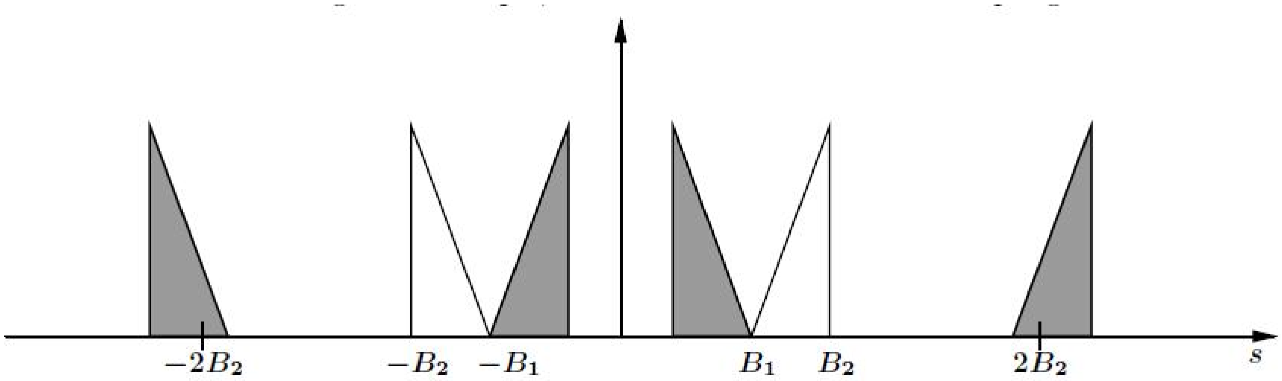
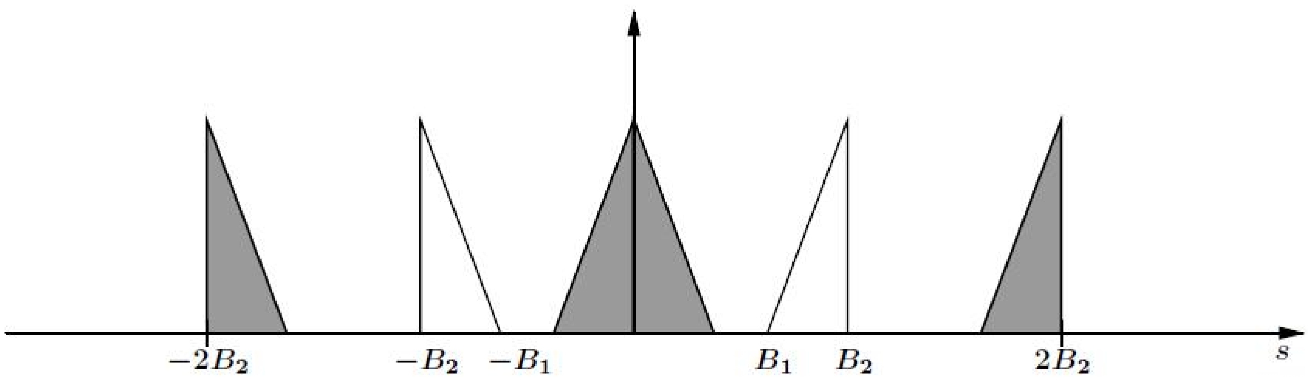
Problem 1

只要没有重合即可，考虑如下两种极限情形：





所以结果为

$$B_2 < s < 2B_1$$

Problem 2

(a)利用III函数对原式进行化简可得

$$\sum_{k=-\infty}^{\infty} Tp(t-kT) = Tp(t) * \mathrm{III}_T(t)$$

取傅里叶变换可得

$$
\begin{aligned}
\mathcal{F}g(s) &= \mathcal{F}f(s) * \left(T\mathcal{F}p(s)\frac{1}{T}\text{III}_{\frac{1}{T}}(t)\right) \\
&= \mathcal{F}f(s) * \left(\mathcal{F}p(s)\,\text{III}_{\frac{1}{T}}(s)\right) \\
&= \mathcal{F}f(s) * \left(\sum_{k=-\infty}^{\infty}\mathcal{F}p(s)\delta\left(s-\frac{k}{T}\right)\right) \\
&= \mathcal{F}f(s) * \left(\sum_{k=-\infty}^{\infty}\mathcal{F}p\left(\frac{k}{T}\right)\delta\left(s-\frac{k}{T}\right)\right) \\
&= \sum_{k=-\infty}^{\infty}\mathcal{F}p\left(\frac{k}{T}\right)\left(\mathcal{F}f(s) * \delta\left(s-\frac{k}{T}\right)\right) \\
&= \sum_{k=-\infty}^{\infty}\mathcal{F}p\left(\frac{k}{T}\right)\mathcal{F}f\left(s-\frac{k}{T}\right)
\end{aligned}
$$

取逆变换可得

$$
g(s) = \sum_{k=-\infty}^{\infty}\mathcal{F}p\left(\frac{k}{T}\right)f\left(s-\frac{k}{T}\right)
$$

对比采样定理

$$
f(t) = \sum_{k=-\infty}^{\infty}f\left(\frac{k}{p}\right)\operatorname{sinc}p\left(t-\frac{k}{p}\right)
$$

只要利用第一个式子求解出第二个式子的系数即可，其条件为采样间隔大于带宽，即$\frac{1}{T}>2B$

Problem 3

因为

$$
f(x) = \frac{1}{2}\left(e^{-2\pi ix}+e^{2\pi ix}\right)
$$

取傅里叶变换可得

$$
\mathcal{F}f(s) = \frac{1}{2}(\delta(s-1)+\delta(s+1))
$$

按$\frac{2}{3}$采样即使用$\text{III}_{\frac{2}{3}}$，所以

$$
\begin{aligned}
\mathcal{F}f(s) * \text{III}_{\frac{2}{3}} &= \frac{1}{2}(\delta(s-1)+\delta(s+1)) * \sum_{i=1}^{\infty}\delta\left(x-\frac{2}{3}k\right) \\
&= \frac{1}{2}\sum_{i=1}^{\infty}\delta\left(s-\frac{2}{3}k-1\right)+\frac{1}{2}\sum_{i=1}^{\infty}\delta\left(s-\frac{2}{3}k+1\right)
\end{aligned}
$$

按照$\frac{2}{3}$赫兹截断，那么

$$
\begin{aligned}
\Pi_{\frac{2}{3}}\left(\mathcal{F}f(s)*\mathrm{III}_{\frac{2}{3}}\right) &= \frac{1}{2}\left(\delta\left(s-\frac{1}{3}\right)+\delta\left(s+\frac{1}{3}\right)\right)+\frac{1}{2}\left(\delta\left(s-\frac{1}{3}\right)+\delta\left(s+\frac{1}{3}\right)\right) \\
&= \delta\left(s-\frac{1}{3}\right)+\delta\left(s+\frac{1}{3}\right)
\end{aligned}
$$

取傅里叶逆变换可得

$$
e^{-2\pi it\frac{1}{3}}+e^{2\pi it\frac{1}{3}}=2\cos\left(\frac{2}{3}\pi t\right)
$$

Problem 4

由采样定理，我们得到

$$
g(t)=\sum_{k=-\infty}^{\infty} g\left(t_k\right)\operatorname{sinc} p'\left(t-t_k\right)
$$

其中

$$
p'\geq p
$$

对上式积分

$$
\begin{aligned}
\int_{-\infty}^{\infty} g(t)dt &= \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} g\left(t_k\right)\operatorname{sinc} p'\left(t-t_k\right)dt \\
&= \sum_{k=-\infty}^{\infty} g\left(t_k\right)\int_{-\infty}^{\infty}\operatorname{sinc} p'\left(t-t_k\right)dt
\end{aligned}
$$

现在计算

$$
\int_{-\infty}^{\infty}\operatorname{sinc} p'\left(t-t_k\right)dt
$$

我们有

$$
\begin{aligned}
\mathcal{F}\operatorname{sinc} p'\left(t-t_k\right) &= e^{-2\pi ip't_ks}\mathcal{F}\operatorname{sinc} p't \\
&= e^{-2\pi ip't_ks}\frac{1}{p'}\Pi_{p'}(s)
\end{aligned}
$$

注意到上述积分为傅里叶变换在$s=0$处的值，所以

$$
\int_{-\infty}^{\infty}\operatorname{sinc} p'\left(t-t_k\right)dt=\frac{1}{p'}
$$

因此

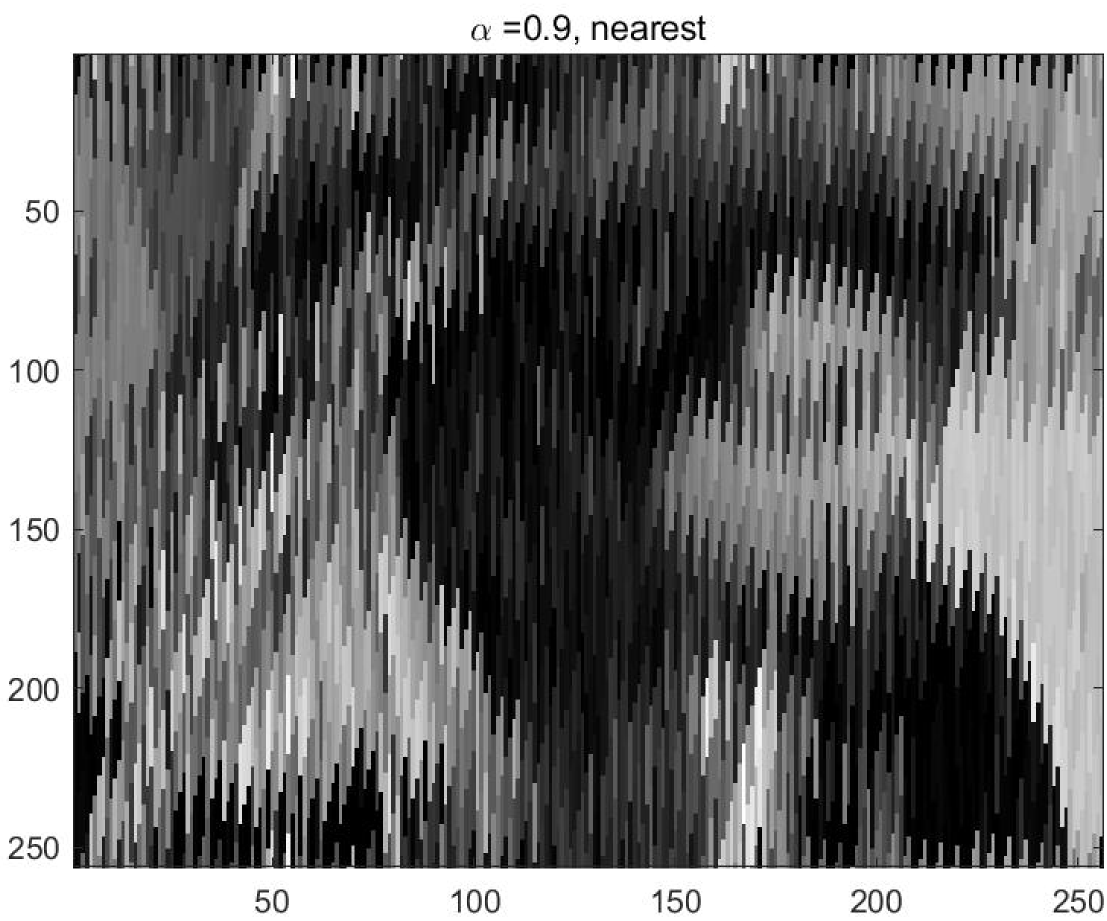
$$
\begin{aligned}
\int_{-\infty}^{\infty} g(t) dt &= \sum_{k=-\infty}^{\infty} g\left(t_k\right) \int_{-\infty}^{\infty} \operatorname{sinc} p^{\prime}\left(t-t_k\right) dt \\
&= \frac{1}{p^{\prime}} \sum_{k=-\infty}^{\infty} g\left(t_k\right)
\end{aligned}
$$

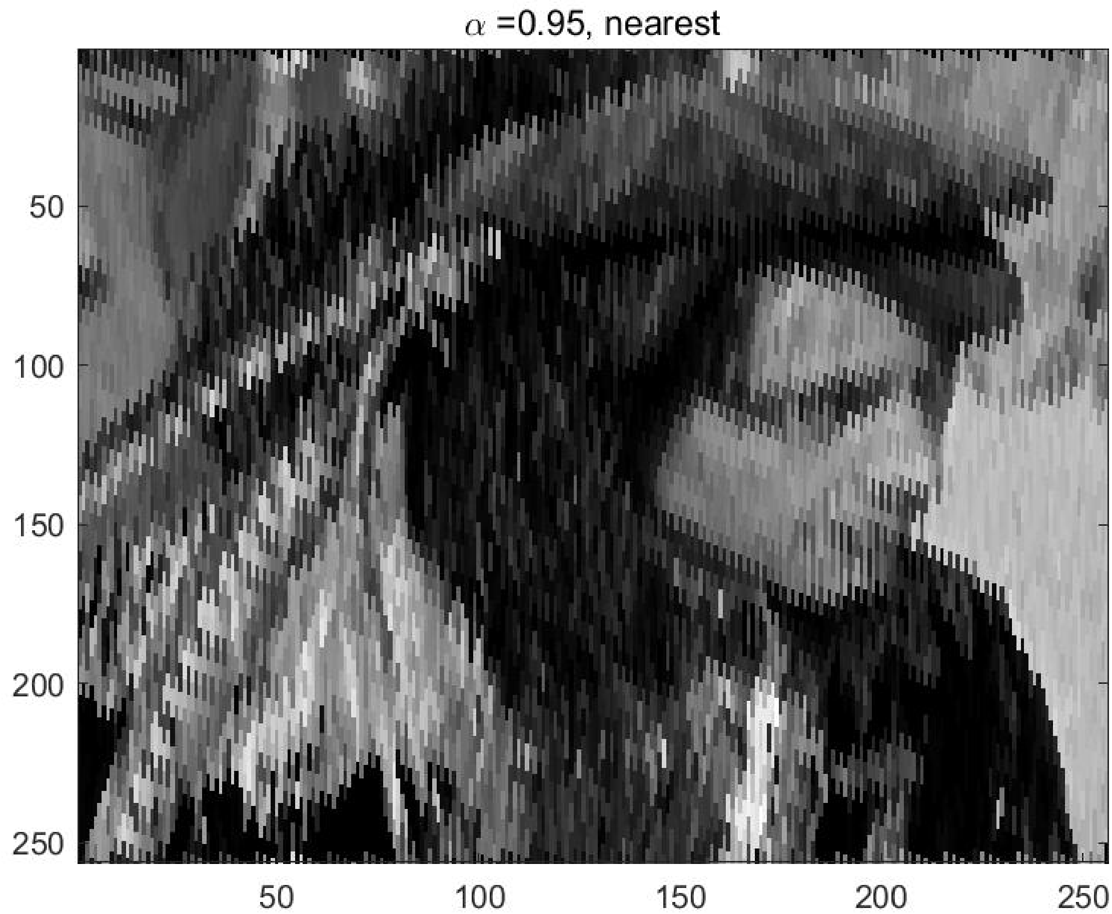
Problem 5

```
% (a)
d1 = imread("man.gif");
%转换类型
%data = im2double(data);
d2 = d1(:);
d3 = double(d2);
d4 = d3 / max(d3);
n = length(d4);

% (b)
x = fft(d4);
l = length(x);
% 中心化
x_center = [x(l/2 + 1: end); x(1: l/2)];
plot([-l/2: l / 2 - 1], abs(x_center))

% (c)
E_total = norm(x_center) ^ 2;
E_partial = zeros(1, l / 2);
E_partial(1) = abs(x(1)) ^ 2;
for i = 2: (l / 2 - 1)
    E_partial(i) = E_partial(i - 1) + 2 * abs(x(i)) ^ 2;
end
%比例
ratio = E_partial / E_total;

Alpha = [0.9, 0.95, 0.99];
%Type = ['nearest', 'linear'];
cnt = 1;
for i = 1: 2
    for j = 1: 3
        %type = Type(i);
        if i == 1
            type = 'nearest';
        else
            type = 'linear';
        end
        alpha = Alpha(j);
        %计算p
        p = double(find_p(ratio, alpha));
        %采样频率
        rate = floor(n / (2 * p));
```

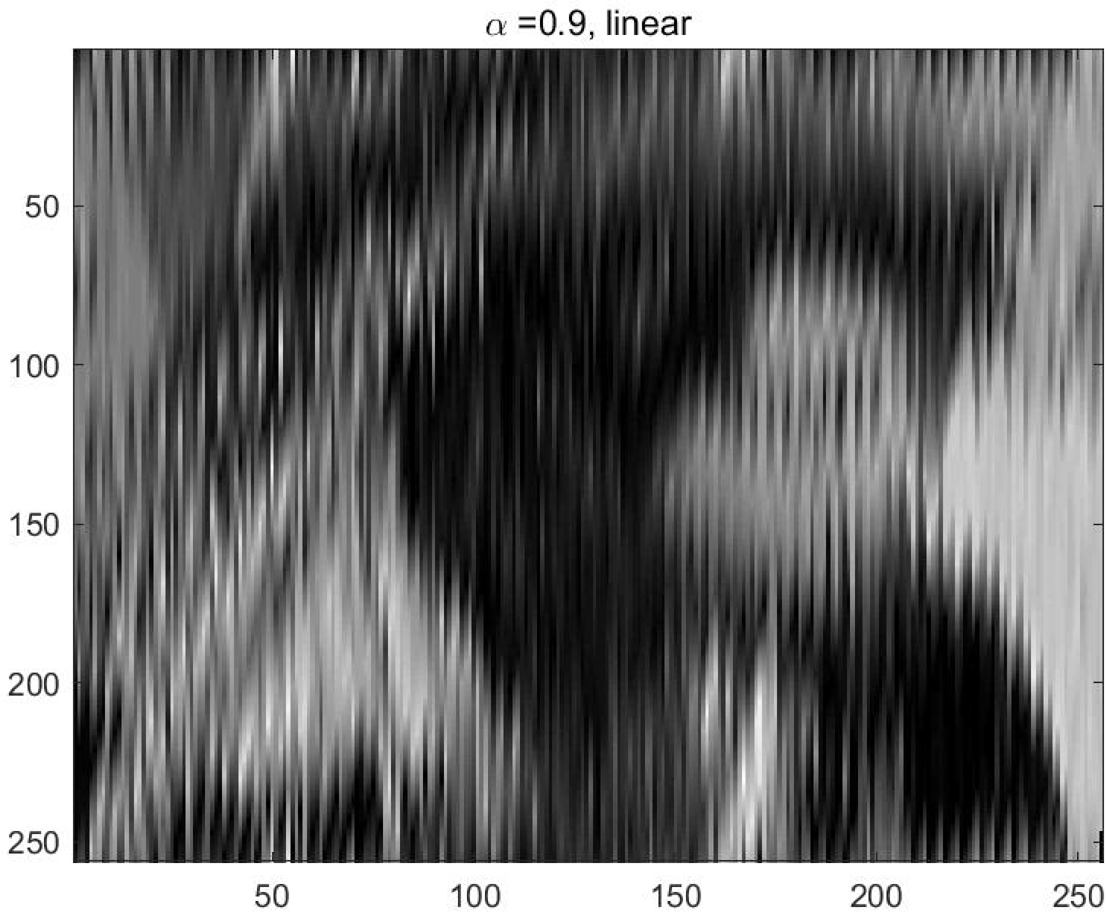
```
        %索引
        index = 1: rate: n;
        %对应的值
        x_res = d4(index);
        %插值
        res = interp1(index, x_res, 1: n, type);
        %恢复图像
        res = max(d3) * res;
        res = reshape(res, 256, 256);
        %作图
        figure(cnt);
        imagesc(res);
        colormap('gray');
        title(['\alpha =' num2str(alpha) ', ', type]);
        cnt = cnt + 1;
    end
end
```

结果如下:

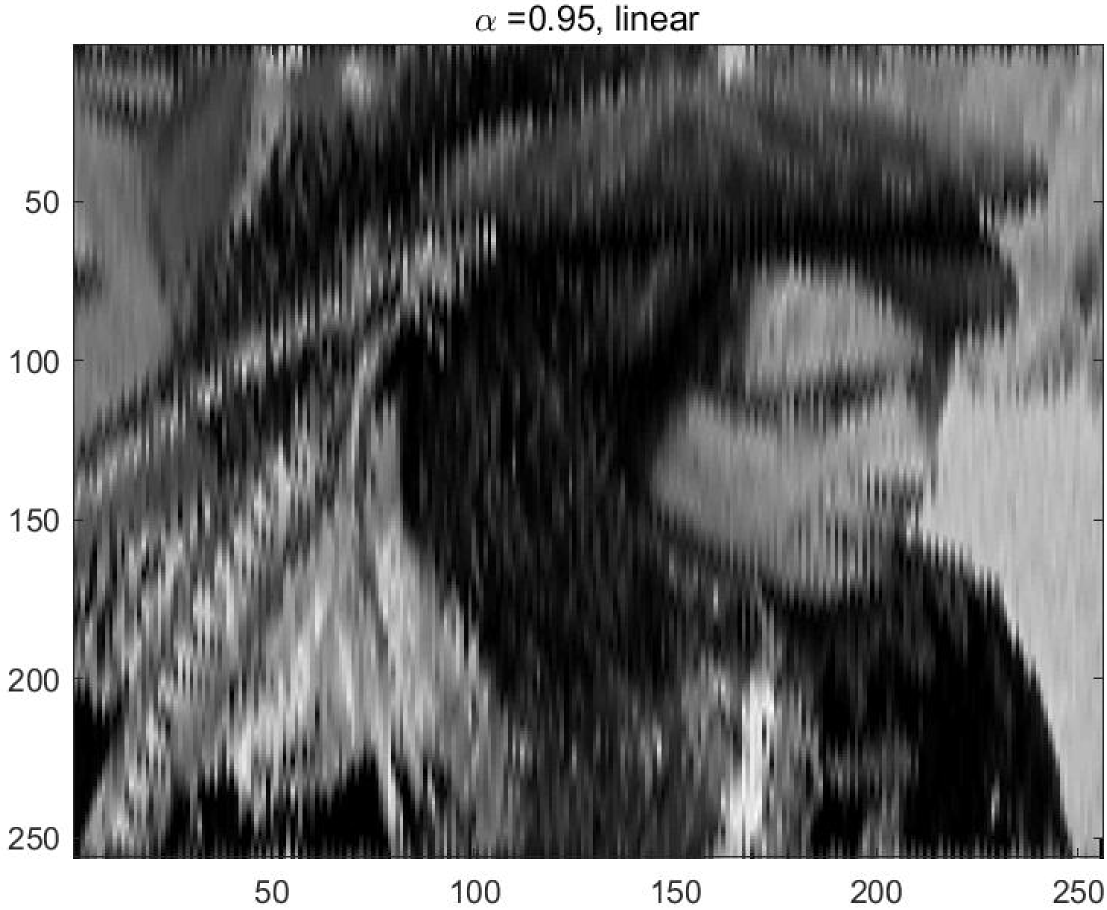



α =0.95, nearest
50
100
150
200
250
50
100
150
200
250


α =0.99, nearest
50
100
150
200
250
50
100
150
200
250



α =0.9, linear
50
100
150
200
250
50
100
150
200
250



α =0.95, linear
50
100
150
200
250
50
100
150
200
250

α =0.99, linear