Problem 1

(a)每行4个格子，每个格子有3种颜色选择，所以每行共有81种模式，因此82行中至少两个模式相同。

(b)选择模式相同的两行即可。

(c)每行4个格子选两个着相同颜色，一共的方法为$C_4^2 \times 3 = 18$。

Problem 2

关于n做数学归纳法，记

$$P(n) = 对于n张红黑卡牌，选择最上面的卡赢的策略为\frac{1}{2}$$

$n = 1$时结论显然，假设$n \leq k$时结论成立，现证$n = k + 1$时结论成立。假设有s个红牌，t个黑牌，那么第一张为红色的概率为

$$p = \frac{2^{s+t-1}}{2^{s+t}} = \frac{1}{2}$$

记k张牌按此策略赢的概率为p_k，所以$k + 1$张牌按此策略赢的概率为

$$p_{k+1} = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times p_k = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

Problem 3

用A_1表示红桃A，A_2表示黑桃A，J表示$Jack$，用(a, b)表示实验结果，其中a为没有选择的卡牌，b为丢弃的卡牌。

(a)实验结果为

$$\begin{gathered}(A_1, A_2), (A_1, J) \\ (A_2, A_1), (A_2, J) \\ (J, A_1), (J, A_2)\end{gathered}$$

并且

$$\mathbb{P}[(a, b)] = \frac{1}{6}$$

1.$[K \geq 1]$为

$$\begin{gathered}(A_1, A_2), (A_1, J) \\ (A_2, A_1), (A_2, J) \\ (J, A_1), (J, A_2)\end{gathered}$$

2.

$$(A_2, A_1), (A_2, J)$$
$$(J, A_1), (J, A_2)$$

3.

$$(A_2, A_1), (J, A_1)$$

4.

$$(A_1, A_2), (A_2, A_1)$$
$$(J, A_1), (J, A_2)$$

(b)$K = 2$表示

$$(J, A_1), (J, A_2)$$

所以

$$\mathbb{P}[K = 2|E_1] = \frac{1}{3}$$
$$\mathbb{P}[K = 2|E_2] = \frac{1}{2}$$
$$\mathbb{P}[K = 2|E_3] = \frac{1}{2}$$
$$\mathbb{P}[K = 2|E_3] = \frac{1}{2}$$

(c)

$$p = \frac{C_{d-1}^{h-1}}{C_d^h} = \frac{h}{d}$$

(d)

$$\begin{aligned}\mathbb{P}[K = 2|A_1\text{is in your hand}] &= \frac{\mathbb{P}[K = 2 \text{ and } A_1\text{is in your hand}]}{\mathbb{P}[A_1\text{is in your hand}]} \\ &= \frac{\mathbb{P}[K = 2]\mathbb{P}[A_1\text{is in your hand}|K = 2]}{\frac{h}{d}} \\ &= \frac{\mathbb{P}[K = 2] \times 2/a}{\frac{h}{d}} \\ &= \mathbb{P}[K = 2] \times \frac{2d}{ah}\end{aligned}$$

(e)因为

$$
\begin{aligned}
\mathbb{P}[\text{the revealed card is an Ace}] &= \frac{\sum_{k=1}^{a} C_a^k \times C_{d-a}^{h-k}}{C_d^h} \times \frac{k}{h} \\
&= \frac{\sum_{k=1}^{a} kC_a^k \times C_{d-a}^{h-k}}{h \times C_d^h} \\
&= \frac{\sum_{k=1}^{a} aC_{a-1}^{k-1} \times C_{d-a}^{h-k}}{h \times C_d^h} \\
&= \frac{aC_{d-1}^{h-1}}{h \times C_d^h} \\
&= \frac{aC_{d-1}^{h-1}}{h \times \frac{d}{h} C_{d-1}^{h-1}} \\
&= \frac{a}{d} \\
\mathbb{P}[\text{the revealed card is an Ace}|K=2] &= \frac{2}{h}
\end{aligned}
$$

所以

$$
\begin{aligned}
\mathbb{P}[K=2|\text{the revealed card is an Ace}] &= \frac{\mathbb{P}[K=2 \text{ and the revealed card is an Ace}]}{\mathbb{P}[\text{the revealed card is an Ace}]} \\
&= \frac{\mathbb{P}[K=2]\mathbb{P}[\text{the revealed card is an Ace}|K=2]}{\mathbb{P}[\text{the revealed card is an Ace}]} \\
&= \frac{\mathbb{P}[K=2] \times \frac{2}{h}}{\frac{a}{d}} \\
&= \mathbb{P}[K=2] \times \frac{2d}{ah} \\
&= \mathbb{P}[K=2|A_1 \text{is in your hand}]
\end{aligned}
$$