HW5

CS40 Summer '24

Due: Monday, July 29, 2024 at 11:59PM on Gradescope

In this assignment,

You will work with recursively defined sets and functions and prove properties about them, practicing induction, contradiction, and other proof strategies. You will also practice applying functions to counting problems and related proofs.

For all HW assignments:

Please see the instructions and policies for assignments on the class website and on the writeup for HW1. In particular, these policies address

- Collaboration policy
- Where to get help
- Typing your solutions
- Expectations for full credit

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW7-Collaborative".

In your proofs and disproofs of statements below, justify each step by reference to the proof strategies we have discussed so far, and/or to relevant definitions and calculations. We include only induction-related strategies here; you can and should refer to past material to identify others.

Assigned Questions

1. Prove the product rule: If A and B are finite sets $|A \times B| = |A||B|$. Hint: Find a witness bijective function $f : A \times B \to \{1, 2, 3, ...|A| \cdot |B|\}$.

2. Given the definition of the set of all linked lists of natural numbers L and the function $toNum : L \to \mathbb{N}$, both defined recursively below, **prove that $toNum$ is one-to-one**.

The set of linked lists of natural numbers L is defined by:

Basis Step:	$[\,] \in L$
Recursive Step:	If $l \in L$ and $n \in \mathbb{N}$, then $(n, l) \in L$

$toNum : L \to \mathbb{N}$ is defined recursively as follows:

Basis Step:	$toNum([]) = 0$
Recursive Step:	If $n \in \mathbb{N}$ and $l \in L$, then $toNum((n, l)) = 2^n \cdot 3^{toNum(l)}$

3. Determine whether the following functions are well defined, and if they are injective (one-to-one) and/or surjective (onto) as well:

   (a) $f : \mathbb{N} \to \{0, 1\}^*$ such that $n \mapsto (n)_2$, where $\{0, 1\}^*$ is the set of all finite length bit strings, and $(n)_2$ is the binary representation of n. *(Hint: use a result you proved in the previous homework.)*

   (b) $f : \mathbb{Q} \to \mathbb{Z}$ given by $\frac{m}{n} \mapsto m^n$;

   (c) $f : \{0, 1\}^3 \to \{0, 1\}^4$ where $(b_3, b_2, b_1) \mapsto (0, b_3, b_2, b_1)$ for $b_i \in \{0, 1\}$;

   (d) $f : \{0, 1\}^4 \to \{0, 1\}^3$ where $(b_4, b_3, b_2, b_1) \mapsto (b_3, b_2, b_1)$ for $b_i \in \{0, 1\}$;

   (e) $f : \mathbb{Z} \to \mathbb{Z}$ where $k \mapsto k \texttt{ mod } 19$.

4. Let $W = \mathcal{P}(\{1, 2, 3, 4, 5\})$.

Sample response that can be used as reference for the detail expected in your answer for parts (a) and (b) below:

To give an example element in the set $\{X \in W \ : \ 1 \in X\} \cap \{X \in W \ : \ 2 \in X\}$, consider $\{1, 2\}$. To prove that this is in the set, by definition of intersection, we need to show that $\{1, 2\} \in \{X \in W \ : \ 1 \in X\}$ and that $\{1, 2\} \in \{X \in W \ : \ 2 \in X\}$.

- By set builder notation, elements in $\{X \in W \ : \ 1 \in X\}$ have to be elements of W which have 1 as an element. By definition of power set, elements of W are subsets of $\{1, 2, 3, 4, 5\}$. Since each element in $\{1, 2\}$ is an element of $\{1, 2, 3, 4, 5\}$, $\{1, 2\}$ is a subset of $\{1, 2, 3, 4, 5\}$ and hence is an element of W. Also, by roster method, $1 \in \{1, 2\}$. Thus, $\{1, 2\}$ satisfies the conditions for membership in $\{X \in W \ : \ 1 \in X\}$.
- Similarly, by set builder notation, elements in $\{X \in W \ : \ 2 \in X\}$ have to be elements of W which have 2 as an element. By definition of power set, elements of W are subsets of $\{1, 2, 3, 4, 5\}$. Since each element in $\{1, 2\}$ is an element of $\{1, 2, 3, 4, 5\}$, $\{1, 2\}$ is a subset of $\{1, 2, 3, 4, 5\}$ and hence is an element of W. Also, by roster method, $2 \in \{1, 2\}$. Thus, $\{1, 2\}$ satisfies the conditions for membership in $\{X \in W \ : \ 2 \in X\}$.

(a) Give two example elements in

$$\mathcal{P}(W)$$

Justify your examples by explanations that include references to the relevant definitions.

(b) Give one example element in

$$\mathcal{P}(W) \times \mathcal{P}(W)$$

that is **not** equal to $(\emptyset, \emptyset)$ or to (W, W). Justify your example by an explanation that includes references to the relevant definitions.

5. Let W be a finite set with n elements. How many pairs (X, Y) are there such that $X \subseteq W$, $Y \subseteq W$, and $X \subseteq Y$? Provide an expression to compute the answer that shows your reasoning. You don't need to express your answer in its most reduced form.

6. Write a recursive algorithm to compute the arithmetic mean of a sequence of integers. Then, use induction on the length of the sequence to prove that your algorithm outputs the correct value for every non-empty input sequence.

Algorithm: Computing the arithmetic mean of a sequence recursively

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procedure mean_recursive( (a_0, ... a_n): a sequence of integers)













return (m){m is the arithmetic mean of (a_0, ... a_n)}
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7. Recall the definition of the set of rational numbers, $Q = \left\{ \frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$. We define the set of **irrational** numbers $\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$. Fill in the blank in the following argument.

Claimed statement: $-\sqrt{2} \in \overline{\mathbb{Q}}$

Proof: Towards a proof by contradiction, we will prove that $\underline{\quad BLANK \quad}$ guarantees $\sqrt{2} \in \overline{\mathbb{Q}} \wedge \sqrt{2} \notin \overline{\mathbb{Q}}$. We proceed by direct proof and assume the hypothesis of the conditional. To prove the conclusion of the conditional, we have two subgoals. Subgoal (1): We need to prove the first conjunct, that $\sqrt{2} \in \overline{\mathbb{Q}}$. This is proved in Chapter 7.2 (Theorem 7.2.1 in Zybook). Subgoal (2): It remains to prove that $\sqrt{2} \notin \overline{\mathbb{Q}}$, in other words (by double negation), that $\sqrt{2} \in \mathbb{Q}$. By our assumption that $-\sqrt{2} \notin \overline{\mathbb{Q}}$ and double negation, we have $-\sqrt{2} \in \mathbb{Q}$. By definition of the set of rational numbers, there are integers p' and q' (with q' nonzero) such that $-\sqrt{2} = \frac{p'}{q'}$. Consider the witnesses $p = -p'$ and $q = q'$. These are integers (since -1 times an integer is an integer) and q is nonzero, so they are candidate witness for the fraction in the definition of rational numbers. Moreover,

$$\frac{p}{q} = \frac{-p'}{q'} = (-1)\frac{p'}{q'} \overset{\text{by def of } p', q'}{=} (-1)(-\sqrt{2}) = \sqrt{2}$$

Thus, we have proved (under our assumption) that $\sqrt{2} \in \mathbb{Q}$, and subgoal (2) is complete. Since the direct proof is complete, we have proved that assuming the negation of the claimed statement leads to a contradiction, and therefore the original statement must be true. QED

8. Consider the binary relation R on the set of integers define as $R_m = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = 3 \cdot m\}$ for some positive integer m. Prove or disprove that R_m is a equivalence relation.

9. Recall that in a movie recommendation system, each user's ratings of movies is represented as a n-tuple (with the positive integer n being the number of movies in the database), and each component of the n-tuple is an element of the collection $\{-1, 0, 1\}$.

   Assume there are five movies in the database, so that each user's ratings can be represented as a 5-tuple. Let R be the set of all ratings, that is, the set of all 5-tuples where each component of the 5-tuple is an element of the collection $\{-1, 0, 1\}$.

   Consider the following two binary relations on R:

$$A_1 = \{(u,v) \in R \times R \mid \text{users } u \text{ and } v \text{ agree about the first movie in the database}\}$$

$$A = \{(u,v) \in R \times R \mid \text{users } u \text{ and } v \text{ don't care or haven't seen the same number of movies}\}$$

   Binary relations that satisfy certain properties (namely, are reflexive, symmetric, and transitive) can help us group elements in a set into categories.

   (a) **True** or **False**: The relation A_1 holds of $u = (1,1,1,1,1)$ and $v = (-1,-1,-1,-1,-1)$.

   (b) **True** or **False**: The relation A holds of $u = (1,0,1,0,-1)$ and $v = (-1,0,1,-1,-1)$.

   (c) **True** or **False**: A_1 is reflexive; namely, $\forall u \in R \ (\ (u,u) \in A_1 \ )$

   (d) **True** or **False**: A_1 is symmetric; namely, $\forall u \in R \ \forall v \in R \ (\ (u,v) \in A_1 \to (v,u) \in A_1 \ )$

   (e) **True** or **False**: A_1 is transitive; namely, $\forall u \in R \ \forall v \in R \ \forall w \in R(\ ((u,v) \in A_1 \wedge (v,w) \in A_1) \to (u,w) \in A_1 \ )$

   (f) **True** or **False**: A is reflexive; namely, $\forall u \in R \ (\ (u,u) \in A \ )$

   (g) **True** or **False**: A is anti-symmetric; namely, $\forall u \in R \ \forall v \in R \ (\ (\ (u,v) \in A \wedge (v,u) \in A \ ) \to (u = v) \ )$

   (h) **True** or **False**: A is transitive; namely, $\forall u \in R \ \forall v \in R \ \forall w \in R(\ ((u,v) \in A \wedge (v,w) \in A) \to (u,w) \in A \ )$

10. In the previous question select any one of parts (c) to (h) that evaluated to true and provide a formal proof using the strategies you have learned in CS40

11. *No justifications are required for credit for this question. It's a good idea to think about how you would explain how you arrived at your examples.* Given the relations A_1 and A in Q9 answer the following questions:

    (a) Give two distinct examples of elements in $[\ (1,0,0,0,0)\ ]_{A_1}$

    (b) Give two distinct examples of elements in $[\ (1,0,0,0,0)\ ]_{A}$

    (c) Find examples $u, v \in R$ where $[u]_{A_1} \neq [v]_{A_1}$ but $[u]_A = [v]_A$

    (d) Find examples $u, v \in R$ (different from the previous part) where $[u]_{A_1} = [v]_{A_1}$ but $[u]_A \neq [v]_A$

12. Consider an old game of matches. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction on the integer j to show that the first player can always win if $n = 4j, 4j + 2$, or $4j + 3$ for some non-negative integer j.

13. (Extra Credit) Define $\mathbb{R}^{(0,1)} = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}^{+} = \{x \in \mathbb{R} \mid x > 0\}$. Here we will show that $|\mathbb{R}^{(0,1)}| = |\mathbb{R}|$.

    (a) Show that the function $f : \mathbb{R}^{(0,1)} \to \mathbb{R}^{+}$ for which $x \mapsto \frac{x}{1-x}$ is bijective.

    (b) Show that the function $g : \mathbb{R}^{(0,1)} \to \mathbb{R}$ is bijective, where

    $$g(x) = \begin{cases} -f(1-2x) & 0 < x < \frac{1}{2}, \\ 0 & x = \frac{1}{2}, \\ f(2x-1) & \frac{1}{2} < x < 1. \end{cases}$$

    and conclude that $|\mathbb{R}^{(0,1)}| = |\mathbb{R}|$.

14. **Bonus - not for credit (but much appreciated):** Please complete the course ESCI and TA evaluations by June 3 (Friday).

Attributions

Thanks to Mia Minnes and Joe Politz for the original version of some of the questions on this homework.