EE263 homework 5 solutions

10.2 *Harmonic oscillator.* The system $\dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x$ is called a *harmonic oscillator.*

(a) Find the eigenvalues, resolvent, and state transition matrix for the harmonic oscillator. Express $x(t)$ in terms of $x(0)$.

(b) Sketch the vector field of the harmonic oscillator.

(c) The state trajectories describe circular orbits, *i.e.*, $\|x(t)\|$ is constant. Verify this fact using the solution from part (a).

(d) You may remember that circular motion (in a plane) is characterized by the velocity vector being orthogonal to the position vector. Verify that this holds for any trajectory of the harmonic oscillator. Use only the differential equation; do not use the explicit solution you found in part (a).

Solution:

(a) We have

$$(sI - A)^{-1} = \frac{1}{s^2 + \omega^2} \begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix}.$$

From this result it follows that the eigenvalues of A are given by $\{\pm j\omega\}$. The inverse Laplace transform gives

$$\Phi(t) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

and we have $x(t) = \Phi(t)x(0)$.

(b) Here is the vector field:

(c) First we note from basic trigonometric relations that $\Phi^T(t)\Phi(t) = I$. From this we conclude that $\Phi(t)$ is *orthogonal*. Now it follows that $x^T(t)x(t) = x^T(0)\Phi^T(t)\Phi(t)x(0) = x^T(0)x(0)$, i.e. $\|x(t)\| = \|x(0)\|$.

(d) Using previous relations we can write

$$\dot{x}^T x = x^T \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x = \begin{bmatrix} -\omega x_2 & \omega x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

This shows that the velocity vector is always orthogonal to the position vector, as claimed.

10.3 *Properties of the matrix exponential.*

(a) Show that $e^{A+B} = e^A e^B$ if A and B commute, *i.e.*, $AB = BA$.

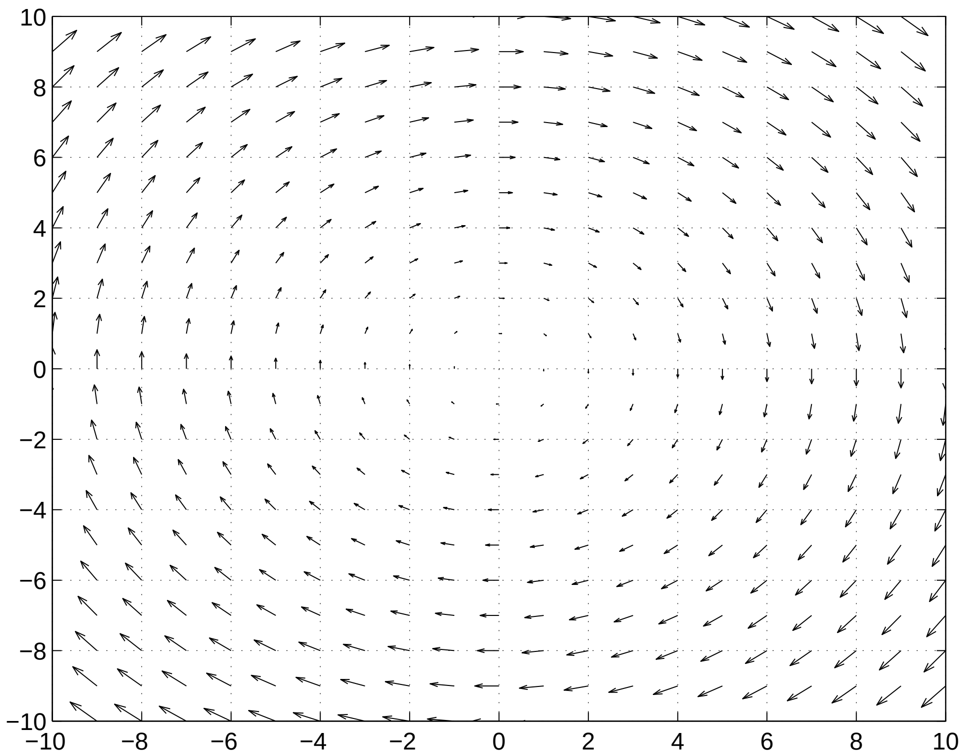

Figure 1: Vector field of harmonic oscillator

(b) Carefully show that $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.

Solution:

(a) We will show that if A and B commute then $e^A e^B = e^{A+B}$. We begin by writing the expressions for e^A and e^B

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots$$

Now we multiply both expressions and get

$$\begin{aligned} e^A e^B &= I + A + B + AB + \frac{A^2}{2!} + \frac{B^2}{2!} + \frac{A^3}{3!} + \frac{A^2 B}{2!} + \frac{AB^2}{2!} + \frac{B^3}{3!} + \cdots \\ &= I + A + B + \frac{A^2 + 2AB + B^2}{2!} + \frac{A^3 + 3A^2B + 3AB^2 + B^3}{3!} + \cdots \end{aligned}$$

Now we note that, if A and B commute, we are able to write things such as $(A+B)^2 = A^2 + 2AB + B^2$. So, if A and B commute we can finally write

$$e^A e^B = I + (A+B) + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} + \cdots = e^{A+B}$$

(b) It suffices to note that A commute with itself. Then one can write

$$\begin{aligned}\frac{de^{At}}{dt} &= A + A^2t + \frac{A^3t^2}{2!} + \cdots \\ &= A(I + At + \frac{(At)^2}{2!} + \cdots) \\ &= (I + At + \frac{(At)^2}{2!} + \cdots)A \\ &= Ae^{At} = e^{At}A\end{aligned}$$

10.4 *Two-point boundary value problem.* Consider the system described by $\dot{x} = Ax$, where $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) Find e^A.

(b) Suppose $x_1(0) = 1$ and $x_2(1) = 2$. Find $x(2)$. (This is called a *two-point boundary value problem*, since we are given conditions on the state at two time points instead of the usual single initial point.)

Solution:

(a) Many methods can be used to find e^A. In this case, power series expansion may be the easiest, since $A^k = A^2 = 0$ for all $k \geq 2$:

$$e^A = I + A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}.$$

(b) Expanding the equation $x(2) = e^A x(1) = e^{2A}x(0)$ yields

$$\begin{aligned}\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1(1) \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}^2\begin{bmatrix} 1 \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} 2x_2(0) - 1 \\ 3x_2(0) - 2 \end{bmatrix}.\end{aligned}$$

Examining the first line yields $x_1(2) = 2$ and so $x_2(0) = 1.5$; substituting into the last yields $x(2) = [\,2 \quad 2.5\,]^T$.

Solutions to additional exercises

1. *Scalar time-varying linear dynamical system.* Show that the solution of $\dot{x}(t) = a(t)x(t)$, where $x(t) \in \mathbf{R}$, is given by

$$x(t) = \exp\left(\int_0^t a(\tau)\, d\tau\right) x(0).$$

(You can just differentiate this expression, and show that it satisfies $\dot{x}(t) = a(t)x(t)$.) Find a specific example showing that the analogous formula does not hold when $x(t) \in \mathbf{R}^n$, with $n > 1$.

Solution. Differentiating the given expression, we obtain

$$\begin{aligned} \dot{x}(t) &= \left(\frac{d}{dt}\int_0^t a(\tau)\, d\tau\right) \exp\left(\int_0^t a(\tau)\, d\tau\right) x(0) \\ &= a(t) \exp\left(\int_0^t a(\tau)\, d\tau\right) x(0) \\ &= a(t)x(t). \end{aligned}$$

For the second part, we look for a counterexample with $x(t) \in \mathbf{R}^2$. We let

$$A(t) = \begin{cases} A_1 & 0 \le t < 1 \\ A_2 & t \ge 1, \end{cases}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned} x(2) &= (\exp A_2)(\exp A_1)x(0) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(0) \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x(0). \end{aligned}$$

The formula above gives

$$\begin{aligned} x(2) &= \exp(A_1 + A_2)x(0) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(0) \\ &= \begin{bmatrix} 1.5431 & 1.1752 \\ 1.1752 & 1.5431 \end{bmatrix} x(0). \end{aligned}$$

Choosing almost any $x(0)$ (*e.g.*, $x(0) = e_1$) will give us a contradiction.