Nonlinear Systems II: Multiple Variables

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

Justin Solomon

Today's Root Problems

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

One Easy Instance

$$f(\vec{x}) = A\vec{x} - \vec{b}$$

Usual Assumption

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$\longrightarrow n \geq m$$

Jacobian

$$(Df)_{ij} \equiv \frac{\partial f_i}{\partial x_j}$$

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How big is Df for $f : \mathbb{R}^n \to \mathbb{R}^m$?

First-Order Approximation of $f : \mathbb{R}^n \to \mathbb{R}^n$

$$f(\vec{x}) \approx f(\vec{x}_k) + Df(\vec{x}_k) \cdot (\vec{x} - \vec{x}_k)$$

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Newton's Method:

$$\vec{x}_{k+1} = \vec{x}_k - [Df(\vec{x}_k)]^{-1} f(\vec{x}_k)$$

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Review: Do we explicitly compute $[Df(\vec{x}_k)]^{-1}$?

Convergence Sketch

1. $\vec{x}_{k+1} = g(\vec{x}_k)$ converges when the maximum-magnitude eigenvalue of Dg is less than 1

2. Extend observations about (quadratic) convergence in multiple dimensions

Two Problems

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2. $Df(\vec{x}_k)$ changes every iteration

Extend Secant Method?

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Direct extensions are not obvious:
Not enough data points!

Observation: Directional Derivative

$$D_{\vec{v}} f = Df \cdot \vec{v}$$

Secant-Like Approximation

$$J \cdot (\vec{x}_k - \vec{x}_{k-1}) \approx f(\vec{x}_k) - f(\vec{x}_{k-1})$$

where $J \approx Df(\vec{x}_k)$

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"Broyden's Method"

Broyden's Method: Outline

- Maintain current iterate $\vec{x}_k$ and approximation J_k of Jacobian near $\vec{x}_k$
- Update $\vec{x}_k$ using Newton-like step
- Update J_k using secant-like formula

Deriving the Broyden Step

$$\begin{aligned}\text{minimize}_{J_k} \; & \|J_k - J_{k-1}\|_{\text{Fro}}^2 \\ \text{such that } & J_k \cdot (\vec{x}_k - \vec{x}_{k-1}) = f(\vec{x}_k) - f(\vec{x}_{k-1})\end{aligned}$$

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$$J_k = J_{k-1} + \frac{(f(\vec{x}_k) - f(\vec{x}_{k-1}) - J_k \cdot \Delta\vec{x})}{\|\vec{x}_k - \vec{x}_{k-1}\|_2^2} (\Delta\vec{x})^\top$$

The Newton Step

$$\vec{x}_{k+1} = \vec{x}_k - J_k^{-1} f(\vec{x}_k)$$

Implementation Details

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- Still have to invert J_k in each step!

Revisiting the Broyden Step

$$J_k = J_{k-1} + \frac{(f(\vec{x}_k) - f(\vec{x}_{k-1}) - J_k \cdot \Delta\vec{x})}{\|\vec{x}_k - \vec{x}_{k-1}\|_2^2}(\Delta\vec{x})^\top$$

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$$J_k = J_{k-1} + \frac{(f(\vec{x}_k) - f(\vec{x}_{k-1}) - J_k \cdot \Delta\vec{x})}{\|\vec{x}_k - \vec{x}_{k-1}\|_2^2}(\Delta\vec{x})^\top$$

Simpler form:

$$J_k = J_{k-1} + \vec{u}_k \vec{v}_k^\top$$

Sherman-Morrison Formula

$$(A + \vec{u}\vec{v}^\top)^{-1} = A^{-1} - \frac{A^{-1}\vec{u}\vec{v}^\top A^{-1}}{1 + \vec{v}^\top A^{-1}\vec{u}}$$

Homework (if I remember): Check this

`#sorrynotsorry`

Broyden Without Inversion

- Start with a J_0 for which you know J_0^{-1} (e.g. identity)
- Update J_0^{-1} directly!

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Question: Limited-memory strategy?

Question: Large-scale strategy?

Automatic Differentiation

$$\left(x, \frac{dx}{dt}\right); \left(y, \frac{dy}{dt}\right) \mapsto$$

$$\left(x + y, \frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$\left(xy, \frac{dx}{dt}y + x\frac{dy}{dt}\right) \qquad \left(\frac{x}{y}, \frac{y\frac{dx}{dt} + x\frac{dy}{dt}}{y^2}\right) \quad \cdots$$

Next