Problem 1

(a)

$$p=\frac{1}{4}$$

(b)

$$p=\frac{3}{4}\times\frac{1}{2}=\frac{3}{8}$$

Problem 2

(a)定义

$$\mathbb{P}[\{1,\ldots,n\}]=p,\mathbb{P}[\varnothing]=1-p$$

以及其余情形发生的概率为

$$0$$

(b)定义

$$\begin{aligned}\mathbb{P}[\{i\}]=p,1\le i\le n\\ \mathbb{P}[\varnothing]=1-np\end{aligned}$$

以及其余情形发生的概率为

$$0$$

(c)显然

$$F=\cup_{i=1}^n F_i$$

所以

$$\begin{aligned}p&=\mathbb{P}[F_i]\\ &\le\mathbb{P}[F]\\ &\le\sum_{i=1}^n\mathbb{P}[F_i]\\ &=np\end{aligned}$$

Problem 3

投币2次赢的概率为

$$
p(1-p)
$$

投币4次赢的概率为

$$
\left(p^{2}+(1-p)^{2}\right) p(1-p)
$$

所以第一个人赢的概率为

$$
\sum_{i=0}^{\infty}\left(p^{2}+(1-p)^{2}\right)^{i} p(1-p)=p(1-p) \times \frac{1}{1-\left(p^{2}+(1-p)^{2}\right)}=\frac{1}{2}
$$

同理可得第二个人赢的概率为

$$
\sum_{i=0}^{\infty}\left(p^{2}+(1-p)^{2}\right)^{i}(1-p) p=(1-p) p \times \frac{1}{1-\left(p^{2}+(1-p)^{2}\right)}=\frac{1}{2}
$$

所以没人赢的概率为

$$
1-\frac{1}{2}-\frac{1}{2}=0
$$

Problem 4

记

$$
B_{k}=\bigcup_{n=1}^{k} A_{n}
$$

那么

$$
\mathbb{P}\left[B_{k}\right] \leq \sum_{n=1}^{k} \mathbb{P}[A_{n}]
$$

因此

$$
\mathbb{P}\left[B_{k}\right] \leq \sum_{n=1}^{\infty} \mathbb{P}[A_{n}]
$$

因为$B_{k} \uparrow$，所以$\mathbb{P}[B_{k}]$单调递增，因此

$$
\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_{n}\right]=\lim_{k \rightarrow \infty} \mathbb{P}\left[B_{k}\right] \leq \sum_{n=1}^{\infty} \mathbb{P}[A_{n}]
$$

Problem 5

因为

$$(A \cap B) \cap (A - B) = \varnothing, (A \cap B) \cup (A - B) = A$$

所以

$$\begin{aligned}\mathbb{P}[A \cap B] + \mathbb{P}[A - B] &= \mathbb{P}[A] \\ \mathbb{P}[A - B] &= \mathbb{P}[A] - \mathbb{P}[A \cap B]\end{aligned}$$

对第一个结论取$A = \Omega, B = A$即可。

$$\begin{aligned}\mathbb{P}[A \cup B] &= \mathbb{P}[A] + \mathbb{P}[(A \cup B) - A] \\ &= \mathbb{P}[A] + \mathbb{P}[B - A] \\ &= \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]\end{aligned}$$

利用上一个结论可得

$$\mathbb{P}[A \cup B] \le \mathbb{P}[A] + \mathbb{P}[B]$$

对第一个等式取$A = B, B = A$可得

$$\mathbb{P}[B] = \mathbb{P}[A] + \mathbb{P}[B - A] \ge \mathbb{P}[A]$$

Problem 6

(a)

$$p = 1 - \left(\frac{3}{5}\right)^2 - \left(\frac{2}{5}\right)^2 = \frac{12}{25}$$

(b)

$$p = \frac{3}{5} \times \left(\frac{2}{5}\right)^2 + \frac{2}{5} \times \left(\frac{3}{5}\right)^2 = \frac{6}{25}$$

(c)应该是指胜率大的队伍赢得比赛的概率：

$$p = \left(\frac{3}{5}\right)^2 + 2 \times \frac{2}{5} \times \left(\frac{3}{5}\right)^2 = \frac{81}{125}$$