Construct a finite field of 8 elements.

Sol: To construct $F_8 = F_{2^3}$ (ie) $GF(8)$

Take the prime field $\{0, 1\}$ & choose an irreducible polynomial of degree 3 over F_2.

A common choice $f(x) = x^3 + x + 1$

Here $n = 3$, the elements of F_{2^3} will be of the form

$a_0 + a_1 x + a_2 x^2$; $a_0, a_1, a_2 \in F_2$.

Thus the elements are $\{0, 1, x, x^2, 1+x, 1+x^2, x+x^2, 1+x+x^2\}$

Addition modulo 2

$+_2$	0	1	x	x^2	$1+x$	$1+x^2$	$x+x^2$	$1+x+x^2$
0	0	1	x	x^2	$1+x$	$1+x^2$	$x+x^2$	$1+x+x^2$
1	1	0	$1+x$	$1+x^2$	x	x^2	$1+x+x^2$	$x+x^2$
x	x	$1+x$	0	$x+x^2$	1	$1+x+x^2$	x^2	$1+x^2$
x^2	x^2	x^2+1	x^2+x	0	$1+x+x^2$	1	x	$1+x$
$1+x$	$1+x$	x	1	$1+x+x^2$	0	$x+x^2$	$1+x^2$	x^2
$1+x^2$	$1+x^2$	x^2	$1+x+x^2$	1	$x+x^2$	0	$1+x$	x
$x+x^2$	x^2+x^2	$1+x+x^2$	x^2	x	$1+x^2$	$1+x$	0	1
$1+x+x^2$	$1+x+x^2$	$x+x^2$	$1+x^2$	$1+x^2$	x^2	x	1	0

$e = 0$

The inverses are

$0 = 0$, $1 = 1$, $x = x$, $x^2 = x^2$, $1+x = 1+x$

$1+x^2 = 1+x^2$, $x+x^2 = x+x^2$ and

$1+x+x^2 = 1+x+x^2$.

Multiplication modulo 2.

$\times_2$	0	1	x	x^2	$1+x$	$1+x^2$	$x+x^2$	$1+x+x^2$
0	0	0	0	0	0	0	0	0
1	0	1	x	x^2	$1+x$	$1+x^2$	$x+x^2$	$1+x+x^2$
x	0	x	x^2	$1+x$	$x+x^2$	1	x^2+x+1	x^2+1
x^2	0	x^2	$1+x$	x^2+x	$1+x+x^2$	x	x^2+1	1
$1+x$	0	$1+x$	x^2+x	x^2+x+1	x^2+1	x^2	1	x
$1+x^2$	0	$1+x^2$	1	x	x^2	$1+x+x^2$	$x+1$	x^2+x
$x+x^2$	0	$x+x^2$	x^2+x+1	x^2+1	1	$x+1$	x	x^2
$1+x+x^2$	0	$1+x+x^2$	x^2+1	1	x	x^2+x	x^2	$x+1$

The multiplication inverses are

$1^{-1} = 1$, $x^{-1} = 1+x^2$ $(x^2)^{-1} = 1+x+x^2$

$(1+x)^{-1} = x+x^2$, $(1+x^2)^{-1} = x$

$(x+x^2)^{-1} = 1+x$, $(1+x+x^2)^{-1} = x^2$.

Unit - 5.

Classical Theorems In Number Theory.

In this chapter, we will discuss three important classical theorems in number theory namely wilson's theorem, Fermat's little theorem and Euler's theorem. These theorems are important milestones in the development of the theory of Congruence and illustrate the significance of Congruence.

Lemma:

A positive integer a is self-invertible modulo p iff $a \equiv \pm 1 \pmod{p}$.

Proof:

part (i) Assume a is self invertible modulo p

(ii) $a \times a \equiv 1 \pmod{p}$

$a^2 \equiv 1 \pmod{p}$.

$\Rightarrow p \mid a^2 - 1 \Rightarrow p \mid (a+1)(a-1)$.

$\Rightarrow p \mid (a+1)$ (or) $p \mid (a-1)$.

$\Rightarrow a+1 \equiv 0 \pmod{p}$ (or) $a-1 \equiv 0 \pmod{p}$

$\Rightarrow a \equiv -1 \pmod{p}$ (or) $a \equiv 1 \pmod{p}$

$\Rightarrow a \equiv \pm 1 \pmod{p}$.

part (ii):

Assume $a \equiv \pm 1 \pmod{p}$

Squaring, $a^2 \equiv 1 \pmod{p}$

$\Rightarrow a$ is self invertible.

Theorem: (Wilson's Theorem)

If p is a prime number, then $(p-1)! \equiv -1 \pmod{p}$

Proof:

If $p=2$, $(2-1)! = 1! \Rightarrow 1 \equiv -1 \pmod{2}$

$\therefore$ The result is true for $p=2$

$\therefore$ Assume p is a prime & $p>2$, [p is odd]

Consider $(p-1)! = 1 \times 2 \times 3 \cdots \times p-1$

Clearly $1 \equiv 1 \pmod{p} \Rightarrow 1$ is self invertible

$p-1 \equiv -1 \pmod{p} \Rightarrow (p-1)$ is self invertible $\hookrightarrow$ ①

wkt, All the elements in the set $\{1, 2, 3, \cdots p-1\}$ has inverse.

Since 1 & $p-1$ have self inverse

$\therefore$ Group remaining elements $\{2, 3, \cdots p-2\}$ with their inverse.

Then, $(2 \times 3 \times 4 \times 5 \cdots \times p-2) \equiv 1 \pmod{p} \longrightarrow$ ②

Consider, $(p-1)! \equiv (1 \times 2 \times \cdots \times p-2 \times p-1) \pmod{p}$

$\equiv (1 \times 1 \times p-1) \pmod{p}$ (∵ From ②)

$\therefore (p-1)! \equiv -1 \pmod{p}$ (∵ From ①)

Eg:

Let $p=11$.

$(p-1)! \equiv -1 \pmod{p}$

$(11-1)! \equiv -1 \pmod{11}$

$10! \equiv -1 \pmod{11} \longrightarrow$ ①

$10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$

$10! = 1 \cdot (2 \cdot 6)(3 \cdot 4)(5 \cdot 9)(7 \cdot 8) \cdot 10$

$\equiv 1 \cdot (1)(1)(1)(1)\, 10 \pmod{11}$

$\equiv 10 \pmod{11}$

$10! \equiv -1 \pmod{11}$ (which illustrate wilson's thm).

Theorem: (Converse of Wilson's Theorem).

If n is a positive integer such that $(n-1)! \equiv -1 \pmod{n}$, then n is a prime.

Proof:

Suppose n is not a prime.

Then $n = ab$ for some $1 < a < n$, $1 < b < n$. → (1)

a is any integer from 2 to $n-1$.

So 'a' divides the product $2 \times 3 \times \cdots \times (n-1)$ → (2)

Suppose $(n-1)! \equiv -1 \pmod{n}$ → (3)

By (2) $a \mid (n-1)!$

By (1) $a \mid n$

By (3) $n \mid (n-1)! + 1$

By transitive property, we get $a \mid (n-1)! + 1$ → (4)

From (2) & (4) 'a' divides their difference

(ie) $a \mid (n-1)! + 1 - (n-1)!$

(ie) $a \mid 1 \Rightarrow a = 1$

which is a contradiction since $1 < a < n$

So, n must be prime.

Lemma:

Let p be a prime number and a be any integer such that $p \nmid a$. Then the least residues of the integers $1.a, 2a, 3a \ldots (p-1)a$ modulo p are a permutation of the integers $1.2.3 \ldots (p-1)$.

Proof: This proof consists of 2 steps.

i) $ia \not\equiv 0 \pmod{p} \quad \forall i = 1, 2, \ldots p-1$.

ii) Suppose $ia \equiv ja \pmod{p} \Rightarrow i = j$

Step (i): Suppose $ia \equiv 0 \pmod{p}$ for some $i = 1, 2 \ldots p-1$

$\Rightarrow p \mid ia$ but $p \nmid a \Rightarrow p \mid i$ but $i < p$

$\Rightarrow i = 0$

$\therefore ia \not\equiv 0 \pmod{p}$

ii) Suppose $ia \equiv ja \pmod{p}$

$\Rightarrow p \mid (ia - ja)$

$\Rightarrow p \mid a(i-j)$ but $p \nmid a$

$\Rightarrow p \mid (i-j) \quad [\because i < p \text{ \& } j < p.$

$\Rightarrow i - j = 0 \qquad i - j < p].$

$\Rightarrow \boxed{i = j}$.

Hence the lemma is proved.

Theorem: (Fermat's little theorem)

Let p be a prime number and a any integer such that $p \nmid a$. Then $a^{p-1} \equiv 1 \pmod{p}$.

Proof:

By lemma,

Consider, $1a \cdot 2a \cdots (p-1)a \equiv (1 \cdot 2 \cdots p-1) \pmod{p}$

$\Rightarrow (1 \cdot 2 \cdot 3 \cdots (p-1))\, a^{p-1} \equiv (1 \cdot 2 \cdots p-1) \pmod{p}$

$\Rightarrow (p-1)!\, a^{p-1} \equiv (p-1)! \pmod{p}$

$\Rightarrow a^{p-1} \equiv 1 \pmod{p}$ since $\gcd((p-1)!, p) = 1$.

Hence theorem is proved.

[If $ac \equiv bc \pmod{m}$ since, $\gcd(c, m) = 1$ then $a \equiv b \pmod{m}$]

Eg:

Let $p = 7$ and $a = 12$.

By the above lemma, $1 \cdot 12, 2 \cdot 12, 3 \cdot 12, 4 \cdot 12, 5 \cdot 12, 6 \cdot 12$ modulo 7 are a permutation of the integers 1 through 6.

So, $(1 \cdot 12)(2 \cdot 12)(3 \cdot 12)(4 \cdot 12)(5 \cdot 12)(6 \cdot 12) \equiv 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \pmod{7}$

(ie) $6!\, 12^6 \equiv 6! \pmod{7}$

$\Rightarrow 12^6 \equiv 1 \pmod{7}$

Problems:

1) Find the remainder when 24^{1947} is divid
by 17.

Sol: W.K.T $24 \equiv 7 \pmod{17}$

By Fermat's theorem $a = 7$, $p = 17$ and $p \nmid a$

$\therefore a^{p-1} \equiv 1 \pmod{p}$

$7^{17-1} \equiv 1 \pmod{17}$

$7^{16} \equiv 1 \pmod{17}$

$24^{1947} \equiv 7^{1947} \pmod{17}$

$\equiv (7^{16})^{121} \times (7^{11}) \pmod{17}$

$\equiv (1)^{121} \times 7^{11} \pmod{17}$

$\equiv 7^{11} \pmod{17}$

$\equiv 7^8 \times 7^3 \pmod{17}$

$\equiv (-1)(343) \pmod{17}$

$\equiv (-3) \pmod{17}$

$24^{1947} \equiv 14 \pmod{17}$

The remainder is 14.

Find the remainder when 30^{2020} is divided by 19.

We know that $30 \equiv 11 \pmod{19}$

By Fermat's theorem,

$a = 11$, $p = 19$ and $p \nmid a$.

$\therefore a^{p-1} \equiv 1 \pmod{p}$

$11^{19-1} \equiv 1 \pmod{19}$

$11^{18} \equiv 1 \pmod{19}$

$30^{2020} \equiv 11^{2020} \pmod{19}$

$\equiv (11^{18})^{112} 11^4 \pmod{19}$

$\equiv (1)^{112} 11^4 \pmod{19}$

$\equiv 11^4 \pmod{19}$

Consider $11^2 = 121 = 7 \pmod{19}$

$11^2 \equiv 7 \pmod{19}$

$$11^4 = 7^2 \pmod{19}$$

$$\equiv 49 \pmod{19}$$

$$\equiv 11 \pmod{19}$$

But, $30^{2020} \equiv 11^4 \pmod{19}$

$$\equiv 11 \pmod{19}$$

$\therefore$ the remainder is 11

Euler's Theorem

Euler's Phi function

Definition:

Let m be a positive integer. Then Euler's Phi function ($\phi(m)$) denotes the no. of positive integers $\leq m$ and relatively prime to m.

Eg: since $1 \leq 1$ and relatively prime to 1.

$\therefore \phi(1) = 1$.

$\phi(2) = 1$. Since 1 is the only integer ≤ 2 & relatively prime to 2.

Illy $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, $\phi(6) = 2$, $\phi(7) = 6$

Lemma:

A positive integer p is a prime if and only if $\phi(p) = p - 1$.

Proof: Let p be a prime. Then there are $p-1$ positive integers $\leq p$ and relatively prime to p.

So, $\phi(p) = p - 1$

Conversely, let p be a positive integer such that $\phi(p) = p - 1$.

Let $d \mid p$ where $1 < d < p$. since there are exactly $p-1$ positive integers $< p$, d is one of them, and $(d, p) \neq 1$, so $\phi(p) < p - 1$, a contradiction.

Thus, p must be prime.

Multiplicative Function:

A number-theoretic function f is multiplicative if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime.

Euler's Theorem:

Let m be a positive integer and 'a' any integer with $(a,m)=1$. Then $a^{\phi(m)} \equiv 1 \pmod{m}$.

Problems:

1) Find the remainder when 245^{1040} is divided 18, using Euler's theorem.

Sol: By Euler's theorem,

If $(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$

Here $a = 245$, $m = 18$ & $(245, 18) = 1$

$\therefore 245^{\phi(18)} \equiv 1 \pmod{18}$

$245^{6} \equiv 1 \pmod{18}$ since $\phi(18) = 6$.

$\therefore (245^{6})^{173} \equiv (1)^{173} \pmod{18}$

$\equiv 1 \pmod{18}$

But $245^{1040} \equiv 245^{1038} \cdot 245^{2}$

since $245 \equiv 11 \pmod{18}$

2 | 18
3 | 9
3 | 3
1

$18 = 2 \times 3 \times 3$

17
6 | 1040
6
44
42
20
18
2

$245^2 \equiv 11^2 \pmod{18}$

$\equiv 121 \pmod{18}$

$\equiv 13 \pmod{18}$

$\therefore 245^{1040} \equiv 1 \times 13 \pmod{18}$

$\equiv 13 \pmod{18}$

$\therefore$ 13 is the remainder when 245^{1040} is divided by 18.

Using Euler's theorem, find the remainder when 7^{1020} is divided by 15.

By Euler's theorem, if $(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$

Here $a = 7$, $m = 15$ and $(7, 15) = 1$

$\therefore 7^{\phi(15)} \equiv 1 \pmod{15}$

$7^8 \equiv 1 \pmod{15}$ $(\because \phi(15) = 8)$

$\therefore 7^{1020} \equiv 7^{(8 \times 127) + 4} \pmod{15}$

$\equiv (7^8)^{127} \cdot 7^4 \pmod{15}$

$\equiv (1)^{127} \cdot 7^4 \pmod{15}$

$\equiv 7^4 \pmod{15}$

$\equiv 7^2 \times 7^2 \pmod{15}$

$\equiv 49 \times 49 \pmod{15}$

$\equiv 4 \times 4 \pmod{15}$

$\equiv 16 \pmod{15}$

$\equiv 1 \pmod{15}$

$\therefore$ 1 is the remainder when 7^{1020} is divided by 15.

To find $\phi(6860)$:

$$\phi(6860) = \phi(2^2)\,\phi(5)\,\phi(7^3)$$

$$= 2^2\left(1-\frac{1}{2}\right)4\cdot 7^3\left(1-\frac{1}{7}\right)$$

$$= 252.$$

Evaluate $\phi(221)$ and $\phi(1105)$.

$\phi(221) = \phi(13 \times 17)$

$= \phi(13) \times \phi(17)$

$= 12 \times 16 = 192$

$\phi(1105) = \phi(5 \times 13 \times 17)$

$= \phi(5) \times \phi(13) \times \phi(17)$

$= 4 \times 12 \times 16 = 768$ ✓

Theorem : ✓

Let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ be cannonical decomposition of a positive integer n. Then $\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_k}\right)$

Proof: Given $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$

since ϕ is multiplicative.

$\therefore \phi(n) = \phi(p_1^{e_1}) \phi(p_2^{e_2}) \phi(p_3^{e_3}) \dots \phi(p_k^{e_k})$

$= p_1^{e_1}\left(1-\frac{1}{p_1}\right) p_2^{e_2}\left(1-\frac{1}{p_2}\right) \dots p_k^{e_k}\left(1-\frac{1}{p_k}\right)$

(since $\phi(p^e) = p^e - p^{e-1}$)

$= p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_k}\right)$

$= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_k}\right)$

Hence the proof.

1) Compute $\phi(666)$ and $\phi(1976)$

Sol: $666 = 2 \times 3^2 \times 37$

$\phi(666) = \cancel{\phi(2)}\ 666\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{37}\right)$

$1976 = 2^3 \times 13 \times 19$

$= 1976\left(1-\frac{1}{2}\right)\left(1-\frac{1}{13}\right)\left(1-\frac{1}{19}\right)$

2) Compute $\phi(600)$ and $\phi(71)$

$600 = 2^3 \times 3 \times 5^2$

$\phi(600) = \phi(2^3)\,\phi(3)\,\phi(5^2)$

$= 2^3\left(1-\frac{1}{2}\right) \times 2 \times 5^2\left(1-\frac{1}{5}\right)$

$= 4 \times 2 \times 5 \times 4 = 160$

$71! = 5040 = 2^4 \times 3^2 \times 5 \times 7$

$\phi(71!) = \phi(2^4)\,\phi(3^2)\,\phi(5)\,\phi(7)$

$= 2^4\left(1-\frac{1}{2}\right) 3^2\left(1-\frac{1}{3}\right) 4 \times 6$

$= 1152.$

TAU AND SIGMA FUNCTIONS

1) Let n be a positive integer. Then $\tau(n)$ denotes the no. of positive divisors of n.

$$\tau(n) = \sum_{d|n} (1)$$

2) Let n be a positive integer. Then $\sigma(n)$ denotes the sum of positive divisors of n.

$$\sigma(n) = \sum_{d|n} d.$$

Theorem: The tau and Sigma functions are multiplicative

The constant function $f(n)=1$ is multiplicative,

we have $F(n) = \sum_{d|n} f(d) = \sum_{d|n} 1 = \tau(n)$ is multiplicative.

only statement with proof.

The function $g(n) = n$ is multiplicative.

$\therefore F(n) = \sum_{d|n} g(d) = \sum_{d|n} d = \sigma(n)$ is multiplicative.

If $(m,n)=1$, then $\tau(mn) = \tau(m)\tau(n)$

$\sigma(mn) = \sigma(m)\sigma(n)$.

Theorem: Let p be any prime and k any positive integer. Then $\tau(p^k) = k+1$

$$\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$$

If p is prime then the divisor are 1 and p.

So, $\tau(p) = 2$.

Now, the divisors of p^k are $1, p, p^2 \ldots p^k$ and there are $k+1$ terms.

$$\tau(p^k) = k+1$$

$\sigma(p^k)$ = sum of divisors of p^k.

$$= 1 + p + p^2 + \ldots + p^k$$

$$= \frac{p^{k+1} - 1}{p - 1} \quad \text{since } p > 1.$$

Theorem: Let $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$. Then

$$\tau(n) = (k_1+1)(k_2+1)\ldots(k_r+1)$$

$$\sigma(n) = \left(\frac{p_1^{k+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{k+1} - 1}{p_2 - 1}\right) \ldots \left(\frac{p_r^{k+1} - 1}{p_r - 1}\right)$$

Compute $\tau(23)$, $\tau(857)$ and $\tau(6491)$

$n = 23$, 857 and 6491 are prime numbers.

If p is prime then $\tau(n) = 2$.

$\therefore \tau(23) = \tau(857) = \tau(6491) = 2$.

Compute $\tau(81)$, $\tau(2187)$ and $\tau(1024)$

$n = 81 = 3^4$

$\tau(81) = \tau(3^4) = 4+1 = 5$

$n = 2187 = 3^7$

$\tau(2187) = \tau(3^7) = 7+1 = 8$.

$\tau(1024) = \tau(2^{10}) = 10+1 = 11$.

Compute $\tau(36)$, $\tau(1560)$, $\tau(6120)$ & $\tau(44982)$

$\tau(36) = \tau(2^2 \times 3^2)$

$= (2+1)(2+1) = 3 \times 3 = 9$

$\tau(1560) = \tau(2^3 \times 3 \times 5 \times 13)$

$= (3+1)(1+1)(1+1)(1+1) = 32$.

$\tau(6120) = \tau(2^3 \times 3^2 \times 5 \times 17)$

$= (3+1)(2+1)(1+1)(1+1)$

$= 48.$

$\tau(44982) = \tau(2 \times 3^3 \times 7^2 \times 17)$

$= (1+1)(3+1)(2+1)(1+1)$

$= 48.$

4) Compute $\sigma(97)$, $\sigma(331)$ and $\sigma(4027)$

Sol: Here $n = 97$, 331 and 4027 are prime numbers

$\therefore \sigma(p) = p+1)$

$\sigma(97) = 98$

$\sigma(331) = 332$

$\sigma(4027) = 4028.$

5) Compute $\sigma(81)$, $\sigma(2187)$ and $\sigma(1024)$

Sol: $\sigma(81) = \sigma(3^4) = \frac{3^{4+1}-1}{3-1} = \frac{3^5-1}{2}$

$= 121$

$\sigma(2187) = \sigma(3^7) = \frac{3^8-1}{3-1} = \frac{6561-1}{2}$

$= 3280$

$\sigma(1024) = \sigma(2^{10}) = \frac{2^{11}-1}{2-1} = 2048-1$

$= 2047$

6) Compute $\sigma(36)$, $\sigma(1560)$, $\sigma(6120)$ & $\sigma(2187)$

$\sigma(36) = \sigma(2^2)\,\sigma(3^2)$

$= \frac{2^{2+1}-1}{2-1} \times \frac{3^{2+1}-1}{3-1}$

$= 7 \times 13 = 91$

$\sigma(1560) = \sigma(2^3 \times 3 \times 5 \times 13)$

$= \frac{2^4-1}{2-1} \times \frac{3^2-1}{3-1} \times \frac{5^2-1}{5-1} \times \frac{13^2-1}{13-1}$

$= 15 \times 4 \times 6 \times 14$

$= 5040.$

$\sigma(6120) = \sigma(2^3 \times 3^2 \times 5 \times 17)$

$= \frac{2^4-1}{2-1} \times \frac{3^3-1}{3-1} \times \frac{5^2-1}{5-1} \times \frac{17^2-1}{17-1}$

$= 15 \times 13 \times 6 \times 18$

$= 21060$

$\sigma(2187) = \sigma(2 \times 3^3 \times 7^2 \times 17)$

$= \frac{2^2-1}{2-1} \times \frac{3^4-1}{3-1} \times \frac{7^3-1}{7-1} \times \frac{17^2-1}{17-1}$

$= 3 \times 40 \times 57 \times 18 = 123120.$

Let n be the product of a pair or twin prime p being the smallest of the two then show that $\sigma(n) = (p+1)(p+3)$.

Given that n is the product of a pair of twin prime.

$n = p(p+2)$

$\sigma(n) = \sigma[p(p+2)]$

$= \sigma(p)\,\sigma(p+2)$

$= (p+1)(p+3)$

If p and q are twin primes then $p < q$ then show that $\sigma(p) = \sigma(p) + 2$ or $\sigma(p+2) = \sigma(p) + 2$.

If p and q are twin primes with $p < q$ then $q = p+2$.

If p is prime then $\sigma(p) = p+1$.

q is also prime then

$$\begin{aligned}\sigma(q) &= \sigma(p+2) \\ &= p+2+1 \\ &= \sigma(p+1)+2 \\ &= \sigma(p)+2 .\end{aligned}$$