Problem 1

(a)E_1测量f和f_0的误差，E_2测量f的光滑性。

(b)该式表示关于g的相对变化率。

(c)

$$
\begin{aligned}
dE_1(f,g) &= \frac{d}{d\epsilon}\int_0^1 \left(f(t)+\epsilon g(t)-f_0(t)\right)^2 dt\Big|_{\epsilon=0} \\
&= \int_0^1 2\left(f(t)+\epsilon g(t)-f_0(t)\right)g(t)dt\Big|_{\epsilon=0} \\
&= 2\int_0^1 \left(f(t)-f_0(t)\right)g(t)dt \\
dE_2(f,g) &= \frac{d}{d\epsilon}\int_0^1 \left(f'(t)+\epsilon g'(t)\right)^2 dt\Big|_{\epsilon=0} \\
&= \int_0^1 2\left(f'(t)+\epsilon g'(t)\right)g'(t)dt\Big|_{\epsilon=0} \\
&= 2\int_0^1 f'(t)g'(t)dt \\
&= 2\int_0^1 f'(t)d(g(t)) \\
&= 2\left(f'(t)g(t)\Big|_{t=0}^{t=1} - \int_0^1 f''(t)g(t)dt\right) \\
&= -2\int_0^1 f''(t)g(t)dt
\end{aligned}
$$

(d)由(c)可得

$$dE(f) = 2\int_0^1 \left(f(t)-f_0(t)-\alpha f''(t)\right)g(t)dt = 0$$

所以可以近似认为

$$f(t)-f_0(t) = \alpha f''(t)$$

(e)这部分感觉有点问题，这里略过。

Problem 2

假设a_i存在数组a中，那么利用如下算法即可在$O(k)$时间内计算出$f(x)$：

```
res = 0
s = 1
for i in a:
    res += i * s
    s *= x
```

Problem 3

(a)设

$$g(x)=p\left(x-\frac{a+b}{2}\right)^2+q\left(x-\frac{a+b}{2}\right)+r$$

那么

$$g'(x)=2p\left(x-\frac{a+b}{2}\right)+q$$

因为

$$\begin{gathered} f'(a)=g'(a) \\ f'(b)=g'(b) \\ g\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right) \end{gathered}$$

所以

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&=r \\ f'(a)=2p\left(a-\frac{a+b}{2}\right)+q \\ f'(b)=2p\left(b-\frac{a+b}{2}\right)+q \end{aligned}$$

解得

$$\begin{cases} p &= \frac{f'(a)-f'(b)}{2(a-b)} \\ q &= \frac{f'(a)+f'(b)}{2} \\ r &= f\left(\frac{a+b}{2}\right) \end{cases}$$

因此

$$g(x)=\frac{f'(a)-f'(b)}{2(a-b)}\left(x-\frac{a+b}{2}\right)^2+\frac{f'(a)+f'(b)}{2}\left(x-\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)$$

(b)对$g(x)$积分可得

$$
\begin{aligned}
\int_a^b g(x)dx &= \frac{f'(a)-f'(b)}{2(a-b)}\int_a^b \left(x-\frac{a+b}{2}\right)^2 dx + \frac{f'(a)+f'(b)}{2}\int_a^b \left(x-\frac{a+b}{2}\right) dx + f\left(\frac{a+b}{2}\right)\int_a^b dx \\
&= -\frac{f'(a)-f'(b)}{2(a-b)} \times \frac{2}{3} \times \left(\frac{a-b}{2}\right)^3 + f\left(\frac{a+b}{2}\right)(b-a) \\
&= (b-a)\left(f\left(\frac{a+b}{2}\right) + \frac{1}{24}\left(f'(b)-f'(a)\right)(b-a)\right)
\end{aligned}
$$

(c)为方便叙述，记

$$
c = \frac{1}{2}(a+b)
$$

那么f在c处的泰勒展开为

$$
f(x) = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + \frac{1}{6}f'''(c)(x-c)^3 + \frac{1}{24}f''''(c)(x-c)^4 + \cdots \tag{1}
$$

对上式积分可得

$$
\int_a^b f(x)dx = (b-a)f(c) + \frac{1}{24}f''(c)(b-a)^3 + \frac{1}{1920}f''''(c)(b-a)^5 + \cdots \tag{2}
$$

注意

$$
\begin{aligned}
\int_a^b g(x)dx &= (b-a)\left(f\left(\frac{a+b}{2}\right) + \frac{1}{24}\left(f'(b)-f'(a)\right)(b-a)\right) \\
&= (b-a)\left(f(c) + \frac{1}{24}f''(\xi)(b-a)^2\right) \qquad a \le \xi \le b \\
&= (b-a)f(c) + \frac{1}{24}f''(\xi)(b-a)^3
\end{aligned}
$$

所以(2)可以化为

$$
\begin{aligned}
\int_a^b f(x)dx &= (b-a)f(c) + \frac{1}{24}f''(\xi)(b-a)^3 - \frac{1}{24}f''(\xi)(b-a)^3 + \frac{1}{24}f''(c)(b-a)^3 + O((b-a)^5) \\
&= \int_a^b g(x)dx + O((b-a)^3)
\end{aligned}
$$

所以精度为3次。

(d)假设

$$
\Delta x = \frac{b-a}{k}, x_i = a + i\Delta x, i = 0, \ldots, k
$$

那么复合求积公式为

$$
\int_a^b f(x)dx \approx \sum_{i=0}^{k-1} \Delta x \left(f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{1}{24}\left(f'(x_{i+1}) - f'(x_i)\right)\Delta x\right)
$$

Problem 4

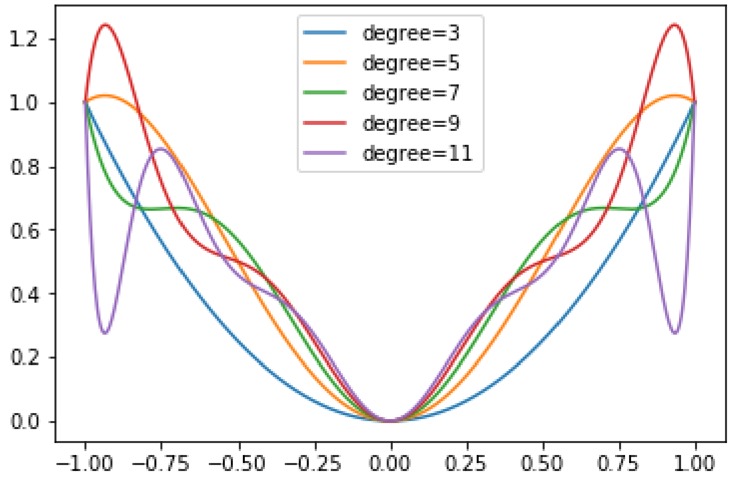
利用范德蒙行列式计算结果：

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix}$$

对应代码如下：

```
import numpy as np
import matplotlib.pyplot as plt

def Vandermon(X, k):
    #维度
    n = X.shape[0]
    #计算结果
    res = np.ones(n).reshape(-1, 1)
    x = np.copy(X)
    for i in range(k):
        res = np.c_[res, x]
        x *= X

    return res

x1 = np.linspace(-1, 1, 500).reshape(-1, 1)

K = [3, 5, 7, 9, 11]
for k in K:
    X = np.linspace(-1, 1, k).reshape(-1, 1)
    y = np.abs(X)
    Van = Vandermon(X, k-1)
    #计算系数
    a = np.linalg.solve(Van, y)
    #计算结果
    y1 = Vandermon(x1, k-1).dot(a)
    plt.plot(x1, y1, label="degree={}".format(k))

plt.legend()
plt.show()
```

图像结果如下：



不难看出，随着次数增加，曲线变化幅度越来越大。