Partial Differential Equations II

CS 205A:
Mathematical Methods for Robotics, Vision, and Graphics

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Almost Done!

- **Homework 7:** 12/2 (two days late!)
- **Homework 8:** 12/9 (optional)
- **Section:** 12/6 (final review)
- **Final exam:** 12/12, 12:15pm (**Gates B03**)

Go to office hours!

Course Reviews

On Axess!

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Request for Help

$$\text{CS 205A notes} \xmapsto{\textit{your help!}} \text{Textbook}$$

- Review text
- Write reference implementations
- Solidify your CS205A knowledge

Final Exam

- Cumulative
- Similar format to midterm
- **Two** sheets of notes

This Week

Couple relationships between derivatives.

- *Pressure gradient* determining fluid flow
- *Image operators* using x and y derivatives

Partial Differential Equations (PDE)

Boundary Value Problems

- *Dirichlet conditions*: Value of $f(\vec{x})$ on $\partial\Omega$
- *Neumann conditions*: Derivatives of $f(\vec{x})$ on $\partial\Omega$
- *Mixed* or *Robin conditions*: Combination

Second-Order Model Equation

$$\sum_{ij} a_{ij} \frac{\partial f}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial f}{\partial x_i} + cf = 0$$

$$(\nabla^\top A \nabla + \nabla \cdot \vec{b} + c) f = 0$$

Classification of Second-Order PDE

$$(\nabla^\top A \nabla + \nabla \cdot \vec{b} + c) f = 0$$

- If A is *positive or negative definite*, system is *elliptic*.
- If A is *positive or negative semidefinite*, the system is *parabolic*.
- If A has only one eigenvalue of different sign from the rest, the system is *hyperbolic*.
- If A satisfies none of the criteria, the system is *ultrahyperbolic*.

Derivative Operator Matrix

$$h^2\vec{w} = L_1\vec{y}$$

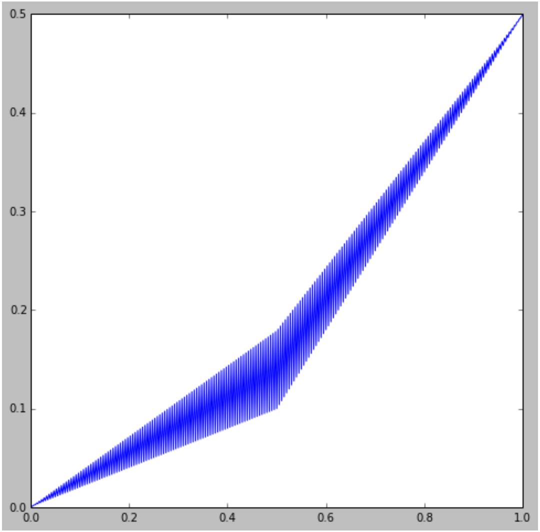
$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

Dirichlet

What About First Derivative?

- Potential for asymmetry at boundary
- Centered differences: Fencepost problem
- Possible resolution: Imitate leapfrog

Fencepost Problem



Big Idea

Derivatives : Functions :: Matrices : Vectors

Elliptic PDE

$$\mathcal{L}f = g \longmapsto L\vec{y} = \vec{b}$$

Elliptic PDE

$$\mathcal{L}f = g \longmapsto L\vec{y} = \vec{b}$$

Example: Laplace's equation on a line

Common Theme

Elliptic PDE $\mapsto$ Positive definite matrix

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$$L = -D^\top D, D = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & -1 & 1 & \\ & & & & -1 & \end{pmatrix}$$

Common Theme

Elliptic PDE $\mapsto$ Positive definite matrix

$$L = -D^\top D, D = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}$$

Review: Name two ways to solve.

Time Dependence

Choice:

1. Treat t separate from $\vec{x}$ ("semidiscrete")
2. Treat all variables democratically ("fully discrete")

Semidiscrete Heat Equation

$$f_t = f_{xx}$$

Semidiscrete Heat Equation

$$f_t = f_{xx} \longmapsto f_t = Lf$$

Semidiscrete Heat Equation

$$f_t = f_{xx} \longmapsto f_t = Lf$$

Stability for elliptic spatial operator (parabolic PDE)

Semidiscrete Time Stepping

Left with a multivariable **ODE problem**!

- Forward/backward Euler, RK, and friends

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  - Implicit vs. explicit (vs. symplectic)

Semidiscrete Time Stepping

Left with a multivariable **ODE problem**!

- Forward/backward Euler, RK, and friends
  - Implicit vs. explicit (vs. symplectic)
    - Alternative: Eigenvector methods (low-frequency approximation)

Fully Discrete PDE

- Discretize $\vec{x}$ and t simultaneously
- Can create larger linear algebra problems
- Philosophical point: What is “fully” discrete?

Gradient Domain Inpainting

http://groups.csail.mit.edu/graphics/classes/CompPhoto06/html/lecturenotes/10_Gradient.pdf

Gradient Domain

Pipeline for image $I(x, y)$:

1. Compute gradient: $\vec{v}(x, y) = \nabla I(x, y)$
2. Edit: $\vec{v} \mapsto \vec{v}'$
3. Reconstruct: $\nabla g \stackrel{?}{=} \vec{v}'$

Gradient Domain Reconstruction

$$\min_g \int_\Omega \|\nabla g - \vec{v}\|_2^2 \, dA$$

Gradient Domain Reconstruction

$$\min_g \int_\Omega \|\nabla g - \vec{v}\|_2^2 \, dA$$

$$\mapsto \nabla^2 g = \nabla \cdot \vec{v}$$

Elliptic!

Incompressible Navier-Stokes

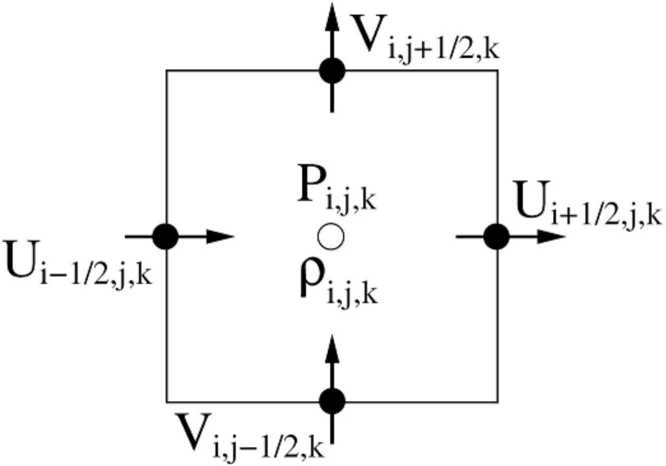
$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right)=-\nabla p+\mu\nabla^2\vec{v}+\vec{f}$$

- $t\in[0,\infty)$: Time
- $\vec{v}(t):\Omega\to\mathbb{R}^3$: Velocity
- $\rho(t):\Omega\to\mathbb{R}$: Density
- $p(t):\Omega\to\mathbb{R}$: Pressure
- $\vec{f}(t):\Omega\to\mathbb{R}^3$: External forces (e.g. gravity)

Lagrangian vs. Eulerian

- **Lagrangian:** Track parcels of fluid
- **Eulerian:** Fluid flows past a point in space

Marker-and-Cell (MAC) Grid



http://students.cs.tamu.edu/hrg/image/MAC.bmp

Splitting for Incompressible Flow

$$\nabla \cdot \vec{u} = 0 \text{ (divergence-free)}$$

$$\rho_t + \vec{u} \cdot \nabla \rho = 0 \text{ (density advection)}$$

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \frac{\nabla p}{\rho} = \vec{g} \text{ (velocity advection)}$$

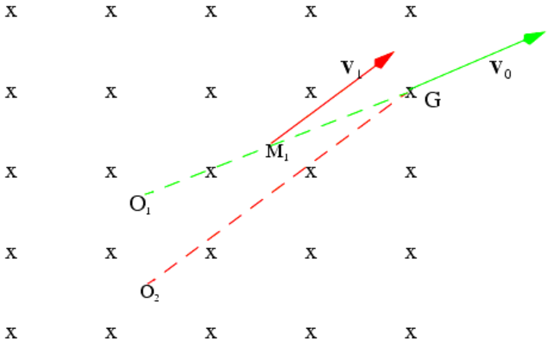
http://www.stanford.edu/class/cs205b/lectures/lecture17.pdf

Steps for Flow (on board)

1. Adjust Δt
2. Advect velocity
3. Apply forces
4. Solve for pressure: $\nabla \cdot \frac{\nabla p}{\rho} = \nabla \cdot \vec{u}$; divergence-free projection
5. Advect density

http://www.proxyarch.com/util/techpapers/papers/Fluidflowfortherestofus.pdf

Semilagrangian Advection



ecmwf.int/newsevents/training/rcourse_notes/NUMERICAL_METHODS/NUMERICAL_METHODS/Numerical_methods6.html

Next