1. Logistic Regression: Training stability

(a)在数据A上训练logistic regression model很快就收敛了，在数据B上训练logistic regression model无法收敛。

(b)观察后可以发现θ的模长来越大，回顾logistic regression model

$$h_\theta(x)=g(\theta^Tx),g(z)=1/(1+e^{-z}),g(z)'=g(z)(1-g(z))$$

当θ的模长很大时，θ^Tx的模长很大，$g(\theta^Tx)\to 0$，$g(z)'=g(z)(1-g(z))\to 0$，从而梯度会越来越小，训练会很慢。

之所以数据B发生这个现象而A没有发生这个现象，是因为数据A线性不可分，数据B线性可分。

由数据B线性可分可的

$$y_i\theta^Tx_i\ge 0$$

我们的目标函数为

$$J(\theta)=-\frac{1}{m}\sum_{i=1}^{m}\log(h_\theta(y^{(i)}x^{(i)}))$$

要使得使得目标函数越小，只要$h_\theta(y^{(i)}x^{(i)})$越大即可，由于$y_i\theta^Tx_i\ge 0$，所以θ的模长越大，$y_i\theta^Tx_i$就会越大，由梯度下降的性质可知，迭代之后会让θ的模长越来越大，就会发生上述现象。

而数据A不是线性可分的，所以存在j，使得

$$y_j\theta^Tx_j<0$$

所以算法不会让θ的模长不停地增加。

(c)要解决上述问题，关键是不能让θ的模长不停地增长，所以(iii),(v)是最好的方法。

(d)SVM不会发生这个问题，因为SVM是最大间隔分类器，即使可分，最大距离分类器也是唯一的，不会无限迭代下去。

而logistic回归实际上是在让函数间隔变大，所以会出现无法收敛的情形。

2.Model Calibration

(a)只要考虑两个分子即可，logistic回归的输出范围为$(0,1)$，题目中的$(a,b)=(0,1)$，所以

$$\sum_{i\in I_{a,b}}P(y^{(i)}=1|x^{(i)};\theta)=\sum_{i=1}^{m}P(y^{(i)}=1|x^{(i)};\theta)$$
$$\sum_{i\in I_{a,b}}1\{y^{(i)}=1\}=\sum_{i=1}^{m}1\{y^{(i)}=1\}$$

接下来证明这两项相等。

回顾损失函数

$$
J(\theta)=-\frac{1}{m}\sum_{i=1}^{m}\Big(y^{(i)}\log(h_\theta(x^{(i)}))+(1-y^{(i)})\log(1-h_\theta(x^{(i)}))\Big)
$$
$$
y^{(i)}\in\{0,1\},h_\theta(x)=g(\theta^Tx),g(z)=1/(1+e^{-z})
$$

回顾课本的讲义可得

$$
\frac{\partial}{\partial\theta_j}J(\theta)=-\frac{1}{m}\sum_{i=1}^{m}(y^{(i)}-h_\theta(x^{(i)}))x_j
$$

那么

$$
\nabla_\theta J(\theta)=-\frac{1}{m}X^TS
$$

其中

$$
x_k=[1,x_1^{(k)},\ldots,x_n^{(k)}]^T\in\mathbb{R}^{n+1}
$$
$$
X=\begin{bmatrix}x_1^T\\ \ldots\\ x_m^T\end{bmatrix}=\begin{bmatrix}1 & x_1^{(1)} & \ldots & x_n^{(1)}\\ \ldots & \ldots & \ldots & \ldots\\ 1 & x_1^{(m)} & \ldots & x_n^{(m)}\end{bmatrix}\in\mathbb{R}^{m\times(n+1)}
$$
$$
S=\begin{bmatrix}y^{(1)}-h_\theta(x^{(1)})\\ \ldots\\ y^{(m)}-h_\theta(x^{(m)})\end{bmatrix}\in\mathbb{R}^m
$$

由θ的选择规则可知

$$
X^TS=0
$$

这里有$n+1$个等式，注意X^T的第一行全为1，所以我们考虑第一个等式

$$
[1,\ldots,1]S=0
$$
$$
\sum_{i=1}^{m}y^{(i)}-h_\theta(x^{(i)})=0
$$
$$
\sum_{i=1}^{m}y^{(i)}=\sum_{i=1}^{m}h_\theta(x^{(i)})
$$

由于$y^{(i)}\in\{0,1\},h_\theta(x^{(i)})=P(y^{(i)}=1|x^{(i)};\theta)$，所以上式即为

$$
\sum_{i=1}^{m}P(y^{(i)}=1|x^{(i)};\theta)=\sum_{i=1}^{m}1\{y^{(i)}=1\}
$$

从而

$$\sum_{i\in I_{a,b}} P(y^{(i)}=1|x^{(i)};\theta)=\sum_{i\in I_{a,b}} 1\{y^{(i)}=1\}$$

命题得证。

(b)考虑两个数据的数据集$x^{(1)},x^{(2)}$，不妨设$y^{(1)}=1,y^{(2)}=0$，如果

$$P(y^{(1)}=1|x^{(1)};\theta)=0.4,P(y^{(2)}=1|x^{(2)};\theta)=0.6$$

那么我们预测$y^{(1)}=0,y^{(2)}=1$，准确率为0，但是

$$\sum_{i\in I_{a,b}} P(y^{(i)}=1|x^{(i)};\theta)=\sum_{i\in I_{a,b}} 1\{y^{(i)}=1\}=1$$

所以perfectly calibrated无法推出perfect accuracy。

反之，如果

$$P(y^{(1)}=1|x^{(1)};\theta)=0.6,P(y^{(2)}=1|x^{(2)};\theta)=0.3$$

那么我们预测$y^{(1)}=1,y^{(2)}=0$，此时准确率为1，但是

$$\sum_{i\in I_{a,b}} P(y^{(i)}=1|x^{(i)};\theta)=0.9\neq\sum_{i\in I_{a,b}} 1\{y^{(i)}=1\}=1$$

所以perfect accuracy无法推出perfectly calibrated。

(c)设损失函数为

$$J(\theta)=-\frac{1}{m}\sum_{i=1}^{m}\Big(y^{(i)}\log(h_\theta(x^{(i)}))+(1-y^{(i)})\log(1-h_\theta(x^{(i)}))\Big)+C\sum_{i=1}^{n+1}\theta_i^2$$

记

$$\theta=\begin{bmatrix}\theta_1\\ \ldots\\ \theta_{n+1}\end{bmatrix}$$

继续使用(a)的记号，那么

$$\begin{aligned}\nabla J(\theta)=-\frac{1}{m}X^TS+2C\theta=0\\ X^TS=2mC\theta\end{aligned}$$

依旧考虑第一个等式

$$
\begin{aligned}
[1,\ldots,1]S &= 2mC\theta_1 \\
\sum_{i=1}^{m} y^{(i)} - h_\theta(x^{(i)}) &= 2mC\theta_1 \\
\sum_{i=1}^{m} y^{(i)} &= \sum_{i=1}^{m} h_\theta(x^{(i)}) + 2mC\theta_1 \\
\sum_{i=1}^{m} 1\{y^{(i)} = 1\} &= \sum_{i=1}^{m} P(y^{(i)} = 1|x^{(i)};\theta) + 2mC\theta_1
\end{aligned}
$$

从而

$$
\sum_{i=1}^{m} 1\{y^{(i)} = 1\} = \sum_{i=1}^{m} P(y^{(i)} = 1|x^{(i)};\theta) + 2mC\theta_1
$$

3.Bayesian Logistic Regression and weight decay

回顾定义

$$
\begin{aligned}
\theta_{\mathrm{MAP}} &= \arg\max_\theta p(\theta) \prod_{i=1}^{m} p(y^{(i)}|x^{(i)},\theta) \\
&= \arg\max_\theta \exp(-\frac{||\theta||^2}{2\tau^2}) \prod_{i=1}^{m} p(y^{(i)}|x^{(i)},\theta)
\end{aligned}
$$

由定义可知

$$
\exp(-\frac{||\theta_{\mathrm{MAP}}||^2}{2\tau^2}) \prod_{i=1}^{m} p(y^{(i)}|x^{(i)},\theta_{\mathrm{MAP}}) \ge \exp(-\frac{||\theta_{\mathrm{ML}}||^2}{2\tau^2}) \prod_{i=1}^{m} p(y^{(i)}|x^{(i)},\theta_{\mathrm{ML}})
$$

因为

$$
\prod_{i=1}^{m} p(y^{(i)}|x^{(i)},\theta_{\mathrm{MAP}}) \le \prod_{i=1}^{m} p(y^{(i)}|x^{(i)},\theta_{\mathrm{ML}})
$$

所以

$$
\exp(-\frac{||\theta_{\mathrm{MAP}}||^2}{2\tau^2}) \ge \exp(-\frac{||\theta_{\mathrm{ML}}||^2}{2\tau^2})
$$

从而

$$
||\theta_{\mathrm{MAP}}||_2 \le ||\theta_{\mathrm{ML}}||_2
$$

4.Constructing kernels

假设K_i对应的矩阵为M_i，K对应矩阵为M，由核函数的定义可知M_i为半正定阵。

(a)$K(x,z)=K_1(x,z)+K_2(x,z)$是核，因为

$$
\begin{aligned}
x^TMx &= x^T(M_1+M_2)x \\
&= x^TM_1x + x^TM_2x \\
&\ge 0
\end{aligned}
$$

(b)$K(x,z)=K_1(x,z)-K_2(x,z)$不是核。取$K_2(x,z)=2K_1(x,z)$

$$
x^TMx = x^T(M_1-M_2)x = -x^TM_1x \le 0
$$

(c)$K(x,z)=aK_1(x,z), a>0$是核

$$
\begin{aligned}
x^TMx &= x^T(aM_1)x \\
&= ax^TMx \\
&\ge 0
\end{aligned}
$$

(d)$K(x,z)=-aK_1(x,z), a>0$不是核

$$
\begin{aligned}
x^TMx &= x^T(-aM_1)x \\
&= -ax^TMx \\
&\le 0
\end{aligned}
$$

(e)$K(x,z)=K_1(x,z)K_2(x,z)$是核

因为K_1,K_2为核，所以设$K_1(x,z)=\Phi_1(x)\Phi_1^T(z), K_2=\Phi_2(x)\Phi_2^T(z)$。

记$\Phi(x)$是$\Phi_1(x)\Phi_2^T(x)$每一行拼接而成的向量，设$\Phi_1(x),\Phi_2(x)\in\mathbb{R}^n$，给出以下记号

$$
\begin{gathered}
\Phi^i(x) = \Phi_1^i(x)\Phi_2^T(x) \in \mathbb{R}^{1\times n} \\
\Phi_1^i(x)\text{为}\ \Phi_1(x)\text{的 第 } i\text{个 分 量}
\end{gathered}
$$

那么

$$
\Phi(x) = \begin{bmatrix} \Phi^1(x) & \Phi^2(x) & \ldots & \Phi^n(x) \end{bmatrix} \in \mathbb{R}^{1\times n^2}
$$

接着计算$\Phi(x)\Phi^T(x')$，注意$\Phi^i(x)$为行向量

$$
\begin{aligned}
(\Phi(x)\Phi^T(x^{'})) &= \sum_{i=1}^{n}(\Phi^i(x))\Phi^i(x^{'})^T \\
&= \sum_{i=1}^{n}(\Phi_1^i(x)\Phi_2(x)^T)(\Phi_1^i(x^{'})\Phi_2(x^{'})^T)^T \\
&= \sum_{i=1}^{n}\Phi_1^i(x)\Phi_1^i(x^{'})\Phi_2(x)^T\Phi_2(x^{'}) \\
&= \sum_{i=1}^{n}\Phi_1^i(x)\Phi_1^i(x^{'})K_2(x,x^{'}) \\
&= K_2(x,x^{'})\sum_{i=1}^{n}\Phi_1^i(x)\Phi_1^i(x^{'}) \\
&= K_2(x,x^{'})K_1(x,x^{'})
\end{aligned}
$$

所以$\Phi(x)$对应的核为$K_1(x,x^{'})K_2(x,x^{'})$，从而$K(x,z)=K_1(x,z)K_2(x,z)$是核。

(f)$K(x,z)=f(x)f(z)$是核，因为符合定义。

(g)$K(x,z)=K_3(\phi(x),\phi(z))$是核，因为

$$
y^TMy=y^TM_3y\ge 0
$$

(h)由(e)可知，如果K_1是核，那么$K_1^i(i\ge 1,i\in N)$也是核，又由(a)(c)可得核函数的正系数的线性组合为核，所以$K(x,z)=p(K_1(x,z))$也是核。

5.Kernelizing the Perceptron

设这里的数据为$x^{(1)},\ldots,x^{(m)}$

(a)根据更新公式

$$
\theta^{(i+1)}:=\theta^{(i)}+\alpha 1\{g({\theta^{(i)}}^T\phi(x^{(i+1)}))y^{(i+1)}<0\}y^{(i+1)}\phi(x^{(i+1)})
$$

如果初始化$\theta^{(0)}=0$，那么$\theta^{(i)}$可以表示为$\phi(x^{(i)})$的线性组合，从而

$$
\theta^{(i)}=\sum_{j=1}^{i}\beta_j\phi(x^{(j)})
$$

(b)计算$g({\theta^{(i)}}^T\phi(x^{(i+1)}))$

$$
g({\theta^{(i)}}^T\phi(x^{(i+1)}))=g\Big(\sum_{j=1}^{i}\beta_j\phi(x^{(j)})^T\phi(x^{(i+1)})\Big)=g\Big(\sum_{j=1}^{i}\beta_jM_{j,i+1}\Big)
$$

(c)由上述公式可知，第i次我们只要更新β_i即可，更新公式如下

$$
\beta_{i+1}=\alpha 1\{g({\theta^{(i)}}^T\phi(x^{(i+1)}))y^{(i+1)}<0\}y^{(i+1)}
$$

6.Spam classification

(a)(b)(c)

代码的注释比较详细，这里只说明一点，在朴素贝叶斯中我们需要计算：

$$\prod_{i=1}^{m}\Big(\prod_{j=1}^{n_i}p(x_j^{(i)}|y^{(i)};\phi_{k|y=0},\phi_{k|y=1})p(y^{(i)};\phi_y)\Big)$$

题目中计算的对数概率，这是为了防止数值过小变为0

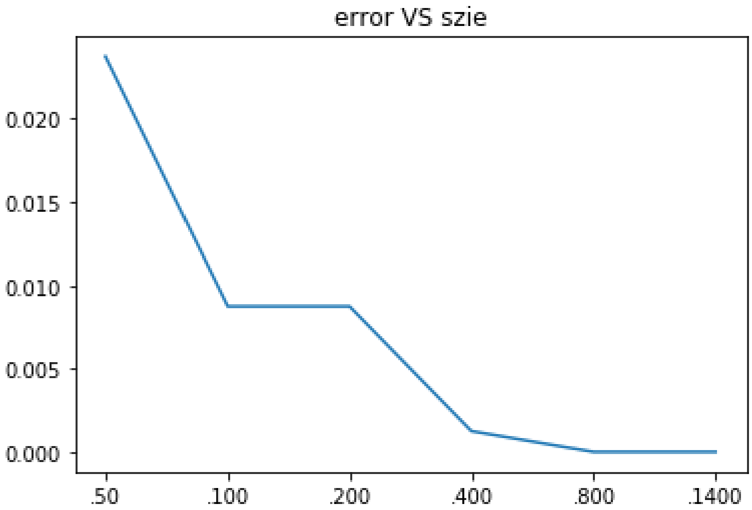
$$\sum_{i=1}^{n}\Big(\sum_{j=1}^{n_i}\log p(x_j^{(i)}|y^{(i)};\phi_{k|y=0},\phi_{k|y=1})+\log p(y^{(i)};\phi_y)\Big)$$

```
# -*- coding: utf-8 -*-
"""
Created on Thu Mar 14 15:58:26 2019

@author: qinzhen
"""

import numpy as np
import matplotlib.pyplot as plt

def readMatrix(file):
    fd = open(file, 'r')
    #读取第一行
    hdr = fd.readline()
    #读取行列
    rows, cols = [int(s) for s in fd.readline().strip().split()]
    #读取单词
    tokens = fd.readline().strip().split()
    #构造空矩阵
    matrix = np.zeros((rows, cols))
    Y = []
    #line为每行的元素
    for i, line in enumerate(fd):
        nums = [int(x) for x in line.strip().split()]
        #第一个元素表示是否为垃圾邮件
        Y.append(nums[0])
        #将后续数据读入
        kv = np.array(nums[1:])
        #从第一个开始每两个累加
        k = np.cumsum(kv[:-1:2])
        #从第二个开始每隔一个取出
        v = kv[1::2]
        #这里应该是一种特殊的存储格式，我们直接使用即可
        matrix[i, k] = v
    return matrix, tokens, np.array(Y)

def nb_train(matrix, category):
    #matrix(i,j)表示第i个邮件中第j个元素出现了几次
```

```
    state = {}
    #token的数量
    N = matrix.shape[1]
    #邮件数量
    M = matrix.shape[0]
    ##################

    #垃圾邮件的数量
    y1 = matrix[category==1]
    n1 = np.sum(y1)
    #非垃圾邮件的数量
    y0 = matrix[category==0]
    n0 = np.sum(y0)

    #P(y=1)
    p1 = y1.shape[0] / M
    #P(y=0)
    p0 = y0.shape[0] / M
    state[-1] = [p0, p1]

    for i in range(N):
        #找到第i个token
        #垃圾邮件中第i个token出现的数量
        s1 = matrix[category==1][:, i]
        #拉普拉斯平滑
        u1 = (s1.sum() + 1) / (n1 + N)
        #非垃圾邮件中第i个token出现的数量
        s0 = matrix[category==0][:, i]
        #拉普拉斯平滑
        u0 = (s0.sum() + 1) / (n0 + N)
        #存入字典
        state[i] = [u0, u1]

    ##################
    return state

def nb_test(matrix, state):
    output = np.zeros(matrix.shape[0])
    ##################
    #邮件数量
    M = matrix.shape[0]
    #token的数量
    N = matrix.shape[1]
    for i in range(M):
        #第i个邮件
        vector = matrix[i]
        s1 = np.log(state[-1][1])
        s0 = np.log(state[-1][0])

        for j in range(N):
            #对第j个token的对数概率做累加
            s1 += vector[j] * np.log(state[j][1])
            s0 += vector[j] * np.log(state[j][0])
```

```
        if s1 > s0:
            output[i] = 1

    ###################
    return output

def evaluate(output, label):
    error = (output != label).sum() * 1. / len(output)
    print('Error: %1.4f' % error)
    return error

def nb(file):
    trainMatrix, tokenlist, trainCategory = readMatrix(file)
    testMatrix, tokenlist, testCategory = readMatrix('MATRIX.TEST')

    state = nb_train(trainMatrix, trainCategory)
    output = nb_test(testMatrix, state)

    return evaluate(output, testCategory)

trainMatrix, tokenlist, trainCategory = readMatrix('MATRIX.TRAIN')
testMatrix, tokenlist, testCategory = readMatrix('MATRIX.TEST')

state = nb_train(trainMatrix, trainCategory)
output = nb_test(testMatrix, state)

evaluate(output, testCategory)

#problem b
b=[]
for i in range(1448):
    b.append((i,np.log(state[i][1])-np.log(state[i][0])))

b.sort(key=lambda i:i[-1],reverse=True)
key = b[:5]

word = []
for i in key:
    word.append(tokenlist[i[0]])

print(word)

#problem c
size = ['.50','.100','.200','.400','.800','.1400']
size1 = [50, 100, 200, 400, 800, 1400]
train = "MATRIX.TRAIN"
error = []
for i in size:
    file = train+i
    error.append(nb(file))

plt.plot(size, error)
plt.title("error vs szie")
```

```
Error: 0.0025
Error: 0.0238
Error: 0.0088
Error: 0.0088
Error: 0.0013
Error: 0.0000
Error: 0.0000
```



(d)这部分老师已经提供，但这里还是详细解释下。

首先这题需要计算高斯核矩阵，所以我们需要计算$[||x^{(i)}-x^{(j)}||^2]_{i,j}$，下面介绍向量化计算的方法：

假设

$$X=\begin{bmatrix}—(x^{(1)})^T— \\ —(x^{(2)})^T— \\ \vdots \\ —(x^{(m)})^T—\end{bmatrix}\in\mathbb{R}^{m\times d},Y=\begin{bmatrix}—(y^{(1)})^T— \\ —(y^{(2)})^T— \\ \vdots \\ —(y^{(n)})^T—\end{bmatrix}\in\mathbb{R}^{n\times d}$$

其中$x^{(i)},y^{(i)}\in\mathbb{R}^d$，现在的问题是如何高效计算矩阵$D\in\mathbb{R}^{m\times n}$，其中

$$D_{i,j}=||x^{(i)}-y^{(j)}||^2$$

首先对$D_{i,j}$进行处理

$$\begin{aligned}D_{i,j}&=||x^{(i)}-y^{(j)}||^2\\&=(x^{(i)}-y^{(j)})^T(x^{(i)}-y^{(j)})\\&=(x^{(i)})^Tx^{(i)}-2(x^{(i)})^Ty^{(j)}+(y^{(j)})^Ty^{(j)}\end{aligned}$$

那么

$$
\begin{aligned}
D &= \begin{bmatrix} D_{1,1} & \ldots & D_{1,n} \\ \ldots & \ldots & \ldots \\ D_{m,1} & \ldots & D_{m,n} \end{bmatrix} \\
&= \begin{bmatrix} (x^{(1)})^T x^{(1)} - 2(x^{(1)})^T y^{(1)} + (y^{(1)})^T y^{(1)} & \ldots & (x^{(1)})^T x^{(1)} - 2(x^{(1)})^T y^{(n)} + (y^{(n)})^T y^{(n)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^T x^{(m)} - 2(x^{(m)})^T y^{(1)} + (y^{(1)})^T y^{(1)} & \ldots & (x^{(m)})^T x^{(m)} - 2(x^{(m)})^T y^{(n)} + (y^{(n)})^T y^{(n)} \end{bmatrix} \\
&= \begin{bmatrix} (x^{(1)})^T x^{(1)} & \ldots & (x^{(1)})^T x^{(1)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^T x^{(m)} & \ldots & (x^{(m)})^T x^{(m)} \end{bmatrix} + \begin{bmatrix} (y^{(1)})^T y^{(1)} & \ldots & (y^{(n)})^T y^{(n)} \\ \ldots & \ldots & \ldots \\ (y^{(1)})^T y^{(1)} & \ldots & (y^{(n)})^T y^{(n)} \end{bmatrix} - 2 \begin{bmatrix} (x^{(1)})^T y^{(1)} & \ldots & (x^{(1)})^T y^{(n)} \\ \ldots & \ldots & \ldots \\ (x^{(m)})^T y^{(1)} & \ldots & (x^{(m)})^T y^{(n)} \end{bmatrix} \\
&= \begin{bmatrix} (x^{(1)})^T x^{(1)} \\ \ldots \\ (x^{(m)})^T x^{(m)} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}}_{1\times n\text{矩阵}} + \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{m\times 1\text{矩阵}} \begin{bmatrix} (y^{(1)})^T y^{(1)} & \ldots & (y^{(n)})^T y^{(n)} \end{bmatrix} - 2XY^T
\end{aligned}
$$

特别的，这里$X=Y$，所以上述代码如下：

```
d1 = np.sum(matrix ** 2, axis=1).reshape(-1, 1)
d2 = d1.T
dist = d1 + d2 - 2 * matrix.dot(matrix.T)
```

带入高斯核的计算公式可得：

```
k = np.exp(- dist / (2 * (tau ** 2)))
```

老师给出的代码为：

```
gram = matrix.dot(matrix.T)
squared = np.sum(matrix*matrix, axis=1)
k = np.exp(-(squared.reshape((-1,1)) + squared.reshape((1,-1)) - 2 * gram) / (2 * (tau ** 2)))
```

为了解释剩余代码，首先介绍SVM对应的合页损失函数（参考统计学习方法113页）：

$$\sum_{i=1}^{N}[1-y^{(i)}(w^Tx^{(i)}+b)]_+ + \lambda||w||^2$$

其中

$$[x]_+ = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$

假设SVM对应的特征转换为$\phi(x)$，那么由课上的讨论可知，w的形式如下：

$$w = \sum_{j=1}^{N}\alpha_j\phi(x^{(j)}) \tag{1}$$

与之对应的损失函数为

$$
\begin{aligned}
\sum_{i=1}^{N}[1-y^{(i)}(w^T\phi(x^{(i)})+b)]_+ +\lambda||w||^2 &= \sum_{i=1}^{N}\Big[1-y^{(i)}\Big(\big(\sum_{j=1}^{N}\alpha_j\phi(x^{(j)})\big)^T\phi(x^{(i)})+b\Big)\Big]_+ \\
&+\lambda\Big(\sum_{j=1}^{N}\alpha_j\phi(x^{(j)})\Big)^T\Big(\sum_{j=1}^{N}\alpha_j\phi(x^{(j)})\Big) \\
&= \sum_{i=1}^{N}\Big[1-y^{(i)}\Big(\sum_{j=1}^{N}\alpha_j\phi^T(x^{(j)})\phi(x^{(i)})+b\Big)\Big]_+ \\
&+\lambda\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j\phi^T(x^{(i)})\phi(x^{(j)})
\end{aligned}
$$

由(1)可知，我们只要求出$\alpha=(\alpha_1,\ldots,\alpha_N)^T$即可，所以可以关于$\alpha$做梯度随机梯度下降法，注意一个样本（不妨设为$x^{(i)}$）对应的损失函数为

$$
L_i \triangleq \Big[1-y^{(i)}\Big(\sum_{j=1}^{N}\alpha_j\phi^T(x^{(j)})\phi(x^{(i)})+b\Big)\Big]_+ +\frac{\lambda}{N}\sum_{s=1}^{N}\sum_{t=1}^{N}\alpha_s\alpha_t\phi^T(x^{(s)})\phi(x^{(t)})
$$

记核矩阵为K，对上式关于α_k求偏导可得

$$
\begin{aligned}
\frac{\partial L}{\partial \alpha_k} &= 1\{[1-y^{(i)}(\sum_{j=1}^{N}\alpha_j\phi^T(x^{(j)})\phi(x^{(i)})+b)]>0\}\times\Big(-y^{(i)}\phi^T(x^{(k)})\phi(x^{(i)})\Big)+2\frac{\lambda}{N}\Big(\sum_{j=1}^{N}\alpha_j\phi^T(x^{(j)})\phi(x^{(k)})\Big) \\
&= 1\{[1-y^{(i)}(w^T\phi(x^{(i)})+b)]>0\}\times\Big(-y^{(i)}\phi^T(x^{(k)})\phi(x^{(i)})\Big)+2\frac{\lambda}{N}\Big(\phi(x^{(i)})w^T\Big) \\
&= 1\{[1-y^{(i)}(w^T\phi(x^{(i)})+b)]>0\}\times\Big(-y^{(i)}K_{k,i}\Big)+2\frac{\lambda}{N}\Big(\phi(x^{(i)})w^T\Big)
\end{aligned}
$$

因此梯度为

$$
\nabla_\alpha L = 1\{[1-y^{(i)}(w^T\phi(x^{(i)})+b)]>0\}(-y^{(i)})\begin{bmatrix}K_{1,i}\\ \vdots \\ K_{N,i}\end{bmatrix}+\frac{2\lambda}{N}K\alpha
$$

所以计算梯度的过程如下，首先随机选个更新的样本（M为样本的数量）：

```
i = int(np.random.rand() * M)
```

接着计算函数间隔（k为之前计算的核矩阵）：

```
margin = category[i] * (k[i, :].dot(alpha))
```

然后计算正则项的梯度，注意这里我和老师给的计算式不大一样，老师给的式子如下：

```
grad = M * L * k[:, i] * alpha[i]
```

我的计算式如下：

```
grad = L * k.dot(alpha)
```

这里系数没有那么重要，实验结果表明两种方法的效果都尚可。

最后根据函数间隔是否大于1决定是否更新前一项的梯度：

```
if(margin < 1):
    grad -= category[i] * k[:, i]
```

代码中的梯度除以了迭代次数j的一个函数，这是为了让梯度逐渐变小：

```
alpha -= grad / ((np.sqrt(j+1)))
```

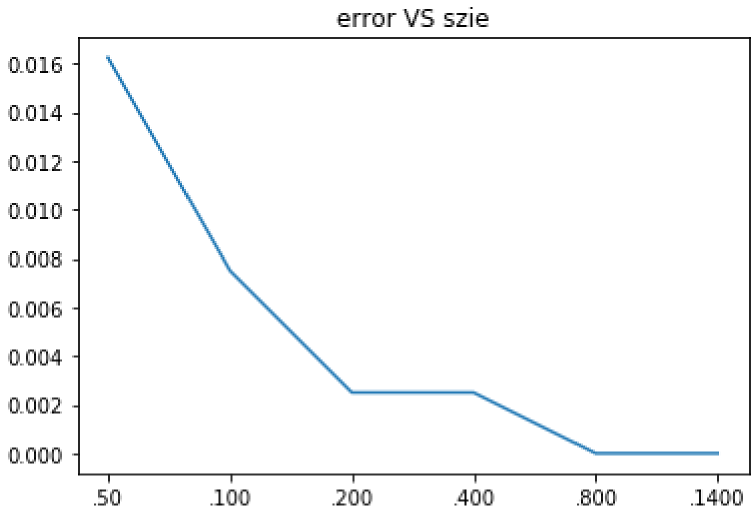
完整代码如下：

```
# -*- coding: utf-8 -*-
"""
Created on Thu Mar 14 16:04:08 2019

@author: qinzhen
"""

import numpy as np
import matplotlib.pyplot as plt

tau = 8.

def readMatrix(file):
    fd = open(file, 'r')
    hdr = fd.readline()
    rows, cols = [int(s) for s in fd.readline().strip().split()]
    tokens = fd.readline().strip().split()
    matrix = np.zeros((rows, cols))
    Y = []
    for i, line in enumerate(fd):
        nums = [int(x) for x in line.strip().split()]
        Y.append(nums[0])
        kv = np.array(nums[1:])
        k = np.cumsum(kv[:-1:2])
        v = kv[1::2]
        matrix[i, k] = v
    #化为正负1
    category = (np.array(Y) * 2) - 1
    return matrix, tokens, category

def svm_train(matrix, category):
    state = {}
    M, N = matrix.shape
    #####################
```

```
    #大于0的化为1
    matrix = 1.0 * (matrix > 0)
    '''
    #构造kernel矩阵
    d1 = np.sum(matrix ** 2, axis=1).reshape(-1, 1)
    d2 = d1.T
    squared = matrix.dot(matrix.T)
    dist = d1 + d2 - 2 * squared
    k = np.exp(- dist / (2 * (tau ** 2)))
    '''
    gram = matrix.dot(matrix.T)
    squared = np.sum(matrix*matrix, axis=1)
    k = np.exp(-(squared.reshape((-1,1)) + squared.reshape((1,-1)) - 2 * gram) / (2 * (tau
** 2)))

    #初始化向量
    alpha = np.zeros(M)
    #循环次数
    n = 40
    #系数
    L = 1. / (64 * M)
    #平均值
    alpha_avg = np.zeros(M)

    for j in range(n * M):
        #随机取一个样本
        i = int(np.random.rand() * M)
        #计算函数间隔
        margin = category[i] * (k[i, :].dot(alpha))
        #grad = M * L * k[:, i] * alpha[i]
        grad = L * k.dot(alpha)
        if(margin < 1):
            grad -= category[i] * k[:, i]
        alpha -= grad / ((np.sqrt(j+1)))
        alpha_avg += alpha

    alpha_avg /= (n * M)

    state['alpha'] = alpha
    state['alpha_avg'] = alpha_avg
    state['Xtrain'] = matrix
    state['Sqtrain'] = squared

    ###################
    return state


def svm_test(matrix, state):
    M, N = matrix.shape
    output = np.zeros(M)
    ###################
    Xtrain = state['Xtrain']
    Sqtrain = state['Sqtrain']
```

```
    #大于0的化为1
    matrix = 1.0 * (matrix > 0)
    #做测试集的kernel
    gram = matrix.dot(Xtrain.T)
    squared = np.sum(matrix * matrix, axis=1)
    k = np.exp(-(squared.reshape((-1, 1)) + Sqtrain.reshape((1, -1)) - 2 * gram) / (2 *
(tau**2)))
    #读取alpha
    alpha_avg = state['alpha_avg']
    #预测
    pred = k.dot(alpha_avg)
    output = np.sign(pred)

    ###################
    return output

def evaluate(output, label):
    error = (output != label).sum() * 1. / len(output)
    print('Error: %1.4f' % error)
    return error

def svm(file):
    trainMatrix, tokenlist, trainCategory = readMatrix(file)
    testMatrix, tokenlist, testCategory = readMatrix('MATRIX.TEST')

    state = svm_train(trainMatrix, trainCategory)
    output = svm_test(testMatrix, state)

    return evaluate(output, testCategory)


trainMatrix, tokenlist, trainCategory = readMatrix('MATRIX.TRAIN.400')
testMatrix, tokenlist, testCategory = readMatrix('MATRIX.TEST')

state = svm_train(trainMatrix, trainCategory)
output = svm_test(testMatrix, state)

evaluate(output, testCategory)

size = ['.50','.100','.200','.400','.800','.1400']
size1 = [50, 100, 200, 400, 800, 1400]
train = "MATRIX.TRAIN"
error = []
for i in size:
    file = train+i
    error.append(svm(file))

plt.plot(size, error)
plt.title("error VS szie")
```

```
Error: 0.0025
Error: 0.0163
Error: 0.0075
Error: 0.0025
Error: 0.0025
Error: 0.0000
Error: 0.0000
```



(e)对比两图可以发现，和NB算法相比，SVM算法的效果更好。