# Solutions To HW1

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## Solution To Question 1

**(a) :** *How tall is Stroke Tower* is not a proposition as a question is neither true or false.

**(b) :** *Storke Tower is as tall as 33 people stacked on top of each other* is a proposition.

The negation is : **Storke Tower is not as tall as 33 people stacked on top of each other.**

## Solution To Question 2

**(a) :** $\neg n$

**(b) :** $t \land m$

**(c) :** $n \to \neg m$

**(d) :** $\neg m \to (n \lor t)$

## Solution To Question 3

### (a) If 51 is an even number, then the sky is green.

- $P$: "51 is an even number." This is False because 51 is an odd number.
- $Q$: "The sky is green." This is also False because, typically, the sky is blue.

Since $P$ is false and $Q$ is also false, the statement $P \to Q$ is True.

### (b) If 51 is an odd number, then the sky is blue.

- $P$: "51 is an odd number." This is true.
- $Q$: "The sky is blue." This is true.

Since both $P$ and $Q$ are true, the statement $P \to Q$ is True.

## Solution To Question 4

### (a) It either rains or it pours.

$$i \vee j$$

### (b) It is raining on our parade.

$$i \wedge l$$

### (c) We are having this parade, come rain or shine.

$$l$$

## Solution To Question 5

### (a) Rafael can ride the elephant only if he is not afraid of heights.

Equivalent statement:

If Rafael rides the elephant, then he is not afraid of heights.

### (b) Rafael can ride the elephant if he is not afraid of heights.

Equivalent statement:

If Rafael is not afraid of heights, then he rides the elephant.

### (c) Being able to swim is a necessary skill needed for Tyra to learn to surf.

Equivalent statement:

If Tyra learns to surf, then she is able to swim.

### (d) Being able to swim is a sufficient skill needed for Tyra to learn to surf.

Equivalent statement:

If Tyra is able to swim, then she learns to surf.

## Solution To Question 6

### (a) $\neg p \to q \equiv \neg q \to p$

**Proof**:

$$
\begin{aligned}
\neg p \to q &\quad \text{(Given)}\\
\neg\neg p \vee q &\quad \text{(Conditional Identities)}\\
p \vee q &\quad \text{(Double Negative Law)}\\
q \vee p &\quad \text{(Commutative Laws)}\\
\neg\neg q \vee p &\quad \text{(Double Negative Law)}\\
\neg q \to p &\quad \text{(Conditional Identities)}
\end{aligned}
$$

### (b) $\neg p \to (q \wedge \neg q) \equiv p$

**Proof**:

$$
\begin{aligned}
\neg p \to (q \wedge \neg q) &\quad \text{(Given)}\\
\neg p \to F &\quad \text{(Complement Laws)}\\
\neg\neg p \vee F &\quad \text{(Conditional Identities)}\\
p \vee F &\quad \text{(Double Negative Laws)}\\
p &\quad \text{(Identity Laws)}
\end{aligned}
$$

### (c) $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \equiv p \wedge \neg r$

**Proof**:

$$
\begin{aligned}
(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) &\quad \text{(Given)}\\
(\neg r \wedge p \wedge q) \vee (\neg r \wedge p \wedge \neg q) &\quad \text{(Commutative Laws)}\\
\neg r \wedge ((p \wedge q) \vee (p \wedge \neg q)) &\quad \text{(Distributive Laws)}\\
\neg r \wedge (p \wedge (q \vee \neg q)) &\quad \text{(Distributive Laws)}\\
\neg r \wedge (p \wedge T) &\quad \text{(Complement Laws)}\\
\neg r \wedge p &\quad \text{(Domination Laws)}\\
p \wedge \neg r &\quad \text{(Commutative Laws)}
\end{aligned}
$$

## Solution To Question 7

| $p$ | $q$ | $r$ | $p \to q$ | $q \to r$ | $(p \to q) \wedge (q \to r)$ | $p \to r$ | $((p \to q) \wedge (q \to r)) \to (p \to r)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

From the truth table, we prove that this is a tautology.

# Solution To Question 8

Given the statements:

- Andromeda: "Clytemnestra is the knave."
- Brunhilda: "Andromeda is the knight."
- Clytemnestra: "I am the spy."

**Case 1: Andromeda is the knight**

If Andromeda is the knight, she tells the truth. Therefore, her statement "Clytemnestra is the knave" is true. Thus:

- Clytemnestra is the knave.
- Brunhilda must be the spy.

We verify the statements:

- Brunhilda says, "Andromeda is the knight." As the spy, she can tell the truth.
- Clytemnestra says, "I am the spy." As the knave, she lies.

**Case 2: Brunhilda is the knight**

If Brunhilda is the knight, she tells the truth. Therefore, her statement "Andromeda is the knight" must be true. This is a contradiction.

**Case 3: Clytemnestra is the knight**

If Clytemnestra is the knight, she tells the truth. Therefore, her statement "I am the spy" must be true. This is a contradiction.

## Conclusion

- Andromeda is the knight.
- Brunhilda is the spy.
- Clytemnestra is the knave.

# Solution To Question 9

$$Q = (\neg p \land q \land r) \lor (\neg p \land \neg q \land r) \lor (p \land q \land r) \lor (p \land \neg q \land \neg r)$$

## Solution To Question 10

### (a) An element from the set $B \times A \times C$

$$(\text{foam}, \text{tall}, \text{non-fat})$$

### (b) The set $B \times C$ in roster notation

$$B \times C = \{(\text{foam}, \text{non-fat}), (\text{foam}, \text{whole}), (\text{no-foam}, \text{non-fat}), (\text{no-foam}, \text{whole})\}$$

## Solution To Question 11

### (a) Negation of $\forall x \exists y (P(x, y) \wedge Q(x, y))$

$$\neg(\forall x \exists y (P(x, y) \wedge Q(x, y)))$$
$$\exists x \neg(\exists y (P(x, y) \wedge Q(x, y)))$$
$$\exists x \forall y \neg(P(x, y) \wedge Q(x, y))$$
$$\exists x \forall y (\neg P(x, y) \vee \neg Q(x, y))$$

### (b) Negation of $\exists x \forall y (P(x, y) \to Q(x, y))$

$$\neg(\exists x \forall y (P(x, y) \to Q(x, y)))$$
$$\forall x \neg(\forall y (P(x, y) \to Q(x, y)))$$
$$\forall x \exists y \neg(P(x, y) \to Q(x, y))$$
$$\forall x \exists y (P(x, y) \wedge \neg Q(x, y))$$

## Solution To Question 12

- $P(x)$: $x$ showed up with a pencil. - $C(x)$: $x$ showed up with a calculator.

### (a) At least one of the students showed up with a pencil.

$$\exists x \, P(x)$$

Negation:

$$\neg(\exists x \, P(x))$$
$$\forall x \, \neg P(x)$$

English translation: *Every student showed up without a pencil.*

### (b) Every student showed up with a pencil or a calculator (or both).

$$\forall x\,(P(x) \vee C(x))$$

Negation:

$$\neg(\forall x\,(P(x) \vee C(x)))$$
$$\exists x\,\neg(P(x) \vee C(x))$$
$$\exists x\,(\neg P(x) \wedge \neg C(x))$$

English translation: *There is at least a student who showed up without a pencil and without a calculator.*

### (c) Every student who showed up with a calculator also had a pencil.

$$\forall x\,(C(x) \rightarrow P(x))$$

Negation:

$$\neg(\forall x\,(C(x) \rightarrow P(x)))$$
$$\exists x\,\neg(C(x) \rightarrow P(x))$$
$$C(x) \rightarrow P(x) \equiv \neg C(x) \vee P(x)$$
$$\exists x\,(\neg(\neg C(x) \vee P(x)))$$
$$\exists x\,(C(x) \wedge \neg P(x))$$

English translation: *There is at least a student who showed up with a calculator and without a pencil.*

### (d) There is a student who showed up with both a pencil and a calculator.

$$\exists x\,(P(x) \wedge C(x))$$

Negation:

$$\neg(\exists x\,(P(x) \wedge C(x)))$$
$$\forall x\,\neg(P(x) \wedge C(x))$$
$$\forall x\,(\neg P(x) \vee \neg C(x))$$

English translation: *Every student showed up without a pencil or without a calculator (or both).*