Problem 1

(a)

$$\begin{aligned}
\underline{\mathcal{F}}\left(\tau_p \mathrm{f}\right)[m] &= \sum_{k=0}^{N-1} \tau_p \mathrm{f}[k] w^{-km} \\
&= \sum_{k=0}^{N-1} \mathrm{f}[k-p] w^{-km} \\
&= w^{-pm} \sum_{k=0}^{N-1} \mathrm{f}[k-p] w^{-(k-p)m} \\
&= w^{-pm} \underline{\mathcal{F}} \mathrm{f}[m] \\
&= (\underline{\omega}^{-p} \underline{\mathcal{F}} \mathrm{f})[m]
\end{aligned}$$

所以

$$\underline{\mathcal{F}}\left(\tau_p \mathrm{f}\right) = \omega^{-p} \underline{\mathcal{F}} \mathrm{f}$$

(b)

$$\begin{aligned}
\underline{\mathcal{F}} \underline{g}[m] &= \sum_{k=0}^{2N-1} \underline{g}[k] e^{-2\pi imk/2N} \\
&= \sum_{k=0}^{N-1} \underline{g}[k] e^{-2\pi imk/2N} + \sum_{k=N}^{2N-1} \underline{g}[k] e^{-2\pi imk/2N} \\
&= \sum_{k=0}^{N-1} \underline{f}[k] e^{-2\pi imk/2N} + \sum_{k=N}^{2N-1} \underline{f}[k-N] e^{-2\pi imk/2N} \\
&= \sum_{k=0}^{N-1} \underline{f}[k] e^{-2\pi imk/2N} + e^{-\pi im} \sum_{k=N}^{2N-1} \underline{f}[k-N] e^{-2\pi im(k-N)/2N} \\
&= \sum_{k=0}^{N-1} \underline{f}[k] e^{-2\pi imk/2N} + e^{-\pi im} \sum_{k=0}^{N-1} \underline{f}[k] e^{-2\pi imk/2N}
\end{aligned}$$

如果$m = 2l$，那么

$$\begin{aligned}
\underline{\mathcal{F}} \underline{g}[m] = \underline{\mathcal{F}} \underline{g}[2l] \\
&= \sum_{k=0}^{N-1} \underline{f}[k] e^{-2\pi i2lk/2N} + \sum_{k=0}^{N-1} \underline{f}[k] e^{-2\pi i2lk/2N} \\
&= 2\underline{\mathcal{F}} \underline{f}[l]
\end{aligned}$$

如果$m = 2l + 1$，那么

$$
\begin{aligned}
\underline{\mathcal{F}g}[m] &= \underline{\mathcal{F}g}[2l+1] \\
&= \sum_{k=0}^{N-1} \underline{f}[k] e^{-2\pi imk/2N} + e^{-\pi i(2l+1)} \sum_{k=0}^{N-1} \underline{f}[k] e^{-2\pi imk/2N} \\
&= 0
\end{aligned}
$$

(c)记

$$
w_k = e^{2\pi i/k}, \omega_k = (1, w_k, \dots, w_k^{k-1})
$$

那么

$$
\begin{aligned}
\underline{X} &= \mathcal{F}\underline{x} \\
&= \sum_{k=0}^{N-1} \underline{\mathrm{x}}[k] \omega_N^{-k} \\
\underline{\tilde{X}} &= \mathcal{F}\underline{\tilde{x}} \\
&= \sum_{k=0}^{N+M-1} \underline{\mathrm{x}}[k] \omega_{N+M}^{-k} \\
&= \sum_{k=0}^{N-1} \underline{\mathrm{x}}[k] \omega_{N+M}^{-k}
\end{aligned}
$$

考虑第m个分量，其中$0 \le m \le N-1$：

$$
\begin{aligned}
\underline{X}[m] &= \sum_{k=0}^{N-1} \underline{\mathrm{x}}[k] \omega_N^{-k}[m] \\
&= \sum_{k=0}^{N-1} \underline{\mathrm{x}}[k] w_N^{-km} \\
\underline{\tilde{X}}[m] &= \sum_{k=0}^{N-1} \underline{\mathrm{x}}[k] \omega_{N+M}^{-k}[m] \\
&= \sum_{k=0}^{N-1} \underline{\mathrm{x}}[k] w_{N+M}^{-km} \\
&= \sum_{k=0}^{N-1} \underline{\mathrm{x}}[k] e^{-2\pi i(km)/(N+M)} \\
&= \sum_{k=0}^{N-1} \underline{\mathrm{x}}[k] e^{-2\pi i(kmN/(N+M))/N} \\
&= \underline{X}[mN/(N+M)]
\end{aligned}
$$

使用zero-pad是为了将信号数量扩充成2的幂，方便使用FFT算法。

Problem 2

$$
\begin{aligned}
(\underline{1} * \underline{\mathrm{f}})[m] &= \sum_{k=0}^{N-1} 1[k]\mathrm{f}[m-k] \\
&= \sum_{k=0}^{N-1} \mathrm{f}[m-k] \\
&= \sum_{k=0}^{N-1} \mathrm{f}[k]
\end{aligned}
$$

所以

$$
\underline{1} * \underline{\mathrm{f}} = \left(\sum_{k=0}^{N-1} \mathrm{f}[k]\right) \underline{1}
$$

特别的，我们有

$$
\begin{aligned}
\underline{1} * \underline{1} &= N\underline{1} \\
\underline{1} * \underline{a} &= N\underline{a}
\end{aligned}
$$

Problem 3

(a)

$$
\begin{aligned}
\underline{\mathcal{F}}\underline{\mathrm{h}}[n] &= \sum_{k=0}^{2N-1} \underline{\mathrm{h}}[k]e^{-2\pi ink/2N} \\
&= \sum_{l=0}^{N-1} \underline{\mathrm{h}}[2l]e^{-2\pi in2l/2N} \\
&= \sum_{l=0}^{N-1} \underline{\mathrm{f}}[l]e^{-2\pi inl/N} \\
&= \underline{\mathcal{F}}\underline{\mathrm{f}}[n]
\end{aligned}
$$

(b)

$$
\begin{aligned}
(\underline{\mathcal{F}}\underline{\mathrm{g}})[n] &= \sum_{k=0}^{N/2-1} \underline{\mathrm{g}}[k]e^{-2\pi ink/(N/2)} \\
&= \sum_{k=0}^{N/2-1} \underline{\mathrm{f}}[2k]e^{-2\pi in(2k)/N}
\end{aligned}
$$

注意到

$$
\begin{aligned}
F[n] &= \sum_{k=0}^{N-1} \underline{\mathrm{f}}[k] e^{-2\pi i n k/N} \\
&= \sum_{k=0}^{N/2-1} \underline{\mathrm{f}}[2k] e^{-2\pi i n(2k)/N} + \sum_{k=0}^{N/2-1} \underline{\mathrm{f}}[2k+1] e^{-2\pi i n(2k+1)/N}
\end{aligned}
$$

那么

$$
\begin{aligned}
F[n-\frac{N}{2}] &= \sum_{k=0}^{N-1} \underline{\mathrm{f}}[k] e^{-2\pi i n k/N} \\
&= \sum_{k=0}^{N/2-1} \underline{\mathrm{f}}[2k] e^{-2\pi i (n-\frac{N}{2})(2k)/N} + \sum_{k=0}^{N/2-1} \underline{\mathrm{f}}[2k+1] e^{-2\pi i (n-\frac{N}{2})(2k+1)/N} \\
&= \sum_{k=0}^{N/2-1} \underline{\mathrm{f}}[2k] e^{-2\pi i n(2k)/N} - \sum_{k=0}^{N/2-1} \underline{\mathrm{f}}[2k+1] e^{-2\pi i n(2k+1)/N}
\end{aligned}
$$

从而

$$
\begin{aligned}
(\mathcal{F}\underline{\mathrm{g}})[n] &= \sum_{k=0}^{N/2-1} \underline{\mathrm{f}}[2k] e^{-2\pi i n(2k)/N} \\
&= \frac{1}{2}\left(F[n] + F[n-\frac{N}{2}]\right)
\end{aligned}
$$

Problem 4

该文件无法读取，略过。