Week 5 highlights

- Classify sets by cardinality into: finite sets, countable sets, uncountable sets.
- Product and sum rules
- Reason about the size of power sets
- Permutations and combinations
- Explain the central idea in Cantor's diagonalization argument.

Cantor-Schroder-Bernstein Theorem: For all nonempty sets,

$$|A| = |B| \quad \text{if and only if} \quad (|A| \leq |B| \text{ and } |B| \leq |A|) \quad \text{if and only if} \quad (|A| \geq |B| \text{ and } |B| \geq |A|)$$

To prove $|A| = |B|$, we can do any **one** of the following

one one & onto

$|A|=|B|$

- Prove there exists a bijection $f : A \to B$;
- Prove there exists a bijection $f : B \to A$;
- Prove there exists two functions $f_1 : A \to B$, $f_2 : B \to A$ where each of f_1, f_2 is one-to-one. $\underbrace{|A|\leq|B| \quad |B|\leq|A|}$ combined $|A|=|B|$
- Prove there exists two functions $f_1 : A \to B$, $f_2 : B \to A$ where each of f_1, f_2 is onto.

$\underbrace{|A|\geq|B| \quad |B|\geq|A|}$ combined $|A|=|B|$

A set A is **finite** means it is empty or it is the same size as $\{1, \ldots, n\}$ for some unique $n \in \mathbb{N}$.
A set A is **countably infinite** means it is the same size as $\mathbb{N}$.

Key insight for proofs involving sizes of finite sets: Use the definition of size of a set S to mean the existence of a bijection from S to $\{1, \ldots, |S|\}$.

Theorem: If A and B are disjoint, finite sets, then $|A \cup B| = |A| + |B|$ sum rule

$\exists$ bijection $f: A \cup B \longrightarrow \{1, \ldots, |A|+|B|\}$

Order A, B

$\forall a \in A \quad f(a) = i$ where a is the i^{th} element in A

$\forall b \in B \quad f(b) = |A| + j$ where b is the j^{th} element in B

f is (i) well defined

(ii) one-one

(iii) onto

Theorem: If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$ product rule

$\exists$ bijection $f: A \times B \longrightarrow \{1, 2, 3, \ldots, |A| \cdot |B|\}$

Corollary: If A is a finite set, then $|A^2| = |A|^2$

$|A \times A| = |A| \cdot |A|$

$\underbrace{A \times \dots \times A}_{n \text{ ti}}$

$|A^n| = |A|^n \longrightarrow$ Induction with this as base case

Applications of the sum and product rule

How many RNA strings of length 5 can be constructed from the basis set $B = \{\mathtt{A}, \mathtt{C}, \mathtt{U}, \mathtt{G}\}$?

✓ 4^5 or 5^4 Plug

1024

$A = B$ $|A| = 4$

$n = 5$ $\therefore |A|^n = 4^5$

How many functions can be defined between two finite sets A and B ?

$|A|^{|B|}$ X $\quad \underbrace{|B| \cdot |B| \cdots |B|}_{A \text{ times}}$ $|B|^{|A|}$ ✓ ?

for each $a \in A$ it can map to a $b \in B$

So $|B|$ options for each $a \in A$

$|A|$ $|B|$

A B

$f: A \longrightarrow B$

#possible f $= |B|^{|A|}$

The CS department is looking for an instructor for CS40 who maybe selected among the faculty and grad students. If there are 35 faculty and 100 grad students, how many choices are there for the instructor?

A B

A & B are disjoint $A \cap B = \phi$

$100 + 35 = 135$

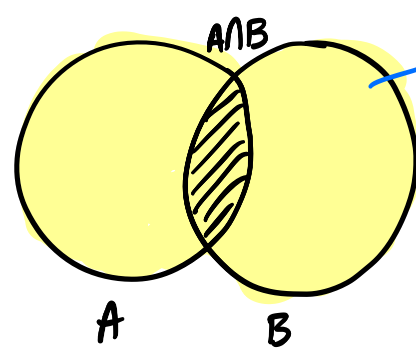
$|B| + |A| = |A \cup B|$

Theorem: If A and B are finite sets, not necessarily disjoint then $|A \cup B| = |A| + |B| - |A \cap B|$

0 or 1 at each bit

How many bit strings of length 8 start with 1 or end with 00?

__ __ __ __ __ __ __ __



yellow $A \cup B$

bit strings of len 8 $= 2^8$

bit strings of len 8 starting with 1 $= 2^7$

bit strings of len 8 end with 00 $= 2^6$

bit strings of len 8 that start with 1 or end with 00 $= 2^7 + 2^6 - 2^5$

start with 1 end with 00 both

Definition: The **power set** of a set S is the set of all the subsets of S and is denoted by $\mathcal{P}(S)$

If $S = \{1, 2, 3\}$, *what is* $\mathcal{P}(S)$ *?*

$$\mathcal{P}(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$$

Construct the power set of S*and reason about its size,* $|\mathcal{P}(S)|$

$$|\mathcal{P}(S)| = 2^{|S|}$$

Think of each element (set) in $\mathcal{P}(S)$ as a bit string of len $|S|$ where bit i says whether ith element of $S \in$ set or not.

$S = \{1, 2, 3\}$

$\{1,2\} = 1\ 1\ 0$

ex:- $\{1, 2\} = 110$

$\{1, 2, 3\} = 111$

$\{2\} = 010$

$\mathcal{P}(S) \longrightarrow$ think as the set of all bit strings of len $|S|$.

$\therefore |\mathcal{P}(S)| = 2^{|S|}$

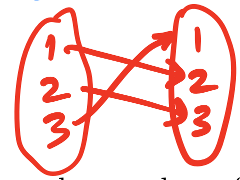
A **permutation** is an ordered arrangement of the elements of a set.

$S = \{1, 2, 3\}$ ex:- permutation of S = 3, 2, 1

$\phi : S \to S$ such that ϕ is one-one & onto

An **r-permutation** is an ordered arrangement of r elements of a set

perm 2 3 1



$S = \{1, 2, 3\}$ 2, 3 $\rightarrow$ 2-permutation

Define $P(n, r)$ as the number of r-permutations of a set with n elements. $P(n, n)$ is then the number of permutations of a set with n elements

Example: List all the r-permutations of $S = \{1, 2, 3\}$.

use case perm: shuffling a deck

r-perm - black jack
only 1/2 the deck is permuted

#permutations = $n!$

(i) $n!$ ✓

Mapping
1st element n choices
2nd element n-1 -
.
.
.
nth element 1 choices

Total = $n \cdot n{-}1 \cdots 1 = n!$

The traveling salesman problem (also TSP) asks the question: *Given a list of n cities, a city of origin, and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?*

Brute force: (i) go over all permutations of cities & compute the cost

There is no efficient solution to TSP. A solution by exhaustion will need to compute the length of all possible routes. How many routes are there? $n!$

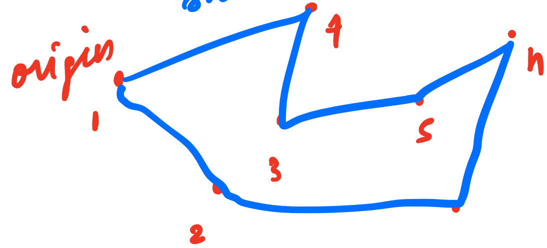
shortest hamiltonian cycle

(ii) return best one

Time: $O(n!)$

Qs: what is the shortest path that visits all cities & returns to origin

Think: mailman deliver letter



NP-complete

$$\# r\text{-permutations} = \frac{n!}{(n-r)!}$$

1^{st} element n choices

2^{nd} element $n-1$ -

.
.
.

r^{th} element 1 choices

Total $= n \cdot n-1 \cdot \cdot \cdot 1 = n!$

same as this but stop at $(n-r+1)$

$$\therefore \text{Total } r\text{-perm} = \underset{1}{n}\,\underset{2}{(n-1)}\,\underset{3}{(n-2)} \dots \underset{r}{(n-r+1)}$$

element num

$$= \frac{n(n-1)(n-2)\dots(n-r+1)\big[(n-r)\dots(1)\big]}{\big[(n-r)\dots(1)\big]}$$

$$= \frac{n!}{(n-r)!}$$

I am playing a card game (with 52 cards) and say there are b bad shuffling options, then what is the probability that we end up with a good shuffle:

$$P[\text{bad shuffle}] = \frac{\#\text{ bad options}}{\#\text{ total options}} = \frac{b}{52!}$$

$$P[\text{good shuffle}] = 1 - \frac{b}{52!}$$

→ we do not care about order!

Definition: An **r-combination** of a set with n elements is a subset of the set of size r and is denoted as

r-combinations ^nC_r $\quad \binom{n}{r}$

n choose r also called binomial coefficients

Two ways to write it

Fill in the table to list all the subsets of a given size of the set $S = \{1, 2, 3\}$

r	Subsets of S of size r
0	ϕ
1	$\{1\}$ $\{2\}$ $\{3\}$
2	$\{1,2\}$ $\{2,3\}$ $\{1,3\}$
3	$\{1,2,3\}$

n-combinations = 1

$^nC_n = 1$

In general, how many subsets of size r can be constructed from a set of size n?

$$^nC_r = \frac{n!}{(n-r)!\, r!}$$

r-perm

fix r-comb 1, 3, 2

how many perm = $r!$

each r-comb appears $r!$ times in the set of all r-permutations

$S = \{1,2,3\}$

$^nC_2$ Perm = 12 21, 23 32, 13 31 $\quad \frac{n!}{(n-2)!} \cdot \frac{1}{2!}$

In general, how many subsets (of any size) are there for a set of size n?

Pascal's Δ^{le} :

Fibonacci Δ^{le}

n=0: 1

n=1: 1 1

n=2: 1 2 1

n=3: 1 3 3 1

n=4: 1 4 6 4 1

n=5: 1 5 10 10 5 1

r=0 r=1 r=2 r=3 r=4 r=5

How do we relate Pascal's Δ^{le} to ^nC_r?

rows → n, cols → r, entry ^nC_r

Binomial Theorem: Let x and y be real numbers, and n a non-negative integer. Then,

$(x + y)^n =$

Properties of Binomial coefficients: ✱

$$\boxed{(i)\ ^nC_r = {}^nC_{n-r}}$$

$$\boxed{(ii)\ ^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r}$$

Proof (i) $^nC_r = \frac{n!}{r!\,(n-r)!}$ $\quad ^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\, r!}$

Binomial Theorem: Let x and y be real numbers, and n a non-negative integer. Then,

$(x + y)^n =$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$ → Can prove by indn (exercise)

$$(1+1)^n = \sum_{i=0}^{n} \binom{n}{i} (1)^i (1)^{n-i}$$

$$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

$\binom{n}{i}$ – # subsets of size of i of set of n elements

↑ #subsets of a set of size n

→ Alternate way to prove size of power set.

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A set A is **countable** means it is either finite or countably infinite.

The set of positive integers ($\mathbb{Z}^+$) is countably infinite — *List*: 1 2 3 4 5 6 7 8 9 10 11...

we need to show $|N| = |\mathbb{Z}^+|$

bijection $f: N \to \mathbb{Z}^+$

$\forall x \in N \; f(x) = x+1$

0 1 2 3 4 ....

1 2 3 4 ...

The set of integers ($\mathbb{Z}$) is countably infinite — *List*: 0 −1 1 −2 2 −3 3 −4 4 −5 5...

$|N| = |\mathbb{Z}|$

Consider the function $f : \quad \to$

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x-1 & \text{if } x < 0 \end{cases}$$

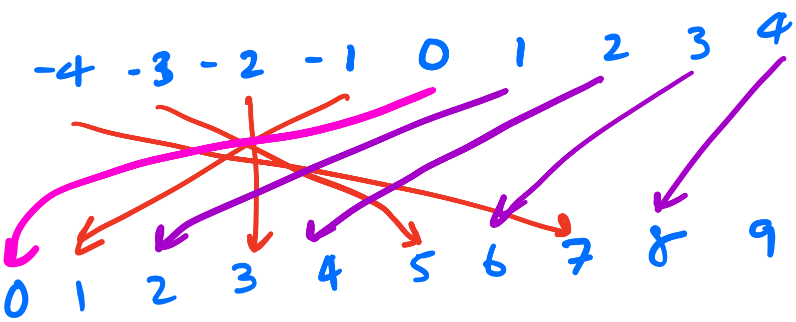
The function f is bijective between which possible domains and co-domains:

A. $f : \mathbb{Z}^+ \to \mathbb{N}$

B. $f : \mathbb{Z} \to \mathbb{N}$

C. $f : \mathbb{N} \to \mathbb{Z}$

C. $f : \mathbb{N} \to \mathbb{Z}^+$



well-defined

show one-one & onto

Properties of cardinality

$$\forall A \; (\; |A| = |A| \;)$$
$$\forall A \, \forall B \; (\; |A| = |B| \;\to\; |B| = |A| \;)$$
$$\forall A \, \forall B \, \forall C \; (\; (|A| = |B| \;\wedge\; |B| = |C|) \;\to\; |A| = |C| \;)$$

FINAL SATURDAY AUG 3, 8am - 11am

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More examples of countably infinite sets

Claim: $|\mathbb{Z}^+ \times \mathbb{Z}^+| = |\mathbb{Z}^+|$

$|\mathbb{Z}^+ \times \mathbb{Z}^+| \le |\mathbb{Z}^+|$ & $|\mathbb{Z}^+| \le |\mathbb{Z}^+ \times \mathbb{Z}^+|$

$f: \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$

s.t: f is one-one

$(x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$

$f(x,y) = 2^x \cdot 3^y$

1

$$\begin{array}{cccc} 1\,1 & 2\,1 & 3\,1 & \cdots \\ 1\,2 & 2\,2 & 3\,2 & \cdots \\ 1\,3 & 2\,3 & 3\,3 & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

$\cancel{f(x,y) = x}$

$1,2 \longrightarrow 1$

$1,3 \longrightarrow 1$

$\cancel{f(x,y) = x^y}$

$(2,2) \longrightarrow 4$

$(4,1) \longrightarrow 4$

$g: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$

s.t: g is one-one

$x \in \mathbb{Z}^+$

$g(x) = (x,x)$

$1 \longrightarrow (1,1)$

$2 \longrightarrow (2,2)$

$3 \longrightarrow (3,3)$

$\vdots$

Show that $f(x,y) = 2^x 3^y$ is one-one

(i) f is well-defined

$\forall (x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$

$2^x 3^y \in \mathbb{Z}^+$

$\therefore$ well defined

(ii) f is One-one

$\forall (x_1, y_1) \in \mathbb{Z}^+ \times \mathbb{Z}^+$

$\forall (x_2, y_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$

$f(x_1, y_1) = f(x_2, y_2) \longrightarrow (x_1, y_1) = (x_2, y_2)$

Let $(x_1, y_1) \in \mathbb{Z}^+ \times \mathbb{Z}^+$

& $(x_2, y_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$

& Let $f(x_1, y_1) = f(x_2, y_2)$

$$2^{x_1} 3^{y_1} = 2^{x_2} 3^{y_2}$$

$$\Rightarrow \frac{2^{x_1}}{2^{x_2}} \cdot \frac{3^{y_1}}{3^{y_2}} = 1$$

$$\Rightarrow 2^{x_1 - x_2} \cdot 3^{y_1 - y_2} = 1$$

$x_1 - x_2 = 0$ $\Rightarrow$ $\boxed{\begin{array}{l} x_1 = x_2 \\ y_1 = y_2 \end{array}}$ *

$y_1 - y_2 = 0$

it should work out even with $4^x 3^y$ $\gcd(4,3) = 1$

$[\,]$, $(2, (1, [\,]))$, $(5, [\,])$

Claim: L is countably infinite

One-to-one function from $\mathbb{N}$ to L

$f : \mathbb{N} \to L$

easy case:

Let $n \in \mathbb{N}$,

$f(n) = \underbrace{(1, (1, (1, \ldots (1, [\,])))}_{n \text{ 1s}}$

one-one

One-to-one function from L to $\mathbb{N}$

$g : L \to \mathbb{N}$

harder case

Use recursively defined fn:

Base case: $g([\,]) = 0$

Let $L = (n, L')$ $\to$ L' smaller than L

$g(L) = 2^n \cdot 3^{g(L')}$ $\to$ Try expanding it out to see how the fn looks.

$2^{n_1} \cdot 3^{2^{n_2} \cdot 3^{2^{n_3} \cdot 3^{2^{n_4} \cdot 1}}}$

T.S.T: g is well defined, one-one

T.S.T: g is one-one, (i.e) $\forall L_1 \in L \, \forall L_2 \in L \;\; g(L_1) = g(L_2) \to L_1 = L_2$

claim

Countable sets

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A set A is **countable** means it is either finite or countably infinite.

All countably infinite sets are the same size as one another! Uncountable sets exist! *Implications: There are different sizes of infinity. Some infinities are smaller than other infinities!!*

Structural Induction: $\exists t \in \mathbb{Z}$ s.t $|L_1|, |L_2| \leq t$

T.P.T: Claim is true when $|L_1|, |L_2| \leq t$

induct on $i \in [0, t]$

Base Case: $|L_1|, |L_2| \leq 0$
$(i=0)$

$$g(L_1) = g(L_2) = g([\,]) = 0$$

$$\Rightarrow L_1 = L_2 = [\,]$$

Inductive Case: $\boxed{|L_1|, |L_2| \leq \underline{\underline{i}}}$

→ Assume that the claim is true for list L_1', L_2' s.t $|L_1'|, |L_2'| \leq \underline{\underline{i-1}}$

(ie) $g(L_1') = g(L_2') \longrightarrow L_1' = L_2'$

since $i \geq 1$

let $\boxed{L_1 = (n_1, L_1')}$, $\boxed{L_2 = (n_2, L_2')}$

$$g(L_1) = 2^{n_1} 3^{g(L_1')}$$

& $$g(L_2) = 2^{n_2} 3^{g(L_2')}$$

given $g(L_1) = g(L_2)$

$$\Rightarrow 2^{n_1} 3^{g(L_1')} = 2^{n_2} 3^{g(L_2')}$$

$$\Rightarrow 2^{n_1 - n_2} \cdot 3^{g(L_1') - g(L_2')} = 1$$

$$\Rightarrow \boxed{n_1 = n_2} \quad \boxed{g(L_1') = g(L_2')}$$

w.k.t $\boxed{|L_1'|, |L_2'| \leq i-1}$

By IH, $L_1' = L_2'$

$\therefore$ since $n_1 = n_2$

$$L_1 = (n_1, L_1') = (n_2, L_2') = L_2$$

$$\therefore \boxed{L_1 = L_2}$$ ☆

Let, $L_1 \in L$, $L_2 \in L$ s.t $g(L_1) = g(L_2)$

$\boxed{\text{Case (i)}}$: $g(L_1) = g(L_2) = 0$

$g : L \to \mathbb{N}$

Since g is well defined, $\Rightarrow$ $2^n \cdot 3^{g(L')} \geq 1$

$g(L') \geq 0 \ \forall$ LL L

Conclude that $L = [\ ]$ is the only linked list with $g(L) = 0$

$\therefore L_1 = L_2$.

$\boxed{\text{Case (ii)}}$ $g(L_1) = g(L_2) \neq 0$

$\therefore g(L_1) = g(L_2) \geq 1$

Let $L_1 = (n_1, L_1')$ $L_2 = (n_2, L_2')$

$$2^{n_1} 3^{g(L_1')} = 2^{n_2} 3^{g(L_2')}$$

$$\Rightarrow \quad 2^{n_1 - n_2} \cdot 3^{g(L_1') - g(L_2')} = 1$$

$$\Rightarrow \quad \boxed{n_1 = n_2}^{*} \;\&\; \boxed{g(L_1') = g(L_2')}^{*}$$

Now, since $g(L_1') = g(L_2')$

case (i) $g(L_1') = g(L_2') = 0$

then w.k.t $l_1' = l_2' = [\,]$

case (ii) $g(L_1') = g(L_2') \geq 1$

$$2^{n_1'} 3^{g(L_1'')} = 2^{n_2'} 3^{g(L_2'')}$$

$\vdots$

$\therefore n_1' = n_2'$

same argument with L_1'', L_2'' until

$\vdots$

$L_1^{'\cdots'}, L_2^{'\cdots'} = [\,]$

Another way (cleaner)

Case (ii) $g(L_1) = g(L_2) \geq 1$

Let $L_1 = (n_1^1, (n_1^2, (n_1^3, \ldots, (n_1^l, [\,]))))$

$L_2 = (n_2^1, n_2^2, n_2^3, \ldots, (n_2^{l'}, [\,])))$

$$g(L_1) = 2^{n_1^1} \cdot 3^{g(n_1^2, n_1^3, \ldots, [\,])}$$

$$g(L_2) = 2^{n_2^1} \cdot 3^{g(n_2^2, n_2^3, \ldots, [\,])}$$

$$g(L_1) = g(L_2) \Rightarrow \boxed{n_1^1 = n_2^1}$$

$$g(n_1^2, n_1^3, \dots, [\,]) = g(n_2^2, n_2^3, \dots [\,])$$

$$\downarrow$$

$$n_1^2 = n_2^2$$

$$\vdots$$

use the same argument until you get to the case $g([\,])$.

(iii) Induction on L_1 & L_2

Let $L_1 = (n_1^1, \underbrace{(n_1^2, (n_1^3, \dots, (n_1^l, [\,])))})$

$L_2 = (n_2^1, \underbrace{n_2^2, n_2^3, \dots, (n_2^{l'}, [\,]))})$

show for, assume by indn

$n_1' = n_2$

Use induction to s.t

$\forall i \qquad n_1^i = n_2^i$

Base case is $i = l$,

$$g(n_1^l, [\,]) = g(n_2^l, [\,])$$

$$2^{n_1} \cdot 3^{2^{n_2} \cdot 3^{2^{n_3} \cdot 3^{2^{n_4} \cdots 3^{2^{n_l}}}}} = 2^{n_1'} \cdot 3^{2^{n_2'} \cdot 3^{2^{n_3'} \cdot 3^{2^{n_4'} \cdots 3^{2^{n_{l'}'}}}}}$$

$\Rightarrow n_1 = n_1'$

$n_2 = n_2'$

$n_3 = n_3'$

Observe $l = l'$

$\forall i \; n_i = n_i'$

Base case: $n_l = n_l'$

$n_i = n_i'$

Extra example Prove or disprove: There is a set Y, $\neg(\ |Y| = |Y \times Y|\ )$

Extra example Prove or disprove: There is a set Y, $\neg(\ |Y| = |\mathcal{P}(Y)|\ )$

$\mathbb{N}$ and its power set

Example elements of $\mathbb{N}$

Example elements of $\mathcal{P}(\mathbb{N})$

Recall: For set A, its power set is $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

Claim: $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$

A set A is **uncountable** means it is not countable.

Claim: There is an uncountable set. Example: ________

Proof: By definition of countable, since ________ is not finite, **to show** is $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$.

Rewriting using the definition of cardinality, **to show** is

Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

To show: f is not a bijection. It's enough to show that f is not onto.

Rewriting using the definition of onto, **to show**:

By logical equivalence, we can write this as an existential statement:

____

In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{n \in \mathbb{N} \mid n \notin f(n)\}$$

By definition of power set, since all elements of D_f are in $\mathbb{N}$, $D_f \in \mathcal{P}(\mathbb{N})$. It's enough to prove the following Lemma:

Lemma: $\forall a \in \mathbb{N}\ (\ f(a) \neq D_f\ )$.

Proof of lemma:

By the Lemma, we have proved that f is not onto, and since f was arbitrary, there are no onto functions from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. QED

Where does D_f come from? The idea is to build a set that would "disagree" with each of the images of f about some element.

$n \in \mathbb{N}$	$f(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	$\ldots$	Is $n \in D_f$?
0	$f(0) = X_0$	**Y / N**	Y / N	Y / N	Y / N	Y / N	$\ldots$	**N / Y**
1	$f(1) = X_1$	Y / N	**Y / N**	Y / N	Y / N	Y / N	$\ldots$	**N / Y**
2	$f(2) = X_2$	Y / N	Y / N	**Y / N**	Y / N	Y / N	$\ldots$	**N / Y**
3	$f(3) = X_3$	Y / N	Y / N	Y / N	**Y / N**	Y / N	$\ldots$	**N / Y**
4	$f(4) = X_4$	Y / N	Y / N	Y / N	Y / N	**Y / N**	$\ldots$	**N / Y**
$\vdots$								

$f_A : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ where $f_A(x) = x^2$

$f_B : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ where $f_B(x) = \{x^2\}$

$n \in \mathbb{N}$	$f_B(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	...	Is $n \in D_{f_B}$?
0	$f_B(0) = \{0\} = X_0$	**Y**	N	N	N	N	...	No
1	$f_B(1) = \{1\} = X_1$	N	**Y**	N	N	N	...	No
2	$f_B(2) = \{4\} = X_2$	N	N	**N**	N	Y	...	Yes
3	$f_B(3) = \{9\} = X_3$	N	N	N	**N**	N	...	Yes
4	$f_B(4) = \{16\} = X_4$	N	N	N	N	**N**	...	Yes
$\vdots$								

Claim: $\{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$ is uncountable.

Proof: By definition of countable, since $\{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$ is not finite, **to show** is $|\mathbb{N}| \neq |\{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}|$.

To show $\forall f : \mathbb{Z}^+ \to \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$ (f is not a bijection) . Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{Z}^+ \to \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$.

To show: f is not a bijection. It's enough to show that f is not onto. Rewriting using the definition of onto, **to show**:

$$\exists x \in \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\} \; \forall a \in \mathbb{Z}^+ \; (f(a) \neq x)$$

List all the images of f for every $a \in \mathbb{Z}^+$

$f(1) = r1 = 0.b_{11}b_{12}b_{13}b_{14}...$

$f(2) = r2 = 0.b_{21}b_{22}b_{23}b_{24}...$

$f(3) = r3 = 0.b_{31}b_{32}b_{33}b_{34}...$

$f(4) = r4 = 0.b_{41}b_{42}b_{43}b_{44}...$

$\vdots$

In search of a witness, define the following real number by defining its binary expansion

$$d_f = 0.b_1 b_2 b_3 \cdots$$

where $b_i = 1 - b_{ii}$ where b_{jk} is the coefficient of 2^{-k} in the binary expansion of $f(j)$. Since $d_f \neq f(a)$ for any positive integer a, f is not onto. ■

Examples of uncountable sets

- $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathbb{Z}^+)$, $\mathcal{P}(\mathbb{Z})$, power set of any countably infinite set.
- The closed interval $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, any other nonempty closed interval of real numbers whose endpoints are unequal, as well as the related intervals that exclude one or both of the endpoints.
- $\mathbb{R}$
- $\overline{\mathbb{Q}}$, the set of irrational numbers