Problem 1

设

$$\sum_{i=1}^{n} i^3 = an^4 + bn^3 + cn^2 + dn + e$$

令$n = 0, 1, 2, 3, 4$可得

$$\begin{cases} 0 = e \\ 1 = a + b + c + d + e \\ 9 = 16a + 8b + 4c + 2d + e \\ 36 = 81a + 27b + 9c + 3d + e \\ 100 = 144a + 64b + 16c + 4d + e \end{cases}$$

解得

$$\begin{cases} a = \frac{1}{4} \\ b = \frac{1}{2} \\ c = \frac{1}{4} \\ d = 0 \\ e = 0 \end{cases}$$

所以

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Problem 2

由斯特林公式可得

$$\begin{aligned} \ln(n^2!) &= \ln\left(\sqrt{2\pi n^2}\left(\frac{n^2}{e}\right)^{n^2} e^{\epsilon(n^2)}\right) \\ &= n^2 \ln(n^2) - n^2 + \frac{1}{2}\ln(2\pi n^2) + \epsilon(n^2) \end{aligned}$$

其中

$$\frac{1}{12n+1} \leq \epsilon(n) \leq \frac{1}{12n}$$

所以

$$\lim_{n\to\infty}\frac{\ln(n^2!)}{n^2\ln(n^2)}=1$$

因此

$$\ln\left(n^2!\right)=\Theta\left(n^2\ln n\right)$$

Problem 3

记

$$f(x)=x^6$$

那么

$$I=\int_1^n x^6dx=\frac{1}{7}(n^7-1)$$

因此

$$I+f(1)\le\sum_{k=1}^n k^6\le I+f(n)$$
$$\frac{1}{7}(n^7-1)+1\le\sum_{k=1}^n k^6\le\frac{1}{7}(n^7-1)+n^6$$
$$\frac{1}{7}=\lim_{n\to\infty}\frac{\frac{1}{7}(n^7-1)+1}{n^7}\le\lim_{n\to\infty}\frac{\sum_{k=1}^n k^6}{n^7}=\lim_{n\to\infty}\frac{\frac{1}{7}(n^7-1)+n^6}{n^7}=\frac{1}{7}$$

因此

$$\sum_{k=1}^n k^6=\Theta\left(n^7\right)$$