Problem 1

因为

$$\mathcal{F}f(s)=\int_{-\infty}^{\infty}e^{-2\pi ist}f(t)dt$$

所以

$$\begin{aligned}W_{\mathcal{F}f}&=\frac{1}{\mathcal{F}f(0)}\int_{-\infty}^{\infty}\mathcal{F}f(s)ds\\&=\frac{1}{\int_{-\infty}^{\infty}f(t)dt}\mathcal{F}(\mathcal{F}f)(0)\\&=\frac{1}{\int_{-\infty}^{\infty}f(t)dt}f^{-}(0)\\&=\frac{1}{\int_{-\infty}^{\infty}f(t)dt}f(0)\\&=\frac{1}{W_f}\end{aligned}$$

Problem 2

因为

$$\sin c(x)=\frac{\sin\pi x}{\pi x}$$

所以图像中的函数为

$$f(x)=a\sin c\left(b(x-c)\right)$$

其傅里叶变换为

$$\mathcal{F}f=ae^{2\pi is(-c)}\frac{1}{|b|}\Pi\left(\frac{s}{b}\right)=\frac{ae^{-2\pi ics}}{|b|}\Pi\left(\frac{s}{b}\right)$$

Problem 3

(a)

$$g_1(t)=f(-t)$$

所以

$$\mathcal{F}g_1=\mathcal{F}f^{-}=(\mathcal{F}f)^{-}\Rightarrow\mathcal{F}g_1(s)=F(-s)$$

(b)

$$g_2(t) = g_1(t-2)$$

所以

$$\mathcal{F}g_2(s) = e^{-4\pi is}\mathcal{F}g_1(s) \Rightarrow \mathcal{F}g_2(s) = e^{-4\pi is}F(-s)$$

(c)

$$g_3(t) = f(t-1)$$

所以

$$\mathcal{F}g_3(s) = e^{-2\pi is}\mathcal{F}f(s) \Rightarrow \mathcal{F}g_3(s) = e^{-2\pi is}F(s)$$

(d)

$$g_4(t) = f\left(\frac{t}{2}\right)$$

所以

$$\mathcal{F}g_4(s) = 2\mathcal{F}f(2s) \Rightarrow \mathcal{F}g_4(s) = 2F(2s)$$

(e)

$$g_5(t) = f(t) + f(t+2)$$

所以

$$\mathcal{F}g_5(s) = \mathcal{F}f(s) + e^{-2\pi is(-2)}\mathcal{F}f(s) \Rightarrow \mathcal{F}g_5(s) = (1 + e^{4\pi is})F(s)$$

(f)

$$g_6(t) = g_1(t) + f(t)$$

所以

$$\mathcal{F}g_6(s) = \mathcal{F}f(s) + \mathcal{F}f(-s) \Rightarrow \mathcal{F}g_6(s) = F(s) + F(-s)$$

Problem 4

(a)因为

$$\Pi_a(t) = \begin{cases} 1 & |t| < a/2 \\ 0 & |t| \geq a/2 \end{cases}$$

所以如果$0 < t < a$，那么

$$
\begin{aligned}
(\Pi_a * \Pi_a)(t) &= \int_{-\infty}^{\infty} \Pi_a(t-x)\Pi_a(x)dx \\
&= \int_{t-\frac{a}{2}}^{\frac{a}{2}} dx \\
&= a-t
\end{aligned}
$$

如果$-a < t < 0$，那么

$$
\begin{aligned}
(\Pi_a * \Pi_a)(t) &= \int_{-\infty}^{\infty} \Pi_a(t-x)\Pi_a(x)dx \\
&= \int_{-\frac{a}{2}}^{t+\frac{a}{2}} dx \\
&= a+t
\end{aligned}
$$

其余情形都有

$$
(\Pi_a * \Pi_a)(t) = 0
$$

所以

$$
(\Pi_a * \Pi_a)(t) = \begin{cases} a-|t| & |t| < a \\ 0 & |t| \geq a \end{cases}
$$

(b)因为

$$
\begin{aligned}
(f*f)(-t) &= \int_{-\infty}^{\infty} f(-t-x)f(x)dx \\
&= \int_{-\infty}^{\infty} f(t+x)f(-x)dx & \text{偶函数} \\
&= \int_{\infty}^{-\infty} f(t-x)f(x)d(-x) & \text{令}\ x=-x \\
&= \int_{-\infty}^{\infty} f(t-x)f(x)dx \\
&= (f*f)(t)
\end{aligned}
$$

即$(f*f)(t)$是偶函数，所以只需讨论$t \geq 0$的情形。

如果$t \geq 0$，那么

$$
\begin{aligned}
(f*f)(t) &= \int_{-\infty}^{\infty} f(t-x)f(x)dx \\
&= \int_{-\infty}^{\infty} e^{-|t-x|-|x|}dx \\
&= \int_{-\infty}^{0} e^{-t+2x}dx + \int_{0}^{t} e^{-t}dx + \int_{t}^{\infty} e^{t-2x}dx \\
&= e^{-t}\frac{1}{2} + e^{-t}t + e^{t}\frac{1}{2}e^{-2t} \\
&= e^{-t}(t+1)
\end{aligned}
$$

利用对称性可得当$t<0$时，

$$
(f*f)(t) = (f*f)(-t) = e^{t}(-t+1)
$$

因此

$$
(f*f)(t) = e^{-|t|}(|t|+1)
$$

(c)

$$
\begin{aligned}
(g*g)(t) &= \int_{-\infty}^{\infty} g(t-x)g(x)dx \\
&= \int_{-\infty}^{\infty} e^{-\pi(t-x)^2}e^{-\pi x^2}dx \\
&= \int_{-\infty}^{\infty} e^{-\pi(2x^2-2tx+t^2)}dx \\
&= \int_{-\infty}^{\infty} e^{-\pi\left(2(x-\frac{t}{2})^2+\frac{t^2}{2}\right)}dx \\
&= e^{-\frac{\pi t^2}{2}} \int_{-\infty}^{\infty} e^{-2\pi(x-\frac{t}{2})^2}dx \\
&= \frac{1}{\sqrt{2}}e^{-\frac{\pi t^2}{2}}
\end{aligned}
$$

(e)推广：

$$
(g*g*\ldots*g)(t) = \frac{1}{\sqrt{n}}e^{-\frac{\pi t^2}{n}}
$$

Problem 5

(a)

$$
\begin{aligned}
(f^- * g^-)(t) &= \int_{-\infty}^{\infty} f^-(t-x)g^-(x)dx \\
&= \int_{-\infty}^{\infty} f(x-t)g(-x)dx \\
&= \int_{\infty}^{-\infty} f(-t-x)g(x)d(-x) \qquad x=-x \\
&= \int_{-\infty}^{\infty} f(-t-x)g(x)dx \\
&= (f*g)(-t) \\
&= (f*g)^-(t)
\end{aligned}
$$

所以

$$
(f^- * g^-) = (f*g)^-
$$

另一方面

$$
\begin{aligned}
(f * g^-)(t) &= \int_{-\infty}^{\infty} f(t-x)g^-(x)dx \\
&= \int_{-\infty}^{\infty} f(t-x)g(-x)dx \\
&= \int_{\infty}^{-\infty} f^-(-t+x)g(-x)dx \\
&= \int_{\infty}^{-\infty} f^-(-t-x)g(x)d(-x) \qquad x=-x \\
&= \int_{-\infty}^{\infty} f^-(-t-x)g(x)dx \\
&= (f^- * g)^-(t)
\end{aligned}
$$

所以

$$
(f * g^-) = (f^- * g)^-
$$

(b)

$$
\begin{aligned}
((\tau_b f) * g)(t) &= \int_{-\infty}^{\infty} \tau_b f(t-x) g(x) dx \\
&= \int_{-\infty}^{\infty} f(t-x-b) g(x) dx \\
&= \int_{-\infty}^{\infty} f(t-b-x) g(x) dx \\
&= (f * g)(t-b) \\
&= (\tau_b (f * g))(t) \\
&= \int_{-\infty}^{\infty} f(t-x) g(x-b) dx \qquad x = x+b \\
&= \int_{-\infty}^{\infty} f(t-x) \tau_b g(x) dx \\
&= (f * (\tau_b g))(t)
\end{aligned}
$$

等式的含义为位移函数的卷积等于卷积的位移。

只需证明f的周期为T的情形：

$$
\begin{aligned}
(f * g)(t-T) &= \tau_T (f * g)(t) \\
&= ((\tau_T f) * g)(t) \\
&= (f * g)(t)
\end{aligned}
$$

(c)

$$
\begin{aligned}
((\sigma_a f) * g)(t) &= \int_{-\infty}^{\infty} f(at-ax) g(x) dx \\
&= \int_{-\infty}^{\infty} f(at-ax) \sigma_{1/a} g(ax) dx \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} f(at-x) \sigma_{1/a} g(x) dx \qquad x = ax \\
&= \frac{1}{|a|} \sigma_a \left(f * \left(\sigma_{1/a} g \right) \right) (t)
\end{aligned}
$$

所以

$$
(\sigma_a f) * g = \frac{1}{|a|} \sigma_a \left(f * \left(\sigma_{1/a} g \right) \right)
$$

另一方面

$$
\begin{aligned}
((\sigma_a f) * (\sigma_a g))(t) &= \int_{-\infty}^{\infty} f(at-ax) g(ax) dx \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} f(at-x) g(x) dx \qquad x = ax \\
&= \frac{1}{|a|} \sigma_a \left(f * g \right) (t)
\end{aligned}
$$

所以

$$(\sigma_a f) * (\sigma_a g) = \frac{1}{|a|}\sigma_a(f * g)$$

Problem 6

首先计算卷积

$$\int_{-\infty}^{\infty} \sin 2\pi(t-x)\sin 2\pi x dx = \frac{1}{2}\int_{-\infty}^{\infty}(\cos(2\pi(t-2x)) + \cos(2\pi t))\,dx$$

所以上述积分并不存在，上述例子说明两个周期函数的卷积并不存在。

Problem 7

(a)

$$p_n = \sum_{i+j+k=n} p_i p_j p_k$$

代码如下：

```
import numpy as np
import matplotlib.pyplot as plt

X = np.ones(6) / 6
Y = np.array([1, 1, 2, 2, 3, 3]) / 12
Z = np.array([4, 3, 2, 1, 1, 1]) / 12
data = np.arange(1, 7)
N = 6

####(a)
def P(n):
    p = 0
    for i in range(N):
        for j in range(N):
            for k in range(N):
                if i + j + k + 3 == n:
                    p += X[i] * Y[j] * Z[k]

    return p

res = [7, 8, 12]
p = 0
for n in res:
    p += P(n)
print(p)
```

```
0.2997685185185185
```

(b)注意到

$$
\begin{aligned}
\mathbb{E}[X_i] &= \frac{1}{6}\sum_{i=1}^{6} i \\
&= \frac{7}{2} \\
\mathbb{E}[Y_i] &= \frac{1\times 1+2\times 1+3\times 2+4\times 2+5\times 3+6\times 3}{12} \\
&= \frac{50}{12} \\
&= \frac{25}{6} \\
\mathbb{E}[Z_i] &= \frac{1\times 4+2\times 3+3\times 2+4\times 1+5\times 1+6\times 1}{12} \\
&= \frac{31}{12}
\end{aligned}
$$

由大数定律可得

$$
\begin{aligned}
\lim_{N\to\infty}\frac{1}{3N}\sum_{i=1}^{N}(X_i+Y_i+Z_i) &= \mathbb{E}[X_i]+\mathbb{E}[Y_i]+\mathbb{E}[Z_i] \\
&= \frac{1}{3}\left(\frac{7}{2}+\frac{25}{6}+\frac{31}{12}\right) \\
&= \frac{41}{12}
\end{aligned}
$$

(c)代码如下:

```
####(c)
def Print(n, M=100000):
    x = np.random.choice(data, p=X, size=(M, n))
    y = np.random.choice(data, p=Y, size=(M, n))
    z = np.random.choice(data, p=Z, size=(M, n))

    res = np.mean(x + y + z, axis=1) / 3
    plt.hist(res)
    plt.title("N={}".format(n))
    plt.show()

N = [2, 10, 100, 1000]
for n in N:
    Print(n)
```