Problem 1

 $\forall f,g\in C^1(\mathbb{R}), \forall \alpha,\beta\in\mathbb{R}$,我们有 $\alpha f+\beta g$ 连续可导,所以 $\alpha f+\beta g\in C^1(\mathbb{R})$,因此 $C^1(\mathbb{R})$ 是线性空间。 考虑 $C^1(\mathbb{R})$ 的子集多项式全体,显然全体多项式的维度为 ∞ ,因此 $C^1(\mathbb{R})$ 的维度为 ∞ 。

Problem 2

$$A^TA = egin{bmatrix} ec{c}_1^Tec{c}_1 & \dots & ec{c}_1^Tec{c}_n \ \dots & \dots & ec{c}_n^Tec{c}_n \end{bmatrix} \in \mathbb{R}^{n imes n}, AA^T = egin{bmatrix} ec{r}_1^Tec{r}_1 & \dots & ec{r}_1^Tec{r}_m \ \dots & \dots & ec{r}_m^Tec{r}_n \end{bmatrix} \in \mathbb{R}^{m imes m}$$

Problem 3

注意到原问题等价于最小化

$$egin{aligned} f^2(ec{x}) &= ||Aec{x} - ec{b}||^2 \ &= (Aec{x} - ec{b})^T (Aec{x} - ec{b}) \ &= ec{x}^T A^T A ec{x} - ec{b}^T A ec{x} - ec{x}^T A^T ec{b} + ec{b}^T ec{b} \ &= ec{x}^T A^T A ec{x} - 2 ec{x}^T A^T ec{b} + ec{b}^T ec{b} \end{aligned}$$

对上式关于求求梯度可得

$$egin{aligned}
abla_{ec{x}}f^2(ec{x}) &=
abla_{ec{x}}(ec{x}^TA^TAec{x} - 2ec{x}^TA^Tec{b} + ec{b}^Tec{b}) \ &= 2A^TAec{x} - 2A^Tec{b} \end{aligned}$$

令上式为0可得

$$A^TAec{x}=A^Tec{b} \ ec{x}=(A^TA)^{-1}A^Tec{b}$$

Problem 4

注意到原问题等价于最小化

$$||A\vec{x}||^2 = \vec{x}^T A^T A \vec{x}$$

约束条件等价于

$$\left|\left|B\vec{x}
ight|
ight|^2=ec{x}^TB^TBec{x}=1$$

根据该条件构造拉格朗日乘子:

$$L(ec{x},\lambda) = ec{x}^T A^T A ec{x} - \lambda (ec{x}^T B^T B ec{x} - 1)$$

求梯度可得

$$egin{aligned}
abla_{ec{x}} L(ec{x},\lambda) &= 2A^T A ec{x} - 2\lambda B^T B ec{x} \
abla_{\lambda} L(ec{x},\lambda) &= -ec{x}^T B^T B ec{x} + 1 \end{aligned}$$

令上式为0可得

$$A^{T}A\vec{x} = \lambda B^{T}B\vec{x} \tag{1}$$

$$\vec{x}^{T}B^{T}B\vec{x} = 1 \tag{2}$$

将(1), (2)带入目标函数可得

$$\vec{x}^T A^T A \vec{x} = \lambda \vec{x}^T B^T B \vec{x} = \lambda$$

所以接下来只要求出λ即可,对等式(1)稍作变形可得

$$(A^T A - \lambda B^T B)\vec{x} = 0$$

由约束条件可知 $\vec{x} \neq 0$,所以上述线性方程有非零解,因此

$$|A^T A - \lambda B^T B| = 0$$

解该n次方程即可求出 $\lambda_1,\ldots,\lambda_n$,记最小的正根为 λ_i ,最大的正根为 λ_i ,所以

$$\left|\left|Aec{x}
ight|
ight|^2=\lambda\in\left[\lambda_i,\lambda_j
ight]$$

Problem 5

注意约束条件等价于

$$ec{x}^Tec{x}=1$$

根据该条件构造拉格朗日乘子:

$$L(ec{x},\lambda) = ec{a}.\,ec{x} - \lambda(ec{x}^Tec{x} - 1) \ = ec{a}^Tec{x} - \lambda(ec{x}^Tec{x} - 1)$$

求梯度可得

$$abla_{ec{x}} L(ec{x},\lambda) = ec{a} - \lambda ec{x} \
abla_{\lambda} L(ec{x},\lambda) = ec{x}^T ec{x} - 1$$

令上式为0可得

$$egin{aligned} ec{x} &= rac{1}{\lambda} ec{a} \ ec{x}^T ec{x} &= rac{1}{\lambda^2} ec{a}^T ec{a} = 1 \ \lambda &= \pm ||ec{a}|| \end{aligned}$$

将 $ec{x}=rac{1}{\lambda}ec{a}$ 带入可得

$$f(ec{x}) = rac{1}{\lambda} ec{a}^T ec{a} = rac{1}{\lambda} ||ec{a}||^2 = \pm ||ec{a}||$$

所以

$$\max f(\vec{x}) = ||\vec{a}||$$