2.1

(a)首先化简(1)可得

$$
\begin{aligned}
p_i(t+1) &= p_i(t) \times \frac{\alpha\gamma}{S_i(t)} \\
&= \alpha\gamma \times \frac{p_i(t) q_i(t)}{s_i(t)} \\
&= \alpha\gamma \frac{\left(\sigma + \sum_{j\neq i} G_{ij} p_j(t)\right) p_i(t)}{G_{ii} p_i(t)} \\
&= \alpha\gamma \sum_{j\neq i} \frac{G_{ij}}{G_{ii}} p_j(t) + \frac{\alpha\gamma\sigma}{G_{ii}} \\
&= \alpha\gamma \left(\sum_{j=1}^{n} \frac{G_{ij}}{G_{ii}} p_j(t)\right) - \alpha\gamma p_i(t) + \frac{\alpha\gamma\sigma}{G_{ii}}
\end{aligned}
$$

记

$$
\begin{aligned}
\Lambda &= \operatorname{diag}(G_{11}, \ldots, G_{nn}) \\
1_n &= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}
\end{aligned}
$$

所以

$$
\begin{aligned}
p(t+1) &= \alpha\gamma \left(\Lambda^{-1} G - I_n\right) p(t) + \alpha\gamma\sigma\Lambda^{-1} 1_n \\
&\triangleq Ap(t) + b
\end{aligned}
$$

其中

$$
\begin{aligned}
A &= \alpha\gamma \left(\Lambda^{-1} G - I_n\right) \\
b &= \alpha\gamma\sigma\Lambda^{-1}
\end{aligned}
$$

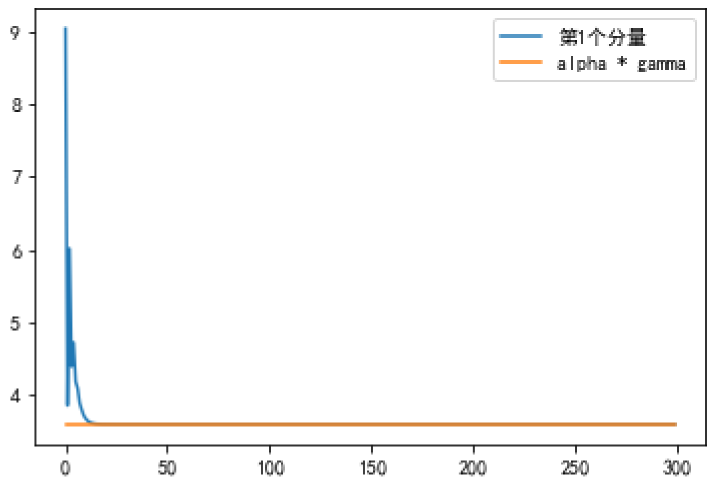
(b)首先给出$s_i(t), q_i(t)$的矩阵形式

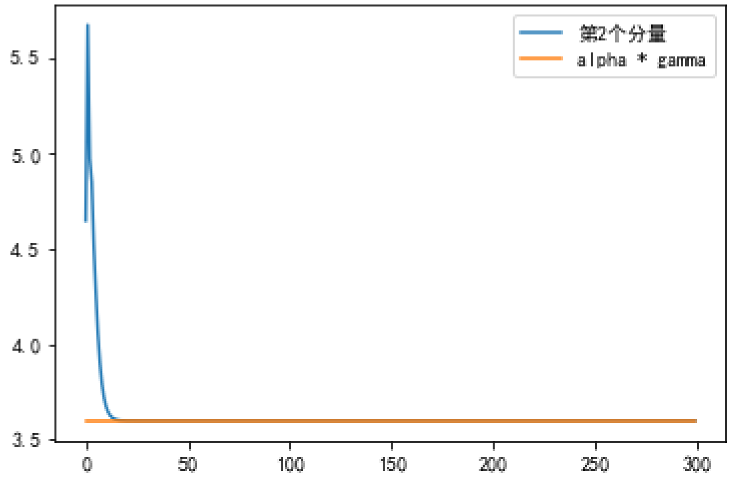
$$
\begin{aligned}
s(t) &= \Lambda^{-1} p(t) \\
q(t) &= \sigma I_n + (G - \Lambda) p(t)
\end{aligned}
$$

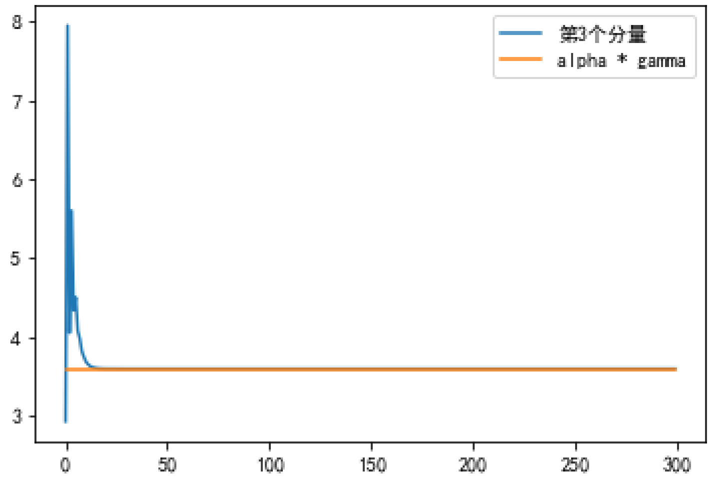
对应代码如下:

```
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams['font.sans-serif']=['SimHei'] #用来正常显示中文标签
plt.rcParams['axes.unicode_minus']=False #用来正常显示负号
```

```python
G = np.array([[1, 0.2, 0.1], [0.1, 2, 0.1], [0.3, 0.1, 3]])
gamma = 3
alpha = 1.2
sigma = 0.01
p = np.random.rand(3).reshape(-1, 1)
N = 300

def f(G, gamma, alpha, sigma, p, N):
    #数据维度
    n = G.shape[0]

    Lambda = np.diag(G).reshape(-1, 1)
    Res = np.array([])
    for i in range(N):
        s = Lambda * p
        q = sigma + G.dot(p) - Lambda * p
        S = s / q
        if i == 0:
            Res = np.copy(S)
        else:
            Res = np.c_[Res, S]
        #更新
        p = alpha * gamma * ((G / Lambda).dot(p) - p) + \
            alpha * gamma * sigma / Lambda

    target = np.ones(N) * alpha * gamma
    for i in range(n):
        si = Res[i, :]
        label = "第" + str(i+1) + "个分量"
        plt.plot(si, label=label)
        plt.plot(target, label="alpha * gamma")
        plt.legend()
        plt.show()

f(G, gamma, alpha, sigma, p, N)
```







利用该算法，最后每个分量都会收敛到$\alpha\gamma$。

2.2

因为

$$M\ddot{q} + D\dot{q} + Kq = f$$

并且M可逆，所以

$$\ddot{q} = M^{-1}f - M^{-1}Kq - M^{-1}D\dot{q}$$

回顾动力系统的形式：

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

结合题目可得

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} \\
&= \begin{bmatrix} \dot{q} \\ M^{-1} f - M^{-1} K q - M^{-1} D \dot{q} \end{bmatrix} \\
&= \begin{bmatrix} 0 & I_k \\ -M^{-1} K & -M^{-1} D \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} f \\
&= \begin{bmatrix} 0 & I_k \\ -M^{-1} K & -M^{-1} D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} u \\
y &= q \\
&= \begin{bmatrix} I_k & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \\
&= \begin{bmatrix} I_k & 0 \end{bmatrix} x
\end{aligned}
$$

2.3

回顾离散时间的线性动力系统形式:

$$
x(t+1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)
$$

MA模型:

记

$$
\begin{aligned}
x(k) &= \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-r) \end{bmatrix} \\
u(k) &= u(k) \\
C &= [a_1, \ldots, a_r] \in \mathbb{R}^{1\times r} \\
D &= a_0
\end{aligned}
$$

那么

$$
y(k) = Cx(k) + Du(k)
$$

AR模型:

记

$$
\begin{aligned}
x(k) &= \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-p) \end{bmatrix} \\
u(k) &= \begin{bmatrix} u(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^p \\
b &= [b_1, \ldots, b_p] \in \mathbb{R}^{1\times p} \\
A &= \begin{bmatrix} & b & \\ I_{p-1} & & 0 \end{bmatrix} \\
B &= 1
\end{aligned}
$$

那么

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-p+1) \end{bmatrix} \\
&= \begin{bmatrix} u(k) + b_1 y(k-1) + \cdots + b_p y(k-p) \\ y(k-1) \\ \vdots \\ y(k-p+1) \end{bmatrix} \\
&= \begin{bmatrix} b_1 y(k-1) + \cdots + b_p y(k-p) \\ y(k-1) \\ \vdots \\ y(k-p+1) \end{bmatrix} + \begin{bmatrix} u(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} & b & \\ I_{p-1} & & 0 \end{bmatrix} x(k) + u(k) \\
&= Ax(k) + Bu(k)
\end{aligned}
$$

ARMA模型:

记

$$
\begin{aligned}
x(k) &= \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-p) \\ u(k-1) \\ \vdots \\ u(k-r) \end{bmatrix} \in \mathbb{R}^{p+r} \\
u(k) &= u(k) \\
a &= [a_1, \ldots, a_r] \in \mathbb{R}^{1\times r} \\
b &= [b_1, \ldots, b_p] \in \mathbb{R}^{1\times p} \\
A_1 &= [a, b] \in \mathbb{R}^{1\times p} \\
A_2 &= [I_{p-1}, 0] \in \mathbb{R}^{(p-1)\times p} \\
A_3 &= 0 \in \mathbb{R}^{1\times p} \\
A_4 &= [0, I_{r-1}, 0] \in \mathbb{R}^{(r-1)\times p}
\end{aligned}
$$

那么

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-p+1) \\ u(k) \\ u(k-1) \\ \vdots \\ u(k-r+1) \end{bmatrix} \\
&= \begin{bmatrix} b_1 y(k-1) + \cdots + b_p y(k-p) + a_0 u(k) + \cdots + a_r u(k-r) \\ y(k-1) \\ \vdots \\ y(k-p+1) \\ u(k) \\ u(k-1) \\ \vdots \\ u(k-r+1) \end{bmatrix} \\
&= \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} x(k) + \begin{bmatrix} a_0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)
\end{aligned}
$$

记

$$
A=\begin{bmatrix}A_1\\A_2\\A_3\\A_4\end{bmatrix}\in\mathbb{R}^{(p+r)\times(p+r)}
$$

$$
B=\begin{bmatrix}a_0\\0\\\vdots\\0\\1\\0\\\vdots\\0\end{bmatrix}\in\mathbb{R}^{p+r}
$$

其中

$$
B_i=\begin{cases}a_0 & i=1\\1 & i=p+1\\0 & \text{其他}\end{cases}
$$

那么模型为

$$
x(k+1)=Ax(k)+Bu(k)
$$

2.4

定义n维标准单位列向量$u_i\in\mathbb{R}^n$:

$$
u_1=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},u_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\ldots,u_n=\begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}
$$

那么

$$
x=\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=\sum_{i=1}^n x_iu_i
$$

定义m维标准单位列向量$v_i\in\mathbb{R}^m$:

$$
v_1=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},v_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\ldots,v_m=\begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}
$$

假设

$$
f(u_i)=\sum_{j=1}^m a_{ji}v_j,A=[a_{ji}]\in\mathbb{R}^{m\times n}
$$

那么

$$
\begin{aligned}
f(x)&=f\left(\sum_{i=1}^n x_iu_i\right)\\
&=\sum_{i=1}^n x_if(u_i)\\
&=\sum_{i=1}^n x_i\sum_{j=1}^m a_{ji}v_j\\
&=\sum_{j=1}^m\sum_{i=1}^n a_{ji}x_iv_j\\
&=\begin{bmatrix}\sum_{i=1}^n a_{1i}x_i\\\ldots\\\sum_{i=1}^n a_{mi}x_i\end{bmatrix}\\
&=Ax
\end{aligned}
$$

如果还存在$\tilde A\in\mathbb{R}^{m\times n}$，使得

$$
f(x)=\tilde Ax
$$

那么取$x=u_i,i=1,\ldots,n$可得

$$
\begin{aligned}
f(u_i)&=Au_i\\
&=\begin{bmatrix}a_{1i}\\\ldots\\a_{mi}\end{bmatrix}\\
&=\tilde Au_i\\
&=\begin{bmatrix}\tilde a_{1i}\\\ldots\\\tilde a_{mi}\end{bmatrix}
\end{aligned}
$$

因此

$$A=\tilde{A}$$

2.6

由

$$\mathcal{D}p=\sum_{i=1}^{n-1} i a_i x^{i-1}$$

可得

$$\begin{bmatrix} a_1 \\ 2a_2 \\ \vdots \\ (n-1)a_{n-1} \end{bmatrix}=\begin{bmatrix} 0 & 1 & & & \\ 0 & & 2 & & \\ 0 & & & \ddots & \\ 0 & & & & n-1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

因此

$$D=\begin{bmatrix} 0 & 1 & & & \\ 0 & & 2 & & \\ 0 & & & \ddots & \\ 0 & & & & n-1 \end{bmatrix}$$

2.9

(a)不难看出

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

所以

$$A=\begin{bmatrix} 2 & 0 \\ 0.5 & 1 \end{bmatrix}$$

(b)利用(a)不难得出

$$
\begin{aligned}
B &= A^4 \\
&= \left(\begin{bmatrix} 2 & 0 \\ 0.5 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0.5 & 1 \end{bmatrix}\right)^2 \\
&= \left(\begin{bmatrix} 4 & 0 \\ 1.5 & 1 \end{bmatrix}\right)^2 \\
&= \begin{bmatrix} 4 & 0 \\ 1.5 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1.5 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 16 & 0 \\ 7.5 & 1 \end{bmatrix}
\end{aligned}
$$

另一方面，直接考虑该问题，我们知道x_1到z_1的路线只有一条，所以$b_{11}=2^4=16$；x_2到z_2的路线只有一条，所以$b_{22}=1^4=1$；x_2到z_1没有路线，所以$b_{12}=0$；x_1到y_2的路线一共有4条，总权重为

$$
0.5+2\times 0.5+2^2\times 0.5+2^3\times 0.5=7.5
$$

2.12

首先考虑$(A^2)_{ij}$：

$$
(A^2)_{ij}=\sum_{k=1}^{n} A_{ik}A_{kj}=i\text{到}j\text{长度为}2\text{的路径数量}
$$

递推可得

$$
B_{ij}=(A^k)_{ij}=i\text{到}j\text{长度为}k\text{的路径数量}
$$

补充题

1

(a)$\forall \alpha,\beta,\alpha+\beta=1$：

$$
\begin{aligned}
f(\alpha x+\beta y) &= A(\alpha x+\beta y)+b \\
&= A(\alpha x+\beta y)+b(\alpha+\beta) \\
&= \alpha(Ax+b)+\beta(Ax+b) \\
&= \alpha f(x)+\beta f(y)
\end{aligned}
$$

(b)考虑

$$
g(x)=f(x)-f(0)
$$

下面证明该函数为线性函数，首先证明

$$
g(kx)=kg(x)
$$

由定义，这等价于

$$
\begin{aligned}
f(kx)-f(0)=k(f(x)-f(0)) \Leftrightarrow \\
f(kx)+(k-1)f(0)=kf(x) \Leftrightarrow \\
\frac{1}{k}f(kx)+\frac{k-1}{k}f(0)=f(x)
\end{aligned}
$$

最后一行由$f(x)$的性质即可得到。

接着证明

$$
g(x+y)=g(x)+g(y)
$$

事实上，我们有

$$
\begin{aligned}
g(x+y) &= f(x+y)-f(0) \\
&= f\left(\frac{1}{2}\times 2x+\frac{1}{2}\times 2y\right)-\frac{1}{2}f(0)-\frac{1}{2}f(0) \\
&= \frac{1}{2}f\left(2x\right)+\frac{1}{2}f\left(2y\right)-\frac{1}{2}f(0)-\frac{1}{2}f(0) && \text{由 } f(x)\text{的性质} \\
&= \frac{1}{2}\left(f\left(2x\right)-f(0)\right)+\frac{1}{2}\left(f\left(2y\right)-f(0)\right) \\
&= \frac{1}{2}g(2x)+\frac{1}{2}g(2y) \\
&= \frac{1}{2}\times 2\times g(x)+\frac{1}{2}\times 2\times g(y) && \text{由 } g(kx)=kg(x) \\
&= g(x)+g(y)
\end{aligned}
$$

结合以上两点，$g(x)$是线性函数，所以存在唯一的A，使得

$$
g(x)=f(x)-f(0)=Ax
$$

现在记

$$
b=f(0)
$$

那么

$$
g(x)=Ax+b
$$

A的唯一性已经说明，现在说明b的唯一性即可。若存在$\tilde{b}$同样满足条件，那么

$$
\tilde{b}=g(0)=b
$$

因此b唯一。