Eigenproblems III: Computation, Conditioning

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Our Story So Far

$$A = QR$$
$$Q^{-1}AQ = RQ$$

Recall: QR Iteration

$$\begin{aligned} A_1 &= A \\ \text{Factor } A_k &= Q_k R_k \\ \text{Multiply } A_{k+1} &= R_k Q_k \end{aligned}$$

Convergence: More Detail

$$A_\infty = Q_\infty R_\infty = R_\infty Q_\infty$$

Commutativity

Lemma

If $A_\infty = Q_\infty R_\infty = R_\infty Q_\infty$ with no repeated eigenvalues, then A_∞ is diagonal.

Proof.

$\lambda\vec{x} = A\vec{x} \implies \lambda Q\vec{x} = QA\vec{x} = Q(QR)\vec{x} = (QR)Q\vec{x} = AQ\vec{x} \implies Q\vec{x} = \pm\vec{x}$ by orthogonality and uniqueness of $\vec{x} \implies Q$ is diagonal since $\vec{x}$'s span $\mathbb{R}^n$. Statement follows by symmetry of A_∞ and upper triangular shape of R_∞. $\square$

Intuition

$$A^k = A^{k-1} \cdot A = \begin{pmatrix} | & | & & | \\ A^{k-1}\vec{a}_1 & A^{k-1}\vec{a}_2 & \cdots & A^{k-1}\vec{a}_n \\ | & | & & | \end{pmatrix}$$

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Questions:

1. What do these look like?

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Questions:

1. What do these look like?
2. What if you do Gram-Schmidt on the columns?

Intuition for Convergence

$$A^k = Q_1 Q_2 \cdots Q_k R_k R_{k-1} \cdots R_1$$

Q from QR of A^k looks a lot like QR of A^{k-1}, so $Q_i \to I$. We conjugate A_k by Q_k each time, so A_k converges.

Krylov Subspace Methods

Krylov matrix:

$$K_k = \begin{pmatrix} | & | & | & & | \\ \vec{b} & A\vec{b} & A^2\vec{b} & \cdots & A^{k-1}\vec{b} \\ | & | & | & & | \end{pmatrix}$$

Column space related to eigenstructure of A.

Starting Point

$$(A + \delta A)(\vec{x} + \delta\vec{x}) = (\lambda + \delta\lambda)(\vec{x} + \delta\vec{x})$$

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Approximation:

$$A\delta\vec{x} + \delta A \cdot \vec{x} \approx \lambda\delta\vec{x} + \delta\lambda \cdot \vec{x}$$

Trick: Left Eigenvector

$$A\vec{x} = \lambda\vec{x}, \vec{x} \neq \vec{0} \implies$$
$$\exists \vec{y} \neq \vec{0} \text{ such that } A^\top \vec{y} = \lambda \vec{y}$$

Change in Eigenvalue

$$|\delta\lambda| \lessapprox \frac{\|\delta A\|_2}{|\vec{y} \cdot \vec{x}|}$$

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What about symmetric A?

Next