HW6

CS40 Summer '24

Due: Monday, August 5, 2024 at 11:59PM on Gradescope

In this assignment,

You will have more practice with induction and other proof strategies.

For all HW assignments:

Please see the instructions and policies for assignments on the class website and on the writeup for HW1. In particular, these policies address

- Collaboration policy
- Where to get help
- Typing your solutions
- Expectations for full credit

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW6".

In your proofs and disproofs of statements below, justify each step by reference to the proof strategies we have discussed so far, and/or to relevant definitions and calculations. We include only induction-related strategies here; you can and should refer to past material to identify others.

Proof by Mathematical Induction: To prove a universal quantification over the set of all integers greater than or equal to some base integer b:

Basis Step:	Show the statement holds for b.
Recursive Step:	Consider an arbitrary integer n greater than or equal to b, assume (as the **induction hypothesis**) that the property holds for n, and use this and other facts to prove that the property holds for $n+1$.

Proof by Strong Induction To prove that a universal quantification over the set of all integers greater than or equal to some base integer b holds, pick a fixed nonnegative integer j and then:

Basis Step:	Show the statement holds for b, $b+1$, $\ldots$, $b+j$.
Recursive Step:	Consider an arbitrary integer n greater than or equal to $b+j$, assume (as the **strong induction hypothesis**) that the property holds for **each of** b, $b+1$, $\ldots$, n, and use this and other facts to prove that the property holds for $n+1$.

Assigned questions

1. Write the first 6 terms of the sequence that is described by each of the recurrence relations below:

   (a) $f_1 = 0, f_2 = 2$, and $f_n = 5f_{n-1} - 2f_{n-2}$ for $n \geq 3$.

   (b) $g_1 = 2$ and $g_2 = 1$. The rest of the terms are given by the formula $g_n = ng_{n-1} + g_{n-2}$.

2. Prove the following upper bound for the given recurrence relation using strong induction.

   Define the sequence $\{a_n\}$ as follows:

$$
\begin{aligned}
a_0 &= a_1 = 2 \\
a_n &= a_{n-1}^2 a_{n-2}, \qquad n \geq 2
\end{aligned}
$$

   Prove that

$$\forall n \in \mathbb{Z}^{\geq 0} \; (\; a_n \leq 2^{3^n} \;)$$

3. Write a recursive algorithm to compute the maximum of a sequence of numbers. Then, use induction to prove that your algorithm outputs the correct value for every non-empty input sequence.

4. Define $P(n)$ to be the assertion that:

$$\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$$

   Answer the questions that follow:

   (a) Verify that $P(3)$ is true, and then express $P(k)$ and $P(k+1)$

   (b) What is the basis step for an inductive proof of $\forall n \in \mathbb{Z}^{+}(P(n))$

   (c) What would be the inductive hypothesis? What must be proven in the inductive step?

   (d) Write the complete inductive proof for the provided assertion by combining all your answers from the previous parts

5. (*Graded for correctness*) Prove the following statement:

$$\exists n_0 \in \mathbb{N} \, \forall n \in \mathbb{Z}^{\geq n_0} \, (n^2 < 2^n)$$

   In your proof, you may use the following lemma:

$$\exists n_0 \in \mathbb{N} \, \forall n \in \mathbb{Z}^{\geq n_0} \, (1 + 2n \; < n^2)$$

Proof of lemma. This proof can also be used as reference for a possible approach for the statement you are trying to prove:

To prove the existential claim, consider the witness $n_0 = 3$. We will prove that

$$\forall n \in \mathbb{Z}^{\geq 3}\, (1 + 2n \ < n^2)$$

using mathematical induction.

Basis step For the basis step, we need to show that $1+2 \cdot 3 \ < \ 3^2$. Evaluating: $1+2\cdot 3 = 1+6 = 7$ and $3^2 = 9$. Since $7 < 9$, the basis step is complete.

Recursive step Consider arbitrary integer n that is greater than or equal to 3. Assume, as the induction hypothesis, that $1+2n \ < \ n^2$. We need to show that $1+2(n+1) \ < (n+1)^2$. Calculating:

$$\begin{aligned}
(n+1)^2 &= (n+1)(n+1) = n^2 + 2n + 1 \\
&> (1+2n) + 2n + 1 && \text{by the induction hypothesis} \\
&> 2n + 2n + 1 && \text{since } 1 > 0 \\
&> 2n + 2 \cdot 1 + 1 && \text{since } n > 1 \text{ by assumption that } n \geq 3 \\
&= 2(n+1) + 1 = 1 + 2(n+1) && \text{as required to complete the recursive step.}
\end{aligned}$$

Thus, the universal quantification was proved using mathematical induction and so the witness $n_0 = 3$ proves the existential. ■

6. (*Graded for fair effort completeness*) Can the statement you proved above be used to prove or disprove the following statement? Why or why not?

$$\exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{Z}^{\geq n_0}\, (2^n < n^2)$$

7. Use induction to prove that the following algorithm is correct. Binary strings are the set of all strings of length 0 or more made up of characters from the set $\{0, 1\}$. We define λ to be the empty string (string of length zero).

Recursively computing the set of all binary strings of a fixed length n

```
procedure StringSet(n: a non-negative integer)

S := ∅
if n = 0
    Add λ to S
    return (S)
end if
T := StringSet(n − 1)
for every x ∈ T
    y := 0x
    Add y to S
    y := 1x
    Add y to S
end for
return (S) {output is S}
```

8. We define the following function:

$$f : \{0,1\}^4 \to \{0,1\}^3,$$

where the output of f is obtained by taking the input string and dropping the first bit. For example $f(1011) = 011$. Indicate whether the f is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

Attributions

Thanks to Mia Minnes and Joe Politz for the original version of Q2. All materials created by them is licensed under a Creative Commons Attribution-Non Commercial 4.0 International License.