Problem 1

(a)

$$\begin{aligned}\mathbb{P}[B] &= \frac{1}{1000}\\ \mathbb{P}[Y|B] &= 0.99\\ \mathbb{P}[\bar{Y}|\bar{B}] &= 0.97\end{aligned}$$

(b)

$$\begin{aligned}\mathbb{P}[\bar{B}] &= 1-\mathbb{P}[B]\\ &= 0.999\\ \mathbb{P}[Y|\bar{B}] &= 1-\mathbb{P}[\bar{Y}|\bar{B}]\\ &= 0.03\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{P}[Y] &= \mathbb{P}[B]\mathbb{P}[Y|B]+\mathbb{P}[\bar{B}]\mathbb{P}[Y|\bar{B}]\\ &= \frac{1}{1000}\times 0.99+0.999\times 0.03\\ &= 0.03096\end{aligned}$$

(d)

$$\begin{aligned}p &= \mathbb{P}[B|Y]\\ &= \frac{\mathbb{P}[B]\mathbb{P}[Y|B]}{\mathbb{P}[B]\mathbb{P}[Y|B]+\mathbb{P}[\bar{B}]\mathbb{P}[Y|\bar{B}]}\\ &= \frac{\frac{1}{1000}\times 0.99}{0.03096}\\ &= 0.03197674418604651\end{aligned}$$

(e)该比例即为

$$\begin{aligned}p &= \mathbb{P}[Y|B]\\ &= 0.99\end{aligned}$$

Problem 2

$$
\begin{aligned}
\mathbb{P}[S|\text{" }F\text{ "}] &= \frac{\mathbb{P}[S\bigcap\text{" }F\text{ "}]}{\mathbb{P}[\text{" }F\text{ "}]} \\
&= \frac{\mathbb{P}[\text{" }F\text{ "}|S]\mathbb{P}[S]}{\mathbb{P}[\text{" }F\text{ "}|S]\mathbb{P}[S]+\mathbb{P}[\text{" }F\text{ "}|\bar{S}]\mathbb{P}[\bar{S}]} \\
&= \frac{\frac{1}{2}\times\frac{2}{3}}{\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}} \\
&= \frac{2}{3}
\end{aligned}
$$

Problem 3

$$
p = \frac{\frac{1}{2}\times\frac{1}{54}}{\frac{1}{2}\times\frac{1}{54}+\frac{1}{2}\times\frac{1}{53}} = \frac{53}{107}
$$

Problem 4

定义HHT出现在HTT之前的事件为A，那么

$$
\begin{aligned}
\mathbb{P}[A] &= \mathbb{P}[A|H]\mathbb{P}[H]+\mathbb{P}[A|T]\mathbb{P}[T] \\
&= \frac{1}{2}\mathbb{P}[A|H]+\frac{1}{2}\mathbb{P}[A]
\end{aligned}
$$

所以

$$
\mathbb{P}[A] = \mathbb{P}[A|H]
$$

另一方面

$$
\begin{aligned}
\mathbb{P}[A|H] &= \mathbb{P}[A|HH]\mathbb{P}[H]+\mathbb{P}[A|HT]\mathbb{P}[T] \\
&= \frac{1}{2}\mathbb{P}[A|HH]+\frac{1}{2}\mathbb{P}[A|HT]
\end{aligned}
$$

现在考虑给定前两位HH，事件A发生的概率，这时候后事件A发生的情形为

$$
HHT, HHHT, HHHHT, \ldots
$$

所以

$$
\mathbb{P}[A|HH] = \sum_{i=1}^{n}\frac{1}{2^i} = 1
$$

带入可得

$$
\begin{aligned}
\mathbb{P}[A|H] &= \mathbb{P}[A|HH]\mathbb{P}[H]+\mathbb{P}[A|HT]\mathbb{P}[T] \\
&= \frac{1}{2}+\frac{1}{2}\mathbb{P}[A|HT]
\end{aligned}
$$

但是显然有

$$\begin{aligned}\mathbb{P}[A|HT] &= \mathbb{P}[A|HTH]\mathbb{P}[H] + \mathbb{P}[A|HTT]\mathbb{P}[T] \\ &= \frac{1}{2} \times \mathbb{P}[A|HTH] + \frac{1}{2} \times 0 \\ &= \frac{1}{2} \times \mathbb{P}[A|H]\end{aligned}$$

综合之前的讨论得到

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{P}[A|H] \\ &= \frac{1}{2} + \frac{1}{2}\mathbb{P}[A|HT] \\ &= \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \mathbb{P}[A|H] \\ &= \frac{1}{2} + \frac{1}{4} \times \mathbb{P}[A]\end{aligned}$$

因此

$$\mathbb{P}[A] = \frac{2}{3}$$

所以题目中事件发生的概率为

$$1 - \frac{2}{3} = \frac{1}{3}$$

参考资料:

https://dicedcoins.wordpress.com/2012/07/19/flip-hhh-before-htt/