Problem 1

(1)

对于$n \neq 0$

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_0^T e^{-2\pi int/T} f(t)dt \\
&= \frac{1}{2}\int_0^2 e^{-\pi int} t^2 dt \\
&= \frac{1}{2}\left(t^2 \frac{e^{-\pi int}}{-\pi in}\Big|_{t=0}^{t=2} - \int_0^2 \frac{e^{-\pi int}}{-\pi in} \times 2tdt\right) \\
&= -\frac{2}{\pi in} + \frac{1}{\pi in}\int_0^2 e^{-\pi int} \times tdt \\
&= -\frac{2}{\pi in} + \frac{1}{\pi in}\left(t\frac{e^{-\pi int}}{-\pi in}\Big|_{t=0}^{t=2} - \int_0^2 \frac{e^{-\pi int}}{-\pi in} dt\right) \\
&= -\frac{2}{\pi in} + \frac{2}{\pi^2 n^2}
\end{aligned}
$$

对于$n = 0$

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_0^T f(t)dt \\
&= \frac{1}{2}\int_0^2 t^2 dt \\
&= \frac{4}{3}
\end{aligned}
$$

(2)

$$
t^2 = \frac{4}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{2}{\pi in} + \frac{2}{\pi^2 n^2}\right) e^{\pi int}
$$

取$t = 0$，得到

$$
\begin{aligned}
\frac{0+2^2}{2} = 2 \\
&= \frac{4}{3} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{2}{\pi i n} + \frac{2}{\pi^2 n^2}\right) \\
&= \frac{4}{3} + \sum_{n=-\infty}^{-1}\left(-\frac{2}{\pi i n} + \frac{2}{\pi^2 n^2}\right) + \sum_{n=1}^{\infty}\left(-\frac{2}{\pi i n} + \frac{2}{\pi^2 n^2}\right) \\
&= \frac{4}{3} + 4\sum_{n=1}^{\infty}\frac{1}{\pi^2 n^2}
\end{aligned}
$$

所以

$$
\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}
$$

取$t=1$，得到

$$
\begin{aligned}
1 &= \frac{2}{3} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\frac{2}{\pi i n} + \frac{2}{\pi^2 n^2}\right)(-1)^{n+1} \\
&= \frac{2}{3} + \sum_{n=-\infty}^{-1}\left(-\frac{2}{\pi i n} + \frac{2}{\pi^2 n^2}\right)(-1)^{n+1} + \sum_{n=1}^{\infty}\left(-\frac{2}{\pi i n} + \frac{2}{\pi^2 n^2}\right)(-1)^{n+1} \\
&= \frac{2}{3} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi i n} + \frac{2}{\pi^2 n^2}\right)(-1)^{-n+1} + \sum_{n=1}^{\infty}\left(-\frac{2}{\pi i n} + \frac{2}{\pi^2 n^2}\right)(-1)^{n+1} \\
&= \frac{2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\pi^2 n^2}
\end{aligned}
$$

所以

$$
\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}
$$

将之前两个等式相加得到

$$
\frac{3\pi^2}{12} = \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \sum_{n=0}^{\infty}\frac{2}{(2n+1)^2}
$$

所以

$$
\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}
$$

Problem 2+

记

$$e_n(t) = \frac{1}{\sqrt{T}} e^{2\pi int/T}$$

不难得到

$$(e_n, e_m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

以及

$$(f, e_n) = \frac{1}{\sqrt{T}} \int_0^T f(t) e^{-2\pi int/T} dt = \sqrt{T} c_n$$

所以

$$f = \sum_{n=-\infty}^{\infty} (f, e_n)\, e_n$$

两边取模得到

$$\begin{aligned} \int_0^1 |f(t)|^2 dt &= \|f\|^2 \\ &= (f, f) \\ &= \left(\sum_{n=-\infty}^{\infty} (f, e_n)\, e_n, \sum_{m=-\infty}^{\infty} (f, e_m)\, e_m \right) \\ &= \sum_{n,m} (f, e_n)\, \overline{(f, e_m)}\, (e_n, e_m) \\ &= \sum_{n,m=-\infty}^{\infty} (f, e_n)\, \overline{(f, e_m)} \delta_{nm} \\ &= \sum_{n=-\infty}^{\infty} (f, e_n)\, \overline{(f, e_n)} \\ &= \sum_{n=-\infty}^{\infty} |(f, e_n)|^2 \\ &= \sum_{n=-\infty}^{\infty} T|c_n|^2 \end{aligned}$$

因此

$$\frac{1}{T} \int_0^1 |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Problem 3

略过

Problem 4

(a)回顾

$$\Lambda(x)=\begin{cases}1-|x| & |x|\le 1\\ 0 & \text{otherwise}\end{cases}$$

记

$$\Lambda_a(x)=\Lambda(\frac{x}{a})=\begin{cases}1-|\frac{x}{a}| & |\frac{x}{a}|\le 1\\ 0 & \text{otherwise}\end{cases}$$

不难看出上图的函数为

$$2\Lambda_2(x-2)+2.5\Lambda_2(x-4)$$

所以该函数的傅里叶变换为

$$2e^{-2\pi is2}2\operatorname{sinc}^2(2s)+2.5e^{-4\pi is2}2\operatorname{sinc}^2(2s)=\operatorname{sinc}^2(2s)\left(4e^{-4\pi is}+5e^{-8\pi is}\right)$$

(b)同(a)可得，考虑$[kT,(k+2)T]$区间，该区间对应的Λ函数为

$$f((k+1)T)\Lambda_T(x-(k+1)T)$$

所以

$$\begin{aligned}g(x)&=\sum_{k=0}^{n-2}f((k+1)T)\Lambda_T(x-(k+1)T)\\ \mathcal{F}g(s)&=\sum_{k=0}^{n-2}f((k+1)T)e^{-2(k+1)T\pi is}T\operatorname{sinc}^2(Ts)\\ &=T\operatorname{sinc}^2(Ts)\sum_{k=1}^{n-1}f(kT)e^{-2kT\pi is}\end{aligned}$$

Problem 5

(a)

$$
\begin{aligned}
\mathcal{F}g(s) &= \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\cos(2\pi s_0 t)dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-2\pi ist} f(t)\left(e^{2\pi is_0 t} + e^{-2\pi is_0 t}\right)dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-2\pi i(s-s_0)t} f(t)dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-2\pi i(s+s_0)t} f(t)dt \\
&= \frac{1}{2}\mathcal{F}f\left(s-s_0\right) + \frac{1}{2}\mathcal{F}f\left(s+s_0\right)
\end{aligned}
$$

(b)图形对应的函数为

$$\Lambda_2(x-4)+\Lambda_2(x+4)=\frac{1}{2}\times 2\Lambda_2(x-4)+\frac{1}{2}\times 2\Lambda_2(x+4)$$

注意到

$$\mathcal{F}\operatorname{sinc}^2=\Lambda$$

记

$$f(t)=4\operatorname{sinc}^2(2t)$$

那么

$$\mathcal{F}\left(f\right)=4\times\frac{1}{2}\Lambda_2=2\Lambda_2$$

结合(a)可得

$$g(t)=4\operatorname{sinc}^2(2t)\cos(8\pi t)$$

Problem 6

$$
\begin{aligned}
\hat{g}(n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi int} g(t)dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi int} \sum_{k=-\infty}^{\infty} f(t-k)dt \\
&= \sum_{k=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi int} f(t-k)dt \\
&= \sum_{k=-\infty}^{\infty} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} e^{-2\pi in(t+k)} f(t)dt \\
&= \sum_{k=-\infty}^{\infty} e^{-2\pi ink} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} e^{-2\pi int} f(t)dt \\
&= \sum_{k=-\infty}^{\infty} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} e^{-2\pi int} f(t)dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi int} f(t)dt \\
&= \mathcal{F}f(n)
\end{aligned}
$$

Problem 7

(a)因为

$$
\lim_{s\to 0} \frac{\sin(2\pi s) - 2\pi s}{4\pi^2 s^2} = \lim_{s\to 0} \frac{-\frac{1}{6}(2\pi s)^3}{4\pi^2 s^2} = 0
$$

所以$s=0$不是奇异点。

(b)因为$h(x)$是实，奇函数，所以$\mathcal{F}h(s)$是纯虚数，因此$\angle \mathcal{F}h(s)$为$\frac{\pi}{2}$或$-\frac{\pi}{2}$

(c)

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{F}g(s)\cos(\pi s)ds &= \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{F}g(s)\left(e^{i\pi s} + e^{-i\pi s}\right)ds \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\mathcal{F}g(s) + \mathcal{F}g(-s)\right)e^{i\pi s}ds
\end{aligned}
$$

注意到

$$
\begin{aligned}
\mathcal{F}g + (\mathcal{F}g)^- &= \mathcal{F}g + \mathcal{F}g^- \\
&= \mathcal{F}\left(g + g^-\right) \\
&= \mathcal{F}h
\end{aligned}
$$

因此

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{F}g(s)\cos(\pi s)ds &= \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{F}he^{2i\pi s\times\frac{1}{2}}ds \\
&= \frac{1}{2}\mathcal{F}^{-1}\mathcal{F}h(\frac{1}{2}) \\
&= \frac{1}{2}h(\frac{1}{2}) \\
&= \frac{1}{4}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{F}h(s)e^{4i\pi s}ds &= \int_{-\infty}^{\infty} \mathcal{F}h(s)e^{2i\pi s\times 2}ds \\
&= \mathcal{F}^{-1}\mathcal{F}h(2) \\
&= h(2) \\
&= 0
\end{aligned}
$$

(e)由对称性可得

$$
\begin{aligned}
\Re\mathcal{F}g(s) &= \frac{1}{2}\Re\mathcal{F}\Pi(s) \\
&= \frac{1}{2}\mathrm{sinc}^2(s)
\end{aligned}
$$

(f)依然由对称性，我们有

$$
\begin{aligned}
\Im\mathcal{F}h(s) &= 2\Im\mathcal{F}g(s) \\
&= 2\frac{\sin(2\pi s)-2\pi s}{4\pi^2 s^2} \\
&= \frac{\sin(2\pi s)-2\pi s}{2\pi^2 s^2}
\end{aligned}
$$

(g)

$$
\begin{aligned}
c_k &= \frac{1}{2}\int_{-1}^{1} e^{-\pi ikx}h(x)dx \\
&= \frac{1}{2}\int_{-1}^{1} e^{-\pi i2x(\frac{k}{2})}h(x)dx \\
&= \frac{1}{2}\mathcal{F}h(\frac{k}{2}) \\
&= i\frac{\sin(\pi k)-\pi k}{\pi^2 k^2} && \mathcal{F}h(s)\text{是纯虚数} \\
&= -\frac{i}{\pi k}
\end{aligned}
$$