Problem 1

(a)代码如下：

```
M = zeros(nParticles);
[m, n] = size(edges);
for i = 1: m
    x = edges(i, 1);
    y = edges(i, 2);
    M(x, x) = M(x, x) - 1;
    M(x, y) = M(x, y) + 1;
    M(y, x) = M(y, x) + 1;
    M(y, y) = M(y, y) - 1;
end
```

(b)令

$$Y_1 = X, Y_2 = X', Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

那么方程为

$$Y' = \begin{bmatrix} X' \\ X'' \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ F & 0 \end{bmatrix} \begin{bmatrix} X \\ X' \end{bmatrix} \triangleq AY$$

所以代码如下：

```
[m, n] = size(secondOrderMatrix);
matrix = zeros(2 * m);
matrix(1: m, m+1: 2*m) = eye(m);
matrix(m+1: 2*m, 1: m) = secondOrderMatrix;
```

(c)

(i)前向欧拉

$$y_{k+1} = y_k + hF(y_k) = (I_n + hA)y_k$$

代码如下

```
x = x + dt * firstOrder * x;
```

(ii)后向欧拉

$$\begin{aligned} y_k &= y_{k+1} - hF(y_{k+1}) = (I_n - hA)y_{k+1} \\ y_{k+1} &= (I_n - hA)^{-1} y_k \end{aligned}$$

代码如下

```
x = inv(eye(2 * n) - dt * firstOrder) * x;
```

(iii)梯形法

$$
\begin{aligned}
y_{k+1} &= y_k + h\frac{F\left(y_{k+1}\right) + F\left(y_k\right)}{2} = (I_n + \frac{1}{2}hA)y_k + \frac{1}{2}hAy_{k+1} \\
y_{k+1} &= (I_n - \frac{1}{2}hA)^{-1}(I_n + \frac{1}{2}hA)y_k
\end{aligned}
$$

代码如下

```
x = inv(eye(2 * n) - dt * firstOrder / 2) * (eye(2 * n) + dt * firstOrder / 2) * x;
```

(d)leapfrog

$$
\begin{aligned}
\vec{y}_{k+1} &= \vec{y}_k + h\vec{v}_{k+1/2} \\
\vec{a}_{k+1} &= F\left[t_{k+1}, \vec{y}_{k+1}\right] = A\vec{y}_{k+1} \\
\vec{v}_{k+3/2} &= \vec{v}_{k+1/2} + h\vec{a}_{k+1} = \vec{v}_{k+1/2} + A\vec{y}_{k+1}
\end{aligned}
$$

代码如下

```
positions = positions + dt * velocities;
velocities = velocities + dt * force * positions;
```

Problem 2

(a)求导可得

$$
\begin{aligned}
E'\left(t\right) &= \theta'\theta'' + \theta'\sin\theta \\
&= \theta'\left(\theta'' + \sin\theta\right) \\
&= 0
\end{aligned}
$$

所以$E(t)$关于t是常数。

(b)

$$
\begin{aligned}
E_{k+1} - E_k &= \frac{1}{2}w_{k+1}^2 - \cos\theta_{k+1} - \frac{1}{2}w_k^2 + \cos\theta_k \\
&= \frac{1}{2}(w_k - h\sin\theta_{k+1})^2 - \cos(\theta_k + hw_k) - \frac{1}{2}w_k^2 + \cos\theta_k \\
&= \frac{1}{2}(-h\sin\theta_{k+1})(2w_k - h\sin\theta_{k+1}) + 2\sin\left(\frac{hw_k}{2}\right)\sin\left(\theta_k + \frac{hw_k}{2}\right) \\
&= \frac{1}{2}h^2\sin^2\theta_{k+1} - hw_k\sin\theta_{k+1} + 2\sin\left(\frac{hw_k}{2}\right)\left(\sin\theta_k\cos\left(\frac{hw_k}{2}\right) + \cos\theta_k\sin\left(\frac{hw_k}{2}\right)\right) \\
&= \frac{1}{2}h^2\sin^2\theta_{k+1} + 2\sin^2\left(\frac{hw_k}{2}\right)\cos\theta_k + \sin\theta_k\sin(hw_k) - hw_k\sin(\theta_k + hw_k)
\end{aligned}
$$

因为

$$
\begin{aligned}
\sin\theta_k\sin(hw_k)-hw_k\sin(\theta_k+hw_k)&=\sin\theta_k\sin(hw_k)-hw_k\sin\theta_k\cos(hw_k)-hw_k\cos\theta_k\sin(hw_k)\\
&=-hw_k\cos\theta_k\sin(hw_k)+\sin\theta_k\cos(hw_k)\left(\tan(hw_k)-hw_k\right)\\
&=O(h^2)
\end{aligned}
$$

以及

$$
\frac{1}{2}h^2\sin^2\theta_k+2\sin^2\left(\frac{hw_k}{2}\right)\cos\theta_k=O(h^2)
$$

所以

$$
E_{k+1}=E_k+O(h^2)
$$

(c)因为

$$
\begin{aligned}
w_{k+1}&=w_k-h\theta_{k+1}\\
&=w_k-h(\theta_k+hw_k)\\
&=-h\theta_k+(1-h^2)w_k
\end{aligned}
$$

所以

$$
\begin{pmatrix}\theta_{k+1}\\ w_{k+1}\end{pmatrix}=\begin{pmatrix}1 & h\\ -h & 1-h^2\end{pmatrix}\begin{pmatrix}\theta_k\\ w_k\end{pmatrix}
$$

(d)

$$
\begin{aligned}
E_{k+1}&=w_{k+1}^2+hw_{k+1}\theta_{k+1}+\theta_{k+1}^2\\
&=\begin{pmatrix}\theta_{k+1}\\ w_{k+1}\end{pmatrix}^\top\begin{pmatrix}1 & \frac{1}{2}h\\ \frac{1}{2}h & 1\end{pmatrix}\begin{pmatrix}\theta_{k+1}\\ w_{k+1}\end{pmatrix}\\
&=\begin{pmatrix}\theta_k\\ w_k\end{pmatrix}^\top\begin{pmatrix}1 & -h\\ h & 1-h^2\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2}h\\ \frac{1}{2}h & 1\end{pmatrix}\begin{pmatrix}1 & h\\ -h & 1-h^2\end{pmatrix}\begin{pmatrix}\theta_k\\ w_k\end{pmatrix}\\
&=\begin{pmatrix}\theta_k\\ w_k\end{pmatrix}^\top\begin{pmatrix}1-\frac{1}{2}h^2 & -\frac{1}{2}h\\ \frac{3}{2}h-\frac{1}{2}h^3 & 1-\frac{1}{2}h^2\end{pmatrix}\begin{pmatrix}1 & h\\ -h & 1-h^2\end{pmatrix}\begin{pmatrix}\theta_k\\ w_k\end{pmatrix}\\
&=\begin{pmatrix}\theta_k\\ w_k\end{pmatrix}^\top\begin{pmatrix}1 & \frac{1}{2}h\\ \frac{1}{2}h & 1\end{pmatrix}\begin{pmatrix}\theta_k\\ w_k\end{pmatrix}\\
&=E_k
\end{aligned}
$$