Problem 1

(a)记

$$S=\sum_{n=p}^{q} w^n$$

那么

$$wS=\sum_{n=p}^{q} w^{n+1}=\sum_{n=p+1}^{q+1} w^n$$

相减得到

$$\begin{aligned}(1-w)S=\sum_{n=p}^{q} w^n-\sum_{n=p+1}^{q+1} w^n=w^p-w^{q+1}\\ S=\frac{w^p-w^{q+1}}{1-w}\end{aligned}$$

接着讨论级数收敛的问题。

如果$p=-\infty, q<\infty$，那么当$|w|>1$时，级数收敛，并且

$$S=\frac{-w^{q+1}}{1-w}=\frac{w^{q+1}}{w-1}$$

如果$q=\infty, p>-\infty$，那么当$|w|<1$时，级数收敛，并且

$$S=\frac{w^p}{1-w}$$

如果$q=\infty, p=-\infty$，那么由之前讨论可得，该级数必然发散。

最后将收敛情形总结如下：

$$\sum_{n=p}^{q} w^n=\begin{cases}\frac{w^{q+1}}{w-1} & |w|>1, p=-\infty, q<\infty\\ \frac{w^p}{1-w} & |w|<1, p>-\infty, q=\infty\\ \frac{w^p-w^{q+1}}{1-w} & w\neq 1, -\infty<p, q<\infty\end{cases}$$

(b)对上式中取

$$\begin{aligned}w&=e^{2\pi in/N}\\ p&=0\\ q&=N-1\end{aligned}$$

得到

$$
\begin{aligned}
\sum_{n=0}^{N-1} e^{2\pi in/N} &= \frac{1-e^{2\pi in}}{1-e^{2\pi in/N}} \\
&= 0
\end{aligned}
$$

注意$e^{2\pi in/N}$为1的N次单位根，所以上式的几何解释为1的N次单位根的重心为原点。

(c)对上式中取

$$
\begin{aligned}
w &= e^{2\pi it} \\
p &= -N \\
q &= N
\end{aligned}
$$

得到

$$
\begin{aligned}
\sum_{k=-N}^{N} e^{2\pi ikt} &= \frac{e^{-2\pi iNt}-e^{2\pi i(N+1)t}}{1-e^{2\pi it}} \\
&= \frac{(e^{-2\pi iNt}-e^{2\pi i(N+1)t})(1-e^{-2\pi it})}{(1-e^{2\pi it})(1-e^{-2\pi it})} \\
&= \frac{e^{-2\pi iNt}-e^{2\pi i(N+1)t}-e^{-2\pi i(N+1)t}+e^{2\pi iNt}}{1-e^{2\pi it}-e^{-2\pi it}+1} \\
&= \frac{2\cos(2\pi Nt)-2\cos(2\pi(N+1)t)}{2-2\cos(2\pi t)} \\
&= \frac{\cos(2\pi Nt)-\cos(2\pi(N+1)t)}{1-\cos(2\pi t)} \\
&= \frac{2\sin(2\pi t(N+1/2))\sin(\pi t)}{2\sin^2(\pi t)} \\
&= \frac{\sin(2\pi t(N+1/2))}{\sin(\pi t)}
\end{aligned}
$$

Problem 2

(a)直线段必然为两部分的叠加，所以方程为

$$
\Lambda_2(t)+\Lambda_2(t-2)
$$

(b)依然由直线段为两部分的叠加，得到方程为

$$
2\Lambda_2(t)+2\Lambda_2(t-3)
$$

(c)区间$[1,3]$为第一个波的左半部分，根据此确定第二个波即可：

$$
6\Lambda_2(t-3)+3\Lambda_2(t-5)
$$

(d)类似上一题得到

$$y_2\Lambda_{x_2-x_1}(t-x_2)+\frac{y_2}{2}\Lambda_{\frac{x_3-x_2}{2}}\left(t-\frac{x_2+x_3}{2}\right)$$

由几何关系，我们得到如下约束

$$\begin{aligned}x_2-x_1&=\frac{x_3-x_2}{2}\\x_3-3x_2+x_1&=0\end{aligned}$$

Problem 3

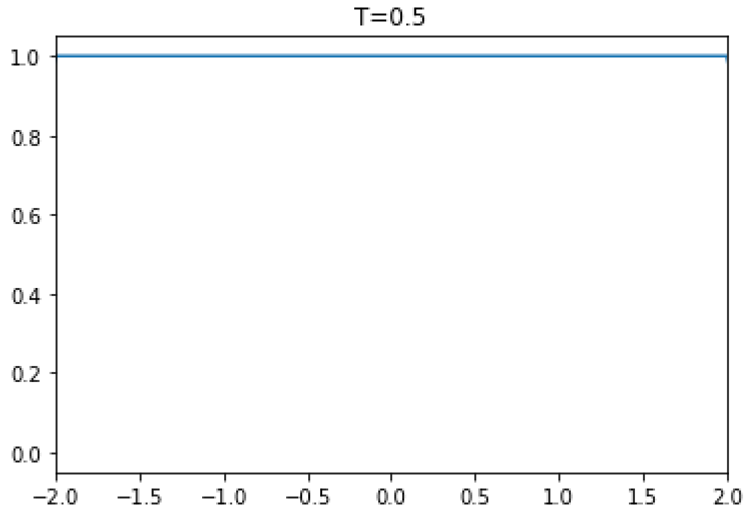
(a)

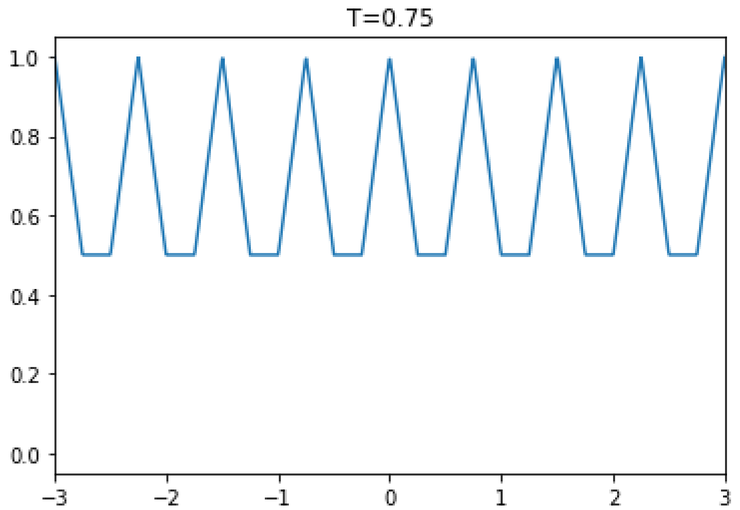
$$\begin{aligned}g(t+T)&=\sum_{n=-\infty}^{\infty}f(t+T-nT)\\&=\sum_{n=-\infty}^{\infty}f\left(t-(n-1)T\right)\\&=\sum_{m=-\infty}^{\infty}f(t-mT)\qquad\qquad m=n-1\\&=g(t)\end{aligned}$$

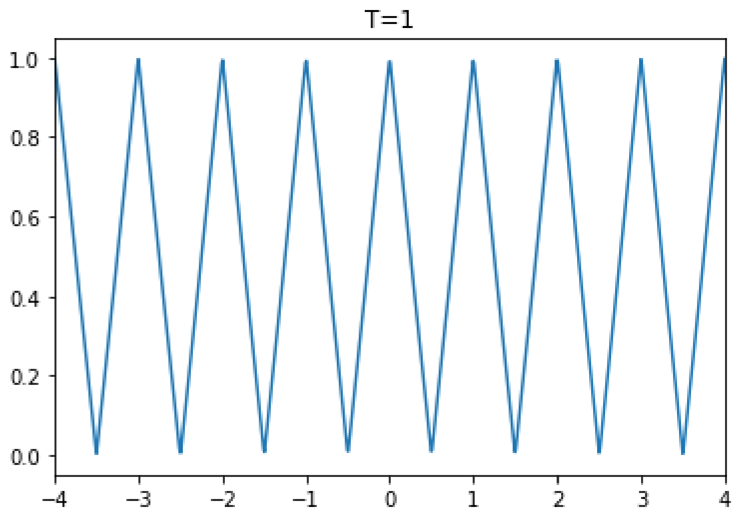
(b)

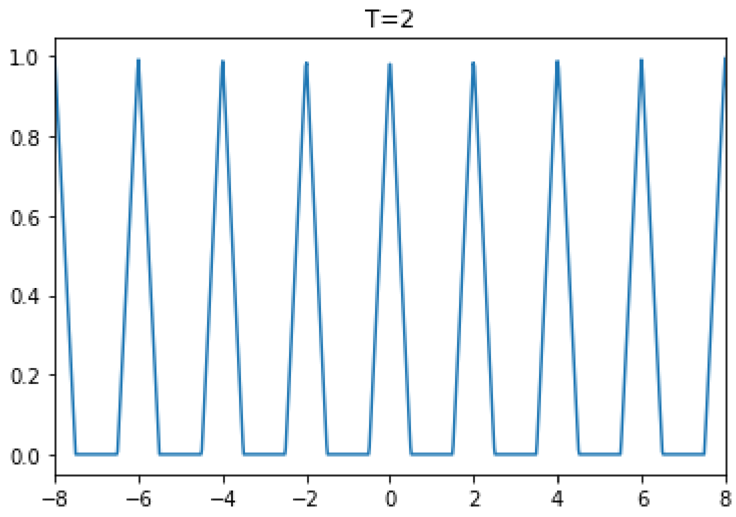
```
import numpy as np
import matplotlib.pyplot as plt

def Lambda(t, a):
    t1 = np.abs(t)
    r = 1 - t1 / a
    index = (t1 > a)
    r[index] = 0

    return r

def f(T, n=5):
    #10个周期的点
    t = np.linspace(-n * T, n * T, 1000)
    r = Lambda(t, 1 / 2)
    for i in range(1, n):
        r1 = Lambda(t - i * T, 1 / 2)
        r2 = Lambda(t + i * T, 1 / 2)
        r += r1
        r += r2
    plt.plot(t, r)
    plt.title("T={}".format(T))
    plt.xlim(-(n-1) * T, (n-1) * T)
    plt.show()
```

```
T = [1/2, 3/4, 1, 2]
for t in T:
    f(t)
```









(c)如果$f(t)=0$，那么结论成立；否则由$f(t)$在任意一点有定义，累加项$\sum_{n=-\infty}^{\infty} f(t-nT)$必然不收敛，所以

$$g(t) \neq f(t)$$

Problem 4

符号解释：gcd表示最大公约数，lcm表示最小公倍数。

(a)因为

$$\begin{aligned}
f(x+1) &= \sin(2\pi m(x+1)) + \sin(2\pi n(x+1)) \\
&= \sin(2\pi mx + 2\pi m) + \sin(2\pi nx + 2\pi n) \\
&= \sin(2\pi mx) + \sin(2\pi nx) \\
&= f(x)
\end{aligned}$$

所以1是$f(x)$的周期。接着求最小正周期，假设T为周期，那么

$$\begin{aligned}
f(x+T) &= \sin(2\pi m(x+T)) + \sin(2\pi n(x+T)) \\
&= \sin(2\pi mx + 2\pi mT) + \sin(2\pi nx + 2\pi nT) \\
&= \sin(2\pi mx) + \sin(2\pi nx)
\end{aligned}$$

要使得最后一个等号成立，那么必然要有mT为整数，nT为整数，所以必然有T为有理数，因此不妨设

$$T = \frac{t_1}{t_2}, \text{gcd}(t_1, t_2) = 1$$

那么

$$\frac{t_1 m}{t_2}, \frac{t_1 n}{t_2} \in \mathbb{Z}$$

所以必然有

$$t_2 | m, t_2 | n \Rightarrow t_2 | \text{gcd}(m, n)$$

设

$$\gcd(m,n)=t_2t_3$$

那么

$$T=\frac{t_1t_3}{\gcd(m,n)}\triangleq\frac{k}{\gcd(m,n)},k\in\mathbb{N}^+$$

所以最小正周期为

$$T=\frac{1}{\gcd(m,n)}$$

(b)因为

$$\begin{aligned}g(x+rs)&=\sin(2\pi p(x+rs))+\sin(2\pi q(x+rs))\\&=\sin\left(2\pi px+2\pi\frac{m}{r}rs\right)+\sin\left(2\pi qx+2\pi\frac{n}{s}rs\right)\\&=\sin(2\pi px+2\pi ms)+\sin(2\pi qx+2\pi nr)\\&=\sin(2\pi px)+\sin(2\pi qx)\end{aligned}$$

所以$g(x)$是周期函数。接着求最小正周期，假设T为周期，那么

$$\begin{aligned}g(x+rs)&=\sin(2\pi p(x+T))+\sin(2\pi q(x+T))\\&=\sin\left(2\pi px+2\pi\frac{m}{r}T\right)+\sin\left(2\pi qx+2\pi\frac{n}{s}T\right)\\&=\sin(2\pi px)+\sin(2\pi qx)\end{aligned}$$

要使得最后一个等号成立，那么必然要有mT为整数，nT为整数，所以T为有理数，因此不妨设

$$T=\frac{t_1}{t_2},\gcd(t_1,t_2)=1$$

那么

$$\begin{aligned}\frac{m}{r}T&=\frac{mt_1}{rt_2}\in\mathbb{Z}\\\frac{n}{s}T&=\frac{nt_1}{st_2}\in\mathbb{Z}\end{aligned}$$

所以

$$\begin{aligned}t_2|m,r|t_1\\t_2|n,s|t_1\end{aligned}$$

因此

$$t_2|\gcd(m,n),\operatorname{lcm}(r,s)|t_1$$

设

$$t_1=a_1\operatorname{lcm}(r,s),\gcd(m,n)=t_2a_2$$

那么

$$T=\frac{a_1a_2\text{lcm}(r,s)}{\text{gcd}(m,n)}\triangleq k\frac{\text{lcm}(r,s)}{\text{gcd}(m,n)},k\in\mathbb{N}^+$$

所以最小正周期为

$$T=\frac{\text{lcm}(r,s)}{\text{gcd}(m,n)}$$

(c)反证法，假设$f(t)$是周期函数，并且周期为T，那么

$$\begin{aligned}f(t+T)&=\cos(t+T)+\cos(\sqrt{2}(t+T))\\&=\cos(t)+\cos(\sqrt{2}t)\\&=f(t)\end{aligned}$$

令$t=0$，那么

$$\cos(T)+\cos(\sqrt{2}T)=2$$

那么必然有

$$\begin{aligned}T&=2k_1\pi.k_1\in\mathbb{Z}\\\sqrt{2}T&=2k_2\pi.k_2\in\mathbb{Z}\end{aligned}$$

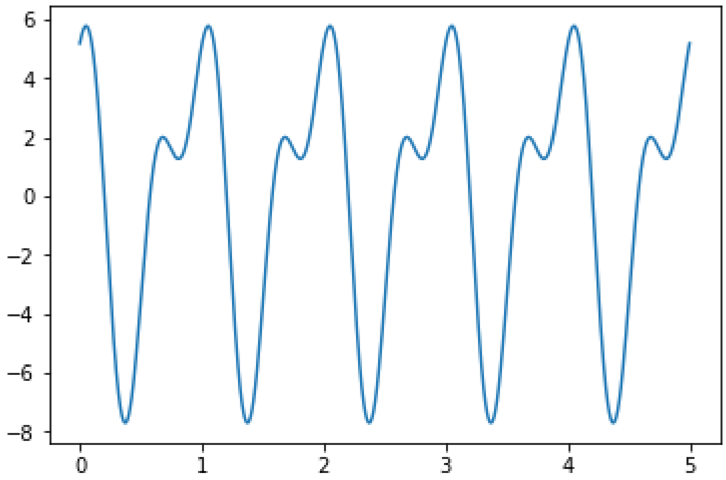
相除得到

$$\sqrt{2}=\frac{k_2}{k_1}$$

这就与$\sqrt{2}$是无理数产生了矛盾。

(d)

```
import numpy as np
import matplotlib.pyplot as plt

#参数
nu1 = 2
nu2 = 1
start = 0
end = 5
step = 0.0001
t = np.arange(start, end, step)
v = 3 * np.cos(2 * np.pi * nu1 * t - 1.3) + 5 * np.cos(2 * np.pi * nu2 * t + 0.5)

plt.plot(t, v)
plt.show()
print(np.max(v))
```




```
5.781103145757635
```

Problem 5

(a)

$$\begin{aligned}(f^-, g^-) &= \int_{-\infty}^{\infty} f(-t)\overline{g(-t)}dt \\ &\overset{t=-t}{=} \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt \\ &= (f, g)\end{aligned}$$

(b)

$$\begin{aligned}(f, g^-) &= \int_{-\infty}^{\infty} f(t)\overline{g(-t)}dt \\ &\overset{t=-t}{=} \int_{-\infty}^{\infty} f(-t)\overline{g(t)}dt \\ &= (f^-, g)\end{aligned}$$

(c)

$$\begin{aligned}(\tau_a f, \tau_a g) &= \int_{-\infty}^{\infty} f(t-a)\overline{g(t-a)}dt \\ &\overset{t=t-a}{=} \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt \\ &= (f, g)\end{aligned}$$

(d)

$$
\begin{aligned}
(\tau_a f, g) &= \int_{-\infty}^{\infty} f(t-a)\overline{g(t)}dt \\
&\overset{t=t-a}{=} \int_{-\infty}^{\infty} f(t)\overline{g(t+a)}dt \\
&= (f, \tau_{-a} g)
\end{aligned}
$$

(e)

$$
\begin{aligned}
(\tau_a f, \tau_b g) &= \int_{-\infty}^{\infty} f(t-a)\overline{g(t-b)}dt \\
&\overset{t=t-a}{=} \int_{-\infty}^{\infty} f(t)\overline{g(t+a-b)}dt \\
&= (f, \tau_{b-a} g) \\
&\overset{t=t-b}{=} \int_{-\infty}^{\infty} f(t+b-a)\overline{g(t)}dt \\
&= (\tau_{a-b} f, g)
\end{aligned}
$$

(f)重新计算之前结论。

首先如果$h(x)$周期为1，那么

$$
\int_a^{a+1} h(t)dt = \int_0^1 h(t)dt
$$

(a)

$$
\begin{aligned}
(f^-, g^-) &= \int_0^1 f(-t)\overline{g(-t)}dt \\
&\overset{t=-t}{=} \int_{-1}^{0} f(t)\overline{g(t)}dt \\
&= \int_0^1 f(t)\overline{g(t)}dt && \text{由周期性} \\
&= (f, g)
\end{aligned}
$$

(b)

$$
\begin{aligned}
(f, g^-) &= \int_0^1 f(t)\overline{g(-t)}dt \\
&\overset{t=-t}{=} \int_{-1}^{0} f(-t)\overline{g(t)}dt \\
&= \int_0^1 f(-t)\overline{g(t)}dt && \text{由周期性} \\
&= (f^-, g)
\end{aligned}
$$

(c)

$$
\begin{aligned}
(\tau_a f, \tau_a g) &= \int_0^1 f(t-a)\overline{g(t-a)}dt \\
&\overset{t=t-a}{=} \int_{-a}^{-a+1} f(t)\overline{g(t)}dt \\
&= \int_0^1 f(t)\overline{g(t)}dt \qquad \text{由周期性} \\
&= (f, g)
\end{aligned}
$$

(d)

$$
\begin{aligned}
(\tau_a f, g) &= \int_0^1 f(t-a)\overline{g(t)}dt \\
&\overset{t=t-a}{=} \int_{-a}^{1-a} f(t)\overline{g(t+a)}dt \\
&= \int_0^1 f(t)\overline{g(t+a)}dt \\
&= (f, \tau_{-a} g)
\end{aligned}
$$

(e)

$$
\begin{aligned}
(\tau_a f, \tau_b g) &= \int_0^1 f(t-a)\overline{g(t-b)}dt \\
&\overset{t=t-a}{=} \int_{-a}^{-a+1} f(t)\overline{g(t+a-b)}dt \\
&= \int_0^1 f(t)\overline{g(t+a-b)}dt \\
&= (f, \tau_{b-a} g) \\
&\overset{t=t-b}{=} \int_{-b}^{1-b} f(t+b-a)\overline{g(t)}dt \\
&= \int_0^1 f(t+b-a)\overline{g(t)}dt \\
&= (\tau_{a-b} f, g)
\end{aligned}
$$

Problem 6

(a)首先验证收敛性。

假设

$$
|c_n| \le c, n \in \mathbb{N}^+
$$

那么

$$
\begin{aligned}
\left|c_0+2\sum_{n=1}^{\infty}c_nr^ne^{in\theta}\right| &\le |c_0|+2\sum_{n=1}^{\infty}|c_n|\left|e^{in\theta}\right|r^n \\
&\le |c_0|+2c\sum_{n=1}^{\infty}r^n
\end{aligned}
$$

因为$0\le r<1$，所以上述级数绝对收敛，因此有定义。

记

$$
h(r,\theta)=c_0+2\sum_{n=1}^{\infty}c_nr^ne^{in\theta}
$$

那么

$$
\begin{aligned}
\frac{\partial h}{\partial r} &= 2\sum_{n=1}^{\infty}nc_nr^{n-1}e^{in\theta} \\
\frac{\partial h}{\partial \theta} &= 2i\sum_{n=1}^{\infty}nc_nr^ne^{in\theta} \\
\frac{\partial^2 h}{\partial r^2} &= 2\sum_{n=1}^{\infty}n(n-1)c_nr^{n-2}e^{in\theta} \\
\frac{\partial^2 h}{\partial \theta^2} &= -2\sum_{n=1}^{\infty}n^2c_nr^ne^{in\theta}
\end{aligned}
$$

所以

$$
\begin{aligned}
\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} &= 2\sum_{n=1}^{\infty}n(n-1)c_nr^{n-2}e^{in\theta}+2\sum_{n=1}^{\infty}nc_nr^{n-2}e^{in\theta}-2\sum_{n=1}^{\infty}n^2c_nr^{n-2}e^{in\theta} \\
&= 0
\end{aligned}
$$

所以

$$
\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}=\mathrm{Re}\left\{\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}\right\}=0
$$

(b)注意$f(\theta)$为实数，所以

$$
c_{-n}=\overline{c_n},c_0\text{是 实 数}
$$

那么

$$
\begin{aligned}
f(\theta) &= \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=-\infty}^{-1} c_n e^{in\theta} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=1}^{\infty} c_{-n} e^{-in\theta} \\
&= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{in\theta} + \overline{c_n} e^{-in\theta}\right) \\
&= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{in\theta} + \overline{c_n e^{in\theta}}\right) \\
&= \mathrm{Re}\left\{c_0 + 2\sum_{n=1}^{\infty} c_n e^{in\theta}\right\} \\
&= u(1,\theta)
\end{aligned}
$$

(c)由傅里叶系数的计算公式可得

$$
c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)dt
$$

带入$u(r,\theta)$的计算公式得到

$$
\begin{aligned}
u(r,\theta) &= \mathrm{Re}\left\{c_0 + 2\sum_{n=1}^{\infty} c_n r^n e^{in\theta}\right\} \\
&= \mathrm{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{2\pi}\sum_{n=1}^{\infty} 2\int_0^{2\pi} r^n e^{in\theta} e^{-int} f(t)dt\right\} \\
&= \mathrm{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{2\pi} 2\int_0^{2\pi}\sum_{n=1}^{\infty} r^n e^{in(\theta-t)} f(t)dt\right\} \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty} r^n e^{in(\theta-t)} f(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty} r^n e^{-in(\theta-t)} f(t)dt \qquad \text{由共轭复数的性质} \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i(\theta-t)}}{1-re^{i(\theta-t)}} f(t)dt + \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{-i(\theta-t)}}{1-re^{-i(\theta-t)}} f(t)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{re^{i(\theta-t)}}{1-re^{i(\theta-t)}} + \frac{re^{-i(\theta-t)}}{1-re^{-i(\theta-t)}}\right) f(t)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{re^{i(\theta-t)}(1-re^{-i(\theta-t)}) + re^{-i(\theta-t)}(1-re^{i(\theta-t)})}{(1-re^{-i(\theta-t)})(1-re^{i(\theta-t)})}\right) f(t)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)dt + \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{re^{i(\theta-t)} + re^{-i(\theta-t)} - 2r^2}{1-re^{-i(\theta-t)} - re^{i(\theta-t)} + r^2}\right) f(t)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(1 + \frac{2r\cos(\theta-t) - 2r^2}{1-2r\cos(\theta-t)+r^2}\right) f(t)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} f(t)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)P(r,\theta-\phi)d\phi
\end{aligned}
$$

(d)回顾之前的计算过程，我们有

$$
P(r,\theta-t) = \mathrm{Re}\left\{c_0 + 2\sum_{n=1}^{\infty} r^n e^{in(\theta-t)}\right\}
$$

所以

$$
P(r,\theta) = \mathrm{Re}\left\{c_0 + 2\sum_{n=1}^{\infty} r^n e^{in\theta}\right\}
$$

由(a)可知$P(r,\theta)$是简谐函数。