10.2

(a)

特征值：

$$\det(\lambda I - A) = \lambda^2 + w^2 = 0 \Rightarrow \lambda = \pm iw$$

resolvent：

$$\begin{aligned}(sI - A)^{-1} &= \begin{bmatrix} s & -\omega \\ \omega & s \end{bmatrix} \\ &= \begin{bmatrix} \frac{s}{s^2+w^2} & \frac{\omega}{s^2+w^2} \\ -\frac{\omega}{s^2+w^2} & \frac{s}{s^2+w^2} \end{bmatrix}\end{aligned}$$

状态转移矩阵：

$$\begin{aligned}\Phi(t) &= \mathcal{L}^{-1}\left((sI - A)^{-1}\right) \\ &= \begin{bmatrix} \cos wt & \sin wt \\ -\sin wt & \cos wt \end{bmatrix}\end{aligned}$$

所以

$$x(t) = \begin{bmatrix} \cos wt & \sin wt \\ -\sin wt & \cos wt \end{bmatrix} x(0)$$

(b)略过

(c)因为$\begin{bmatrix} \cos wt & \sin wt \\ -\sin wt & \cos wt \end{bmatrix}$是正交矩阵，所以结论成立。

(d)

$$\begin{aligned}\frac{d}{dt}\|x(t)\|^2 &= \frac{d}{dt}\left(x(t)^T x(t)\right) \\ &= 2\dot{x}(t)^T x(t) \\ &= x(t)^T \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x(t) \\ &= 0\end{aligned}$$

所以

$$\dot{x}(t)^T x(t) = 0$$

结论成立。

10.3

(a)

$$
\begin{aligned}
e^A e^B &= \left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right) \\
&= \sum_{k=0}^{\infty} \sum_{i+j=k} \frac{1}{i!j!} A^i B^j \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i+j=k} \frac{k!}{i!j!} A^i B^j \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} (A+B)^k \qquad\qquad \text{由 } AB = BA \\
&= e^{(A+B)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{d}{dt} e^{At} &= \frac{d}{dt}\left(\sum_{i=0}^{\infty} \frac{(At)^i}{i!}\right) \\
&= \sum_{i=1}^{\infty} \frac{A^i t^{i-1}}{(i-1)!} \\
&= A \sum_{i=1}^{\infty} \frac{(At)^{i-1}}{(i-1)!} \\
&= A e^{At} \\
&= \left(\sum_{i=1}^{\infty} \frac{(At)^{i-1}}{(i-1)!}\right) A \\
&= e^{At} A
\end{aligned}
$$

10.4

(a)因为

$$
A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = 0
$$

所以

$$
\begin{aligned}
e^{tA} &= \sum_{i=0}^{\infty} \frac{(tA)^i}{i!} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t \\
&= \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}
\end{aligned}
$$

取$t=1$得到

$$e^A=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}$$

(b)由结论可得

$$\begin{aligned}x(t)&=e^{tA}x(0)\\&=\begin{bmatrix}1-t & t\\ -t & 1+t\end{bmatrix}\begin{bmatrix}1\\ a\end{bmatrix}\end{aligned}$$

令$t=1$得到

$$\begin{aligned}x(1)&=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}1\\ a\end{bmatrix}\\&=\begin{bmatrix}a\\ 2a-1\end{bmatrix}\end{aligned}$$

由题意可得

$$2a-1=2\Rightarrow a=\frac{3}{2}$$

因此

$$\begin{aligned}x(t)&=\begin{bmatrix}1-t & t\\ -t & 1+t\end{bmatrix}\begin{bmatrix}1\\ \frac{3}{2}\end{bmatrix}\\&=\begin{bmatrix}1+\frac{1}{2}t\\ \frac{3}{2}+\frac{1}{2}t\end{bmatrix}\\x(2)&=\begin{bmatrix}2\\ \frac{5}{2}\end{bmatrix}\end{aligned}$$

补充题

1

求导可得

$$\begin{aligned}\dot{x}(t)&=a(t)\exp\left(\int_0^t a(\tau)d\tau\right)x(0)\\&=a(t)x(t)\end{aligned}$$

作为反例，考虑（参考自解答）

$$A(t)=\begin{cases}A_1 & 0\le t<1\\ A_2 & t\ge 1\end{cases}$$

那么

$$
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
$$

那么

$$
\begin{aligned}
x(2) &= (\exp A_2)\,(\exp A_1)\,x(0) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(0) \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x(0)
\end{aligned}
$$

但是，在上述公式中

$$
\begin{aligned}
\int_0^2 A(t)dt &= \int_0^1 A_1 dt + \int_1^2 A_2 dt \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&\triangleq B
\end{aligned}
$$

不难验证

$$
B^{2k} = I, B^{2k+1} = B
$$

所以

$$
\begin{aligned}
\exp(tB) &= \sum_{i=0}^{\infty} \frac{B^i}{i!} \\
&= \sum_{k=0}^{\infty} \frac{B^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{B^{2k+1}}{(2k+1)!} \\
&= I \sum_{k=0}^{\infty} \frac{1}{(2k)!} + B \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \\
&= I \frac{e + e^{-1}}{2} + B \frac{e - e^{-1}}{2} \\
&= \begin{bmatrix} 1.5431 & 1.1752 \\ 1.1752 & 1.5431 \end{bmatrix}
\end{aligned}
$$

因此

$$
x(2) = \begin{bmatrix} 1.5431 & 1.1752 \\ 1.1752 & 1.5431 \end{bmatrix} x(0)
$$

所以上述事实对高维情形不成立。