Problem 1

(a)$f\star g$很大说明g平移后和f很接近；很小说明不接近；正表示平移后增长趋势接近；负表示平移后增长趋势不同

(b)首先

$$\begin{aligned}(f\star g)(x)&=\int_{-\infty}^{\infty}f(y)g(x+y)dy\\&=\int_{-\infty}^{\infty}f^{-}(-y)g(x+y)dy\\&=(f^{-}*g)(x)\end{aligned}$$

另一方面

$$\begin{aligned}(f\star g)(x)&=\int_{-\infty}^{\infty}f(y)g(x+y)dy\\&=\int_{-\infty}^{\infty}f(y)g^{-}(-x-y)dy\\&=\left(f*g^{-}\right)(-x)\\&=\left(f*g^{-}\right)^{-}(x)\end{aligned}$$

注意到

$$\begin{aligned}(g\star f)(x)&=\left(g*f^{-}\right)^{-}\\&=\left(f^{-}*g\right)^{-}\\&\neq(f^{-}*g)\end{aligned}$$

所以

$$f\star g\neq g\star f$$

(c)

$$\begin{aligned}(f\star(\tau_b g))(x)&=\int_{-\infty}^{\infty}f(y)g(x+y-b)dy\\&=\int_{-\infty}^{\infty}f(y)g(x-b+y)dy\\&=(f\star g)(x-b)\\&=(\tau_b(f\star g))(x)\end{aligned}$$

所以

$$f\star(\tau_b g)=\tau_b(f\star g)$$

另一方面

$$
\begin{aligned}
((\tau_b f) \star g)(x) &= \int_{-\infty}^{\infty} f(y-b) g(x+y) dy \\
&= \int_{-\infty}^{\infty} f(y-b) g(x+b+y-b) dy \\
&= \int_{-\infty}^{\infty} f(y) g(x+b+y) dy \\
&= (f \star g)(x+b) \\
&= (\tau_{-b}(f \star g))(x)
\end{aligned}
$$

Problem 2

(a)

$$
\begin{aligned}
\Pi \star \Pi &= \Pi^{-} * \Pi \\
&= \Pi * \Pi \\
&= \Lambda
\end{aligned}
$$

(b)只要证明如下事实即可

$$
\int_{-\infty}^{\infty} f(y) f(x+y) dy \leq \int_{-\infty}^{\infty} f(y) f(y) dy = \int_{-\infty}^{\infty} f(y)^2 dy
$$

回顾柯西不等式

$$
\int_{-\infty}^{\infty} f(y) g(y) dy \leq \left\{ \int_{-\infty}^{\infty} f(y)^2 dy \right\}^{1/2} \left\{ \int_{-\infty}^{\infty} g(y)^2 dy \right\}^{1/2}
$$

取

$$
\begin{aligned}
f(y) &= f(y) \\
g(y) &= f(x+y)
\end{aligned}
$$

那么

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(y) f(x+y) dy &\leq \left\{ \int_{-\infty}^{\infty} f(y)^2 dy \right\}^{1/2} \left\{ \int_{-\infty}^{\infty} f(x+y)^2 dy \right\}^{1/2} \\
&= \int_{-\infty}^{\infty} f(y)^2 dy
\end{aligned}
$$

等号成立当且仅当

$$
f(y) = k f(x+y)
$$

(c)

$$
\begin{aligned}
\mathcal{F}(f \star f) &= \mathcal{F}(f^- * f) \\
&= \mathcal{F}f^- \mathcal{F}f \\
&= (\mathcal{F}f)^- \mathcal{F}f \\
&= \overline{\mathcal{F}f}\mathcal{F}f \\
&= |\mathcal{F}f|^2
\end{aligned}
$$

(d)

$$
\begin{aligned}
(f \star f_r)(t) &= (f \star (\alpha\,(\tau_{2T} f) + n))\,(t) \\
&= \alpha(f \star (\tau_{2T} f))(t) + (f \star n)t \\
&= \alpha\tau_{2T}(f \star f)(t) + C \\
&= \alpha(f \star f)(t - 2T) + C \\
&\leq \alpha(f \star f)(0) + C
\end{aligned}
$$

当满足如下条件时取极大值

$$
t_0 - 2T = 0 \Rightarrow T = \frac{t_0}{2}
$$

Problem 3

Rectangle Window

$$
\mathcal{F}\Pi = \operatorname{sinc}
$$

Triangular Window

记

$$
g(t) = 2\Lambda(2t)
$$

那么

$$
\mathcal{F}g(s) = 2 \times \frac{1}{2} \times \operatorname{sinc}^2\left(\frac{s}{2}\right) = \operatorname{sinc}^2\left(\frac{s}{2}\right)
$$

Hamming Window

$$
\begin{aligned}
\mathcal{F}w(s) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi ist}\cos^2(\pi t)dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi ist}\left(\frac{e^{-i\pi t}+e^{i\pi t}}{2}\right)^2 dt \\
&= \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi ist}\left(e^{-2\pi it}+e^{2\pi it}+2\right)dt \\
&= \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(e^{-2\pi it(1+s)}+e^{2\pi it(1-s)}+2e^{-2\pi ist}\right)dt \\
&= \frac{1}{4}\left(\frac{\sin(\pi(1+s))}{\pi(1+s)}+\frac{\sin(\pi(1-s))}{\pi(1-s)}+2\frac{\sin(\pi s)}{\pi s}\right) \\
&= \frac{1}{4}\mathrm{sinc}(1+s)+\frac{1}{4}\mathrm{sinc}(s-1)+\frac{1}{2}\mathrm{sinc}(s)
\end{aligned}
$$

Problem 4

现在对方程两边关于x做傅里叶变换，首先对右边做傅里叶变换得到

$$
D\mathcal{F}f_{xx}(s,t) = D(2\pi is)^2\mathcal{F}f(s,t) = -4D\pi^2s^2\mathcal{F}f(s,t)
$$

其次对左边做傅里叶变换得到

$$
\begin{aligned}
\mathcal{F}f_t(s,t) &= \int_{-\infty}^{\infty} f_t(x,t)e^{-2\pi isx}dx \qquad \text{(Fourier transform in } x\text{)} \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}f(x,t)e^{-2\pi isx}dx \\
&= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} f(x,t)e^{-2\pi isx}dx \\
&= \frac{\partial}{\partial t}\hat{f}(s,t)
\end{aligned}
$$

所以原方程可以化为

$$
\frac{\partial\mathcal{F}f(s,t)}{\partial t} = -4D\pi^2s^2\mathcal{F}f(s,t)
$$

因此

$$
\mathcal{F}f(s,t) = \mathcal{F}f(s,0)e^{-4D\pi^2s^2t}
$$

最后计算初值$\mathcal{F}f(s,0)$:

$$
\begin{aligned}
\mathcal{F}f(s,0) &= \mathcal{F}\delta(s) \\
&= 1
\end{aligned}
$$

从而

$$\mathcal{F}f(s,t)=\mathcal{F}\delta(s)e^{-4D\pi^2s^2t}=e^{-4D\pi^2s^2t}$$

设

$$h(x)=e^{-\pi x^2}$$

回顾之前的结论，我们有

$$\mathcal{F}h(s)=h(s)$$

设

$$g(x,t)=ah\left(bx\right)$$

从而

$$\begin{aligned}\mathcal{F}g(s,t)&=\mathcal{F}ah\left(bx\right)\\&=\frac{a}{|b|}e^{-\frac{\pi s^2}{b^2}}\end{aligned}$$

因此对于

$$e^{-4D\pi^2s^2t}$$

我们有

$$\begin{gathered}4D\pi^2s^2t=\frac{\pi s^2}{b^2}\\\frac{a}{|b|}=1\end{gathered}$$

即

$$\begin{aligned}b&=\frac{1}{2\sqrt{D\pi t}}\\a&=\frac{1}{2\sqrt{D\pi t}}\end{aligned}$$

所以

$$g(x,t)=\frac{1}{2\sqrt{D\pi t}}h\left(\frac{x}{2\sqrt{D\pi t}}\right)=\frac{1}{2\sqrt{D\pi t}}e^{-\frac{x^2}{4Dt}}$$

因此

$$f(x,t)=\frac{1}{2\sqrt{D\pi t}}e^{-\frac{x^2}{4Dt}}$$

(b)期望为

$$0$$

方差为

$$2Dt$$

(c)将

$$D=\frac{kT}{6\pi\eta R}$$

代入

$$f(x,t)=\frac{1}{2\sqrt{D\pi t}}e^{-\frac{x^2}{4Dt}}$$

可得

$$f(x,t)=\frac{\sqrt{3\eta R}}{\sqrt{2KTt}}e^{-\frac{3\pi\eta R}{2KTt}x^2}$$

从上式可以可以看出，温度越高，例子波动幅度越大；半径越大，粒子波动幅度越小。

Problem 5

这部分参考了解答的代码。

```
% C B D A
% | | | |
% A B C D
[y, Fs] = audioread("PS-4-scramble.wav");
%变换为频域
z = fft(y);
plot(abs(z))
%plot(imag(z))
%步长，注意有对称性
n = floor(length(y) / 8);
C = z(1: n);
B = z(n + 1: 2 * n);
D = z(2 * n + 1: 3 * n);
A = z(3 * n + 1: 4 * n);
res = [A; B; C; D; flipud([conj(A);conj(B);conj(C);conj(D)])];
%取实部
r = real(ifft(res));
audiowrite('res.wav', r, Fs);
```