HW4 Individual

CS40 Summer '24

Due: Friday, July 22, 2024 at 11:59PM on Gradescope

Integrity reminders for individual homeworks

- "Individual homeworks" must be solely your own work.
- You may not collaborate on individual homeworks with anyone or seek help from online tutors or entities outside the class.
- You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza. However, the staff will only answer clarifying questions on these homeworks. You *cannot* use any online resources about the course content other than the text book and class material from this quarter.
- Do not share written solutions or partial solutions for homework with other students. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW4".

Proof by Strong Induction To prove that a universal quantification over the set of all integers greater than or equal to some base integer b holds, pick a fixed nonnegative integer j and then:

Basis Step:	Show the statement holds for b, $b+1$, $\ldots$, $b+j$.
Recursive Step:	Consider an arbitrary integer n greater than or equal to $b+j$, assume (as the **strong induction hypothesis**) that the property holds for **each of** b, $b+1$, $\ldots$, n, and use this and other facts to prove that the property holds for $n+1$.

Assigned Questions

1. (*Graded for correctness*) Prove that

$$\exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{Z}^{\geq n_0} \ (\ n^3 \leq (n+2)!\ )$$

2. (*Graded for correctness in evaluating statement and for fair effort completeness in the justification*) Consider the functions $f_a : \mathbb{N} \to \mathbb{N}$ and $f_b : \mathbb{N} \to \mathbb{N}$ defined recursively by

$$f_a(0) = 0 \qquad \text{and for each } n \in \mathbb{N}, \qquad f_a(n+1) = f_a(n) + 2n + 1$$

$$f_b(0) = 0 \qquad \text{and for each } n \in \mathbb{N}, \qquad f_b(n+1) = 2f_b(n)$$

   Which of these two functions (if any) equals 2^n and which of these functions (if any) equals n^2? Use induction to prove the equality or use counterexamples to disprove it.

3. Prove that any amount of postage worth 24 cents or more can be made from 7-cent or 5-cent stamps. Hint: Use mathematical induction

4. Group the following numbers according to congruence mod 19:

$$\{22, 15, -35, 34, 72, 79, -111, -42\}$$

5. Prove by contradiction that $a^2 = b^2 + 1$ has no solutions a, b in the positive integers.

6. Give a recursive definition for each of the following sets S. Each set S will be a subset of the set containing all binary strings. A string x belongs to the recursively defined set S if and only if x has each of the following properties (provide a different definition for each part). Note that in each case, you may provide multiple rules for the recursive step of your definition.

   (a) The set S consists of all strings (including the empty string) that have an even number of 1's but may have an even or odd number of zeros.

   (b) The set S consists of all strings (including the empty string) that have the same number of 0's and 1's.

7. RNA is made up of strands of four different bases that match up in specific ways. The bases are elements of the set $B = \{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{U}\}$.

   Definition The set of RNA strands S is defined (recursively) by:

Basis Step:	$\mathtt{A} \in S, \mathtt{C} \in S, \mathtt{U} \in S, \mathtt{G} \in S$
Recursive Step:	If $s \in S$ and $b \in B$, then $sb \in S$

   A function *rnalen* that computes the length of RNA strands in S is defined by:

		$rnalen : S \to \mathbb{Z}^+$	
Basis Step:	If $b \in B$ then	$rnalen(b)$	$= 1$
Recursive Step:	If $s \in S$ and $b \in B$, then	$rnalen(sb)$	$= 1 + rnalen(s)$

   Prove by structural induction that $\forall s \in S\ \forall t \in S\ (rnalen(st) = (\ rnalen(s) + rnalen(t)\ )$ by using the recursive definition of *rnalen*.

8. Prove the gcd lemma: For any positive integers x, y, not both zero, $y \geq x$, $gcd(y, x) = gcd(y - x, x)$

9. Use the gcd lemma from the previous question and strong induction to prove the gcd theorem:

   For any positive integers x, y, not both zero, $y \geq x$, $gcd(y, x) = gcd(x, y \textbf{ mod } x)$.

   Note: We proved the theorem in lecture using a different method. For the homework we will only accept solutions that use induction.

10. For positive integers a, b, and c prove that if $gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

11. Write proofs and algorithms related to finding base 2 expansions

    (a) Use strong induction to prove the theorem: Every positive integer is a sum of (one or more) distinct powers of 2. *You are essentially proving that binary expansions exist!*

    (b) In lecture you were presented with two algorithms for finding the base 2 expansion of any positive integer. Complete the outline of the algorithm below to recursively compute the base 2 expansion of a positive integer.

Algorithm: Calculating base 2 expansion recursively

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procedure base2recursive(n: a positive integer)














return (a_{k-1}, ..., a_0){(a_{k-1} ... a_0)_b is the base 2 expansion of n}
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12. Prove that $n \in \mathbb{N}$ is divisible by 3 if and only if the alternating sum of the bits of n in binary representation is divisible by 3. The alternating sum of any sequence $a_0, a_1, ..., a_m$ is $\sum_{i=0}^{m}(-1)^i a_i$.

Attributions

Thanks to Mia Minnes and Joe Politz for the original version of Q7.