## 5.2

(a)

因为

$$\begin{aligned}\langle u, v\rangle &= u^* v \\ &= (\Re u - i\Im u)^T(\Re v + i\Im v) \\ &= (\Re u)^T\Re v + (\Im u)^T\Im v + ((\Re u)^T\Im v - (\Re v)^T\Im u)i \\ \langle \tilde{u}, \tilde{v}\rangle &= \begin{bmatrix}\Re u \\ \Im u\end{bmatrix}^T \begin{bmatrix}\Re v \\ \Im v\end{bmatrix} \\ &= (\Re u)^T\Re v + (\Im u)^T\Im v\end{aligned}$$

所以

$$\langle \tilde{u}, \tilde{v}\rangle = \Re\langle u, v\rangle \tag{1}$$

(b)

$$\begin{aligned}\|u\| &= \left(|u_1|^2 + \cdots + |u_n|^2\right)^{1/2} \\ &= \left(|\Re u_1|^2 + |\Im u_1|^2 + \cdots + |\Re u_n|^2 + |\Im u_n|^2\right)^{1/2} \\ &= \left(|\Re u|^2 + |\Im u|^2\right)^{1/2} \\ &= \|\tilde{u}\|\end{aligned} \tag{2}$$

另解：对(1)中取

$$u = v$$

那么由于

$$(\Re u)^T\Im v - (\Re v)^T\Im u = (\Re u)^T\Im u - (\Re u)^T\Im u = 0$$

即虚部为0，所以

$$\|\tilde{u}\|^2 = \langle \tilde{u}, \tilde{u}\rangle = \langle u, u\rangle = \|u\|^2$$

(c)

$$
\begin{aligned}
Au &= (\Re A + i\Im A)\,(\Re u + i\Im u) \\
&= \Re A\Re u - \Im A\Im u + (\Re A\Im u + \Im A\Re u)\, i \\
\tilde{A}\tilde{u} &= \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \begin{bmatrix} \Re u \\ \Im u \end{bmatrix} \\
&= \begin{bmatrix} \Re A\Re u - \Im A\Im u \\ \Re A\Im u + \Im A\Re u \end{bmatrix} \\
&= \widetilde{(Au)}
\end{aligned} \tag{3}
$$

(d)

$$
\begin{aligned}
\tilde{A}^T &= \begin{bmatrix} \Re A^T & \Im A^T \\ -\Im A^T & \Re A^T \end{bmatrix} \\
A^* &= \Re A^T - i\Im A^T \\
\tilde{A^*} &= \begin{bmatrix} \Re A^T & \Im A^T \\ -\Im A^T & \Re A^T \end{bmatrix} \\
&= \tilde{A}^T
\end{aligned} \tag{4}
$$

(e)补充证明如下结论：

$$
\widetilde{(A^{-1})} = (\tilde{A})^{-1} \tag{5}
$$

首先，我们有

$$
\begin{aligned}
I_n &= A^{-1}A \\
&= \left(\Re(A^{-1}) + i\Im(A^{-1})\right)(\Re A + i\Im A) \\
&= \left(\Re(A^{-1})\Re A - \Im(A^{-1})\Im A\right) + i\left(\Im(A^{-1})\Re A + \Re(A^{-1})\Im A\right)
\end{aligned}
$$

所以

$$
\begin{gathered}
\Re(A^{-1})\Re A - \Im(A^{-1})\Im A = I_n \\
\Im(A^{-1})\Re A + \Re(A^{-1})\Re A = 0
\end{gathered}
$$

其次我们有

$$
\widetilde{(A^{-1})} = \begin{bmatrix} \Re(A^{-1}) & -\Im(A^{-1}) \\ \Im(A^{-1}) & \Re(A^{-1}) \end{bmatrix}
$$

所以

$$
\begin{aligned}
\widetilde{(A^{-1})}\tilde{A} &= \begin{bmatrix} \Re(A^{-1}) & -\Im(A^{-1}) \\ \Im(A^{-1}) & \Re(A^{-1}) \end{bmatrix} \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \\
&= \begin{bmatrix} \Re(A^{-1})\Re A - \Im(A^{-1})\Im A & -\Re(A^{-1})\Im A - \Im(A^{-1})\Re A \\ \Re(A^{-1})\Im A + \Im(A^{-1})\Re A & \Re(A^{-1})\Re A - \Im(A^{-1})\Im A \end{bmatrix} \\
&= \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \\
&= I_{2n}
\end{aligned}
$$

因此结论成立。

由之前的记号可得

$$
\begin{aligned}
\|Ax-y\|^2 &= \|\widetilde{Ax-y}\|^2 && \text{由}\,(2) \\
&= \langle \widetilde{Ax-y}, \widetilde{Ax-y} \rangle \\
&= \langle \tilde{A}\tilde{x}-\tilde{y}, \tilde{A}\tilde{x}-\tilde{y} \rangle && \text{由}\,(3) \\
&= \|\tilde{A}\tilde{x}-\tilde{y}\|^2
\end{aligned}
$$

由最小二乘法，我们知道该问题的解为

$$
\tilde{x} = (\tilde{A}^T\tilde{A})^{-1}\tilde{A}^T\tilde{y}
$$

下面证明

$$
\tilde{x}_{\mathrm{ls}} = \tilde{x}
$$

即

$$
(A^*\widetilde{A)^{-1}}A^*y = (\tilde{A}^T\tilde{A})^{-1}\tilde{A}^T\tilde{y}
$$

证明如下：

$$
\begin{aligned}
(A^*\widetilde{A)^{-1}}A^*y &= (\widetilde{A^*A)}^{-1}\tilde{A}^*\tilde{y} && \text{由}\,(3) \\
&= (\widetilde{A^*A})^{-1}\tilde{A}^*\tilde{y} && \text{由}\,(5) \\
&= \left(\tilde{A}^*\tilde{A}\right)^{-1}\tilde{A}^*\tilde{y} && \text{由}\,(3) \\
&= (\tilde{A}^T\tilde{A})^{-1}\tilde{A}^T\tilde{y} && \text{由}\,(4)
\end{aligned}
$$

## 6.2

(a)首先目标函数为

$$
\begin{aligned}
\int_{t=0}^{10} f(t)^2 dt &= \sum_{i=1}^{10} \int_{i-1}^{i} f(t)^2 dt \\
&= \sum_{i=1}^{10} \int_{i-1}^{i} x_i^2 dt \\
&= \sum_{i=1}^{10} x_i^2 \\
&= \|x\|^2
\end{aligned}
$$

$\forall k, j-1 < k \le j$，那么由牛顿第二定律可得，

$$
\begin{aligned}
\dot{p}(k) &= \int_0^k f(t)dt \\
&= \sum_{i=1}^{j-1} \int_{i-1}^{i} f(t)dt + \int_{j-1}^{k} f(t)dt \\
&= \sum_{i=1}^{j-1} \int_{i-1}^{i} x_i dt + \int_{j-1}^{k} x_j dt \\
&= \sum_{i=1}^{j-1} x_i + (k-j+1)x_j
\end{aligned}
$$

任取整数$k$，我们有

$$
\begin{aligned}
p(k) &= \int_0^k \dot{p}(t)dt \\
&= \sum_{i=1}^{k} \int_{i-1}^{i} \dot{p}(t)dt \\
&= \sum_{i=1}^{k} \int_{i-1}^{i} \left( \sum_{k=1}^{i-1} x_k + (t-i+1)x_i \right) dt \\
&= \sum_{i=1}^{k} \left( \sum_{k=1}^{i-1} x_k + \frac{1}{2} x_i \right) dt \\
&= \sum_{i=1}^{k} \left( k - i + \frac{1}{2} \right) x_i
\end{aligned}
$$

所以题目中的约束条件为

$$
\begin{aligned}
\sum_{i=1}^{10}\left(10-i+\frac{1}{2}\right)x_i &= 1 \\
\sum_{i=1}^{9}x_i+(10-10+1)x_j=\sum_{i=1}^{10}x_i &= 0 \\
\sum_{i=1}^{5}\left(5-i+\frac{1}{2}\right)x_i &= 0
\end{aligned}
$$

写成矩阵的形式为

$$
Ax=b
$$

其中

$$
\begin{aligned}
A &= \begin{bmatrix} 9.5 & 8.5 & 7.5 & 6.5 & 5.5 & 4.5 & 3.5 & 2.5 & 1.5 & 0.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4.5 & 3.5 & 2.5 & 1.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
b &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

所以优化问题为

$$
\begin{aligned}
\min \quad & \|x\|^2 = x^Tx \\
\text{subject to} \quad & Ax=b
\end{aligned}
$$

构造拉格朗日乘子：

$$
L(x,\lambda)=x^Tx-\lambda^T(Ax-b)
$$

求梯度并令梯度为0得到：

$$
\begin{aligned}
\nabla_x L(x,\lambda) &= 2x-2A^T\lambda=0 \\
\nabla_\lambda L(x,\lambda) &= Ax-b=0
\end{aligned}
$$

解得

$$
\begin{aligned}
\lambda &= \left(AA^T\right)^{-1}y \\
x &= A^T\left(AA^T\right)^{-1}y
\end{aligned}
$$

matlab中使用如下命令计算即可：

```
x = pinv(A) * b
```

完整代码如下：

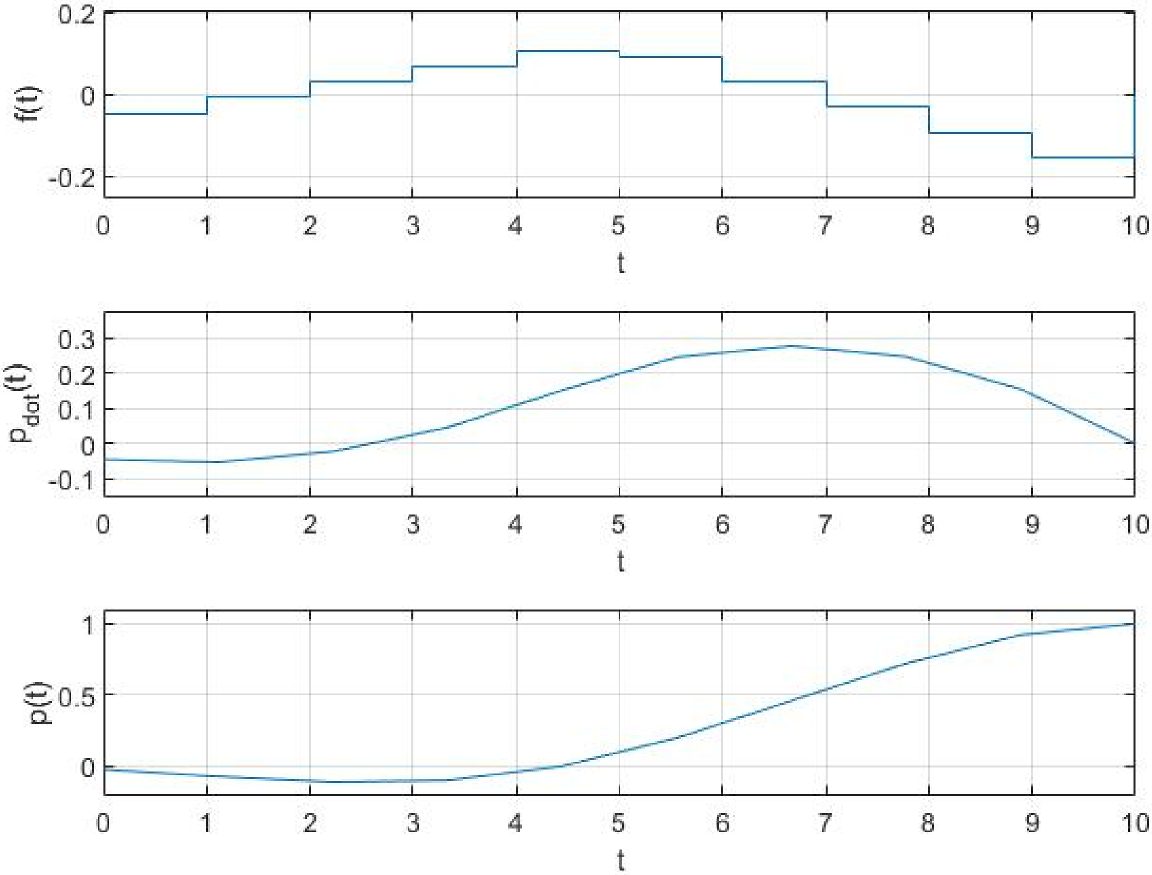
```
% (a)
a1 = linspace(9.5, 0.5, 10);
a2 = ones(1,10);
a3 = [linspace(4.5, 0.5, 5), 0 0 0 0 0];

A = [a1; a2; a3];
b = [1; 0; 0];
x = pinv(A) * b
```

```
x =
   -0.0455
   -0.0076
    0.0303
    0.0682
    0.1061
    0.0939
    0.0318
   -0.0303
   -0.0924
   -0.1545
```

作图代码：

```
% 作图
% f(t)
figure(1);
subplot(3,1,1);
stairs(0:10, [x; 0]);
grid on;
xlabel('t');
ylabel('f(t)');
axis([0, 10, min(x) - 0.1, max(x) + 0.1]);

% p_dot(t)
T1 = toeplitz(ones(10,1), [1,zeros(1,9)]);
p_dot = T1 * x;
subplot(3,1,2);
plot(linspace(0, 10, 10), p_dot);
grid on;
xlabel('t');
ylabel('p_{dot}(t)');
axis([0, 10, min(p_dot) - 0.1, max(p_dot) + 0.1]);

% p(t)
T2 = toeplitz(linspace(0.5, 9.5, 10)', [0.5, zeros(1,9)]);
p = T2 * x;
subplot(3,1,3);
plot(linspace(0, 10, 10), p);
grid on;
xlabel('t');
ylabel('p(t)');
axis([0, 10, min(p) - 0.1, max(p) + 0.1]);
```

结果如下:




(b)首先

$$
\begin{aligned}
J_2 &= \int_{t=0}^{10} f(t)^2 dt \\
&= \sum_{i=1}^{10} \int_{i-1}^{i} f(t)^2 dt \\
&= \sum_{i=1}^{10} \int_{i-1}^{i} x_i^2 dt \\
&= \sum_{i=1}^{10} x_i^2 \\
&= \|x\|^2
\end{aligned}
$$

其次$\forall k, j-1<k\le j$，那么由牛顿第二定律可得，

$$
\begin{aligned}
\dot p(k) &= \dot p(0) + \int_0^k f(t)dt \\
&= 1 + \sum_{i=1}^{j-1}\int_{i-1}^{i} f(t)dt + \int_{j-1}^{k} f(t)dt \\
&= 1 + \sum_{i=1}^{j-1}\int_{i-1}^{i} x_i dt + \int_{j-1}^{k} x_j dt \\
&= \sum_{i=1}^{j-1} x_i + (k-j+1)x_j + 1
\end{aligned}
$$

以及任取整数$k$:

$$
\begin{aligned}
p(k) &= \int_0^k \dot p(t)dt \\
&= \sum_{i=1}^{k}\int_{i-1}^{i} \dot p(t)dt \\
&= \sum_{i=1}^{k}\int_{i-1}^{i} \left(\sum_{k=1}^{i-1} x_k + (t-i+1)x_i + 1\right) dt \\
&= \sum_{i=1}^{k} \left(\sum_{k=1}^{i-1} x_k + \frac{1}{2}x_i + 1\right) dt \\
&= k + \sum_{i=1}^{k} \left(k - i + \frac{1}{2}\right) x_i
\end{aligned}
$$

此时的约束条件为

$ \begin{aligned} p(10)& =0\\ \dot p(10)& =0 \end{aligned} $

而

$$
\begin{bmatrix} p(10) \\ \dot p(10) \end{bmatrix} = \begin{bmatrix} 9.5 & 8.5 & 7.5 & 6.5 & 5.5 & 4.5 & 3.5 & 2.5 & 1.5 & 0.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 10 \\ 1 \end{bmatrix}
$$

记

$$
\begin{aligned}
A &= \begin{bmatrix} 9.5 & 8.5 & 7.5 & 6.5 & 5.5 & 4.5 & 3.5 & 2.5 & 1.5 & 0.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\
y &= \begin{bmatrix} -10 \\ -1 \end{bmatrix}
\end{aligned}
$$

那么

$$
\begin{aligned}
J_1 &= p(10)^2 + \dot p(10)^2 \\
&= \|Ax - y\|^2
\end{aligned}
$$

所以优化函数为

$$J_1 + \mu J_2 = \|Ax - y\|^2 + \mu \|x\|^2$$

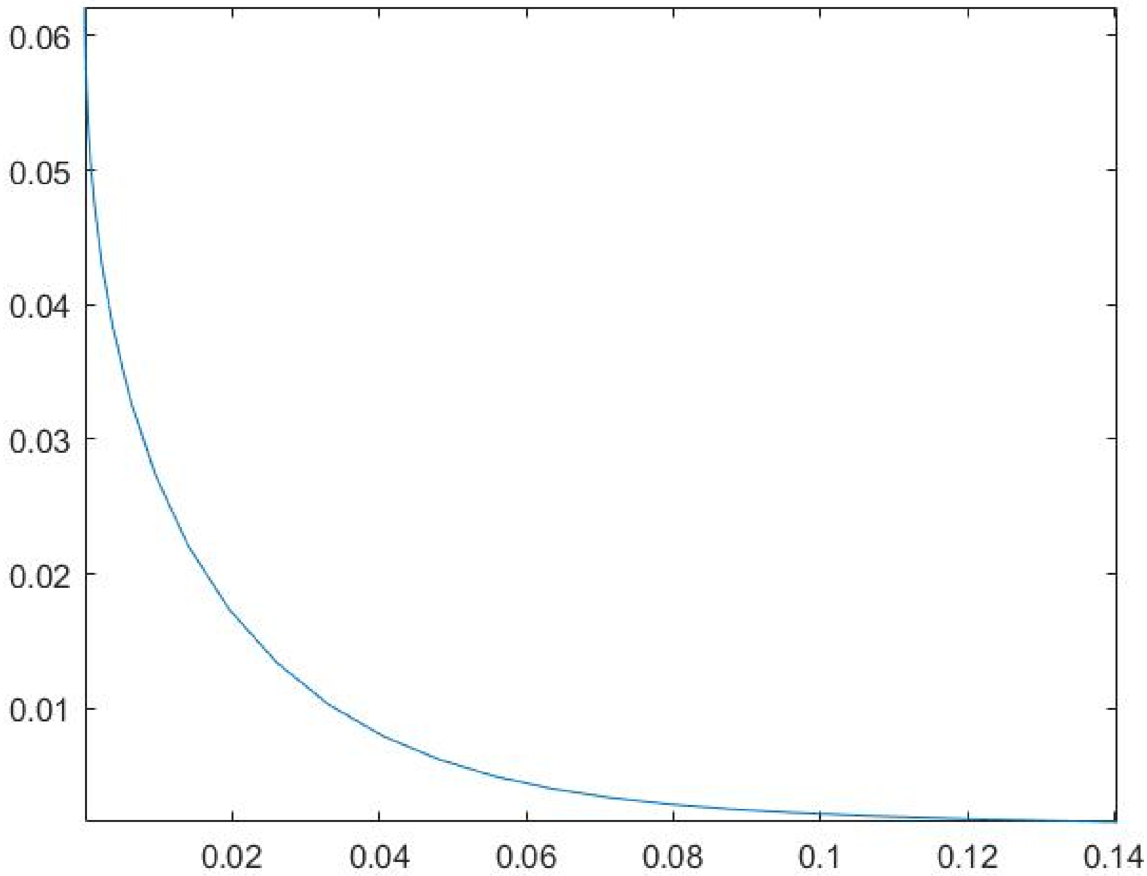
因此

$$x_\mu = \left(A^T A + \mu I\right)^{-1} A^T y$$

代码如下:

```
% (b)
% 作图
N = 50;
[d, m] = size(A);
Mu = logspace(-5, 2, N);
J1 = zeros(N, 1);
J2 = zeros(N, 1);

for i = 1:N
    mu = Mu(i);
    %x = inv(A' * A + mu * eye(m))  * A' * b;
    x = (A' * A + mu * eye(m)) \ (A' * b);
    j1 = norm(A * x - b) ^ 2;
    j2 = norm(x) ^ 2;
    J1(i) = j1;
    J2(i) = j2;
end

figure(2);
plot(J1, J2);
axis tight;
```

结果如下:




### 6.5

将$f(x)$的定义代入

$$f(x_i) = y_i$$

得到

$$\frac{\sum_{j=0}^{m} a_j x_i^j}{1+\sum_{j=1}^{m} b_j x_i^j} = y_i$$

$$\sum_{j=0}^{m} a_j x_i^j = y_i\left(1+\sum_{j=1}^{m} b_j x_i^j\right)$$

$$a_0 + \sum_{j=1}^{m} a_j x_i^j + \sum_{j=1}^{m} b_j(-y_i x_i^j) = y_i$$

记

$$
X_1=\begin{bmatrix}x_1 & x_1^2 & \cdots & x_1^m\\ \vdots & \vdots & \ddots & \vdots\\ x_N & x_N^2 & \cdots & x_N^m\end{bmatrix}
$$
$$
X_2=\begin{bmatrix}-y_1x_1 & -y_1x_1^2 & \cdots & -y_1x_1^m\\ \vdots & \vdots & \ddots & \vdots\\ -y_Nx_N & -y_Nx_N^2 & \cdots & -y_Nx_N^m\end{bmatrix}=-\mathrm{diag}(y)X_1
$$
$$
X=\begin{bmatrix}1_N & X_1 & X_2\end{bmatrix}\in\mathbb{R}^{N\times(2m+1)}
$$
$$
c=\begin{bmatrix}a_0\\ \vdots\\ a_m\\ b_1\\ \vdots\\ b_m\end{bmatrix}\in\mathbb{R}^{2m+1}
$$

所以线性方程组为

$$
Xc=y
$$

要使得该方程有解，应该找到最小的$m$，使得

$$
\mathrm{rank}(X)=\mathrm{rank}(X,y)
$$

代码如下：

```
% 初始化
m = 1;
X1 = x;

while 1
    X1 = [X1, x .* X1(:, m)];
    X2 = - diag(y) * X1;
    X = [ones(N, 1), X1, X2];
    m = m + 1;
    % 判断
    res = (rank(X) == rank([X, y]));

    if res
        break
    end
end

m
```

```
5
```

计算系数:

```
%计算结果
c = X \ y;
a = c(1: m + 1)
b = c(m + 2: 2 * m + 1)
```

```
a =
    0.2742
    1.0291
    1.2906
   -5.8763
   -2.6738
    6.6845
b =
   -1.2513
   -6.5107
    3.2754
   17.3797
    6.6845
```

### 6.12

记

$$y=\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix},v=\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix},w=\begin{bmatrix}w_1\\ \vdots\\ w_m\end{bmatrix},s=\begin{bmatrix}1\\ \vdots\\ m\end{bmatrix}$$

所以

$$v=\alpha 1_n+\beta Ts+w$$

所以模型为

$$y=Ax+v=Ax+\alpha 1_m+\beta Ts+w$$

注意$\alpha,\beta$未知，所以记

$$\tilde x=\begin{bmatrix}x\\ \alpha\\ \beta\end{bmatrix}\in\mathbb R^{n+2},\tilde A=\begin{bmatrix}A & 1_m & Ts\end{bmatrix}\in\mathbb R^{m\times(n+2)}$$

模型修改为

$$y=\tilde A\tilde x+w$$

利用最小二乘法求解该问题即可，正规方程为

$$\tilde{A}^T\tilde{A}\tilde{x}=\tilde{A}^Tw$$

如果$\tilde{A}$满秩，那么该问题的解为

$$\tilde{x}=\left(\tilde{A}^T\tilde{A}\right)^{-1}\tilde{A}^Tw$$

## 6.14

(a)

$$\begin{aligned}
J&=\sum_{k=1}^{N}\left\|Ax^{(k)}-y^{(k)}\right\|^2\\
&=\sum_{k=1}^{N}\left(Ax^{(k)}-y^{(k)}\right)^T\left(Ax^{(k)}-y^{(k)}\right)\\
&=\sum_{k=1}^{N}\left(\left(x^{(k)}\right)^TA^TAx^{(k)}+\left(y^{(k)}\right)^Ty^{(k)}-2\left(y^{(k)}\right)^TAx^{(k)}\right)\\
&=\operatorname{trace}\left(X^TA^TAX+Y^TY-2Y^TAX\right)
\end{aligned}$$

接着利用如下公式求梯度：

$$\begin{aligned}
\nabla_X\operatorname{trace}(AXB)&=A^TB^T\\
\nabla_X\operatorname{trace}\left(AXX^TB\right)&=A^TB^TX+BAX
\end{aligned}$$

我们有

$$\begin{aligned}
\nabla_AJ&=\nabla_A\operatorname{trace}\left(X^TA^TAX\right)-2\nabla_A\operatorname{trace}\left(Y^TAX\right)\\
&=\left(\nabla_{A^T}\operatorname{trace}\left(X^TA^TAX\right)\right)^T-2YX^T\\
&=\left(XX^TA^T+XX^TA^T\right)^T-2YX^T\\
&=2AXX^T-2YX^T
\end{aligned}$$

令梯度为0得到

$$\begin{aligned}
AXX^T&=YX^T\\
A&=YX^T\left(XX^T\right)^{-1}
\end{aligned}$$

(b)实现上述算法：

```
A = Y * X' / (X * X')
```

```
A =
    2.0299    5.0208    5.0104
    0.0114    6.9999    1.0106
    7.0424   -0.0025    6.9448
    6.9977    3.9759    4.0024
    9.0130    1.0449    6.9980
    4.0119    3.9649    9.0267
    4.9871    6.9723    8.0336
    7.9425    6.0875    3.0174
    0.0094    8.9722   -0.0385
    1.0612    8.0208    7.0285
```

## 6.26

(a)题目的意思是利用$q_i, p_i$估计$\alpha, d$，题目中的记号有些重复，这里假设

$$p_i = a\alpha \cos \varphi_i + v_i$$

其中

$$\begin{aligned}\cos \varphi_i &= \frac{d.\, q_i}{\|d\|.\, \|q_i\|} \\ &= q_i^T d \\ &= (\cos \phi_i \cos \theta_i) d_1 + (\cos \phi_i \sin \theta_i) d_2 + (\sin \phi_i) d_3\end{aligned}$$

假设

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix} \in \mathbb{R}^m, q = \begin{bmatrix} q_1^T \\ \vdots \\ q_m^T \end{bmatrix} \in \mathbb{R}^{m\times 3}. v = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^m$$

s

$$p = \alpha q(ad) + v$$

注意$a, d$未知，并且$d$的模长为$1$，所以可以设

$$x = ad$$

那么模型为

$$p = \alpha q x + v$$

估计$x$后，利用下式计算$a, d$即可：

$$\begin{aligned}a &= \|x\| \\ d &= \frac{x}{\|x\|}\end{aligned}$$

(b)代码如下：

```matlab
% Data for beam estimation problem
m = 5;
alpha = 0.5;
det_az =[ 3  10  80  150  275];
det_el =[ 88  34  30  20  50];
p =[ 1.58  1.50  2.47  1.10  0.001];

q1 = [cosd(det_el'), cosd(det_el'), sind(det_el')];
q2 = [cosd(det_az'), sind(det_az'), ones(m, 1)];
q = q1 .* q2;

x = (alpha * q) \ p';
a = norm(x)
d = x / a;

elevation = asind(d(3))
azimuth = asind(d(2) / cosd(elevation))
```

```
a =
    5.0107
elevation =
   38.7174
azimuth =
   77.6623
```

## 7.3

(a)首先考虑特殊情形。

如果$\mu=0$，那么取

$$g=f$$

如果$\mu\to\infty$，那么取

$$g_i=ai+b$$

所以此时的优化问题为

$$\left\|f-A\begin{bmatrix}a\\b\end{bmatrix}\right\|$$

其中

$$A=\begin{bmatrix}1&1\\2&1\\3&1\\\vdots&\vdots\\n&1\end{bmatrix}$$

因此

$$
\begin{bmatrix} a \\ b \end{bmatrix} = \left(A^T A\right)^{-1} A^T f
$$

接着考虑一般情形。

对均方曲率进行处理，令

$$
\begin{aligned}
h &= \begin{bmatrix} g_3 - 2g_2 + g_1 \\ \vdots \\ g_n - 2g_{n-1} + g_{n-2} \end{bmatrix} \\
&= \begin{bmatrix} g_3 \\ \vdots \\ g_n \end{bmatrix} - 2 \begin{bmatrix} g_2 \\ \vdots \\ g_{n-1} \end{bmatrix} + \begin{bmatrix} g_1 \\ \vdots \\ g_{n-2} \end{bmatrix}
\end{aligned}
$$

记

$$
\begin{aligned}
D_1 &= \begin{bmatrix} I_{n-2} & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n-2)\times n} \\
D_2 &= \begin{bmatrix} 0 & I_{n-2} & 0 \end{bmatrix} \in \mathbb{R}^{(n-2)\times n} \\
D_3 &= \begin{bmatrix} 0 & 0 & I_{n-2} \end{bmatrix} \in \mathbb{R}^{(n-2)\times n} \\
D &= D_1 + D_3 - 2D_2
\end{aligned}
$$

那么

$$
h = Dg
$$

所以

$$
\begin{aligned}
d &= \frac{1}{n} \|f - g\|^2 \\
c &= \frac{n^4}{n-2} \|h\|^2 \\
&= \frac{n^4}{n-2} \|Dg\|^2
\end{aligned}
$$

所以目标函数为

$$
\begin{aligned}
d + \mu c &= \frac{1}{n} \|f - g\|^2 + \mu \frac{n^4}{n-2} \|Dg\|^2 \\
&= \left\| \frac{1}{\sqrt{n}} g - \frac{1}{\sqrt{n}} f \right\|^2 + \mu \left\| \frac{n^2\sqrt{n}}{\sqrt{n-2}} D \frac{1}{\sqrt{n}} g \right\|^2
\end{aligned}
$$

对比课件中的标准形式：

$$
J_1 + \mu J_2 = \|Ax - y\|^2 + \mu \|Fx - h\|^2
$$

我们有

$$
\begin{aligned}
x &= \frac{1}{\sqrt{n}} g \\
y &= \frac{1}{\sqrt{n}} f \\
A &= I_n \\
F &= \frac{n^2\sqrt{n}}{\sqrt{n-2}} D \\
h &= 0
\end{aligned}
$$

所以问题的解为

$$
\begin{aligned}
x &= \frac{1}{\sqrt{n}} g \\
&= \left(A^T A + \mu F^T F\right)^{-1} \left(A^T y + \mu F^T h\right) \\
&= \left(I_n + \mu \frac{n^5}{n-2} D^T D\right)^{-1} \left(\frac{1}{\sqrt{n}} f\right)
\end{aligned}
$$

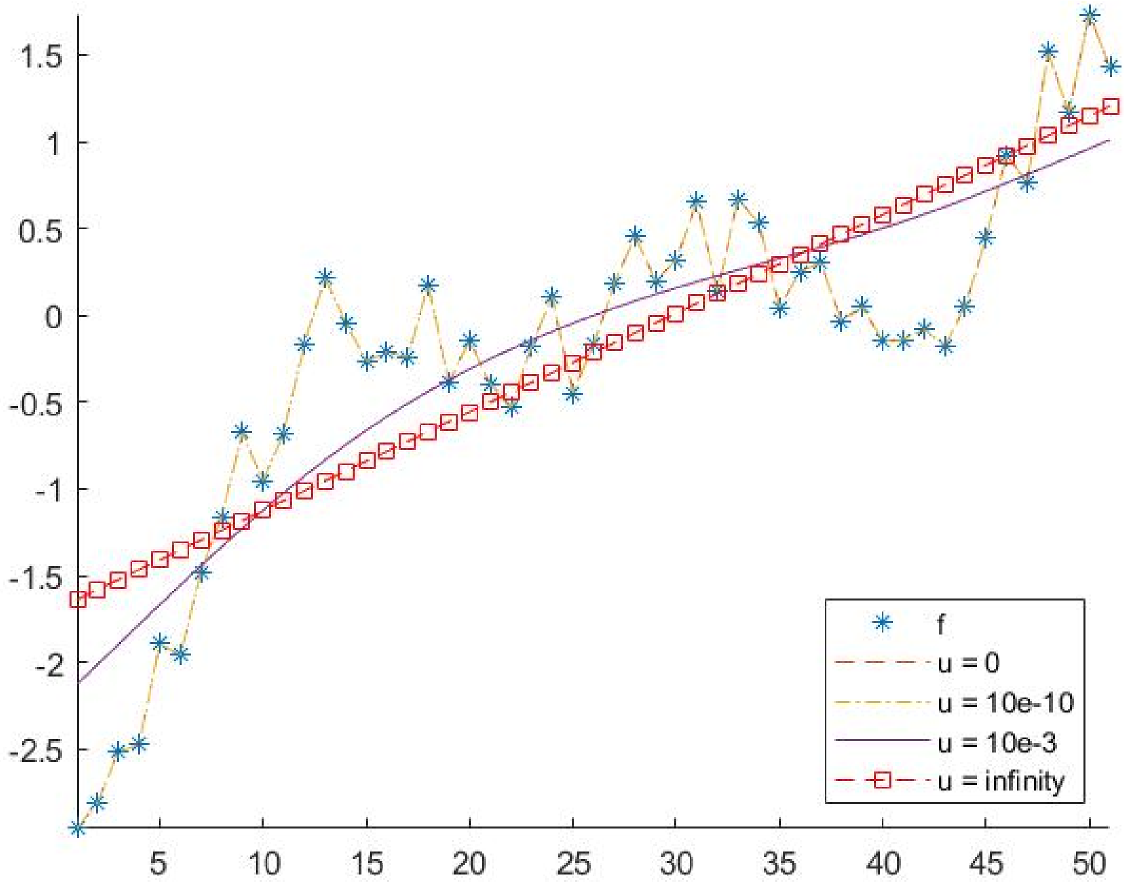
即

$$
g = \left(I_n + \mu \frac{n^5}{n-2} D^T D\right)^{-1} f
$$

(b)对不同的$\mu$作图：

```
% 矩阵D
D1 = [eye(n - 2), zeros(n - 2, 2)];
D2 = [zeros(n - 2, 1), eye(n - 2), zeros(n - 2, 1)];
D3 = [zeros(n - 2, 2), eye(n - 2)];
D = D1 + D3 - 2 * D2;

% 作出不同的mu
% mu = 0
mu = 0;
g1 = (eye(n) + mu * n ^ 5 / (n - 2) * (D' * D)) \ f';

% mu = 1e-10
mu = 1e-10;
g2 = (eye(n) + mu * n ^ 5 / (n - 2) * (D' * D)) \ f';

% mu = 1e-3
mu = 1e-3;
g3 = (eye(n) + mu * n ^ 5 / (n - 2) * (D' * D)) \ f';

% mu = infty
A = [(1:n)', ones(n, 1)];
coef = A \ f';
g4 = A * coef;
```

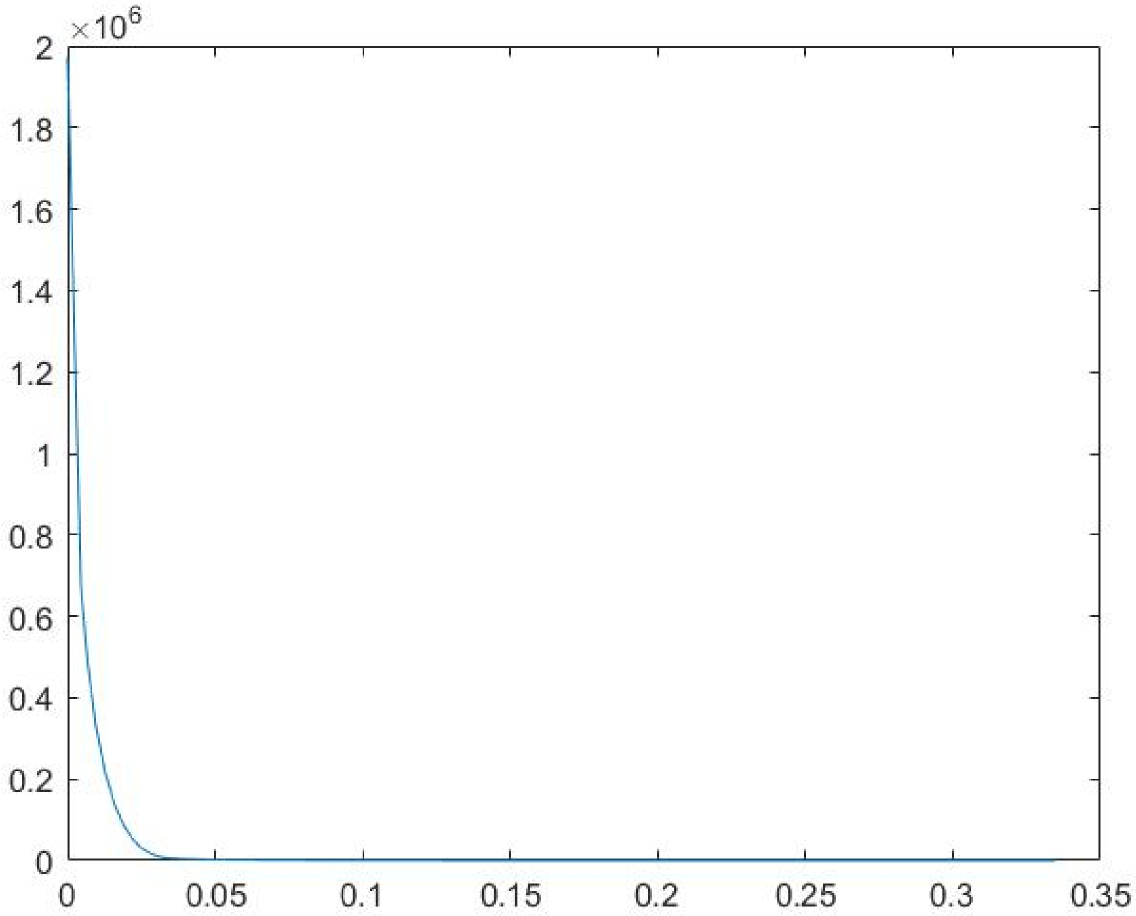
```
% 作图
figure(1);
hold;
plot(f, '*');
plot(g1, '--');
plot(g2, '-.');
plot(g3, '-');
plot(g4, '--rs');
axis tight;
legend('f','u = 0', 'u = 10e-10', 'u = 10e-3', 'u = infinity', 'location', 'SouthEast');
```




作出权衡曲线:

```
% 权衡曲线
m = 30;
Mu = logspace(-8, -3, m);
x = zeros(m+2, 1);
y = zeros(m+2, 1);

x(1) = norm(f' - g1) ^ 2 / n;
y(1) = norm(D * g1) ^ 2 * n ^ 4 / (n - 2);
for i = 1: m
    mu = Mu(i);
    g = (eye(n) + mu * n ^ 5 / (n - 2) * (D' * D)) \ f';
```

```
    x(i+1) = norm(f' - g) ^ 2 / n;
    y(i+1) = norm(D * g) ^ 2 * n ^ 4 / (n - 2);
end

x(m+2) = norm(f' - g4) ^ 2 / n;
y(m+2) = norm(D * g4) ^ 2 * n ^ 4 / (n - 2);

figure(2)
plot(x, y);
```




## 8.2

题目的要求等价于求

$$h\,[\,G \quad \tilde{G}\,] = [\,I_n \quad I_n\,]$$

转置后得到

$$\begin{bmatrix} G^T \\ \tilde{G}^T \end{bmatrix} h^T = \begin{bmatrix} I_n \\ I_n \end{bmatrix}$$

对于例子中的情形，我们需要求

$$\begin{bmatrix} G^T \\ \tilde{G}^T \end{bmatrix} h^T = \begin{bmatrix} I_2 \\ I_2 \end{bmatrix}$$

求解该线性方程组即可，由于方程数量大于未知数数量，所以使用左逆即可

$$A^\dagger = \left(A^T A\right)^{-1} A^T$$

代码如下：

```
G = [2 3; 1 0; 0 4; 1 1; -1 2];
G_tilde = [-3 -1; -1 0; 2 -3; -1 -3; 1 2];

A = [G'; G_tilde'];
b = [eye(2); eye(2)];

X = pinv(A) * b;
h = X';

%验证结果
h * G
h * G_tilde
```

```
ans =
    1.0000   -0.0000
   -0.0000    1.0000
ans =
    1.0000         0
    0.0000    1.0000
```