EE263 homework 7 solutions

10.9 *Characteristic polynomial.* Consider the characteristic polynomial $\mathcal{X}(s) = \det(sI - A)$ of the matrix $A \in \mathbf{R}^{n\times n}$.

(a) Show that $\mathcal{X}$ is *monic*, which means that its leading coefficient is one: $\mathcal{X}(s) = s^n + \cdots$.

(b) Show that the s^{n-1} coefficient of $\mathcal{X}$ is given by $-\mathbf{Tr}\, A$. ($\mathbf{Tr}\, X$ is the *trace* of a matrix: $\mathbf{Tr}\, X = \sum_{i=1}^n X_{ii}$.)

(c) Show that the constant coefficient of $\mathcal{X}$ is given by $\det(-A)$.

(d) Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A, so that

$$\mathcal{X}(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n).$$

By equating coefficients show that $a_{n-1} = -\sum_{i=1}^n \lambda_i$ and $a_0 = \prod_{i=1}^n (-\lambda_i)$.

Solution:

(a) Expand the determinant expression to get

$$\det(sI - A) = (s - a_{11}) \det \tilde{A} + \text{ other terms},$$

where $\tilde{A}$ is the A matrix without the first row and first column. The *other terms* are similar, except for the fact that the determinant is multiplied by a scalar. Expanding $\det \tilde{A}$ we will reach a similar equation, and after expanding all terms you will reach something like

$$\det(sI - A) = \prod_{i=1}^n (s - a_{ii}) + \text{ other terms}.$$

The *other terms* contribute with polynomials whose order is less than n, and since the first term is a monic polynomial with order n it follows that $\det(sI - A)$ is also monic.

(b) Let's take a closer look at the relation

$$\det(sI - A) = \prod_{i=1}^n (s - a_{ii}) + \text{ other terms}.$$

A little reasoning will show us that the *other terms* in fact are polynomials whose degree is less than $n-1$ (provided that $n > 1$, and for $n = 1$ we have the trivial case). This is so because in the first expression of item (a) we have that $\tilde{A}$ is the only matrix that has $n-1$ entries with s, and the same applies to other expansions of the expression. Then it follows that the s^{n-1} term of $\mathcal{X}$ is the s^{n-1} term of $\prod(s - a_{ii})$. But this term is $\sum -a_{ii}$, which is equal to $-\operatorname{Tr} A$.

(c) The constant coefficient is given by $\mathcal{X}(0)$. But $\mathcal{X}$ is simply $\det(sI - A)$. By taking $s = 0$ it follows that $\mathcal{X}(0) = \det(-A)$.

(d) First we note that, if $n = 1$, the relations are valid for the polynomial $s - \lambda_1$. Now suppose the relations are valid for a monic polynomial $P(s)$. Multiply $P(s)$ by $s - \lambda_i$ and expand as

$$P(s)(s - \lambda_i) = sP(s) - \lambda_i P(s).$$

Suppose $P(s)$ has degree n. Then $sP(s)$ is monic with degree $n + 1$ and the constant coefficient is zero. The polynomial $-\lambda_i P(s)$ has degree n, the s^n coefficient is $-\lambda_i$ and the constant coefficient is $\prod(-\lambda_j)$. Since the constant coefficient of $sP(s)$ is zero we conclude by induction that $a_0 = \prod_{i=1}^n(-\lambda_i)$. Since $P(s)$ satisfies the properties, the s^n term of $P(s)$ is $\sum \lambda_j$ and we conclude, again by induction, that $a_{n-1} = -\sum_{i=1}^n \lambda_i$.

10.11 *Spectral resolution of the identity.* Suppose $A \in \mathbf{R}^{n\times n}$ has n linearly independent eigenvectors $p_1, \ldots, p_n$, $p_i^T p_i = 1$, $i = 1, \ldots, n$, with associated eigenvalues λ_i. Let $P = [p_1 \ \cdots \ p_n]$ and $Q = P^{-1}$. Let q_i^T be the ith row of Q.

(a) Let $R_k = p_k q_k^T$. What is the range of R_k? What is the rank of R_k? Can you describe the null space of R_k?

(b) Show that $R_i R_j = 0$ for $i \neq j$. What is R_i^2?

(c) Show that

$$(sI - A)^{-1} = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}.$$

Note that this is a partial fraction expansion of $(sI - A)^{-1}$. For this reason the R_i's are called the *residue* matrices of A.

(d) Show that $R_1 + \cdots + R_n = I$. For this reason the residue matrices are said to constitute a *resolution of the identity.*

(e) Find the residue matrices for

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$$

both ways described above (*i.e.*, find P and Q and then calculate the R's, and then do a partial fraction expansion of $(sI - A)^{-1}$ to find the R's).

Solution:

(a) Note that because R_k may be complex, the linear spaces we work with assume that scalar multiplication is *complex*. Using $R_k = p_k q_k^T$,

$$\begin{aligned} \mathcal{R}(R_k) &= \{\ R_k x \mid x \in \mathbf{C}^n\} = \{\ p_k q_k^T x \mid x \in \mathbf{C}^n\} \\ &= \{\ \alpha p_k \mid \alpha \in \mathbf{C}\} = \operatorname{span}\{p_k\}. \end{aligned}$$

Note that we have used the fact that $q_k \neq 0$ so $q_k^T x$ can be any complex number α. Since $\mathcal{R}(R_k) = \text{span}\{p_k\}$ we have

$$\mathbf{rank}(R_k) = \mathbf{dim}\mathcal{R}(R_k) = 1.$$

The null space of R_k consists of all the vectors orthogonal to q_k. Its dimension is obviously

$$\mathbf{dim}\mathcal{N}(R_k) = n - \mathbf{dim}\mathcal{R}(R_k) = n - 1.$$

Since we know that vectors p_i, $i \neq k$ are $n-1$ linearly independent members of that set, it follows that

$$\mathcal{N}(R_k) = \text{span}\{p_i,\ i \neq k\}.$$

(b) $R_i R_j = p_i q_i^T p_j q_j^T = p_i \delta_{ij} q_j^T = \begin{cases} 0 & i \neq j \\ R_i & i = j \end{cases}$. Note that we have used the fact that $q_i^T p_j = \delta_{ij}$ which is nothing but $QP = I$.

(c) Since $P^{-1}AP = \Lambda$, we obtain

$$\begin{aligned}
(sI - A)^{-1} &= P(sI - P^{-1}AP)^{-1}P^{-1} \\
&= \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \begin{bmatrix} s-\lambda_1 & & \\ & \ddots & \\ & & s-\lambda_n \end{bmatrix}^{-1} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \\
&= \frac{p_1 q_1^T}{s-\lambda_1} + \cdots + \frac{p_n q_n^T}{s-\lambda_n} = \sum_{k=1}^{n} \frac{R_k}{s-\lambda_k}.
\end{aligned}$$

(d) There are several methods for solving this problem. Here we present three. *Method 1,* the simplest:

$$\sum_{k=1}^{n} R_k = \sum_{k=1}^{n} p_k q_k^T = \begin{bmatrix} p_1 & \cdots p_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} = PQ = I.$$

Method 2: Multiply both sides of

$$(sI - A)^{-1} = \sum_{k=1}^{n} \frac{R_k}{s-\lambda_k}$$

by s and take the limit as $s \to \infty$ to obtain

$$I = \lim_{s\to\infty} s(sI - A)^{-1} = \lim_{s\to\infty} \sum_{k=1}^{n} \frac{R_k s}{s-\lambda_k} = \sum_{k=1}^{n} R_k.$$

Method 3:

$$\begin{aligned} I &= (sI-A)^{-1}(sI-A) = \sum_{k=1}^{n} \frac{R_k(sI-A)}{s-\lambda_k} = \sum_{k=1}^{n} \frac{sR_k - R_kA}{s-\lambda_k} \\ &= \sum_{k=1}^{n} \frac{sp_kq_k^T - p_kq_k^TA}{s-\lambda_k} = \sum_{k=1}^{n} p_kq_k^T\frac{s-\lambda_k}{s-\lambda_k} = \sum_{k=1}^{n} R_k. \end{aligned}$$

(e) The eigenvalues of A are are 1 and -2. The corresponding (normalized) eigenvectors are p_1 and p_2, given by

$$p_1 = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}, \quad p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} p_1 & p_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{10}} & 1 \end{bmatrix}, \quad Q = P^{-1} = \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{10}}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} R_1 &= p_1q_1^T = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{10}}{3} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 0 \end{bmatrix}, \quad \text{and} \\ R_2 &= p_2q_2^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}. \end{aligned}$$

To check, let us do the partial fraction expansion of $(sI-A)^{-1}$.

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right) & \frac{1}{s+2} \end{bmatrix} = \frac{1}{s-1}\begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 0 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

as before.

10.19 *Output response envelope for linear system with uncertain initial condition.* We consider the autonomous linear dynamical system $\dot{x} = Ax$, $y(t) = Cx(t)$, where $x(t) \in \mathbf{R}^n$ and $y(t) \in \mathbf{R}$. We do not know the initial condition exactly; we only know that it lies in a ball of radius r centered at the point x_0:

$$\|x(0) - x_0\| \leq r.$$

We call x_0 the *nominal* initial condition, and the resulting output, $y_{\text{nom}}(t) = Ce^{tA}x_0$, the *nominal output*. We define the *maximum output* or *upper output envelope* as

$$\overline{y}(t) = \max\{y(t) \mid \|x(0) - x_0\| \leq r\},$$

i.e., the maximum possible value of the output at time t, over all possible initial conditions. (Here you can choose a different initial condition for each t; you are not required to find a single initial condition.) In a similar way, we define the *minimum output* or *lower output envelope* as

$$\underline{y}(t) = \min\{y(t) \mid \|x(0) - x_0\| \leq r\},$$

i.e., the minimum possible value of the output at time t, over all possible initial conditions.

(a) Explain how to find $\overline{y}(t)$ and $\underline{y}(t)$, given the problem data A, C, x_0, and r.

(b) Carry out your method on the problem data in `uie_data.m`. On the same axes, plot y_{nom}, $\overline{y}$, and $\underline{y}$, versus t, over the range $0 \leq t \leq 10$.

Solution. We have

$$y(t) = Cx(t) = Ce^{tA}x(0) = Ce^{tA}(x(0) - x_0 + x_0) = y_{\text{nom}}(t) + Ce^{tA}(x(0) - x_0).$$

Using the Cauchy-Schwarz inequality, we get

$$-\|(e^{tA})^T C^T\| \|x(0) - x_0\| \leq Ce^{tA}(x(0) - x_0) \leq \|(e^{tA})^T C^T\| \|x(0) - x_0\|.$$

However, since $\|x(0) - x_0\| \leq r$, we can deduce that

$$y_{\text{nom}}(t) - \|(e^{tA})^T C^T\| r \leq y(t) \leq y_{\text{nom}}(t) + \|(e^{tA})^T C^T\| r.$$

In fact, these inequalities are tight. This is so, since if

$$x(0) = x_0 + \frac{r}{\|(e^{tA})^T C^T\|}(e^{tA})^T C^T$$

then

$$y(t) = y_{\text{nom}}(t) + \|(e^{tA})^T C^T\| r.$$

Similarly, if

$$x(0) = x_0 - \frac{r}{\|(e^{tA})^T C^T\|}(e^{tA})^T C^T$$

then

$$y(t) = y_{\text{nom}}(t) - \|(e^{tA})^T C^T\| r.$$

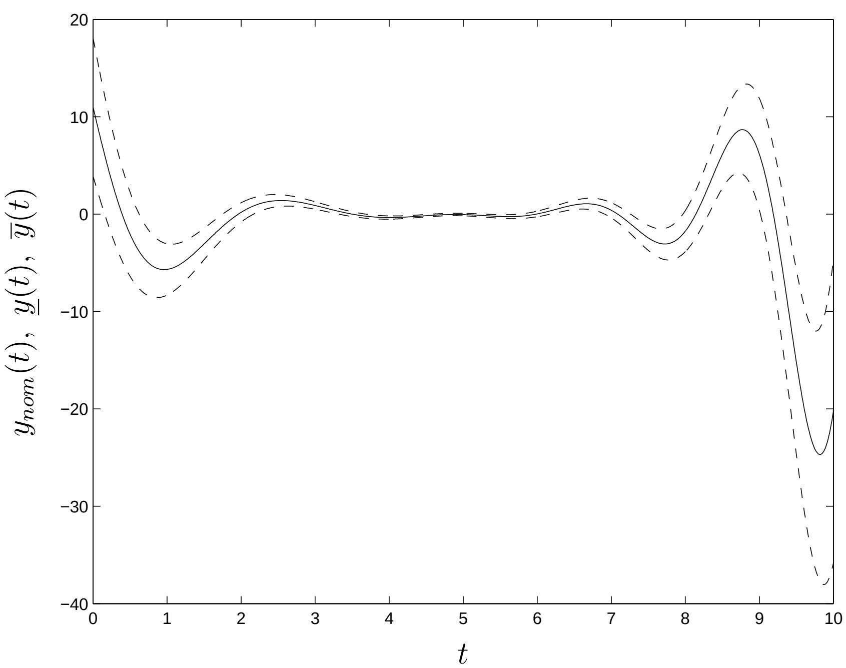
Therefore we finally have

$$\underline{y}(t) = y_{\text{nom}}(t) - \|(e^{tA})^T C^T\| r, \qquad \overline{y}(t) = y_{\text{nom}}(t) + \|(e^{tA})^T C^T\| r.$$

The following Matlab code performs this for the given data:

```
clear all
uie_data
t = linspace(0,10,1000);
y_nom = [];
y_over = [];
y_under = [];
for k = t
v = C*expm(k*A);
y_c = v*x_0;
y_nom = [y_nom y_c];
y_under = [y_under y_c-norm(v)*r];
y_over = [y_over y_c+norm(v)*r];
end
figure
plot(t,y_nom,'b-',t,y_under,'r--',t,y_over,'r--')
print -deps uie_fig.eps
```

This code generates the following figure:



Note that the bounds actually move towards each other near $t \approx 5$, but then diverge again.

11.13 *Real modal form.* Generate a matrix A in $\mathbf{R}^{10\times 10}$ using `A=randn(10)`. (The entries of A will be drawn from a unit normal distribution.) Find the eigenvalues of A. If by chance they are all real, please generate a new instance of A. Find the real modal form of A, *i.e.*, a matrix S such that $S^{-1}AS$ has the real modal form given in lecture 11. Your solution should include a clear explanation of how you will find S, the source code that you use to find S, and some code that checks the results (*i.e.*, computes $S^{-1}AS$ to verify it has the required form).

Solution.
Assuming A is diagonalizable, it can be written as $A = T\Lambda T^{-1}$, Here

$$\Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_r, \lambda_{r+1}, \dots, \lambda_n),$$

where $\lambda_1, \dots, \lambda_r$ are the real eigenvalues of A and $\lambda_{r+1}, \dots, \lambda_n$ are the complex eigenvalues of A and come in complex conjugate pairs. Let t_i be the ith column of T. Take S to be

$$S = [t_1 \ \cdots \ t_r \quad \Re(t_{r+1}) \quad \Im(t_{r+1}) \ \cdots \ \Re(t_{n-1}) \quad \Im(t_{n-1})].$$

Let us now prove why constructing S in this way will give us the desired result. Let $v = \Re(v) + i\Im(v)$ be a complex eigenvector of A associated with the eigenvalue $\lambda = \sigma + i\omega$. We must have $Av = \lambda v$, *i.e.*,

$$A(\Re(v) + i\Im(v)) = (\sigma + i\omega)(\Re(v) + i\Im(v)).$$

This implies that

$$A\Re(v) = \sigma\Re(v) - \omega\Im(v), \quad A\Im(v) = \omega\Re(v) + \sigma\Im(v),$$

or equivalently

$$A\,[\Re(v) \quad \Im(v)] = [\Re(v) \quad \Im(v)] \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}.$$

If we do the same derivation for the other complex conjugate eigenvalue pairs, we get the construction of S that was presented above.

Here is a short MATLAB script that checks that our solution is correct.

```
randn('seed', 21395);
A = randn(10);
[V, D] = eig(A);
% Take eigenvectors of complex eigenvalues and arrange them in
pairs.
S = zeros(10);
S(:,1) = V(:,5);
S(:,2) = V(:,6);
S(:,3) = V(:,7);
S(:,4) = V(:,10);
S(:,5) = real(V(:,1));
S(:,6) = imag(V(:,1));
```

```
S(:,7) = real(V(:,3));
S(:,8) = imag(V(:,3));
S(:,9) = real(V(:,8));
S(:,10) = imag(V(:,8));

% Inspect S^{-1}AS.
inv(S)*A*S
```

12.1 *Some true/false questions.* Determine if the following statements are true or false. No justification or discussion is needed for your answers. What we mean by "true" is that the statement is true for all values of the matrices and vectors that appear in the statement. You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that the dimensions are such that all expressions make sense. For example, the statement "$A + B = B + A$" is true, because no matter what the dimensions of A and B are (they must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because there are (square) matrices for which this doesn't hold. (There are also matrices for which it does hold, *e.g.*, an identity matrix. But that doesn't make the statement true.) "False" means the statement isn't true, in other words, it can fail to hold for some values of the matrices and vectors that appear in it.

(a) If $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{n\times p}$ are both full rank, and $AB = 0$, then $n \geq m + p$. *Solution:* **True.** First observe that A must be fat, *i.e.*, $n > m$, since otherwise A is one-to-one and $AB = 0$ would imply that $B = 0$, and therefore not full rank. Similarly, B must be skinny, *i.e.*, $n > p$; if B were fat, then $AB = 0$ would imply that $A = 0$. Now $AB = 0$ means $\mathcal{R}(B) \subseteq \mathcal{N}(A)$. Since the dimension of $\mathcal{R}(B)$ is p, this implies that the dimension of $\mathcal{N}(A)$ is at least p. By the basic dimension conservation theorem, applied to A, we have $n = m + \dim \mathcal{N}(A)$. Putting this together with the inequality above yields $n \geq p + m$.

(b) If $A \in \mathbf{R}^{3\times 3}$ satisfies $A + A^T = 0$, then A is singular. *Solution:* **True**. A general 3×3 skew symmetric matrix (*i.e.*, one that satisfies $A^T = -A$) has the form

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

Evidently we have $Ax = 0$, with $x = (c, -b, a)$.

(c) If $A^k = 0$ for some integer $k \geq 1$, then $I - A$ is nonsingular. *Solution:* **True**. We first observe that all eigenvalues of A must be zero. The eigenvalues of $I - A$ are each one minus an eigevalue of A, *i.e.*, they are all equal to one. In particular, 0 is not an eigenvalue of $I - A$, so it is nonsingular.

(d) If $A, B \in \mathbf{R}^{n\times n}$ are both diagonalizable, then AB is diagonalizable. *Solution:* **False**. Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$. Clearly both A and B are diagonalizable, but $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

(e) If $A, B \in \mathbf{R}^{n\times n}$, then every eigenvalue of AB is an eigenvalue of BA. *Solution:* **True**. Take any eigenvalue λ of AB, and let v be an eigenvector, *i.e.*, $ABv = \lambda v$. Suppose $\lambda \neq 0$, then

$$BA(Bv) = B(ABv) = \lambda(Bv).$$

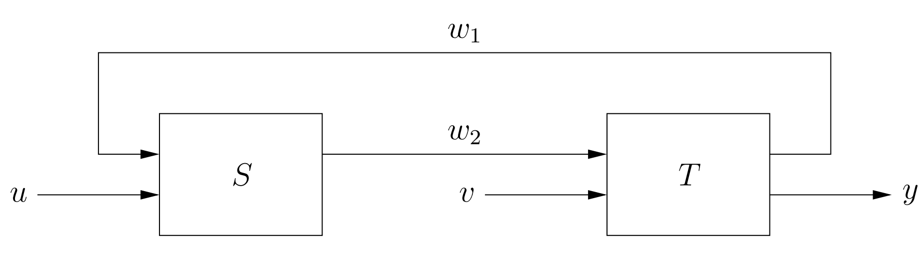
Since $Bv \neq 0$ (otherwise $\lambda = 0$), Bv is an eigenvector of BA associated with the eigenvalue λ. Now suppose $\lambda = 0$ and we need to show that BA is also singular. Suppose BA is nonsingular, then both A and B is full rank. But this will imply that $Bv = 0$ and thus $v = 0$, a contradiction.

(f) If $A, B \in \mathbf{R}^{n\times n}$, then every eigenvector of AB is an eigenvector of BA. *Solution:* **False.** Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ has $(1, 1)$ as an eigenvector, which is clearly not an eigenvector of $BA = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

(g) If A is nonsingular and A^2 is diagonalizable, then A is diagonalizable. *Solution:* **True**. Take any Jordan block J of A. Since A^2 is diagonalizable, J^2 must be a diagonal matrix. Because A has no zero eigenvalue, this is true only if J has block size 1, *i.e.*, A is diagonalizable.

13.1

1. *Interconnection of linear systems.* Often a linear system is described in terms of a block diagram showing the interconnections between components or subsystems, which are themselves linear systems. In this problem you consider the specific interconnection shown below:



Here, there are two subsystems S and T. Subsystem S is characterized by

$$\dot{x} = Ax + B_1u + B_2w_1, \qquad w_2 = Cx + D_1u + D_2w_1,$$

and subsystem T is characterized by

$$\dot{z} = Fz + G_1 v + G_2 w_2, \qquad w_1 = H_1 z, \qquad y = H_2 z + J w_2.$$

We don't specify the dimensions of the signals (which can be vectors) or matrices here. You can assume all the matrices are the correct (*i.e.*, compatible) dimensions. Note that the subscripts in the matrices above, as in B_1 and B_2, refer to different matrices. Now the problem. Express the overall system as a single linear dynamical system with input, state, and output given by

$$\begin{bmatrix} u \\ v \end{bmatrix}, \qquad \begin{bmatrix} x \\ z \end{bmatrix}, \qquad y,$$

respectively. Be sure to explicitly give the input, dynamics, output, and feedthrough matrices of the overall system. If you need to make any assumptions about the rank or invertibility of any matrix you encounter in your derivations, go ahead. But be sure to let us know what assumptions you are making. *Solution:*
This one is easier than it might appear. All we need to do is write down all the equations for this system, and massage them to be in the form of a linear dynamical system with the given input, state, and output. Substituting the expression for w_1 into the first set of equations gives

$$\dot{x} = Ax + B_1 u + B_2 H_1 z, \qquad w_2 = Cx + D_1 u + D_2 H_1 z.$$

Similarly, substituting the expression for w_2 into the second set of equations yields

$$\dot{z} = Fz + G_1 v + G_2(Cx + D_1 u + D_2 H_1 z), \qquad y = H_2 z + J(Cx + D_1 u + D_2 H_1 z).$$

Now we just put this together into the required form:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B_2 H_1 \\ G_2 C & F + G_2 D_2 H_1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ G_2 D_1 & G_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

and

$$y = \begin{bmatrix} JC & H_2 + J D_2 H_1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} J D_1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Solution to additional exercise

1. *Spectral mapping theorem.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is analytic, *i.e.*, given by a power series expansion

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots$$

(where $a_i = f^{(i)}(0)/(i!)$). (You can assume that we only consider values of u for which this series converges.) For $A \in \mathbf{R}^{n \times n}$, we define $f(A)$ as

$$f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots$$

(again, we'll just assume that this converges).

Suppose that $Av = \lambda v$, where $v \neq 0$, and $\lambda \in \mathbf{C}$. Show that $f(A)v = f(\lambda)v$ (ignoring the issue of convergence of series). We conclude that if λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$. This is called the *spectral mapping theorem*.

To illustrate this with an example, generate a random 3×3 matrix, for example using `A=randn(3)`. Find the eigenvalues of $(I + A)(I - A)^{-1}$ by first computing this matrix, then finding its eigenvalues, and also by using the spectral mapping theorem. (You should get very close agreement; any difference is due to numerical round-off errors in the various compuations.)

Solution. Assuming $Av = \lambda v$, $v \neq 0$, we have

$$\begin{aligned} f(A)v &= a_0 Iv + a_1 Av + a_2 A^2 v + \cdots \\ &= a_0 v + a_1 \lambda v + a_2 \lambda^2 v + \cdots \\ &= f(\lambda)v, \end{aligned}$$

using $A^k v = \lambda^k v$.

The Matlab code that illustrates this is shown below.

```
randn('state',0);  % makes it repeatable
A=randn(3);
% eigenvalues of A
lambdas=eig(A);
% create matrix f(A)=(I+A)(I-A)^(-1)
fA=(eye(3)+A)*inv(eye(3)-A);
% eigenvalues of f(A)
lambdas_fA=eig(fA);
% eigenvalues of B via spectral mapping theorem
lambdas_fA_smt = (1+lambdas)./(1-lambdas);
% compare (need not be in same order!)
[lambdas_fA lambdas_fA_smt]
```