Problem 1

$$\begin{aligned}\mathcal{F}g(x,y) &= \mathcal{F}\left(\Pi(x)\Pi(y)\right)\mathcal{F}\left(\Pi(x)\Pi(y)\right)\\ &= \mathcal{F}^2(\Pi(x))\mathcal{F}^2(\Pi(y))\\ &= \operatorname{sinc}^2(\xi)\operatorname{sinc}^2(\eta)\\ &= \mathcal{F}(\Lambda(x)\Lambda(y))\end{aligned}$$

取逆变换可得

$$g(x,y) = \Lambda(x)\Lambda(y)$$

Problem 2

注意汉高变换为

$$G(\rho) = 2\pi\int_0^\infty g(r)J_0(2\pi r\rho)rdr$$

考虑$g(ar)$的汉高变换

$$\begin{aligned}2\pi\int_0^\infty g(ar)J_0(2\pi r\rho)rdr &= \frac{2\pi}{a}\int_0^\infty g(r')J_0(2\pi(r'/a)\rho)r'/adr'\\ &= \frac{2\pi}{a^2}\int_0^\infty g(r')J_0(2\pi r'(\rho/a))r'dr'\\ &= \frac{1}{|a|^2}G\left(\frac{\rho}{a}\right)\end{aligned}$$

Problem 3

(a)

$$\begin{aligned}\mathcal{F}\left(\sin 2\pi ax_1\sin 2\pi bx_2\right) &= \mathcal{F}(\sin 2\pi ax_1)\mathcal{F}(\sin 2\pi bx_2)\\ &= \frac{1}{2i}(\delta(\xi_1-a)-\delta(\xi_1+a))\,\frac{1}{2i}(\delta(\xi_2-b)-\delta(\xi_2+b))\\ &= -\frac{1}{4}(\delta(\xi_1-a,\xi_2-b)-\delta(\xi_1+a,\xi_2-b)-\delta(\xi_1-a,\xi_2+b)+\delta(\xi_1+a,\xi_2+b))\end{aligned}$$

(b)注意到我们有

$$\mathcal{F}e^{-\pi x^2} = \mathcal{F}e^{-\pi s^2}$$

所以

$$
\begin{aligned}
\mathcal{F}e^{-ax^2} &= \mathcal{F}e^{-\pi\left(\frac{\sqrt{a}}{\sqrt{\pi}}x\right)^2} \\
&= \sqrt{\frac{\pi}{a}}e^{-as^2}
\end{aligned}
$$

从而

$$
\begin{aligned}
\mathcal{F}e^{-ar^2} &= \mathcal{F}e^{-a(x^2+y^2)} \\
&= \mathcal{F}e^{-ax^2}\mathcal{F}e^{-ay^2} \\
&= \frac{\pi}{a}e^{-a\xi_1^2}e^{-a\xi_2^2}
\end{aligned}
$$

(c)

$$
\cos(2\pi cx) = \frac{1}{2}\left(e^{2\pi icx} + e^{-2\pi icx}\right)
$$

所以

$$
\begin{aligned}
\mathcal{F}\left(e^{-2\pi i(ax+by)}\cos(2\pi cx)\right) &= \mathcal{F}\left(e^{-2\pi i(ax+by)}\frac{1}{2}\left(e^{2\pi icx} + e^{-2\pi icx}\right)\right) \\
&= \frac{1}{2}\left(\mathcal{F}(e^{-2\pi i((a-c)x+by)}) + e^{-2\pi i((a+c)x+by)}\right) \\
&= \frac{1}{2}\left(\mathcal{F}(e^{-2\pi i(a-c)x})\mathcal{F}(e^{-2\pi iby}) + \mathcal{F}(e^{-2\pi i(a+c)x})\mathcal{F}(e^{-2\pi iby})\right) \\
&= \frac{1}{2}(\delta(\xi_1+a-c)\delta(\xi_2+b) + \delta(\xi_1+a+c)\delta(\xi_2+b)) \\
&= \frac{1}{2}(\delta(\xi_1+a-c,\xi_2+b) + \delta(\xi_1+a+c,\xi_2+b))
\end{aligned}
$$

(d)

$$
\cos(2\pi(ax+by)) = \cos(2\pi ax)\cos(2\pi by) - \sin(2\pi ax)\sin(2\pi by)
$$

取傅里叶变换可得

$$
\begin{aligned}
\mathcal{F}\left(\cos(2\pi(ax+by))\right) &= \mathcal{F}\left(\cos(2\pi ax)\cos(2\pi by)\right) - \mathcal{F}\left(\sin(2\pi ax)\sin(2\pi by)\right) \\
&= \frac{1}{4}\left(\left(\delta(\xi_1-a)+\delta(\xi_1+a)\right)\left(\delta(\xi_2-b)+\delta(\xi_2+b)\right)\right) \\
&+ \frac{1}{4}\left(\left(\delta(\xi_1-a)-\delta(\xi_1+a)\right)\left(\delta(\xi_2-b)-\delta(\xi_2+b)\right)\right) \\
&= \frac{1}{2}(\delta(\xi_1-a,\xi_2-b) + \delta(\xi_1+a,\xi_2+b))
\end{aligned}
$$

Problem 4

(a)

$$
\begin{aligned}
\mathcal{F}g(\xi_1,\xi_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1+\xi_2x_2)} g(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1+\xi_2x_2)} f(-x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(-\xi_1x_1+\xi_2x_2)} f(x_1,x_2)dx_1dx_2 \\
&= \mathcal{F}f(-\xi_1,\xi_2)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathcal{F}h(\xi_1,\xi_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1+\xi_2x_2)} h(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1+\xi_2x_2)} f(x_1,-x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1-\xi_2x_2)} f(x_1,x_2)dx_1dx_2 \\
&= \mathcal{F}f(\xi_1,-\xi_2)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathcal{F}k(\xi_1,\xi_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1+\xi_2x_2)} k(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1+\xi_2x_2)} f(x_2,x_1)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_2x_1+\xi_1x_2)} f(x_1,x_2)dx_1dx_2 \\
&= \mathcal{F}f(\xi_2,\xi_1)
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathcal{F}m(\xi_1,\xi_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1+\xi_2x_2)} m(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(\xi_1x_1+\xi_2x_2)} f(-x_2,-x_1)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(-\xi_2x_1-\xi_1x_2)} f(x_1,x_2)dx_1dx_2 \\
&= \mathcal{F}f(-\xi_2,-\xi_1)
\end{aligned}
$$

Problem 5

(a)考虑f第m行$f_m[l]$，我们有

$$
\begin{aligned}
\mathcal{F}f_m[l] &= \sum_{n=0}^{N-1} f_m[n]\omega_N^{-ln} \\
&= \sum_{n=0}^{N-1} f[m,n]\omega_N^{-ln}
\end{aligned}
$$

那么

$$
\begin{aligned}
\mathcal{F}f[k,l] &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\omega_N^{-ln}\omega_M^{-km} \\
&= \sum_{m=0}^{M-1} \mathcal{F}f_m[l]\omega_M^{-km}
\end{aligned}
$$

所以可以先利用FFT计算$\mathcal{F}f_m[l]$，然后再利用FFT计算二维的结果。

(b)

$$
\begin{aligned}
\mathcal{F}g[k,l] &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g[m,n]\omega_N^{-ln}\omega_M^{-km} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\omega_M^{mk_0}\omega_N^{nl_0}\omega_N^{-ln}\omega_M^{-km} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\omega_N^{n(l_0-l)}\omega_M^{m(k_0-k)} \\
&= \mathcal{F}[k-k_0, l-l_0]
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathcal{F}(f*g)[k,l] &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(f*g)[m,n]\omega_N^{-ln}\omega_M^{-km} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f[u,v]g[m-u,n-v]\omega_N^{-ln}\omega_M^{-km} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f[u,v]\omega_N^{-lu}\omega_M^{-kv}g[m-u,n-v]\omega_N^{-l(n-u)}\omega_M^{-k(m-v)} \\
&= \left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f[u,v]\omega_N^{-lu}\omega_M^{-kv}\right)\left(\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g[m-u,n-v]\omega_N^{-l(n-u)}\omega_M^{-k(m-v)}\right) \\
&= \mathcal{F}f[k,l]\times\mathcal{F}g[k,l]
\end{aligned}
$$

Problem 6

```
%(a)
```

```
data = imread("dog.jpg");
Max = 255;
data = im2double(data);
imshow(data);

%(b)
data_new = treat(data);
figure(1)
imshow(data_new);

%(c)
i = 2;
F = 0.5: -0.05: 0.1;
[Xmax ,Ymax] = size(data);
for f = 0.5: -0.05: 0.1
    h = LP_filter(Xmax, Ymax, f);
    data1 = real(ifft2(fft2(data) .* h));
    data_new = treat(data1);
    figure(i);
    suptitle(sprintf('Initial Image with Object Boundaries (f=%g)', f));
    imshowpair(data1, data_new, 'montage');
    i = i + 1;
end
```

函数treat:

```
function data_new = treat(data)

Max = 255;
Bx = [0, 0, 0; 1, -1, 0; 0, 0, 0];
By = [0, 1, 0; 0, -1, 0; 0, 0, 0];
datax = Max * conv2(data, Bx, 'same');
datay = Max * conv2(data, By, 'same');
data_new = data;
data_new(abs(datax) > 10) = 1;
data_new(abs(datay) > 10) = 1;
```