Problem 1

(a)反证法，如果有超过$\frac{3}{4}$的牛存活，那么平均体温大于

$$\frac{3}{4}\times 90+\frac{1}{4}\times 70=85$$

(b)300头牛是的体温为90；其余100头牛的体温是70即可。

(c)记牛的体温为T，那么

$$\begin{aligned} T &\ge 70\\ \mathbb{E}[T]&=85\end{aligned}$$

所以

$$\begin{aligned}\mathbb{P}[T\ge 90]&=\mathbb{P}[T-70\ge 20]\\ &\le \frac{\mathbb{E}[T-70]}{20}\\ &=\frac{15}{20}\\ &=\frac{3}{4}\end{aligned}$$

Problem 2

(a)记draw poker每天赢的胜场为X_1，black jack每天赢的胜场为X_2，stud poker每天赢的胜场为X_3。那么每天赢的胜场为

$$X=X_1+X_2+X_3$$

因此

$$\begin{aligned}\mathbb{E}[X]&=\sum_{i=1}^{3}\mathbb{E}[X_i]\\ &=20+30+4\\ &=54\end{aligned}$$

(b)

$$\begin{aligned}\mathbb{E}[X\ge 108]&\le\frac{\mathbb{E}[X]}{108}\\ &=\frac{1}{2}\end{aligned}$$

(c)由两两独立可得

$$
\begin{aligned}
\mathrm{Var}(X) &= \sum_{i=1}^{3} \mathrm{Var}(X_i) \\
&\approx 34.87
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathbb{E}[X \geq 108] &= \mathbb{E}[X - 54 \geq 54] \\
&\leq \frac{\mathrm{Var}(X)}{54^2} \\
&\approx 0.0119570
\end{aligned}
$$

Problem 3

(a)

$$
\begin{aligned}
\mathbb{E}[S_n] &= \sum_{i=1}^{n} \mathbb{E}[X_i] \\
&= \sum_{i=1}^{n} \frac{1}{n} \\
&= 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbb{E}[X_i X_j] &= \mathbb{P}[X_i = 1]\mathbb{P}[X_j = 1 | X_i = 1] \\
&= \frac{1}{n} \times \frac{1}{n-1} \\
&= \frac{1}{n(n-1)}
\end{aligned}
$$

(c)因为

$$
\mathbb{P}[X_i = 1, X_j = 1] = \frac{1}{n(n-1)} \neq \frac{1}{n} \times \frac{1}{n}
$$

(d)

$$
\begin{aligned}
\mathbb{E}[S_n^2] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] \\
&= \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i\neq j} \mathbb{E}[X_i X_j] \\
&= \sum_{i=1}^{n} \mathbb{E}[X_i] + \sum_{i\neq j} \mathbb{E}[X_i X_j] \\
&= 1 + n(n-1) \times \frac{1}{n(n-1)} \\
&= 2
\end{aligned}
$$

(e)

$$
\begin{aligned}
\mathrm{Var}(S_n) &= \mathbb{E}[S_n^2] - \mathbb{E}[S_n]^2 \\
&= 2 - 1 \\
&= 1
\end{aligned}
$$

(f)

$$
\begin{aligned}
\mathbb{P}[S_n \geq 11] &= \mathbb{P}[S_n - 1 \geq 10] \\
&\leq \frac{\mathrm{Var}(S_n)}{10^2} \\
&= \frac{1}{100}
\end{aligned}
$$

Problem 4

(a)由独立性可得

$$
m = 6rbg
$$

(b)

$$
\begin{aligned}
\mathbb{E}[I_T] &= m \\
&= 6rgb \\
\mathrm{Var}(I_t) &= 6m(1-6m) \\
&= 6rgb(1-6rgb)
\end{aligned}
$$

(c)如果不共享边，那么该概率为

$$
p_1 = m^2 = 36r^2b^2g^2
$$

如果共享边，那么该概率为

$$
p_2 = r \times (2bg)^2 + g \times (2br)^2 + b \times (2rg)^2
$$

(d)如果

$$r=b=g=\frac{1}{3}$$

那么

$$\begin{aligned}\mathbb{E}[I_T]&=m\\&=\frac{2}{9}\\\mathrm{Var}(I_t)&=m(1-m)\\&=\frac{14}{81}\\p_1&=\frac{4}{81}\\&=p^2\\p_2&=12\times\frac{1}{3^5}\\&=\frac{4}{81}\\&=p_1\\&=p^2\end{aligned}$$

所以独立。

(e)显然有

$$M=\sum_T I_T$$

而三角形的数量为

$$\binom{n}{3}=\frac{n(n-1)(n-2)}{6}$$

所以

$$
\begin{aligned}
\mathbb{E}[M] &= \sum_T \mathbb{E}[I_T] \\
&= \frac{mn(n-1)(n-2)}{6} \\
&= \frac{n(n-1)(n-2)}{6} \times \frac{2}{9} \\
&= \frac{n(n-1)(n-2)}{27} \\
\mathrm{Var}(M) &= \sum_T \mathrm{Var}(I_T) \\
&= \frac{m(1-m)n(n-1)(n-2)}{6} \\
&= \frac{n(n-1)(n-2)}{6} \times \frac{14}{81} \\
&= \frac{7n(n-1)(n-2)}{243}
\end{aligned}
$$

(f)注意我们有

$$
\begin{aligned}
\mathrm{Var}(M) &= \frac{m(1-m)n(n-1)(n-2)}{6} \\
&< \frac{mn(n-1)(n-2)}{6} \\
&= \mathbb{E}[M]
\end{aligned}
$$

所以

$$
\begin{aligned}
\mathbb{P}[|M-\mu| > \sqrt{\mu \log \mu}] &\leq \frac{\mathrm{Var}(M)}{\mu \log \mu} \\
&< \frac{\mathbb{E}[M]}{\mu \log \mu} \\
&= \frac{\mu}{\mu \log \mu} \\
&= \frac{1}{\log \mu}
\end{aligned}
$$

(g)当$n \to \infty$时，我们有

$$
\mu = \mathbb{E}[M] = \frac{n(n-1)(n-2)}{27} \to \infty
$$

所以

$$
\frac{1}{\log \mu} \to 0
$$

因此

$$\lim_{n\to\infty}\mathbb{P}[|M-\mu|>\sqrt{\mu\log\mu}]=0$$

Problem 5

(a)答案是可以趋于无穷，构造如下：

$$\mathbb{P}[R=i]=\frac{4}{i(i+1)(i+2)}, i\in\mathbb{N}$$

下面分别验证条件：

$$\begin{aligned}
\sum_{i=1}^{\infty}\mathbb{P}[R=i] &= \sum_{i=1}^{\infty}\frac{4}{i(i+1)(i+2)} \\
&= 2\sum_{i=1}^{\infty}\left(\frac{1}{i(i+1)}-\frac{1}{(i+1)(i+2)}\right) \\
&= 2\sum_{i=1}^{\infty}\left(\frac{1}{i(i+1)}-\frac{1}{(i+1)(i+2)}\right) \\
&= 2\times\frac{1}{2} \\
&= 1 \\
\mathbb{E}[R] &= \sum_{i=1}^{\infty}\mathbb{P}[R=i]\times i \\
&= \sum_{i=1}^{\infty}\frac{4}{(i+1)(i+2)} \\
&= 4\sum_{i=1}^{\infty}\left(\frac{1}{i+1}-\frac{1}{i+2}\right) \\
&= 4\times\frac{1}{2} \\
&= 2 \\
\mathbb{E}[R^2] &= \sum_{i=1}^{\infty}\mathbb{P}[R=i]\times i^2 \\
&= \sum_{i=1}^{\infty}\frac{4i}{(i+1)(i+2)} \\
&\to\infty
\end{aligned}$$

(b)

$$\mathbb{E}\left[\frac{1}{R}\right]\le\mathbb{E}\left[1\right]=1$$

(c)设

$$
\begin{aligned}
\mathbb{P}[R=1] &= p \\
\mathbb{P}[R=2] &= q \\
p+q &= 1
\end{aligned}
$$

那么

$$
\begin{aligned}
\operatorname{Var}(R) &= \mathbb{E}[R^2] - \mathbb{E}[R]^2 \\
&= p + 4q - (p+2q)^2 \\
&= 1 + 3q - (1+q)^2 \\
&= 1 + 3q - 1 - 2q - q^2 \\
&= -q^2 + q \\
&\le \frac{1}{4}
\end{aligned}
$$

当且仅当$q=\frac{1}{2}$时取等号。