(dx) \\
&= \int_{-\infty}^{0} x^r \mu_X(dx) + \int_{0}^{+\infty} x^r \mu_X(dx) \\
&= \int_{0}^{+\infty} x^r \mu_X(dx) \\
&= \int_{0}^{+\infty} \mu_X(dx) \int_0^x r y^{r-1} dy \\
&= \int_{0}^{+\infty} r y^{r-1} dy \int_y^{\infty} \mu_X(dx) \\
&= \int_{0}^{+\infty} r y^{r-1} \mathbb{P}(X \geq y) dy
\end{aligned}
$$

例3

$$
\text{如果 } S(x) = \sum_{n=1}^{\infty} a_n(x), a_n(x) \geq 0
$$

$$
\text{那么 } \int_a^b S(x)dx = \int_a^b \sum_{n=1}^{\infty} a_n(x)dx = \sum_{n=1}^{\infty} \int_a^b a_n(x)dx
$$

证明：首先将$a_n(x)$改写为$a(n,x)$，那么

$$
\sum_{n=1}^{\infty} a_n(x) = \sum_{n=1}^{\infty} a(n,x)
$$

a的映射关系如下

$$
a: \mathbb{N} \times [a,b] \to \mathbb{R}^+
$$

考虑$\mathbb{N}$上的计数测度

$$
\mu(B) = \sharp B = |B|
$$

所以

$$
\int_{\{m\}} a(n,x)\mu(dn) = a(m,x) \int_{\{m\}} \mu(dn) = a(m,x)
$$

从而

$$
\sum_{n=1}^{\infty} a(n,x) = \int_{\mathbb{N}} a(n,x)\mu(dn)
$$

因此由Fubini定理可得

$$
\begin{aligned}
\int_a^b \sum_{n=1}^{\infty} a_n(x)dx &= \int_a^b \int_{\mathbb{N}} a(n,x)\mu(dn)dx \\
&= \int_{\mathbb{N}} \int_a^b a(n,x)\mu(dn)dx \\
&= \sum_{n=1}^{\infty} \int_a^b a_n(x)dx
\end{aligned}
$$

定理4.2.3

$$
\begin{gathered}
(E_1,\Sigma_1),(E_2,\Sigma_2)\text{是两个可测空间，} \\
\mu_1\text{是}(E_1,\Sigma_1)\text{上概率测度，}\mathbb{P}(X,B)\text{是}E_1\times\Sigma_2\text{上转移概率测度，} \\
\mu\text{是}\Sigma_1\times\Sigma_2\text{上如下定义的概率测度：} \\
\mu(B)=\int_{E_1}\mathbb{P}(x,B)\mu_1(dx), B\in\Sigma_1\times\Sigma_2 \\
f\text{是}\Sigma_1\times\Sigma_2\text{上可测函数，若}\iint_{E_1\times E_2} fd\mu\text{存在，则} \\
\iint_{E_1\times E_2} fd\mu=\int_{E_1}\mu_1(dx)\int_{E_2} f(x,y)\mathbb{P}(x,dy)
\end{gathered}
$$

证明思路依旧是从示性函数到简单函数再到非负可测函数最后到一般可测函数的步骤处理。

3.无穷乘积转移概率测度

定理4.3.1

$$
\begin{gathered}
\text{设}(E_n,\Sigma_n,\mu_n)(n\geq 1)\text{是一列概率空间，} \\
\text{则在}(\times_{n=1}^{\infty}E_n,\times_{n=1}^{\infty}\Sigma_n)\text{上存在唯一的概率测度}\mu\text{，满足} \\
\mu(B_1\times\ldots\times B_n\times E_{n+1}\times E_{n+2}\ldots)=\mu_1(B_1)\times\ldots\times\mu_n(B_n) \\
\forall B_i\in\Sigma_i, 1\leq i\leq n, \forall n\geq 1 \\
\text{或等价地} \\
\mu(B^{(n)}\times E_{n+1}\times E_{n+2}\ldots)=(\times_{k=1}^{n}\mu_k)(B^{(n)}) \\
\forall B^{(n)}\in\times_{i=1}^{n}E_i \\
\text{称}\mu\text{为}\{\mu_n\}\text{的乘积测度，记为} \\
\mu=\mu_1\times\mu_2\ldots=\times_{n=1}^{\infty}\mu_n
\end{gathered}
$$

例1

$$
\begin{gathered}
\text{设}(E_n,\Sigma_n,\mu_n)(n\geq 1)\text{是一列概率空间，} \\
\text{定义}(\Omega,\mathcal{F},\mathbb{P})=(\times_{n=1}^{\infty}E_n,\times_{n=1}^{\infty}\Sigma_n\text{，}\ \times_{n=1}^{\infty}\mu_n) \\
\text{第}n\text{个分量}X_n(w)=w_n \\
\text{则}\{X_n\}\text{在}\mathbb{P}\text{下相互独立，}X_n\text{取值于}E_n\text{，分布是}\mu_n
\end{gathered}
$$

证明：取值于E_n以及独立性显然，计算概率

$$
p=\mathbb{P}((w_1,\ldots,w_n,w_{n+1},\ldots)\in(E_1,\ldots,E_{n-1},B_n,E_{n+1},\ldots))
$$

该概率等于

$$\begin{aligned}
p &= \mu_1(E_1)\dots\mu_{n-1}(E_{n-1})\mu_n(B_n)\mu_{n+1}(E_{n+1})\dots \\
&= \mu_n(B_n)
\end{aligned}$$

习题

习题1

（课本P153/6.2/1）

证明：

$$\begin{aligned}
\mathbb{E}[\xi^n] &= \int_{-\infty}^{+\infty} x^n d(F(x)) \\
&= \int_{-\infty}^{+\infty} x^n \mu_X(dx) \\
&= \int_0^{+\infty} x^n \mu_X(dx) + \int_{-\infty}^0 x^n \mu_X(dx) \\
&= \int_0^{+\infty} \mu_X(dx) \int_0^x nt^{n-1}dt - \int_{-\infty}^0 \mu_X(dx) \int_x^0 nt^{n-1}dt \\
&= \int_0^{+\infty} \mu_X(dx) \int_0^{+\infty} nt^{n-1}1_{t\le x}dt - \int_{-\infty}^0 \mu_X(dx) \int_{-\infty}^0 nt^{n-1}1_{x\le t}dt \\
&= \int_0^{+\infty} dt \int_0^{+\infty} nt^{n-1}1_{t\le x}\mu_X(dx) - \int_{-\infty}^0 dt \int_{-\infty}^0 nt^{n-1}1_{x\le t}\mu_X(dx) \\
&= \int_0^{+\infty} dt \int_t^{+\infty} nt^{n-1}\mu_X(dx) - \int_{-\infty}^0 dt \int_{-\infty}^t nt^{n-1}\mu_X(dx) \\
&= n\int_0^{+\infty} t^{n-1}(1-F(t))dt - n\int_{-\infty}^0 t^{n-1}F(t)dt
\end{aligned}$$

习题2

（课本P153/6.2/2）

证明：

$$
\begin{aligned}
\mathbb{E}[|X|] &= c\mathbb{E}[|\frac{X}{c}|] \\
&= c\int_0^{+\infty} |\frac{x}{c}|\mu_X(dx) + c\int_{-\infty}^0 |\frac{x}{c}|\mu_X(dx) \\
&= c\int_0^{+\infty} \frac{x}{c}\mu_X(dx) - c\int_{-\infty}^0 \frac{x}{c}\mu_X(dx) \\
&= c\int_0^{+\infty} \mu_X(dx)\int_0^{\frac{x}{c}} dt + c\int_{-\infty}^0 \mu_X(dx)\int_{\frac{x}{c}}^0 dt \\
&= c\int_0^{+\infty} dt\int_{ct}^{+\infty} \mu_X(dx) - c\int_{-\infty}^0 dt\int_{-\infty}^{ct} \mu_X(dx) \\
&= c\int_0^{+\infty} \mathbb{P}(X \geq ct)dt - c\int_{-\infty}^0 \mathbb{P}(X \leq ct)dt \\
&\overset{\text{对第二个式子令}t=-t}{=} c\int_0^{+\infty} \mathbb{P}(X \geq ct)dt - c\int_0^{+\infty} \mathbb{P}(X \leq -ct)dt \\
&= c\int_0^{+\infty} \mathbb{P}(|X| \geq ct)dt \\
&= c\sum_{n=0}^{\infty}\int_n^{n+1} \mathbb{P}(|X| \geq ct)dt
\end{aligned}
$$

注意到

$$
\mathbb{P}(|X| \geq c(n+1)) \leq \int_n^{n+1} \mathbb{P}(|X| \geq ct)dt \leq \mathbb{P}(|X| \geq cn)
$$

所以

$$
\sum_{n=1}^{\infty}\mathbb{P}(|X| \geq cn) = \sum_{n=0}^{\infty}\mathbb{P}(|X| \geq c(n+1)) \leq \mathbb{E}[|X|] \leq \sum_{n=1}^{\infty}\mathbb{P}(|X| \geq cn) + 1
$$

从而结论成立。

习题3

$$
\mu(\prod_{i=1}^{\infty} B_i) = \prod_{i=1}^{\infty} \mu(B_i)
$$

证明：记$A_n = \prod_{i=1}^n B_i$，则$A_n \downarrow \prod_{i=1}^{\infty} B_i$，由定义可知

$$
\mu(A_n) = \prod_{i=1}^n \mu(B_i)
$$

因为μ是概率测度，所以

$$
\begin{aligned}
\lim_{n\to\infty}\mu(A_n) &= \lim_{n\to\infty}\prod_{i=1}^n \mu(B_i) \\
&= \mu(\lim_{n\to\infty} A_n) \\
&= \mu(\prod_{i=1}^{\infty} B_i)
\end{aligned}
$$

因此

$$\mu(\prod_{i=1}^{\infty} B_i) = \prod_{i=1}^{\infty} \mu(B_i)$$