Solutions To HW5

Zixuan Chen
X313572
zixuanchen@ucsb.edu

July 2024

Solution To Question 1

Prove: If A and B are finite sets, $|A \times B| = |A|\,|B|$.

We will try to find a bijective function f with domain: $A \times B$ and target: $\{1, 2, 3, ..., |A|\,|B|\}$.

As both A and B are finite sets, we assume that $|A|$ is n and $|B|$ is m. $(n, m \in \mathbb{Z}^{\geq 0})$

Now we set the subscript of every element in set A and B, respectively from 0 to $n-1$ and $m-1$. For every tuple $(a, b) \in A \times B$, it has a corresponding index pair (x, y).

Therefore, we can use the index pair to build the bijective function:

For index pair (x, y), $f : (x, y) \mapsto (x|B| + y + 1)$

Proof:

Well-defined:

Let x and y be arbitrary integers. As they both are greater than or equal to 0, $x|B| + y + 1 \geq 1$. And if we let $x = n - 1$ and $y = m - 1$, we get $(n-1)m + m - 1 + 1 = mn - m + m - 1 + 1 = mn$. So the max value will be in the target, thus every index pair maps to a unique value in the target.

One to one:

We assume that $f(x, y) = f(i, j)$, and we expand it so we have:

$$xm + y + 1 = im + j + 1$$

$$(x - i)m = j - y$$

Case1: If $x = i$, then we have $j - y = 0, j = y$, thus the function is one to one.

Case2: If $x \neq i$. WLOG we can assume that $x > i$, so $x - i$ will be greater than 1, thus $LHS \geq m$. Now we have $j - y \geq m$. However, both j and y are non-negative integers less than $m - 1$, so $j - y < m$. Due to this contridiction, Case2 is not possible.

Onto:

Let e be an arbitrary number in the target, we know that $e = ab + c$ by the property of modular operation. Let$a = |B|, b = x, c = y + 1$. Therefore, we can map every element in the target to an fixed element in the domain.

Solution To Question 2

- Basis Step: $[] \in L$
- Recursive Step: If $l \in L$ and $n \in \mathbb{N}$, then $(n, l) \in L$

$$\mathit{toNum} : L \to \mathbb{N} \text{ is defined recursively as follows:}$$

- Basis Step: $\mathit{toNum}([]) = 0$
- Recursive Step: If $n \in \mathbb{N}$ and $l \in L$, then $\mathit{toNum}((n, l)) = 2^n \cdot 3^{\mathit{toNum}(l)}$

PROOF:

Choose arbitrary l and l'.

Assume $toNum(l) = toNum(l') \equiv 2^a \cdot 3^{2^b} \cdot 3^c = 2^{a'} \cdot 3^{2^{b'}} \cdot 3^{c'}$, where c and c' continue the recursive steps.

Dividing both sides by the RHS, we get $2^{a-a'} \cdot 3^{2^b - 2^{b'}} \cdot 3^{c-c'} = 1$.

For the product of two relatively prime positive integers each raised to a power to be 1, both must be raised to the 0^{th} power. Therefore,

$$a - a' = 0 \to a = a'$$
$$b - b' = 0 \to b = b'$$
$$c - c' = 0 \to c = c'$$

Since each individual node in l is equivalent to the node in the same position of l', $l = l'$ and $toNum$ is one-to-one.

Solution To Question 3

(a)

$$f : \mathbb{N} \to \{0, 1\}^* \text{ such that } n \mapsto (n)_2$$

where $\{0, 1\}^*$ is the set of all finite length bit strings, and $(n)_2$ is the binary representation of n.

We will use strong induction to prove that every integer can be represented by distinct powers of 2.

Base Step: $B(1) = 1$

Inductive Step: We can express any decimal number $n + 1$ as $2m$ or $2m + 1$ for some integer $m < n$. By strong induction, we know that all decimal numbers 1 through n can be represented by distinct powers of 2, therefore we know that m (which is less than n) can be represented in binary. To form $2m$ with the collection of powers of 2 equal to m, we can multiply every power of 2 by 2. To form $2m + 1$,we can do this and then add 2_0.

Thus if we can represent every integer from 1 through n as distinct powers of 2, we can also represent $n+1$ as a distinct power of 2.

Since the powers of 2 are all distinct, we know binary representations of integers consist of only 1s and 0s.

Goal 1: Prove f is well defined.

Since binary representations of integers are just collections of 0s and 1s and binary strings are just collections of 0s and 1s, we know the function f is well defined because for every binary string, there will always be only one corresponding binary representation(the binary with the same number and order of 0s and 1s). Corollarily, we know that every every binary representation must also map to a unique binary string. We will use this to prove f is injective.

Goal 2: Prove f is injective

We must prove that if two binary strings map to the same binary representation, they are the same binary string. We know that f and f^{-1} are both well defined. This means every mapping from binary strings to representations and vice versa are unique. This is the guarantees injectivity since if $f(a) = f(b)$, $a = b$ must be true(since every mapping is unique).

Goal 3: Prove f is surjective

Earlier, we proved every binary representation consisted of only 1s and 0s. This means every representation can be mapped from its corresponding binary string that just contains the same number and order of 0s and 1s. We have proved f is well-defined, injective, and surjective.

(b)

$$f : \mathbb{Q} \to \mathbb{Z} \text{ given by } \frac{m}{n} \mapsto m^n$$

This function is not well defined, so it can not be one to one or onto. This function is not well defined because any rational number inputted can be expressed as any fraction that shares the correct ratio between the numerator and denominator. For example, 1/2 and 2/4 will map to different elements in the codomain even though they both represent the same element in the domain.

(c)

$$f : \{0,1\}^3 \to \{0,1\}^4 \text{ where } (b_3, b_2, b_1) \mapsto (0, b_3, b_2, b_1) \text{ for } b_i \in \{0,1\}$$

This function is well-defined and injective but not surjective.

The function is well-defined. Every tuple in $f : \{0,1\}^3$ maps to some single tuple in $\{0,1\}^4$ as the function f only prepends 0 to the given tuple of (b_3, b_2, b_1).

For the arbitrary tuple $s = \{m_3, m_2, m_1\}$, $f(s) = \{0, s\}$.

The function is one-to-one. Every tuple in the codomain has a unique mapping from the domain.

$$\forall s \in \{0,1\}^3 \forall t \in \{0,1\}^3, (f(s) = f(t) \to s = t)$$

Towards a direct proof, assume $f(s) = f(t)$.

$$f(s) = f(t)$$

$$s = t$$

The function is not onto. We will disprove the claim that it is onto through counterexample $\{1, 0, 0, 0\}$. It is in the codomain but has no mapping from the domain.

(d)

$$f : \{0,1\}^4 \to \{0,1\}^3 \text{ where } (b_4, b_3, b_2, b_1) \mapsto (b_3, b_2, b_1) \text{ for } b_i \in \{0,1\}$$

Well-defined: Yes, every bit string of length 4 maps to a bit string of length 3 as the function f only removes the initial element of the given tuple $s \in \{0,1\}^4$

Injective: No, different inputs can map to the same output. For example, $(0, b_3, b_2, b_1)$ and $(1, b_3, b_2, b_1)$ map to the same output.

Surjective: Yes, every bit string of length 3 can be obtained from some bit string of length 4.

$$\forall g \in \{0,1\}^3, \exists h \in \{0,1\}^4, f(h) = g$$

For arbitrary $g = (i_3, i_2, i_1)$, witness $b_4 = 0, (f(h) = g) \to (h = (0, i_3, i_2, i_1))$

(e)

$$f : \mathbb{Z} \to \mathbb{Z} \text{ where } k \mapsto k \mod 19$$

This function is well defined but not one-to-one or onto.

This function is well defined.

$$\forall k \in \mathbb{Z}, \exists l \in \mathbb{Z}, f(k) = l$$

Since k is an integer, l must also be an integer. Therefore, f is well defined.

This function is not one-to-one.

We will disprove the claim that f is one-to-one through a counterexample.

f being one-to-one claims:

$$\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (f(m) = f(n) \rightarrow m = n)$$

We take values m_1 and n_1 as a counterexample:

$$m_1 \in \mathbb{Z}, n_1 \in \mathbb{Z}$$

$$m_1 = 40, n_1 = 59$$

$$f(m_1) = f(n_1), \quad m_1 \neq n_1$$

Values m_1 and n_1 disprove the claim (the predicate) that f is one-to-one. Therefore, f is not one-to-one.

The function is not onto.

Not every element(e.g. 20) in the codomain is mapped to from some element in the domain.

Solution To Question 4

(a) Give two example elements in $P(W)$. Justify your examples by explanations that include references to the relevant definitions.

- **Example Element:** $\emptyset$
  - Justification:
    * $P(W)$ is the power set of W, which means $P(W)$ is the set of all subsets of W.
    * By definition, the empty set $\emptyset$ is a subset of any set, including W.
    * Therefore, $\emptyset \in P(W)$.
- **Example Element:** $\{\{1\}, \{2, 3\}\}$
  - Justification:
    * To verify that $\{\{1\}, \{2, 3\}\} \in P(W)$, we need to show that $\{\{1\}, \{2, 3\}\}$ is a subset of W.
    * Since $W = P(\{1, 2, 3, 4, 5\})$, each element of W is a subset of $\{1, 2, 3, 4, 5\}$. For example, $\{1\} \in W$ and $\{2, 3\} \in W$.
    * Thus, $\{\{1\}, \{2, 3\}\}$ is a set of subsets of $\{1, 2, 3, 4, 5\}$, making it a subset of W and hence an element of $P(W)$.

(b) Give one example element in $P(W) \times P(W)$ that is not equal to $(\emptyset, \emptyset)$ or (W, W). Justify your example by an explanation that includes references to the relevant definitions.

- **Example Element:** $(\emptyset, \{\emptyset\})$
  - Justification:
    * $P(W) \times P(W)$ is the Cartesian product of $P(W)$ with itself. Elements of this Cartesian product are ordered pairs where each component is an element of $P(W)$.
    * $\emptyset \in P(W)$ because the empty set is a subset of any set, including W.
    * $\{\emptyset\} \in P(W)$ because $\{\emptyset\}$ is a subset of W. To see this, note that $\emptyset \in W$, so the set containing $\emptyset$ (i.e., $\{\emptyset\}$) is a subset of W.
    * Therefore, $(\emptyset, \{\emptyset\})$ is an element of $P(W) \times P(W)$.

Solution To Question 5

$|W| = n, X \subseteq W, Y \subseteq W, X \subseteq Y$. How many pairs (X, Y)?

ANSWER:

$$\dot{Y} = P(W), \dot{X} = P(\dot{Y})$$
$$|\dot{Y}| = 2^n |\dot{X}| = 2^{2^n}$$

Therefore, the number of pairs (X, Y) is equal to

$$|\dot{X}| \times |\dot{Y}| = 2^{2^n} \cdot 2^n$$

PS: Here the $\dot{X}$ stands for the set containing all possible Xs , so does $\dot{Y}$

Solution To Question 6

Write a recursive algorithm to compute the arithmetic mean of a sequence of integers. Then, use induction on the length of the sequence to prove that your algorithm outputs the correct value for every non-empty input sequence.

Algorithm: Computing the arithmetic mean of a sequence recursively

```
procedure mean recursive((a_0, ...a_n): a sequence of integers)

seq = [a_0, ..., a_n]
if length(seq) == 1 then
    return seq[0]
end if

previous_mean = meanrecursive(seq[0 : length(seq) - 1])
n = length(seq)
current_sum = previous_mean * (n - 1) + seq[n - 1]
m = current_sum / n

return (m){m is the arithmetic mean of (a_0, ...a_n)}
```

PROOF:

Base Case

For $n = 1$:

When the sequence has only one element a_0, the mean is simply a_0. The algorithm correctly returns a_0 when the length of the sequence is 1.

Inductive Step

Assume that the algorithm correctly computes the mean for any sequence of length k. That is, for a sequence $(a_0, a_1, \ldots, a_{k-1})$, the algorithm returns the correct mean:

$$\text{mean\_recursive}(a_0, a_1, \ldots, a_{k-1}) = \frac{a_0 + a_1 + \cdots + a_{k-1}}{k}$$

Now, consider a sequence of length $k+1$: $(a_0, a_1, \ldots, a_k)$.

The algorithm computes:

1. The mean of the first k elements:

$$\text{previous\_mean} = \text{mean\_recursive}(a_0, a_1, \ldots, a_{k-1})$$

   By the induction hypothesis, we know:

$$\text{previous\_mean} = \frac{a_0 + a_1 + \cdots + a_{k-1}}{k}$$

2. The current sum including the k-th element:

$$\text{current\_sum} = \text{previous\_mean} \cdot k + a_k$$

   Substituting previous_mean:

$$\text{current\_sum} = \left(\frac{a_0 + a_1 + \cdots + a_{k-1}}{k}\right) \cdot k + a_k$$

$$\text{current\_sum} = a_0 + a_1 + \cdots + a_{k-1} + a_k$$

3. The mean of the sequence of length $k+1$:

$$\text{mean\_recursive}(a_0, a_1, \ldots, a_k) = \frac{\text{current\_sum}}{k+1}$$

$$\text{mean\_recursive}(a_0, a_1, \ldots, a_k) = \frac{a_0 + a_1 + \cdots + a_k}{k+1}$$

Thus, the algorithm correctly computes the arithmetic mean for a sequence of length $k+1$.

By induction, the algorithm correctly computes the arithmetic mean for any non-empty input sequence.

Solution To Question 7

$Q = \left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
$\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$

ANSWER:

$$-\sqrt{2} \notin \overline{\mathbb{Q}}$$

Solution To Question 8

Consider the binary relation R on the set of integers define as $R_m = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = 3 \cdot m\}$ for some positive integer m. Prove or disprove that R_m is a equivalence relation.

R_m is not symmetric.

Let's take $a = 4, b = 1$ as a counterexample.

$$a - b = 4 - 1 = 3 = 3 \cdot 1$$

Here we can have $m = 1$ thus (a, b) should be in the relation R_m.

If it is an equivalence, (b, a) should alse be in it.

So now we consider (b, a):

$$b - a = 1 - 4 = -3 = 3 \cdot (-1)$$

However, m must be some **positive** integers.

Therefore, we can't find a valid m to satisfy the condition.

$\therefore R_m$ is not an equivalence relation.

Solution To Question 9

(a) True or False: The relation A_1 holds for $u = (1, 1, 1, 1, 1)$ and $v = (-1, -1, -1, -1, -1)$.

False beacuse u likes the 1st movie but v hates it.

(b) True or False: The relation A holds for $u = (1, 0, 1, 0, -1)$ and $v = (-1, 0, 1, -1, -1)$.

False because u doesn't care about 2 movies but v doesn't care about only 1.

(c) True or False: A_1 is reflexive; namely, $\forall u \in R\,((u, u) \in A_1)$

True. Everyone will agree with themselves on whether they like or dislike a movie because one can't both like and dislike a movie at the same time.

(d) True or False: A_1 is symmetric; namely, $\forall u \in R\,\forall v \in R\,((u, v) \in A_1 \to (v, u) \in A_1)$

True. If u and v agree on the first movie, then v and u also agree on it.

(e) True or False: A_1 is transitive; namely, $\forall u \in R\,\forall v \in R\,\forall w \in R\,(((u, v) \in A_1 \wedge (v, w) \in A_1) \to (u, w) \in A_1)$

True. If u and v agree on the first movie, v and w also agree on the first movie. Then u, v, w all agree on the first movie, thus u and w agree on the first movie.

(f) True or False: A is reflexive; namely, $\forall u \in R\,((u, u) \in A)$

True. Any user will not care or has not seen the same number of movies as themselves because they are themselves.

(g) True or False: A is anti-symmetric; namely, $\forall u \in R\, \forall v \in R\, (((u,v) \in A \land (v,u) \in A) \rightarrow (u = v))$

False. There can be 2 different people that neither of them have seen any movie. Two people having the same number of movies that they dont care or haven't seen doesn't mean that they are the same person.

(h) True or False: A is transitive; namely, $\forall u \in R\, \forall v \in R\, \forall w \in R\, (((u,v) \in A \land (v,w) \in A) \rightarrow (u,w) \in A)$

True. If user u and v haven't seen or don't care about the same number of movies and users v and arbitrary user w haven't seen or don't care about the same number of movies, then all of them must have not seen or don't care about the same number of movies, meaning users u and w have not seen or don't care about the same number of movies.

Solution To Question 10

(c) We need to show that A_1 is reflexive, which means proving

$$\forall u \in R\, \big((u,u) \in A_1\big).$$

Proof:

Take an arbitrary $u \in R$. By definition, each user's rating can be represented as a 5-tuple:

$$u = (u_1, u_2, u_3, u_4, u_5).$$

According to the definition of A_1:

$$(u,v) \in A_1 \text{ if users } u \text{ and } v \text{ agree about the first movie in the database.}$$

In other words:

$$(u,v) \in A_1 \text{ if } u_1 = v_1.$$

Now, consider the pair (u,u). For this pair:

$$(u,u) \in A_1 \text{ if and only if } u_1 = u_1.$$

Since $u_1 = u_1$ is always true, it follows that:

$$(u,u) \in A_1$$

for any $u \in R$.

Therefore, we have shown that:

$$\forall u \in R\, \big((u,u) \in A_1\big).$$

Thus, A_1 is reflexive.

Solution To Question 11

All of the examples are given with the format (u, v)

(a) Give two distinct examples of elements in $[(1,0,0,0,0)]_{A_1}$

$((1,0,1,0,1),(1,1,1,1,1))$ and $((1,1,1,1,-1),(1,-1,-1,-1,-1))$

(We only need the first element of v and u to be 1)

(b) Give two distinct examples of elements in $[(1,0,0,0,0)]_{A}$

$((0,0,1,0,0),(0,0,-1,0,0))$ and $((0,1,0,0,0),(1,0,0,0,0))$

(Make sure there are 4 0s in u and v)

(c) Find examples $u, v \in R$ where $[u]_{A_1} \neq [v]_{A_1}$ but $[u]_A = [v]_A$

$((1,0,1,0,0),(-1,0,-1,0,0))$ and $((0,-1,0,0,1),(-1,0,0,0,1))$

(The first element different and contain the same numver of 0s)

(d) Find examples $u, v \in R$ (different from the previous part) where $[u]_{A_1} = [v]_{A_1}$ but $[u]_A \neq [v]_A$

$((1,0,1,0,1),(1,1,0,1,1))$ and $((-1,1,1,1,-1),(-1,-1,0,1,0))$

(the same first element but different numbers of 0s)

Solution To Question 12

The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction on the integer j to show that the first player can always win if $n = 4j, 4j+2$, or $4j+3$ for some non-negative integer j.

We will use A and B to call the first player and the other.

Base Case:

$j = 0$: we skip the case when $n = 0$, so start off by $n = 2$: A can take 1 and B must take the left 1 match. A wins. $n = 3$: A takes 2 and B must take the left 1 match and A wins.

$j = 1$: $n = 4$: A takes 3 and B takes 1, A wins. $n = 6$: A takes 1, and after B takes, the left matches will be 2 or 3 or 4. A can make only 1 match left next turn and A wins. $n = 7$: A takes 2, and the following steps are identical as the case $n = 6$.

Inductive Steps:

We define $W(i)$: A wins for $j = i$.

Now we assume that: $W(0) \wedge W(1) \wedge ... \wedge W(k)$.

And we want to prove $W(k+1)$.

We define $M(i)$: A wins when $n = i$.

From definition, we have $M(4k-4) \wedge M(4k-2) \wedge M(4k-1) \wedge M(4k) \wedge M(4k+2) \wedge M(4k+3)$ from $W(k-1) \wedge W(k)$. (In the base case we have proved $k = 1$, so $k - 1 \geq 0$.)

Now we will prove: $M(4k+4) \wedge M(4k+6) \wedge M(4k+7)$.

Case 1: Now we have $4k+4$ matches, we will find a solution to let A win this game. A will take 3 matches and $4k+1$ matches are left. After B takes matches, there are 3 possibilities: $4k$ or $4k-1$ or $4k-2$. And next turn comes. By assumption, we have $M(4k-2) \wedge M(4k-1) \wedge M(4k)$. A wins.

Case 2: Now we have $4k+6$ matches. A will take 1 match. And after B takes, we have 3 possibilities: $4k+4$ or $4k+3$ or $4k+2$. In case 1, we have proved $M(4k+4)$ and by assumption we have $M(4k+2) \wedge M(4k+3)$. A wins.

Case 3: Now we have $4k+7$ matches. If A takes 2 matches, then the situation will be totally the same as case2. The proof is identical. A will win.

Now we have proved $W(k+1)$.

Solution To Question 13

To show that $\left|\mathbb{R}^{(0,1)}\right| = |\mathbb{R}|$, we need to demonstrate that there exists a bijective function between the open interval $(0, 1)$ and $\mathbb{R}$.

(a) Bijectivity of $f\colon \mathbb{R}^{(0,1)} \to \mathbb{R}^+$

Consider the function $f(x) = \frac{x}{1-x}$ for $x \in (0, 1)$. We need to show that f is bijective, i.e., both injective and surjective. As every x in (0,1), so $1 - x > 0$, $\frac{x}{1-x} > 0$. Thus every element will be mapped to a certain value in the codomain. f is **well-defined**.

1. **Injectivity**: - Assume $f(x_1) = f(x_2)$. Then,

$$\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$$

- Cross-multiplying gives:

$$x_1(1 - x_2) = x_2(1 - x_1)$$
$$x_1 - x_1x_2 = x_2 - x_1x_2$$
$$x_1 = x_2$$

- Thus, f is injective.

2. **Surjectivity**: - For any $y \in \mathbb{R}^+$, we need to find $x \in (0, 1)$ such that $f(x) = y$.

$$y = \frac{x}{1-x}$$
$$y(1-x) = x$$
$$y - yx = x$$
$$y = x(1+y)$$
$$x = \frac{y}{1+y}$$

- Since $y \in \mathbb{R}^+$, $x = \frac{y}{1+y} \in (0, 1)$. Thus, f is surjective.

Hence, $f\colon \mathbb{R}^{(0,1)} \to \mathbb{R}^+$ is bijective.

(b) Bijectivity of $g\colon \mathbb{R}^{(0,1)} \to \mathbb{R}$

Define the function $g\colon \mathbb{R}^{(0,1)} \to \mathbb{R}$ by:

$$g(x) = \begin{cases} -f(1-2x) & 0 < x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ f(2x-1) & \frac{1}{2} < x < 1 \end{cases}$$

We need to show that g is bijective.

First, we need to prove that g is well-defined.

When $0 < x < \frac{1}{2}$, $g(x) = -\frac{1-2x}{1-(1-2x)} = \frac{2x-1}{2x} = 1 - \frac{1}{2x}$. As $2x < 1$, $\frac{1}{2x} > 1$, every x will be mapped to a certain negative real number.

$\frac{1}{2}$ is mapped to 0.

When $\frac{1}{2} < x < 1, g(x) = \frac{2x-1}{1-(2x-1)} = \frac{2x-1}{2-2x} = -1 - \frac{1}{2x-2}$. $1 < 2x < 2 \rightarrow -1 < 2x - 2 < 0 \rightarrow \frac{1}{2x-2} < -1 \rightarrow -\frac{1}{2x-2} > 1 \rightarrow g(x) > 0$. So every x will be mapped to a positive real number.

Therefore, g is well-defined.

1. **Injectivity**: - For $0 < x_1, x_2 < \frac{1}{2}$, assume $g(x_1) = g(x_2)$. Then,

$$-f(1 - 2x_1) = -f(1 - 2x_2)$$

$$f(1 - 2x_1) = f(1 - 2x_2)$$

Since f is injective, $1 - 2x_1 = 1 - 2x_2$ implies $x_1 = x_2$.

- For $\frac{1}{2} < x_1, x_2 < 1$, assume $g(x_1) = g(x_2)$. Then,

$$f(2x_1 - 1) = f(2x_2 - 1)$$

Since f is injective, $2x_1 - 1 = 2x_2 - 1$ implies $x_1 = x_2$.

- For $0 < x_1 < \frac{1}{2}$ and $\frac{1}{2} < x_2 < 1$, $g(x_1) = -f(1 - 2x_1)$ and $g(x_2) = f(2x_2 - 1)$ cannot be equal because f maps to $\mathbb{R}^+$ which is always positive, hence $-f(1 - 2x_1) \neq f(2x_2 - 1)$.

- Hence, g is injective.

2. **Surjectivity**: - For any $y \in \mathbb{R}$, we need to find $x \in (0, 1)$ such that $g(x) = y$.

- If $y > 0$, set $y = f(2x - 1)$ for some $\frac{1}{2} < x < 1$.

$$y = -1 - \frac{1}{2x - 2}$$

$$y + 1 = -\frac{1}{2x - 2}$$

$$2x - 2 = -\frac{1}{y + 1}$$

$$x = -\frac{1}{2y + 2} + 1$$

As $y > 0$, $-\frac{1}{2} < -\frac{1}{2y+2} < 0$.

Thus we have $\frac{1}{2} < x < 1$

- If $y < 0$, set $y = -f(1 - 2x)$ for some $0 < x < \frac{1}{2}$.

$$y = 1 - \frac{1}{2x}$$

$$y - 1 = -\frac{1}{2x}$$

$$2x = -\frac{1}{y - 1}$$

$$x = -\frac{1}{2y-2}$$

As $y < 0$, we have $2y - 2 < -2 \to 0 > \frac{1}{2y-2} > -\frac{1}{2}$.

Therefore, we have $0 < x < \frac{1}{2}$.

- If $y = 0$, set $x = \frac{1}{2}$.

- Thus, g is surjective.

Hence, $g\colon \mathbb{R}^{(0,1)} \to \mathbb{R}$ is bijective.

As g is “one to one”, we have $\left|\mathbb{R}^{(0,1)}\right| \leq |\mathbb{R}|$.

As g is “onto”, we have $\left|\mathbb{R}^{(0,1)}\right| \geq |\mathbb{R}|$.

Therefore, $\left|\mathbb{R}^{(0,1)}\right| = |\mathbb{R}|$.