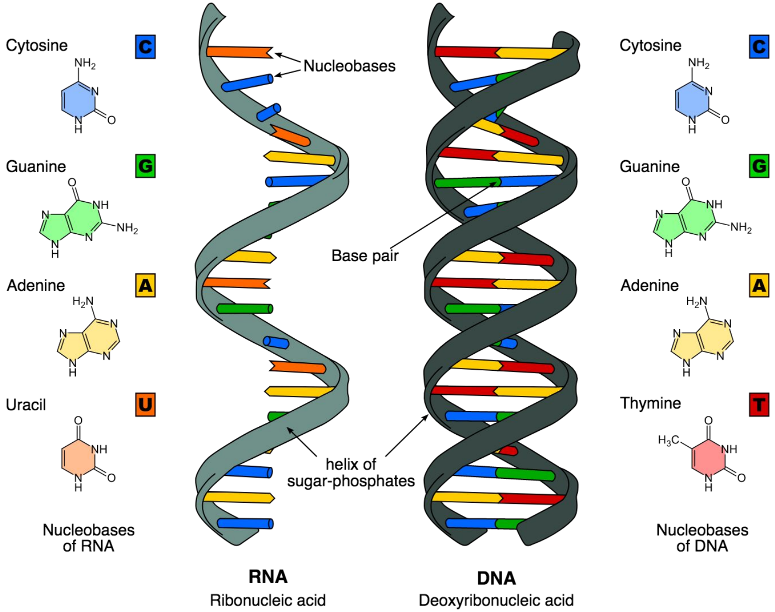
CS 40
FOUNDATIONS OF CS

Summer 2024
Week 2

Adapted for CS40 by Diba Mirza

RNA strands as strings

$S = \{A, C, G, U, AC, AG, AU, GA, \ldots, ACG, \ldots\}$



Each RNA strand is a **string** whose symbols are elements of the set $B = \{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{U}\}$.

Base

Definition by recursion

New! Recursive Definitions of Sets: The set S (pick a name) is defined by:

Basis Step:	Specify finitely many elements of S
Recursive Step:	Give a rule for creating a new element of S from known values existing in S, and potentially other values.

Definition The set of RNA strands S is defined (recursively) by:

$B = \{A, C, U, G\}$

$S = \{A, C, U, G\}$

Basis Step:	$\mathtt{A} \in S, \mathtt{C} \in S, \mathtt{U} \in S, \mathtt{G} \in S$
Recursive Step:	If $s \in S$ and $b \in B$, then $sb \in S$

$\{AA, CA, UA, GA$

Two different RNA strands: AB, CB, UB, GB

$\ldots \}$

Defining functions recursively when domain is recursively defined

Definition (Of a function, recursively) A function *rnalen* that computes the length of RNA strands in S is defined by:

$$rnalen : S \to \mathbb{Z}^+$$

Basis Step:	If $b \in B$ then	$rnalen(b)$	$= 1$
Recursive Step:	If $s \in S$ and $b \in B$, then	$rnalen(s\underline{b})$	$= 1 + rnalen(s)$

$S \to$ set of all RNA strings

The domain of *rnalen* is S. The codomain of *rnalen* is $\mathbb{Z}^+$.

$rnalen(\texttt{ACU}) = 1 + rnalen(AC)$ $= 3$

$\to 1 + rnalen(A)$

$\downarrow$

1

$rnalen : S \longrightarrow \mathbb{Z}^+$

Rule : $\$$

Recall: Each RNA strand is a string whose symbols are elements of the set $B = \{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{U}\}$. The **set of all RNA strands** is called S. The function *rnalen* that computes the length of RNA strands in S is:

$$
\begin{array}{llll}
 & & rnalen : S & \rightarrow \mathbb{Z}^+ \\
\text{Basis Step:} & \text{If } b \in B \text{ then} & rnalen(b) & = 1 \\
\text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B\text{, then} & rnalen(sb) & = 1 + rnalen(s)
\end{array}
$$

Example predicates on S

st S

$H(s) = T$	Truth set of H is S
$L_3(s) = \begin{cases} T & \text{if } rnalen(s) = 3 \\ F & \text{otherwise} \end{cases}$	ACG ✓ Strand where L_3 evaluates to T is e.g. S_3 Strand where L_3 evaluates to F is e.g. AC
$F_{\mathtt{A}}$ is defined recursively by: Basis step: $F_{\mathtt{A}}(\mathtt{A}) = T$, $F_{\mathtt{A}}(\mathtt{C}) = F_{\mathtt{A}}(\mathtt{G}) = F_{\mathtt{A}}(\mathtt{U}) = F$ Recursive step: If $s \in S$ and $b \in B$, then $\boxed{F_{\mathtt{A}}(sb) = F_{\mathtt{A}}(s)}$	Strand where $F_{\mathtt{A}}$ evaluates to T is e.g. A, AC, AAC Strand where $F_{\mathtt{A}}$ evaluates to F is e.g. BC, B
$P_{\mathtt{AUC}}$ is defined as the predicate whose truth set is the collection of RNA strands where the string AUC is a substring (appears inside s, in order and consecutively)	Strand where $P_{\mathtt{AUC}}$ evaluates to T is e.g. AAUC, GAUCD, AUC Strand where $P_{\mathtt{AUC}}$ evaluates to F is e.g. AGUC

Truth set

the set of all RNA strings of len 3

is 1 if s starts with A

~~$F_A(s)$ is T if all except ... A~~

$H(s) = T$

$$L_3(s) = \begin{cases} T & \text{if } rnalen(s) = 3 \\ F & \text{otherwise} \end{cases}$$

$F_{\mathtt{A}}$ is defined recursively by:

Basis step: $F_{\mathtt{A}}(\mathtt{A}) = T$, $F_{\mathtt{A}}(\mathtt{C}) = F_{\mathtt{A}}(\mathtt{G}) = F_{\mathtt{A}}(\mathtt{U}) = F$

Recursive step: If $s \in S$ and $b \in B$, then $F_{\mathtt{A}}(sb) = F_{\mathtt{A}}(s)$

$P_{\mathtt{AUC}}$ is defined as the predicate whose truth set is the collection of RNA strands where the string AUC is a substring (appears inside s, in order and consecutively)

A true universal quantification is:

$\forall s\ H(s)$

A false universal quantification is:

$\forall s\ L_3(s)$, $\sim \forall s\ H(s)$

A true existential quantification is:

$\exists s\ F_A(s)$

A false existential quantification is:

$\exists s\ P_{AUC}(s) \wedge \sim P_{AUC}(s)$

Tuesday’s new learning goals

- Prove identities related to sets using proof strategies learned so far.
- Use predicates with set of tuples as their domain to relate values to one another
- Evaluate nested quantifiers: both alternating and not.
- Use logical equivalence to rewrite quantified statements (including negated quantified statements)
- Counterexample and witness-based arguments for nested quantified statements with finite and infinite domains

Definitions related to sets

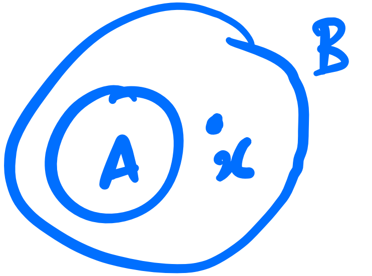
A = B means A is **equal** to B

Formally: $\forall x \; x \in A \longleftrightarrow x \in B$

$A \subseteq B$ means A is a **subset** of B, aka B is a **superset** of A

Formally:

$\forall x \left(x \in A \longrightarrow x \in B \right)$



$A \subsetneq B$ means A is a **proper subset** of B, aka B is a **proper superset** of A

Formally:

$A \subseteq B \wedge A \neq B$

$\boxed{\forall x \left(x \in A \rightarrow x \in B \right)} \wedge \boxed{\cancel{\neg \forall x \; x \in B \rightarrow x \in A}}$

$\exists x \; x \notin A \wedge x \in B$

subset	When A and B are sets, $A \subseteq B$ means $\forall x(x \in A \to x \in B)$	
proper subset	When A and B are sets, $A \subsetneq B$ means $(A \subseteq B) \wedge (A \neq B)$	
empty set	The set that has no elements	$\{\}$, $\emptyset$

ϕ, $\underbrace{\{\ \}}$ no elements

$A = \{\phi, 1, 2, 12\}$

↑ empty set

$B = \{1, 2, 12\}$

Which of the following is true?

A. $\{\mathtt{A}, \mathtt{C}, \mathtt{U}, \mathtt{G}\} \subseteq \{\mathtt{AA}, \mathtt{AC}, \mathtt{AU}, \mathtt{AG}\}$ F

B. The empty set is a proper subset of every set. F $\{\}$ or ϕ, $\cancel{B}$

C. For some set B, $\emptyset \in B$. T $\{\phi\}$, A

D. None of the above.

$\{\phi\} \longrightarrow$ a set that contains an empty set

$\equiv \{\phi\} \rightarrow$ Not a counter example

The empty set is a proper subset of every set. F

→ only counter example is $\{\}$

The empty set is a subset of every set is T:

→ $\forall X \ \phi \subseteq X$

$$\forall X \ \forall e \ \underbrace{\overbrace{e \in \phi \longrightarrow e \in X}^{P(X,e)}}_{\text{True statement}}$$

since

$$e \qquad \underbrace{\overset{F}{e \in \phi} \longrightarrow \overset{T \text{ or } F}{e \in X}}_{T}$$

always T

$P(X,e)$ is T $\forall X$ & $\forall e$ $\quad \therefore$ The claim is T

Note: $\forall X \ \phi \in X$ not true $\qquad \{\{\}\}$

Prove or **disprove** the following claims:

Claim: $\underbrace{\{\texttt{A}, \texttt{C}, \texttt{U}, \texttt{G}\}}_{X} \subseteq \underbrace{\{\texttt{AA}, \texttt{AC}, \texttt{AU}, \texttt{AG}\}}_{Y}$

$\forall x\,(x \in X \rightarrow x \in Y)$

Want to disprove

Counter example

$x = A$ then $\underset{T}{x \in X}$ but $x \notin Y$

so $\underset{F}{x \in Y}$

$\underbrace{T \rightarrow F}_{F}$

Prove or **disprove** the following claims:

Claim: The empty set is a proper subset of every set.	Claim: For some set B, $\emptyset \in B$.
$\forall X\ (\phi \subseteq X \wedge \phi \neq X)$ false choose a counter example ↓ also a counter ex $X = \{\} \to \phi$ $\quad B = \{1, 12, 2\}$	$\exists X\ \phi \in X$ Prove provide a witness $A = \{\phi, 1, 12\}$ $\quad \{\phi\}$

$\phi \subseteq X \to F$

$\therefore \underbrace{\underset{F}{\phi \subseteq X} \wedge \phi \neq X}_{F}$

Defining sets

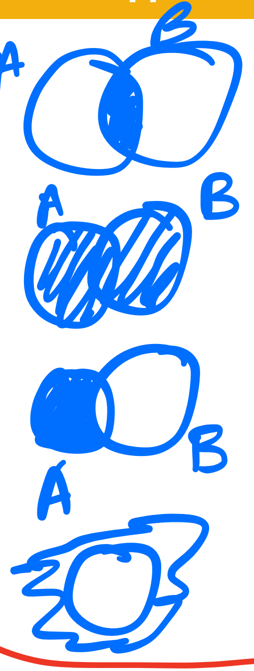
- Roster method, set builder, recursive
- **New** Applying operations to other sets

$A \cap B$

$A \cup B$

$A - B = A \setminus B$

$\bar{A}$



$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$

cardinality or size of set

$|A \times B| = |A| \cdot |B|$

B → A C U G

		A	C	U	G
8	1	(1,A)	(1,C)	(1,U)	(1,G)
	2	(2,A)	(2,C)	(2,U)	(2,G)

Definition (Rosen p. 123) Let A and B be sets. The **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

$B = \{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{U}\}$

Set	Example elements in this set:
Fill in possible set → $B \times B$	$(\mathtt{A}, \mathtt{C})$ $(\mathtt{U}, \mathtt{U})$
$B \times \{-1, 0, 1\}$	Fill in example elements → (A, -1) (A, 0) . . .
$\{-1, 0, 1\} \times B$	Fill in example elements → (-1, A) (0, A) . . .
Fill in possible set → $\{0\}^3 = \{0\} \times \{0\} \times \{0\}$	$(0, 0, 0)$
concatenation $\{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{U}\} \circ \{\mathtt{A}, \mathtt{C}, \mathtt{G}, \mathtt{U}\}$ = AA, AC, CG	Fill in example elements → AA, CG, UU . . .
Fill in possible set → $\cancel{B^4}$ or $\cancel{\{G\}^4}$ $B \circ B \circ B \circ B$ $\{G\} \circ \{GGG\}$	GGGG

Formal definition of a function

Domain A
Codomain B
Rule

Well Defined

$f: A \longrightarrow B$

A fn is a set which is a subset of $A \times B$
$A \to B$
such that for each element $a \in A$
exists exactly one element $b \in B$
s.t $(a, b) \in f$

$a_1 \to b_1$
$a_2 \to b_1$
$a_3 \to b_2$

$\{(a_1, b_1), (a_2, b_1), (a_3, b_2)\} \subseteq A \times B$

Definition (Rosen p. 123) Let A and B be sets. The **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Other important operations on sets

Term	Definition
Cartesian product	When A and B are sets, $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$
union	When A and B are sets, $A \cup B = \{x \mid x \in A \vee x \in B\}$
intersection	When A and B are sets, $A \cap B = \{x \mid x \in A \wedge x \in B\}$
set difference	When A and B are sets, $A - B = \{x \mid x \in A \wedge x \notin B\}$

Which of the following sets are equal?

$\{43, 9\} \times \{9, A\}$

#2 $\{43, 9\} \cup \{9, A\}$

#3 $\{43, 9\} \cap \{9, A\}$

#4 $\{43, 9\} - \{9, A\}$

A. #1, #2
B. #2, #3
C. #3, #4
D. #2, #3
E. None of the above

Cartesian Products and Predicates

Notation: for a predicate P with domain $X_1 \times \cdots \times X_n$ and a n-tuple $(x_1, \ldots, x_n)$ with each $x_i \in X$, we write $P(x_1, \ldots, x_n)$ to mean $P(\ (x_1, \ldots, x_n)\ )$.

$P: X_1 \times \ldots \times X_n \longrightarrow \{T, F\}$

$P(x_1, \ldots, x_n)$

Cartesian Products and Predicates

BC with domain $S \times B \times \mathbb{N}$ is defined by, for $s \in S$ and $b \in B$ and $n \in \mathbb{N}$,

num of
counts #bs in RNA S

T if s contains n instances of b ←

$$BC(s,b,n) = \begin{cases} T & \text{if } basecount(s,b) = n \\ F & \text{otherwise} \end{cases}$$

or b occurs n times s → string

b ∈ B single letter

Which of these is a witness that proves that $\exists t\ BC(t)$ is true?

$t \in S \times B \times \mathbb{N}$

A. G ✗
B. (GA, 2) ✗
C. (GG, C, 0) T   C occurs 0 times in GG
D. None of the above, but something else works.
E. None of the above, because the statement is false.

The **existential quantification** of $P(x)$ is the statement "There exists an element x in the domain such that $P(x)$" and is written $\exists x P(x)$. An element for which $P(x) = T$ is called a **witness** of $\exists x P(x)$.

Cartesian Products and Predicates

BC with domain $S \times B \times \mathbb{N}$ is defined by, for $s \in S$ and $b \in B$ and $n \in \mathbb{N}$,

$$BC(s,b,n) = \begin{cases} T & \text{if } basecount(s,b) = n \\ F & \text{otherwise} \end{cases}$$

Which of these is a counterexample that proves that $\forall (s,b,n)\ (\ BC(s,b,n)\ )$ is false?

A. (G, A, 1) ✓
B. (GC, A, 3) ✓
C. (GG, G, 2) ✗
D. None of the above, but something else works.
E. None of the above, because the statement is true.

The **universal quantification** of $P(x)$ is the statement "$P(x)$ for all values of x in the domain" and is written $\forall x P(x)$. An element for which $P(x) = F$ is called a **counterexample** of $\forall x P(x)$.

Predicate	Domain	Example domain element where predicate is T
$basecount(s, b) = 3$	$S \times B$	(AAA, A) (AAAB, A)
$basecount(s, \texttt{A}) = \underline{n}$	$S \times \mathbb{N}$	(AAA, 3)
$\exists n \in \mathbb{N}\ (basecount(\underline{s}, \underline{b}) = n)$ free variables	$S \times B$	(A, A) (AG, G) s, b n to be num of occurences of b in s
$\forall b \in B\ (basecount(s, b) = 1)$	S	

$B = \{A, C, U, G\}$

Truth Set $\longrightarrow$ ACGU, all permutations of ACGU

Translate to Nested Quantifiers

For **each** strand, there is **some** number such that the strand has that number of A's.

True

$\forall s\, \exists n \;\; \text{basecount}(s, A) = n$

Is this statement true?

select n to num of occurrences of A in s

Proof: let $s \in S$,

choose n_s = num of occurrences of A in s

then $\text{basecount}(s, A) = n_s$

T

$\therefore$ the statement is true $\forall s \in S$.

There exists two unique strands that have the same number of A's

$$\exists s_1 \exists s_2 \neq s_1 \quad \underbrace{basecount(s_1, A) = basecount(s_2, A)}$$

$$\exists s_1 \exists s_2 \ (s_1 \neq s_2) \wedge \quad "$$

$$P(s, b, n) = (basecount(s, b) = n)$$

$$\top \ \forall s \exists n \ P(s, A, n) \Longleftrightarrow \forall s \in S \ \exists n \in \mathbb{N} \ P(s, A, n)$$

$$\forall n \exists s \ P(s, A, n)$$

herefore

Translate to Nested Quantifiers

It is false
How to prove?

for each n you show a counter example
for $\forall s\, P(s, U, n)$.

$\exists n\, \forall s\; P(s, U, n)$

Is it T? NO

For **each** strand, there is **some** number such that the strand has that number of A's.

There is **some** number such that all strands have that number of U's.

Challenge:
"There are (at least) two different strands that have the same number of As"

Nested Quantifiers

$$\forall s \exists n \; BC(s, \mathtt{A}, n)$$

For **each** strand, there is some number such that the strand has that number of A's.

Nested Quantifiers

$$\forall s \exists n \ BC(s, \mathtt{A}, n)$$

For **each** strand, there is some number such that the strand has that number of A’s.

$s = \mathtt{A}$	$s = \mathtt{U}$	$s = \mathtt{G}$	$\cdots$	$s = \mathtt{UGC}$	$\cdots$
$BC(\mathtt{A}, \mathtt{A}, n)$	$BC(\mathtt{U}, \mathtt{A}, n)$	$BC(\mathtt{G}, \mathtt{A}, n)$	$\cdots$	$BC(\mathtt{UGC}, \mathtt{A}, n)$	$\cdots$
$(\mathtt{A}, \mathtt{A}, 0)$	$(\mathtt{U}, \mathtt{A}, 0)$	$(\mathtt{G}, \mathtt{A}, 0)$		$(\mathtt{UGC}, \mathtt{A}, 0)$	
$(\mathtt{A}, \mathtt{A}, 1)$	$(\mathtt{U}, \mathtt{A}, 1)$	$(\mathtt{G}, \mathtt{A}, 1)$	$\cdots$	$(\mathtt{UGC}, \mathtt{A}, 1)$	$\cdots$
$(\mathtt{A}, \mathtt{A}, 2)$	$(\mathtt{U}, \mathtt{A}, 2)$	$(\mathtt{G}, \mathtt{A}, 2)$		$(\mathtt{UGC}, \mathtt{A}, 2)$	
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	

$$(\exists n \ BC(\mathtt{A}, \mathtt{A}, n)) \land (\exists n \ BC(\mathtt{C}, \mathtt{A}, n)) \land \exists n \ BC(\mathtt{G}, \mathtt{A}, n) \land \cdots \land (\exists n \ BC(\mathtt{UGC}, \mathtt{A}, n)) \land \cdots$$

Nested Quantifiers

$$\exists n \forall s \; BC(s, \texttt{U}, n)$$

There is **some** number such that all strands have that number of U's.

Nested Quantifiers

$$\exists n \forall s \; BC(s, \mathtt{U}, n)$$

There is some number such that all strands have that number of U's.

$n=0$ $BC(s,\mathtt{U},0)$		$n=1$ $BC(s,\mathtt{U},1)$		$n=2$ $BC(s,\mathtt{U},2)$		$\cdots$ $\cdots$
$(\mathtt{A},\mathtt{U},0)$	T	$(\mathtt{A},\mathtt{U},1)$	F	$(\mathtt{A},\mathtt{U},2)$	F	
$(\mathtt{U},\mathtt{U},0)$	F	$(\mathtt{U},\mathtt{U},1)$	T	$(\mathtt{U},\mathtt{U},2)$	F	
$(\mathtt{C},\mathtt{U},0)$	T	$(\mathtt{G},\mathtt{U},1)$	F	$(\mathtt{G},\mathtt{U},2)$	F	
$(\mathtt{G},\mathtt{U},0)$	T	$(\mathtt{C},\mathtt{U},1)$	F	$(\mathtt{C},\mathtt{U},2)$	F	
$(\mathtt{AA},\mathtt{U},0)$	T	$(\mathtt{AA},\mathtt{U},1)$	F	$(\mathtt{AA},\mathtt{U},2)$	F	
$(\mathtt{AU},\mathtt{U},0)$	F	$(\mathtt{AU},\mathtt{U},1)$	T	$(\mathtt{AU},\mathtt{U},2)$	F	
$(\mathtt{AG},\mathtt{U},0)$	T	$(\mathtt{AG},\mathtt{U},1)$	F	$(\mathtt{AG},\mathtt{U},2)$	F	
$\vdots$		$\vdots$		$\vdots$		
$(\mathtt{UCU},\mathtt{U},0)$	F	$(\mathtt{UCU},\mathtt{U},1)$	F	$(\mathtt{UCU},\mathtt{U},2)$	T	
$\vdots$		$\vdots$		$\vdots$		

$$(\forall s \; BC(s,\mathtt{U},0)) \vee (\forall s \; BC(s,\mathtt{U},1)) \vee (\forall s \; BC(s,\mathtt{U},2)) \vee \cdots$$

Evaluate each quantified statement as T or F.

$\forall s\ \forall b\ \exists n\ BC(s,b,n)$	$\forall s\ \forall n\ \exists b\ BC(s,b,n)$	$\forall b\ \forall n\ \exists s\ BC(s,b,n)$
$\exists s\ \forall b\ \exists n\ BC(s,b,n)$	$\forall s\ \exists n\ \forall b\ BC(s,b,n)$	$\exists b\ \exists n\ \forall s\ BC(s,b,n)$

Extra example: Write the negation of each of the statements above, and use De Morgan's law to find a logically equivalent version where the negation is applied only to the BC predicate (not next to a quantifier).

Evidence for nested quantifiers

More generally, for predicate P with domain $X \times Y$: The statement

$$\forall x \exists y P(x, y)$$

means every table for an assignment of x ($\forall x$) must have at least one witness row ($\exists y$). One subtable can serve as a counterexample to give evidence that this statement is false. On the other hand, the statement

$$\exists y \forall x P(x, y)$$

means some table for an assignment of y ($\exists y$) must have every row T ($\forall x$). One subtable can serve as a witness to give evidence that this statement is true.

What about $\forall x \forall y P(x, y)$, $\forall y \forall x P(x, y)$, $\exists x \exists y P(x, y)$, $\exists y \exists x P(x, y)$?

Summary

- Cartesian products describe sets as combinations of elements from sets
- Predicates with sets of tuples as their domain can relate values to one another
- When quantifiers are *nested*, the order matters. We read left to right.
- When quantifiers are *nested*, we can visualize and interpret them in several ways
  - As nested tables, one for each value in the outermost quantification
  - As a conjunction or disjunction of other quantified statements

Learning goals contd…

- Translate arguments in English to forms in logic
- Identify the evidence needed for valid mathematical arguments
- Use rules of inference (in propositional and predicate logic) to construct valid arguments
- Distinguish between valid and invalid arguments

Evaluate Nested Quantifiers over Finite Domains

Read each of these and think about what you need to look for in the table to
(i) Prove the claim true
(ii) Prove the claim false

A: $\forall x \, \forall y \, P(x,y)$

B: $\exists x \, \exists y \, P(x,y)$

C: $\exists x \, \forall y \, P(x,y)$

D: $\forall x \, \exists y \, P(x,y)$

y →

x ↓

P	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

(i) Try to prove claim is T
→ each entry of the table is T

(ii) Try to prove is F
→ show one counter example (x,y) s.t. $P(x,y)=F$

(i) Prove T
→ show one witness

. . .

Evaluate Nested Quantifiers over Finite Domains

Which of these quantified statements is True?

A: $\exists x \, \forall y \, P(x,y)$

B: $\exists x \, \forall y \, Q(x,y)$

C: $\exists x \, \exists y \, S(x,y)$

D: More than one of the above

E: None of the above

y

x

P	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

Q	1	2	3
1	F	F	F
2	T	T	T
3	T	F	F

S	1	2	3
1	F	F	F
2	F	F	F
3	F	F	F

Arguments

An **argument** is a sequence of propositions, called **hypotheses**, followed by a final proposition, called the **conclusion**. An argument is valid if the conclusion is true whenever the hypotheses are all true, otherwise the argument is invalid.

Example 1: Prove that the argument below is **valid**

$$\begin{array}{c} p \to q \\ p \vee q \\ \hline \therefore q \end{array}$$

The hypotheses are $p \to q$, $p \vee q$ The conclusion is q

Therefore

$$\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ \hline \therefore C \end{array} \longrightarrow P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \to C$$

$$(p \to q) \wedge (p \vee q) \to q$$

Tautology (always T)

Show that an argument is valid

An **argument** is a sequence of propositions, called **hypotheses**, followed by a final proposition, called the **conclusion**. An argument is valid if the conclusion is true whenever the hypotheses are all true, otherwise the argument is invalid.

Example 1: Prove that the argument below is **valid**

$$\begin{array}{c} p \to q \\ \underline{p \vee q} \\ \therefore q \end{array}$$

The hypotheses are $p \to q$ $p \vee q$ The conclusion is q

We need to show that $((p \to q) \wedge (p \vee q)) \to q$ is a Tautology (Need to show this)?

$$\left.\begin{array}{l} (p \to q) \\ p \end{array}\right\}$$

$(\sim p \vee q) \wedge p$

$(\sim p \wedge p) \vee (q \wedge p)$ (distri)

$F \vee q \wedge p$ (Identity) — must be T

$q \wedge p$ — both q & p must be T

q

$(p \to q) \wedge (p \vee q)$

$(\sim p \vee q) \wedge (p \vee q) \to$ dis

$q \vee (\sim p \wedge p) \to$ complement

$q \vee F \to$ Identity

q

$$\left.\begin{array}{c} p \to q \\ \underline{p} \\ \therefore q \end{array}\right\} \text{Modus Ponens}$$

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

$$(p \to q) \wedge p \equiv q$$

Not this

statement is trying to say this

$$(p \to q) \wedge p \to q$$

q

Show that an argument is invalid

Example 2: Show that the argument below is **invalid**

↳ show that it is invalid

$$\begin{array}{l} \neg p \\ p \to q \\ \hline \therefore \neg q \end{array}$$

An assignment of p and q that make the hypotheses true but the conclusion false is ________________

$$\sim p \wedge (p \to q) \longrightarrow \sim q$$

$$\sim F \qquad \underbrace{F \to T}_{T} \qquad \sim T$$

$$\equiv T \qquad\qquad\qquad F$$

$p = F \qquad q = T$

$$\begin{array}{l} p \to q \\ \underline{p} \\ \hline \therefore q \end{array}$$ Modus Ponens

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$ simplification

$$\begin{array}{l} p \to q \\ \underline{\sim q} \\ \hline \therefore \sim p \end{array}$$ Modus Tollens

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$ Chain rule

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$ conjunction

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$ Disjunctive syllogism

$$\begin{array}{l} p \\ \hline \therefore p \end{array}$$ Addition

$$\begin{array}{l} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array}$$ Resolution

syllogism: 2 statements ly a conclusion

Constructing arguments

Use rules of inference to show that the hypotheses imply the conclusion "It rained." c

$\sim r \quad \vee \quad \sim f$

"If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded"

s l

r : It rained
f : It is foggy
s : sailing race will be held
l : lifesaving demo is held
t : trophy will be awarded

(i) Identify propositions

(2) Identify premises & conclusion : Arg

$(\sim r \vee \sim f) \rightarrow (s \wedge l)$

$s \rightarrow t$

$\sim t$

$\overline{\quad r \quad}$

T

add things to be true

(3) Prove argument

$s \rightarrow t$	Premise
$\sim t$	Premise
$\sim s$	Modus Tollens
$\sim s \vee \sim l$	addition
$\sim(s \wedge l)$	Demorgans
$(\sim r \vee \sim f) \rightarrow s \wedge l$	Premise
$\sim(\sim r \vee \sim f)$	Modus Tollens
$r \wedge f$	Demorgans
$\therefore r$	simplification

Identify valid arguments

Is the following argument valid?
Domain: All living things

All humans are mortal.
Socrates is human.
Therefore, Socrates is mortal.

A. Yes
B. No
C. Seems valid, but I don't know how to justify it.

$H(x)$: x is human

$M(x)$: x is Mortals

$\forall x\ H(x) \rightarrow M(x)$

universal instantiation

true for all living things

T $H(\text{Socrates})$

T $H(\text{Socrates}) \rightarrow M(\text{Socrates})$

$M(\text{Socrates})$ (MP)

Table 3.6.1: Rules of inference for quantified statements.

Rule of Inference	Name	Example
c is an element (arbitrary or particular) $\underline{\forall x\, P(x)}$ $\therefore P(c)$	Universal instantiation	Sam is a student in the class. Every student in the class completed the assignment. Therefore, Sam completed his assignment.
c is an arbitrary element $\underline{P(c)\quad}$ $\therefore \forall x\, P(x)$	Universal generalization	Let c be an arbitrary integer. $c \leq c^2$ Therefore, every integer is less than or equal to its square.
$\underline{\exists x\, P(x)}$ $\therefore$ (c is a particular element) $\land$ P(c)	Existential instantiation*	There is an integer that is equal to its square. Therefore, $c^2 = c$, for some integer c.
c is an element (arbitrary or particular) $\underline{P(c)\quad}$ $\therefore \exists x\, P(x)$	Existential generalization	Sam is a particular student in the class. Sam completed the assignment. Therefore, there is a student in the class who completed the assignment.

Correct use of Universal Generalization

PARTICIPATION ACTIVITY 3.6.3: Correct and incorrect use of generalization and instantiation.

Indicate whether the proof fragment is a correct or incorrect use of the rule of inference.

1)

1.	c is an element	Hypothesis
2.	P(c)	Hypothesis
3.	∀x P(x)	Universal generalization, 1, 2

c is not an arbitrary element.

- ○ Correct
- ○ Incorrect ✓

2)

1.	c is an element	Hypothesis
2.	∀x P(x)	Hypothesis
3.	P(c)	Universal instantiation, 1, 2

- ○ Correct ✓
- ○ Incorrect

Is the argument correct?

3)

1.	c is an element	Hypothesis
2.	P(c)	Hypothesis
3.	d is an element	Hypothesis
4.	Q(d)	Hypothesis
5.	P(c) ∧ Q(d)	Conjunction, 2, 4
6.	∃x (P(x) ∧ Q(x))	Existential generalization, 1, 3, 5

○ Correct

○ Incorrect

c & d can be different elements

2 P(c) : c is even

1 Q(d) : d is odd

$P(2) \wedge Q(1) \not\Rightarrow \exists x(P(x) \wedge Q(x))$

Table 3.6.2: Incorrect use of existential instantiation leading to an erroneous proof of an invalid argument.

$$\begin{array}{l} \exists x\, P(x) \\ \exists x\, Q(x) \\ \hline \therefore \exists x\, (P(x) \land Q(x)) \end{array}$$

1.	$\exists x$ P(x)	Hypothesis
2.	(c is a particular element) $\land$ P(c)	Existential instantiation, 1
3.	P(c) $\land$ (c is a particular element)	Commutative law, 2
4.	$\exists x$ Q(x)	Hypothesis
5.	(c is a particular element) $\land$ Q(c)	Existential instantiation, 4
6.	Q(c) $\land$ (c is a particular element)	Commutative law, 5
7.	P(c)	Simplification, 3
8.	Q(c)	Simplification, 6
9.	P(c) $\land$ Q(c)	Conjunction, 7, 8
10.	c is a particular element	Simplification, 2
11.	$\exists x$ (P(x) $\land$ Q(x))	Existential generalization, 9, 10

saying the existential instantiation is same for both P & Q

$P : D \longrightarrow \mathbb{N}$

$c \in D \land P(c)$

1) The argument below is invalid. Suppose that the domain of x is the set {c, d}. Select the table that proves the argument is invalid.

$$\frac{\forall x\, P(x) \lor \forall x\, Q(x)}{\exists x\, (P(x) \land Q(x))}$$

	P	Q
c	T	F
d	T	F

$\forall x\, P(x)$ ✓ Premise true

$\forall x\, P(x) \lor \forall x\, Q(x)$ addition

$\cancel{\exists x\, P(x) \land Q(x)}$ ✗

	P	Q
c	T	F
d	F	T

	P	Q
c	T	F
d	T	T

PARTICIPATION ACTIVITY 3.6.9: Showing an argument with quantified statements is invalid: integer domain.

The following argument is invalid:

$$\begin{array}{l} \exists x\, P(x) \\ \exists x\, Q(x) \\ \hline \therefore \exists x\, (P(x) \land Q(x)) \end{array}$$

Which definitions for predicates P and Q show that the argument is invalid? In each question the domain of x is the set of positive integers.

1) Suppose that the predicates P and Q are defined as follows:

P(x): x is prime
Q(x): x is even

Do the definitions for P and Q show that the argument is invalid?

- ◯ Yes
- ◯ No

PARTICIPATION ACTIVITY 3.6.9: Showing an argument with quantified statements is invalid: integer domain.

The following argument is invalid:

$$\begin{array}{l} \exists x\, P(x) \\ \exists x\, Q(x) \\ \hline \therefore \exists x\, (P(x) \land Q(x)) \end{array}$$

Which definitions for predicates P and Q show that the argument is invalid? In each question the domain of x is the set of positive integers.

2) Suppose that the predicates P and Q are defined as follows:

P(x): x is prime
Q(x): x is multiple of 4

Do the definitions for P and Q show that the argument is invalid?

- Yes
- No