Week 2 Part A highlights

- Prove identities related to sets using proof strategies learned so far.
- Evaluate nested quantifiers: both alternating and not.
- Use logical equivalence to rewrite quantified statements (including negated quantified statements)
- Counterexample and witness-based arguments for nested quantified statements with finite and infinite domains
- Translate arguments in English to forms in logic
- Identify the evidence needed for valid mathematical arguments
- Use rules of inference (in propositional and predicate logic) to construct valid arguments
- Distinguish between valid and invalid arguments

Term	Definition	Examples
set	an unordered collection of elements	
empty set	set that has no elements	$\{\}$, $\emptyset$
set equality	When A and B are sets, $A = B$ means $\forall x (x \in A \leftrightarrow x \in B)$	$\{43, 7, 9\} = \{7, 43, 9, 7\}$
subset	When A and B are sets, $A \subseteq B$ means $\forall x (x \in A \rightarrow x \in B)$	
proper subset	When A and B are sets, $A \subsetneq B$ means $(A \subseteq B) \wedge (A \neq B)$	

Claim: $\{\texttt{A}, \texttt{C}, \texttt{U}, \texttt{G}\} \subseteq \{\texttt{AA}, \texttt{AC}, \texttt{AU}, \texttt{AG}\}$ **Prove** or **disprove** *Circle one*

Claim: The empty set is a proper subset of every set. **Prove** or **disprove** *Circle one*

Claim: For some set B, $\emptyset \in B$. **Prove** or **disprove** *Circle one*

To define a set we can use the roster method, the set builder notation, a recursive definition, and also we can apply a set operation to other sets.

New! Cartesian product of sets and **set-wise concatenation of sets of strings**

Definition (Rosen p. 123) Let A and B be sets. The **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Definition: Let A and B be sets of strings over the same alphabet. The **set-wise concatenation** of A and B, denoted $A \circ B$, is the set of all results of string concatenation ab where $a \in A$ and $b \in B$

$$A \circ B = \{ab \mid a \in A \text{ and } b \in B\}$$

Set	**Example elements in this set**:
$B = \{\texttt{A}, \texttt{C}, \texttt{G}, \texttt{U}\}$	`A` `C` `G` `U`
	$(\texttt{A}, \texttt{C})$ $(\texttt{U}, \texttt{U})$
$B \times \{-1, 0, 1\}$	
$\{-1, 0, 1\} \times B$	
	$(0, 0, 0)$
$\{\texttt{A}, \texttt{C}, \texttt{G}, \texttt{U}\} \circ \{\texttt{A}, \texttt{C}, \texttt{G}, \texttt{U}\}$	
	`GGGG`

Formal definition of a function:

Definition (Rosen p123): The **Cartesian product** of the sets A and B, $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. That is: $A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$. The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, ..., a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. That is, $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$

Term	**Definition**	**Examples**
complement	When A is a set, $\bar{A} = \{x \mid x \notin A\}$	
Cartesian product	When A and B are sets, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$	$\{43, 9\} \times \{9, A\} =$ $\mathbb{Z} \times \emptyset =$
union	When A and B are sets, $A \cup B = \{x \mid x \in A \vee x \in B\}$	$\{43, 9\} \cup \{9, A\} =$ $\mathbb{Z} \cup \emptyset =$
intersection	When A and B are sets, $A \cap B = \{x \mid x \in A \wedge x \in B\}$	$\{43, 9\} \cap \{9, A\} =$ $\mathbb{Z} \cap \emptyset =$
set difference	When A and B are sets, $A - B = \{x \mid x \in A \wedge x \notin B\}$	$\{43, 9\} - \{9, A\} =$ $\mathbb{Z} - \emptyset =$
disjoint sets	sets A and B are disjoint means $A \cap B = \emptyset$	$\{43, 9\}, \{9, A\}$ are not disjoint $\mathbb{Z}, \emptyset$ are disjoint

Definitions of functions related to RNA strands

Recall: Each RNA strand is a string whose symbols are elements of the set $B = \{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{U}\}$. The **set of all RNA strands** is called S. The function *rnalen* that computes the length of RNA strands in S is:

$$\begin{array}{lll} & & rnalen : S \to \mathbb{Z}^+ \\ \text{Basis Step:} & \text{If } b \in B \text{ then} & rnalen(b) = 1 \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B\text{, then} & rnalen(sb) = 1 + rnalen(s) \end{array}$$

A function *basecount* that computes the number of a given base b appearing in a RNA strand s is:

$$\begin{array}{llll} & & basecount : S \times B & \to \mathbb{N} \\ \text{Basis Step:} & \text{If } b_1 \in B, b_2 \in B & basecount(b_1, b_2) & = \begin{cases} 1 & \text{when } b_1 = b_2 \\ 0 & \text{when } b_1 \neq b_2 \end{cases} \\ \text{Recursive Step:} & \text{If } s \in S, b_1 \in B, b_2 \in B & basecount(sb_1, b_2) & = \begin{cases} 1 + basecount(s, b_2) & \text{when } b_1 = b_2 \\ basecount(s, b_2) & \text{when } b_1 \neq b_2 \end{cases} \end{array}$$

Definition of predicates using the above functions

L with domain $S \times \mathbb{Z}^+$ is defined by, for $s \in S$ and $n \in \mathbb{Z}^+$,

$$L(s, n) = \begin{cases} T & \text{if } rnalen(s) = n \\ F & \text{otherwise} \end{cases}$$

Element where L evaluates to T: ________

Element where L evaluates to F: ________

BC with domain __________ is defined by, for $s \in S$ and $b \in B$ and $n \in \mathbb{N}$,

$$BC(s, b, n) = \begin{cases} T & \text{if } basecount(s, b) = n \\ F & \text{otherwise} \end{cases}$$

Element where BC evaluates to T: ________

Element where BC evaluates to F: ________

Notation: for a predicate P with domain $X_1 \times \cdots \times X_n$ and a n-tuple $(x_1, \ldots, x_n)$ with each $x_i \in X$, we write $P(x_1, \ldots, x_n)$ to mean $P(\ (x_1, \ldots, x_n)\ )$.

$\exists t\ BC(t)$ In English: ____

Witness that proves this existential quantification is true: ____________________

$\forall (s, b, n)\ (\ BC(s, b, n)\ )$ In English: ____

Counterexample that proves this universal quantification is false: ____________________

New predicates from old $BC(s,b,n)$ means $basecount(s,b) = n$.

Predicate	Domain	Example domain element where predicate is T
$basecount(s,b) = 3$		
$basecount(s,\mathtt{A}) = n$		
$\exists n \in \mathbb{N}\ (basecount(s,b) = n)$		
$\forall b \in B\ (basecount(s,b) = 1)$		

Alternating quantifiers

$$\forall s \exists n\ BC(s, \mathtt{A}, n)$$

In English: ______________________________

$$\exists n \forall s\ BC(s, \mathtt{U}, n)$$

In English: ______________________________

Evaluating Predicates over Infinite Domains

Evaluate each quantified statement as T or F.

$\forall s\ \forall b\ \exists n\ BC(s,b,n)$	$\forall s\ \forall n\ \exists b\ BC(s,b,n)$	$\forall b\ \forall n\ \exists s\ BC(s,b,n)$
$\exists s\ \forall b\ \exists n\ BC(s,b,n)$	$\forall s\ \exists n\ \forall b\ BC(s,b,n)$	$\exists b\ \exists n\ \forall s\ BC(s,b,n)$

Extra example: Write the negation of each of the statements above, and use De Morgan's law to find a logically equivalent version where the negation is applied only to the BC predicate (not next to a quantifier).

An **argument** is a sequence of propositions, called **hypotheses**, followed by a final proposition, called the **conclusion**. An argument is valid if the conclusion is true whenever the hypotheses are all true, otherwise the argument is invalid.

Example 1: Prove that the argument below is **valid**

$$\begin{array}{c} p \to q \\ p \lor q \\ \hline \therefore q \end{array}$$

The hypotheses are ______________________ The conclusion is ___________

We need to show that $(\ (p \to q) \land (p \lor q)\ ) \to q$ is a _________________________________ [1]

[1]Zybook Participation activity 3.8.3 uses truth tables to prove this.

Example 2: Show that the argument below is **invalid**

$$\begin{array}{c} \neg p \\ p \to q \\ \hline \therefore \neg q \end{array}$$

An assignment of p and q that make the hypotheses true but the conclusion false is ______________________

Examining arguments involving propositions and quantified statements

Apply the rules of inference for propositions to construct a valid argument form

"If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded", and "The trophy was not awarded"

Use rules of inference for propositions (zyBook Table 3.9.1) to show that the hypotheses imply the conclusion "It rained."

Apply the rules of inference for quantified statements to indicate whether the proof fragment is a correct or incorrect.

1.	c is an element	Hypothesis
2.	$P(c)$	Hypothesis
3.	$\forall x P(x)$	Universal generalization, 1, 2

Is the argument correct?

1.	c is an element	Hypothesis
2.	$\forall x P(x)$	Hypothesis
3.	$P(c)$	Universal instantiation, 1, 2

Is the argument correct?

An argument with quantified statements can be shown to be invalid by defining a domain and predicates for which the hypotheses are all true but the conclusion is false.

Example 1: Choose predicates to show that the argument is invalid over the domain c, d

$$\begin{array}{c} \forall x P(x) \vee \forall x Q(x) \\ \hline \therefore \exists x (P(x) \wedge Q(x)) \end{array}$$

Example 2: Choose predicates and their domain to show that the argument is invalid

$$\begin{array}{c} \exists x P(x) \\ \exists x Q(x) \\ \hline \therefore \exists x (P(x) \wedge Q(x)) \end{array}$$