Problem 1

(a)注意到

$$\mathcal{F}\Lambda = \operatorname{sinc}^2$$

所以

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sinc}^4(t)dt &= \int_{-\infty}^{\infty} (\mathcal{F}\Lambda(t))^2 dt \\ &= \int_{-\infty}^{\infty} \Lambda^2(s)ds \\ &= 2\int_0^1 (1-s)^2 ds \\ &= \frac{2}{3}\end{aligned}$$

(b)设

$$g(t) = e^{-a|t|}$$

那么

$$\mathcal{F}g(s) = \frac{2a}{a^2+4\pi^2 s^2}$$

如果$a=1$，那么

$$\mathcal{F}g(s) = \frac{2}{1+4\pi^2 s^2}$$

另一方面，我们有

$$\mathcal{F}\Pi(s) = \operatorname{sinc}(s)$$

设

$$f(t) = \Pi\left(\frac{t}{2}\right)$$

所以

$$\mathcal{F}f(s) = 2\operatorname{sinc}(2s)$$

因此

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{2}{1+(2\pi t)^2}\operatorname{sinc}(2t)dt &= \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{F}g(t)\mathcal{F}f(t)dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{F}g(t)\overline{\mathcal{F}f(t)}dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} g(s)\overline{f(s)}ds \\
&= \frac{1}{2}\int_{-1}^{1} e^{-|s|}ds \\
&= \int_{0}^{1} e^{-s}ds \\
&= 1-e^{-1}
\end{aligned}
$$

(c)注意到

$$
(\mathcal{F}\Lambda^{(1)})(s) = (2\pi i s)\mathcal{F}\Lambda(s)
$$

所以

$$
\begin{aligned}
\int_{-\infty}^{\infty} t^2 \operatorname{sinc}^4(t)dt &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} |(\mathcal{F}\Lambda^{(1)})(t)|^2 dt \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \left|\Lambda^{(1)}(s)\right|^2 ds \\
&= \frac{1}{4\pi^2}\times 2\times\int_{0}^{1} 1ds \\
&= \frac{1}{2\pi^2}
\end{aligned}
$$

Problem 2

(a)该函数为

$$
f(t) = \operatorname{sign}(t)\times\left(1-\Pi\left(\frac{t}{2}\right)\right)
$$

取傅里叶变换得到

$$
\begin{aligned}
\mathcal{F}f(s) &= \mathcal{F}\operatorname{sign}(t) * \mathcal{F}\left(1-\Pi\left(\frac{t}{2}\right)\right) \\
&= \frac{1}{\pi i t} * (\delta(t) - 2\operatorname{sinc}(2t))
\end{aligned}
$$

设

$$H(x)=\begin{cases}0 & x\le 0\\ 1 & x>0\end{cases}$$

那么

$$\begin{aligned}g(t)&=H(t-1)\\ f(t)&=g(t)-g^{-}(t)\end{aligned}$$

所以

$$\begin{aligned}\mathcal{F}g(s)&=e^{-2\pi is}\mathcal{F}H(s)\\ &=e^{-2\pi is}\frac{1}{2}\left(\delta(s)+\frac{1}{\pi is}\right)\\ \mathcal{F}g^{-}(s)&=(\mathcal{F}g(s))^{-}\\ &=e^{2\pi is}\frac{1}{2}\left(\delta(-s)+\frac{1}{\pi i(-s)}\right)\\ &=e^{2\pi is}\frac{1}{2}\left(\delta(s)-\frac{1}{\pi is}\right)\\ \mathcal{F}f(s)&=\mathcal{F}g(s)-\mathcal{F}g^{-}(s)\\ &=\frac{1}{2}\delta(s)\left(e^{-2\pi is}-e^{2\pi is}\right)+\frac{1}{2}\frac{1}{\pi is}\left(e^{2\pi is}+e^{-2\pi is}\right)\\ &=\frac{1}{2}\delta(s)(-2i)\sin(2\pi s)+\frac{1}{2}\frac{1}{\pi is}2\cos(2\pi s)\\ &=\frac{\cos(2\pi s)}{\pi is}\end{aligned}$$

(b)注意到

$$\sin(2\pi|t|)=\frac{1}{2i}\left(e^{2\pi i|t|}-e^{-2\pi i|t|}\right)$$

设

$$g(t)=e^{-a|t|}$$

那么

$$\mathcal{F}g(s)=\frac{2a}{a^2+4\pi^2s^2}$$

所以

$$\begin{aligned}\mathcal{F}f(s)&=\frac{1}{2i}\left(\frac{2(-2\pi i)}{-4\pi^2+4\pi^2s^2}-\frac{2(2\pi i)}{-4\pi^2+4\pi^2s^2}\right)\\ &=\frac{-4\pi}{4\pi^2(s^2-1)}\\ &=\frac{1}{\pi(1-s^2)}\end{aligned}$$

Problem 3

(a)因为

$$\mathcal{F}\cos(2\pi\nu t) = \frac{1}{2}(\delta(s-\nu)+\delta(s+\nu))$$

注意时域的滤波等价于频域的乘积，所以频域的输出结果为

$$\frac{1}{2}(\delta(s-\nu)+\delta(s+\nu))H(s)$$

利用

$$g(x)\delta_a = g(a)\delta_a$$

取傅里叶逆变换可得

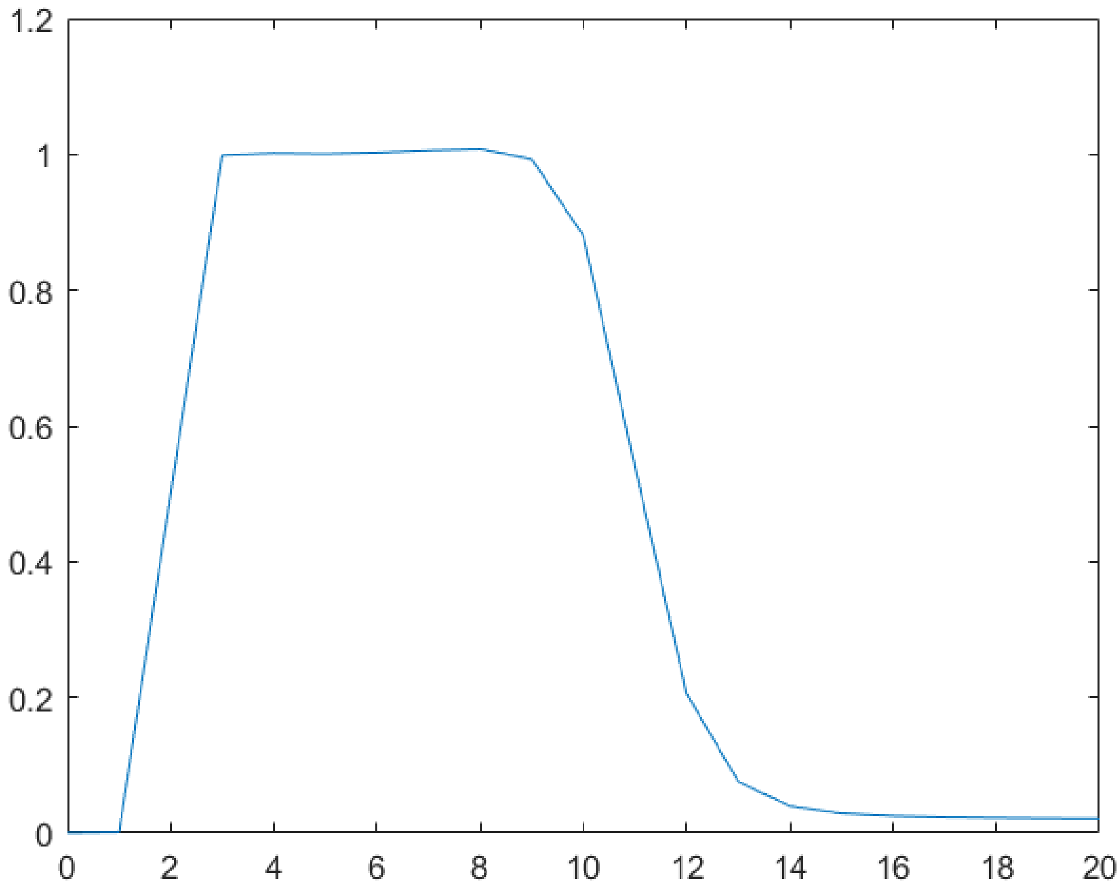
$$\begin{aligned}
\mathcal{F}^{-1}\left(\frac{1}{2}(\delta(s-\nu)+\delta(s+\nu))H(s)\right) &= \mathcal{F}^{-1}\left(\frac{1}{2}(\delta(s-\nu)+\delta(s+\nu))H(s)\right) \\
&= \frac{1}{2}(\mathcal{F}^{-1}\delta(s-\nu)H(\nu)+\mathcal{F}^{-1}\delta(s+\nu)H(-\nu)) \\
&= H(\nu)\mathcal{F}^{-1}\left(\frac{1}{2}(\delta(s-\nu)+\delta(s+\nu))\right) \\
&= H(\nu)\cos(2\pi\nu t)
\end{aligned}$$

(b)算法的思路是利用

$$|H(\nu)| = \max_t |H(\nu)\cos(2\pi\nu t)|$$

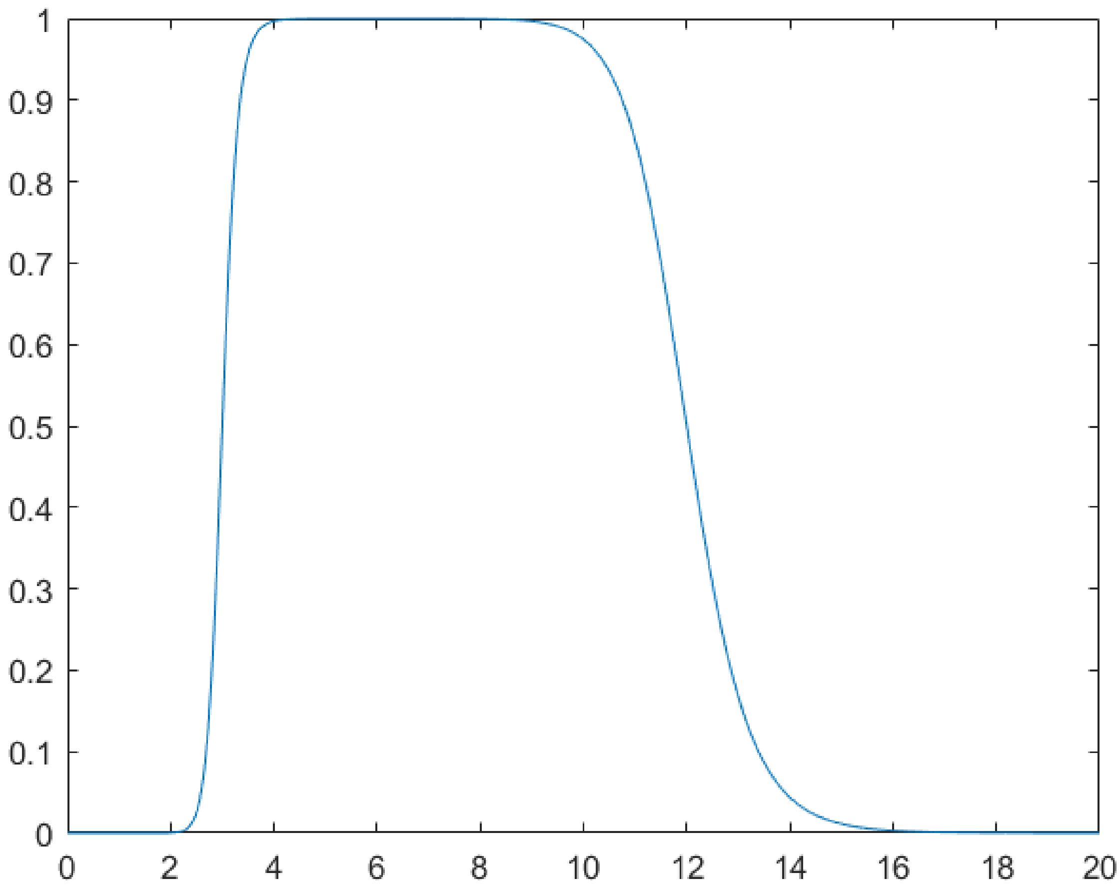
```
%(b)
Nu = 0 : 20;
res = zeros(1, 21);

for nu = 1 : 21
    resolution = 0.001;
    t = 0:resolution:10;
    input = cos(2*pi*nu*t);
    output = identme(input, resolution);
    res(nu) = max(output);
end
figure(1);
plot(Nu, res);
```



(c)

```
%(c)
s = 0: 0.01: 20;
H = transferfcn(s);
figure(2)
plot(s, H);
```



Problem 4

$$\begin{aligned}\mathcal{F}f(s) &= \mathcal{F}\left(1 + \Lambda(3t) * \mathrm{III}_{1/3}(t)\right) \\ &= \delta(s) + \frac{1}{3}\mathrm{sinc}^2\left(\frac{s}{3}\right) \times 3\mathrm{III}_3(s) \\ &= \delta(s) + \sum_{k=-\infty}^{\infty} \mathrm{sinc}^2(k)\,\delta(s-3k)\end{aligned}$$

注意到我们有

$$\mathrm{sinc}^2(k) = \begin{cases} 1 & k = 0 \\ 0 & \text{其他} \end{cases}$$

所以

$$\mathcal{F}f(s) = 2\delta(s)$$

取逆变换得到

$$f(s) = 2$$

Problem 5

如果在整周期的时刻看电风扇，那么其图像不变；特别的，如果该电风扇有三个叶子，那么在$\frac{1}{3}$周期的整数倍看电风扇，其图像依然不变。