Eigenproblems I

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Setup

Given: Collection of data points $\vec{x}_i$

- Age
- Weight
- Blood pressure
- Heart rate

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Find: Correlations between different dimensions

Simplest Model

One-dimensional subspace

$$\vec{x}_i \approx c_i \vec{v}$$

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One-dimensional subspace

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Equivalently:

$$\vec{x}_i \approx c_i \hat{v}$$

Review

What is c_i?

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$$c_i = \vec{x}_i \cdot \hat{v}$$

Variational Idea

$$\text{minimize} \sum_i \|\vec{x}_i - \operatorname{proj}_{\hat{v}} \vec{x}_i\|^2$$
$$\text{such that } \|\hat{v}\| = 1$$

Equivalent Optimization

$$\text{maximize } \|X^\top \hat{v}\|^2$$
$$\text{such that } \|\hat{v}\|^2 = 1$$

End Goal

Eigenvector of $XX^\top$ with largest eigenvalue.

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"First principal component"

Physics (in one slide)

Newton:

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2}$$

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Hooke:

$$\vec{F}_s = k(\vec{x} - \vec{y})$$

First-Order System

$$\frac{d}{dt}\begin{pmatrix}\vec{X}\\ \vec{V}\end{pmatrix} = \begin{pmatrix}0 & I\\ M^{-1}K & 0\end{pmatrix}\begin{pmatrix}\vec{X}\\ \vec{V}\end{pmatrix}$$

General ODE

$$\vec{X}' = A\vec{X}$$

Eigenvector Solution

$$\vec{x}' = A\vec{x}$$

$$A\vec{x}_i = \lambda_i \vec{x}_i$$

$$\vec{x}(0) = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$$

Eigenvector Solution

$$\vec{x}' = A\vec{x}$$

$$A\vec{x}_i = \lambda_i \vec{x}_i$$

$$\vec{x}(0) = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$$

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{x}_1 + \cdots + c_k e^{\lambda_k t}\vec{x}_k$$

Aside: Matrix Inverse

$$\vec{b} = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$$

$$A\vec{x} = \vec{b}$$

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$$\vec{b} = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k$$

$$A\vec{x} = \vec{b}$$

$$\implies \vec{x} = \frac{c_1}{\lambda_1}\vec{x}_1 + \cdots + \frac{c_n}{\lambda_n}\vec{x}_n$$

Setup

Have: n items in a dataset

$w_{ij} \geq 0$ similarity of items i and j

$w_{ij} = w_{ji}$

Want: x_i embedding on $\mathbb{R}$

Quadratic Energy

$$E(\vec{x}) = \sum_{ij} w_{ij}(x_i - x_j)^2$$

Optimization

$$\text{minimize } E(\vec{x})$$

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$$\text{such that } \|\vec{x}\|^2 = 1$$

Optimization

$$\begin{aligned} \text{minimize } & E(\vec{x}) \\ \text{such that } & \|\vec{x}\|^2 = 1 \\ & \vec{1} \cdot \vec{x} = 0 \end{aligned}$$

Simplification

$$E(\vec{x}) = \vec{x}^\top (2A - 2W)\vec{x}$$

Desired

Second smallest eigenvector of $2A - 2W$.

Definitions

Eigenvalue and eigenvector

An *eigenvector* $\vec{x} \neq \vec{0}$ of a matrix $A \in \mathbb{R}^{n\times n}$ is any vector satisfying $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$; the corresponding λ is known as an *eigenvalue*. *Complex* eigenvalues and eigenvectors satisfy the same relationships with $\lambda \in \mathbb{C}$ and $\vec{x} \in \mathbb{C}^n$.

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Scale doesn't matter!

$$\longrightarrow \|\vec{x}\| \equiv 1$$

Definitions

Spectrum and spectral radius

The *spectrum* of A is the set of eigenvalues of A. The *spectral radius* $\rho(A)$ is the eigenvalue λ maximizing $|\lambda|$.

Eigenproblems in the Wild

- ODE/PDE problems
- Minimize/maximize $\|A\vec{x}\|$ such that $\|\vec{x}\| = 1$
- Rayleigh quotient:

$$\frac{\vec{x}^\top A\vec{x}}{\|\vec{x}\|^2}$$

Two Basic Properties

Proved in notes

Lemma

Every matrix $A \in \mathbb{R}^{n \times n}$ has at least one (complex) eigenvector.

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Eigenvectors corresponding to distinct eigenvalues must be linearly independent.

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$\longrightarrow$ at most n eigenvalues

Diagonalizability

Nondefective

$A \in \mathbb{R}^{n \times n}$ is *nondefective* or *diagonalizable* if its eigenvectors span $\mathbb{R}^n$.

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$$D = X^{-1}AX$$

Extending to $\mathbb{C}^{n\times n}$

Complex conjugate

The *complex conjugate* of a number $z = a + bi \in \mathbb{C}$ is $\bar{z} \equiv a - bi$.

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Conjugate transpose

The *conjugate transpose* of $A \in \mathbb{C}^{m\times n}$ is $A^H \equiv \bar{A}^\top$.

Hermitian Matrix

$$A = A^H$$

Properties

Lemma

All eigenvalues of Hermitian matrices are real.

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Lemma

Eigenvectors corresponding to distinct eigenvalues of Hermitian matrices must be orthogonal.

Spectral Theorem

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Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian (if $A \in \mathbb{R}^{n\times n}$, suppose it is symmetric). Then, A has exactly n orthonormal eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$.

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Full set: $D = X^\top A X$

Next