Week 5 Wednesday highlights

Exam
- (i) Aug 3
- (ii) Aug 6

- Compare sets using one-to-one, onto, and invertible functions.
- Define cardinality using one-to-one, onto, and invertible functions.
- Differentiate between important sets of numbers

Reasoning about the cardinality of sets

A **finite** set is one whose distinct elements can be counted by a natural number.

Examples of finite sets: $\emptyset$, $\{\sqrt{2}\}$

An **infinite** set is a set that is not finite. *Examples of infinite sets*:

$\mathbb{Z}$	The set of integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^+$	The set of positive integers	$\{1, 2, \ldots\}$
natural $\mathbb{N}$	The set of nonnegative integers	$\{0, 1, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers	$\left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
$\mathbb{R}$	The set of real numbers	

The **cardinality** of a set A is the size of the set and is denoted by $|A|$.

The cardinality of a finite set is the number of distinct elements in A.

Can we use the subset relationship to reason about the relative cardinality of sets?

$$\mathbb{Z}^+ \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

same as $\subset$

$A = \{1, 2, 3\}$ $\quad$ $B = \{1, 2, 3, 4\}$

$A \subset B \Rightarrow |A| < |B|$ → we can say this since A & B are countably finite

Motivating question: Are some of the above sets *bigger than* others?

Analogy: Musical chairs

People try to sit down when the music stops

Person☼ sits in Chair 1, Person☺ sits in Chair 2,

Person☹ is left standing!

What does this say about the number of chairs and the number of people?

Defining functions A function is defined by its (1) domain, (2) codomain, and (3) rule assigning each element in the domain exactly one element in the codomain. The domain and codomain are nonempty sets. The rule can be depicted as a table, formula, English description, etc.

(zyBooks 2.3, Rosen p139)

well-defined

Example: $f_A : \mathbb{R}^+ \to \mathbb{Q}$ with $f_A(x) = x$ is **not** a well-defined function because

π $\quad f_A(\pi)$ is undefined because $\pi \notin \mathbb{Q}$

Example: $f_B : \mathbb{Q} \to \mathbb{Z}$ with $f_B\left(\frac{p}{q}\right) = p + q$ is **not** a well-defined function because

$\frac{p}{q} = \frac{2}{4} = \frac{1}{2}$ $\quad f_B\left(\frac{2}{4}\right) = f_B\left(\frac{1}{2}\right) = 6$

$f_B\left(\frac{1}{2}\right) = 3$

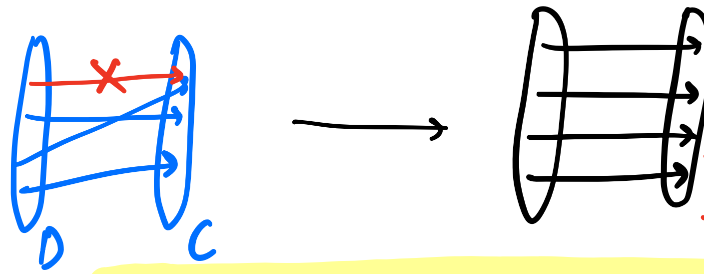
$\frac{1}{2} \to 6$, $\frac{1}{2} \to 3$

Example: $f_C : \mathbb{Z} \to \mathbb{R}$ with $f_C(x) = \frac{x}{|x|}$ is **not** a well-defined function because

$x = 0$ division by 0 is not defined

Definition (zyBooks 9.1, Rosen p141): A function $f : D \to C$ is **one-to-one** (or injective) means for every a, b in the domain D, if $f(a) = f(b)$ then $a = b$.

$\forall a \in D \ \forall b \in D \quad f(a) = f(b) \longrightarrow a = b$



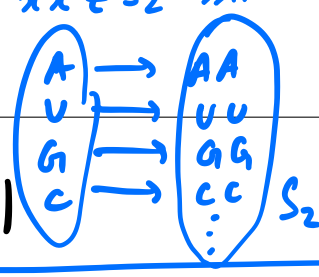
one-one $f: A \to B$ $\Rightarrow |A| \le |B|$

Definition: For sets A, B, we say that **the cardinality of A is no bigger than the cardinality of B**, and write $|A| \le |B|$, to mean there is a one-to-one function with domain A and codomain B.

In the analogy: The function $sitter : \{Chair1, Chair2\} \to \{Person☼, Person☺, Person☹\}$ given by $sitter(Chair1) = Person☼$, $sitter(Chair2) = Person☺$, is one-to-one and witnesses that

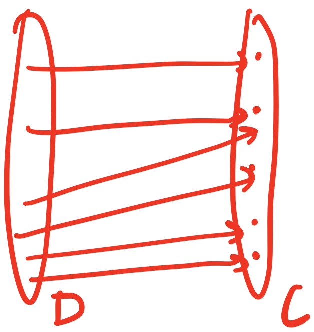
is a one-one fn $\quad |\{Chair1, Chair2\}| \le |\{Person☼, Person☺, Person☹\}|$

Let S_2 be the set of RNA strands of length 2.

Statement	True/~~False~~ , justification
$\lvert\{\texttt{A}, \texttt{U}, \texttt{G}, \texttt{C}\}\rvert \le \lvert S_2\rvert$	Define a one-one fn $f: \{A,U,G,C\} \longrightarrow S_2$ should be well-defined. $\forall x \in \{A,U,G,C\}$ $f(x) = xx$. observe that $xx \in S_2$ since AA, UU, GG, CC belong to S_2. ↳ f is one-one.  ∴ True $\lvert\{A,U,G,C\}\rvert \le \lvert S_2\rvert$
$\lvert\{\texttt{A}, \texttt{U}, \texttt{G}, \texttt{C}\} \times \{\texttt{A}, \texttt{U}, \texttt{G}, \texttt{C}\}\rvert \le \lvert S_2\rvert$	True. $f: \{A,U,G,C\} \times \{A,U,G,C\} \longrightarrow S_2$, $f(x) = x$ since $S_2 = \{A,U,G,C\} \times \{A,U,G,C\}$ by definition.

Definition (Rosen p143): A function $f : D \to C$ is **onto** (or surjective) means for every b in the codomain, there is an element a in the domain with $f(a) = b$.

Formally, $f : D \to C$ is onto means $\forall b \in C \, \exists a \in D \; f(a) = b$ → every element in C is mapped to some element in D.



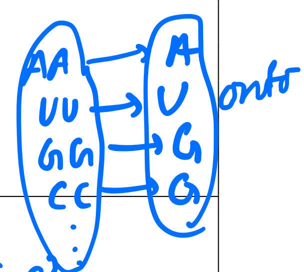
$\exists$ onto $f : A \longrightarrow B \Rightarrow |A| \geq |B|$

Definition: For sets A, B, we say that **the cardinality of A is no smaller than the cardinality of B**, and write $|A| \geq |B|$, to mean there is an onto function with domain A and codomain B.

In the analogy: The function $triedToSit : \{Person☼, Person☺, Person☹\} \to \{Chair1, Chair2\}$ given by $triedToSit(Person☼) = Chair1$, $triedToSit(Person☺) = Chair2$, $triedToSit(Person☹) = Chair2$, is onto and witnesses that

$$|\{Person☼, Person☺, Person☹\}| \geq |\{Chair1, Chair2\}|$$

Let S_2 be the set of RNA strands of length 2.

Statement	True/False , justification
$\|S_2\| \geq \|\{\texttt{A}, \texttt{U}, \texttt{G}, \texttt{C}\}\|$	True onto fn $f : S_2 \longrightarrow \{A, U, G, C\}$ $\forall x$ let $x = c \cdot x'$ where c = first letter in x $f(x) = c$, where $x = c \cdot x'$ 
$\|S_2\| \geq \|\{\texttt{A}, \texttt{U}, \texttt{G}, \texttt{C}\} \times \{\texttt{A}, \texttt{U}, \texttt{G}, \texttt{C}\}\|$	$f : S_2 \longrightarrow \{A, U, G, C\} \times \{A, U, G, C\}$ $\forall x \in S_2 \; f(x) = x$ onto infact f is one-one & onto.

can also say fn is bijective

Definition (Rosen p144): A function $f : D \to C$ is a **bijection** means that it is both one-to-one and onto. The **inverse** of a bijection $f : D \to C$ is the function $g : C \to D$ such that $g(b) = a$ iff $f(a) = b$.

Example of a bijective function

$\{A, U, G, C\} \times \{A, U, G, C\} \longrightarrow S_2$

$\forall x \quad f(x) = x \longrightarrow$ one-one

↳ onto

bijection

For nonempty sets A, B we say

$|A| \leq |B|$ means there is a one-to-one function with domain A, codomain B

$|A| \geq |B|$ means there is an onto function with domain A, codomain B

$|A| = |B|$ means there is a bijection with domain A, codomain B

Cantor-Schroder-Bernstein Theorem: For all nonempty sets,

$$|A| = |B| \quad \text{if and only if} \quad (|A| \leq |B| \text{ and } |B| \leq |A|) \quad \text{if and only if} \quad (|A| \geq |B| \text{ and } |B| \geq |A|)$$

To prove $|A| = |B|$, we can do any **one** of the following

- Prove there exists a bijection $f : A \to B$;
- Prove there exists a bijection $f : B \to A$;
- Prove there exists two functions $f_1 : A \to B$, $f_2 : B \to A$ where each of f_1, f_2 is one-to-one.
- Prove there exists two functions $f_1 : A \to B$, $f_2 : B \to A$ where each of f_1, f_2 is onto.

True or False? $|Z^+| = |Z^-|$ True

Proof: $f : \mathbb{Z}^- \longrightarrow \mathbb{Z}^+$

T.S.T: f is a bijection

$\forall x \in \mathbb{Z}^- \quad f(x) = |x| \quad [\because$ ex $f(-1) = |-1| = 1]$

(i) f is one-one $\quad \forall x_1 \in \mathbb{Z}^- \ \forall x_2 \in \mathbb{Z}^- \ f(x_1) = f(x_2) \longrightarrow x_1 = x_2$

So, $|x_1| = |x_2|$ since x_1, x_2 are $-^{ve}$ integers, $x_1 = x_2$

C D

(ii) f is onto $\forall x_2 \in \mathbb{Z}^+ \; \exists x_1 \in \mathbb{Z}^- \; f(x_1) = x_2$

Let x_2 be an arbitrary element in $\mathbb{Z}^+$

then $f(-x_2) = x_2$

and since $x_2 \in \mathbb{Z}^+$, $-x_2 \in \mathbb{Z}^-$

& $\therefore \exists x_1 = -x_2$ s.t $f(x_1) = x_2$

Using (i) & (ii) f is bijective.

A set A is **finite** means it is empty or it is the same size as $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$. A set A is **countably infinite** means it is the same size as $\mathbb{N}$.

Examples of countably infinite sets:

Negative integers $\mathbb{Z}^-$ $\mathbb{Z}^-$ is countably ∞ *List:* $-1\ -2\ -3\ -4\ -5\ -6\ -7\ -8\ -9\ -10\ -11 \ldots$

$f : \mathbb{N} \to \mathbb{Z}^-$ with $\boxed{f(n) = -n - 1}$ (ii) $|\mathbb{Z}^-| = |\mathbb{N}|$

Claim: f is a bijection. **Corollary:** $|\mathbb{N}| = |\mathbb{Z}^-|$

Proof: We need to show it is a well-defined function that is one-to-one and onto.

- Well-defined?

  Consider an arbitrary element of the domain, $n \in \mathbb{N}$. We need to show it maps to exactly one element of $\mathbb{Z}^-$.

  $f(n) = -n-1$

  $-n-1 \in \mathbb{Z}^-$
  since $n \geq 0$
  $\hookrightarrow -n-1 \leq -1$

  $0 \to -1,\ 1 \to -2,\ 2 \to -3,\ 3 \to -4,\ 4 \to -5,\ 5 \to -6,\ \ldots$

- One-to-one? $\forall a \in \mathbb{N}\ \forall b \in \mathbb{N}\ f(a) = f(b) \longrightarrow a = b$

  Consider arbitrary elements of the domain $a, b \in \mathbb{N}$. We need to show that

  $f(a) = f(b) \Rightarrow -a-1 = -b-1$
  $\Rightarrow -a = -b$
  $\Rightarrow \boxed{a = b}$ ✱

  $\therefore f$ is one-one

- Onto? $\forall b \in \mathbb{Z}^-\ \exists a \in \mathbb{N}\ f(a) = b$ (C, D)

  Consider arbitrary element of the codomain $b \in \mathbb{Z}^-$. We need witness in $\mathbb{N}$ that maps to b.

  Want $f(a) = b$
  (i.e.) $-a-1 = b$
  $\Rightarrow a+1 = -b$
  $\Rightarrow \boxed{a = -b-1}$ ✱

  $b \leq -1 \quad \therefore -b-1 \geq 0$

  $\therefore$ for $a \in \mathbb{N}$
  $a = -b-1$
  $f(a) = b$