HW2: Basic Data Types and Predicate Logic

CS40 Summer'24

Due: Monday, July 8, 2024 at 11:59PM on Gradescope

Collaborative HW assignment:

Collaborative homeworks may be done individually or in **groups of up to 2 students**. You may switch HW partners for different HW assignments. Please ensure your name(s) and PID(s) are clearly visible on the first page of your homework submission.

Integrity reminders

- Problems should be solved together, not divided up between the partners. The homework is designed to give you practice with the main concepts and techniques of the course, while getting to know and learn from your classmates.
- You may not collaborate on homework with anyone other than your group members. You may ask questions about the homework in office hours (of the instructor, TAs, and/or tutors) and on Piazza. You *cannot* use any online resources about the course content other than the text book and class material from this quarter – this is primarily to ensure that we all use consistent notation and definitions we will use this quarter.
- Do not share written solutions or partial solutions for homework with other students in the class who are not in your group. Doing so would dilute their learning experience and detract from their success in the class.

You will submit this assignment via Gradescope (https://www.gradescope.com) in the assignment called "HW2".

Summary of Proof Strategies (so far)

In your proofs and disproofs of statements below, justify each step by reference to a component of the following proof strategies we have discussed so far, and/or to relevant definitions and calculations.

- A counterexample can be used to prove that $\forall x P(x)$ is **false**.
- A witness can be used to prove that $\exists x P(x)$ is **true**.
- **Proof of universal by exhaustion**: To prove that $\forall x\, P(x)$ is true when P has a finite domain, evaluate the predicate at **each** domain element to confirm that it is always T.
- **Proof by universal generalization**: To prove that $\forall x\, P(x)$ is true, we can take an arbitrary element e from the domain and show that $P(e)$ is true, without making any assumptions about e other than that it comes from the domain.
- To prove that $\exists x P(x)$ is **false**, write the universal statement that is logically equivalent to its negation and then prove it true using universal generalization.
- **Proof of Conditional by Direct Proof**: To prove that the implication $p \to q$ is true, we can assume p is true and use that assumption to show q is true.

Assigned Questions

1. Define the following predicates. $O(x, y)$: x is older than y. Define S_t to be the set of students in the third grade at a school and S_f to be the set of students in the fourth grade at a school.

   Translate each English sentence into a quantified logical expression. Define the domain of each variable inline.

   (a) Sam is older than every student in the third grade.

   (b) There is a third grader who is older than at least one fourth grader.

2. Consider the following predicates, each of which has its domain as the set of all integers

   $E(x)$ is T exactly when x is even, and is F otherwise

   $T(x)$ is T exactly when x is a multiple of 3, and is F otherwise

   $M(x)$ is T exactly when x is a multiple of 4, and is F otherwise

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   Sample response that can be used as reference for the detail expected in your answer: To prove that the statement

   $$\forall x \ E(x)$$

   is false, we can use the counterexample $x = 7$, which is not an even number (its odd). So the universal statement is false.

   (a) (*Graded for correctness*) Use a counterexample to prove that the statement

   $$\forall x(\ E(x) \to M(x) \ )$$

   is false.

   (b) (*Graded for correctness*) Use a witness to prove that the statement

   $$\exists x(\ T(x) \land M(x) \ )$$

   is true.

   (c) (*Graded for fair effort completeness*) Translate each of the statements in the previous two parts to English.

3. Imagine a friend suggests the following argument to you because they believe universal quantifier distributes over disjunction: The statement

   $$\forall x(\ P(x) \lor Q(x) \ )$$

   is logically equivalent to

   $$(\forall x P(x)) \lor (\forall x Q(x))$$

(*Graded for correctness*[1]) Prove to your friend that they made a mistake by selecting a specific definition for $P(x)$ and $Q(x)$ and a domain of discourse with finite number of elements. Then show one statement is true while the other statement is false under this set of definitions. (Hint: You may use any of the predicates defined in the previous question). Include enough intermediate steps so that a student in CS 40 who may be struggling with the material can still follow along with your reasoning.

4. For each of these arguments, identify what rule of inference is used.[2]

   (a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

   (b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.

   (c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

   (d) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

5. Is the following argument valid or invalid? Prove your answer by replacing each proposition with a variable to obtain the form of the argument. Then prove that the form is valid or invalid using truth tables.

   I will buy a new stereo system and new sunglasses only if I get a promotion.
   I am not going to get promoted.
   I will buy new sunglasses.
   Therefore, I will not buy a new stereo system.

6. The domain for variables x and y is a group of people. The predicate $F(x, y)$ is true if and only if x is a friend of y. For the purposes of this problem, assume that for any person x and person y, either x is a friend of y or x is an enemy of y. Therefore, $\neg F(x, y)$ means that x is an enemy of y.

   Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until the negation operation applies directly to the predicate and then translate the logical expression back into English.

   (a) Everyone has an enemy.
   (b) Everyone is their own friend.
   (c) At least two different people are friends.
   (d) "The enemy of my enemy is my friend"[3].

7. Assuming that the domains of all quantifiers are the same, use rules of inference to show that

$$\text{if } \forall x(P(x) \vee Q(x)) \wedge \forall x((\neg P(x) \wedge Q(x)) \to R(x)),$$
$$\text{then } \forall x(\neg R(x) \to P(x)).$$

[1]This means your solution will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

[2]See zyBooks 3.5 for list of rules of inference

[3]For all people in the domain, enemies of their enemies are their friends

8. Consider the predicate $F(a, b) =$ "a is a factor of b" over the domain $\mathbb{Z}^{\neq 0} \times \mathbb{Z}$ that was introduced in lecture. Consider the following quantified statements

(i) $\forall x \in \mathbb{Z}\ (F(1, x))$

(ii) $\forall x \in \mathbb{Z}^{\neq 0}\ (F(x, 1))$

(iii) $\exists x \in \mathbb{Z}\ (F(1, x))$

(iv) $\forall x \in \mathbb{Z}^{\neq 0}(\neg F(x, 1))$

(v) $\forall x \in \mathbb{Z}^{\neq 0}\ \exists y \in \mathbb{Z}\ (F(x, y))$

(vi) $\exists x \in \mathbb{Z}^{\neq 0}\ \forall y \in \mathbb{Z}\ (F(x, y))$

(vii) $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}^{\neq 0}\ (F(x, y))$

(viii) $\exists y \in \mathbb{Z}\ \forall x \in \mathbb{Z}^{\neq 0}\ (F(x, y))$

(a) (*Graded for correctness of choice and fair effort completeness in justification*) Which of the statements (i) - (viii) is being **proved** by the following proof?

> By universal generalization, **choose** e to be an **arbitrary** integer. We need to show $\exists y \in \mathbb{Z}^{\neq 0}(F(y, e))$. By definition of the predicate F, we can rewrite this goal as $\exists y \in \mathbb{Z}^{\neq 0} \exists c \in \mathbb{Z}\ (e = c \cdot y)$. We pick the **witnesses** $y = 1$ and $c = e$. y is a non-zero integer and therefore in the domain. Similarly, c is an integer and therefore in the domain. Plugging the value of the witnesses y and c, we get $c \cdot y = e \cdot 1 = e$, as required. Since the predicate $\exists y \in \mathbb{Z}^{\neq 0}(F(y, e))$ evaluates to true for the arbitrary integer e, the claim has been proved.
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Hint: It may be useful to identify the keywords in the proof that indicate proof strategies.

(b) (*Graded for correctness of choice and fair effort completeness in justification*) Which of the statements (i) - (viii) is being **disproved** by the following proof?

> To disprove the statement, we need to find a counterexample. We choose -1, which is a nonzero integer so in the domain. We need to show $F(-1, 1)$. By definition of the predicate F, we can rewrite this goal as $\exists c \in \mathbb{Z}\ (1 = c \cdot -1)$. We pick the **witness** $c = -1$, which is an integer and therefore in the domain. Plugging the value of the witness c, we get $c \cdot -1 = -1 \cdot -1 = 1$, as required. So the counterexample works to disprove the original statement. ■

Hint: It may be useful to identify the key words in the proof that indicate proof strategies.

(c) (*Graded for correctness of evaluation of statement (is it true or false?) and fair effort completeness of the translation and proof*) Translate the following statement to English and then prove or disprove it:

$$\forall x \in \mathbb{Z}^{\neq 0}\ \forall y \in \mathbb{Z}^{\neq 0}\ (\ F(x, y) \to F(x, x + y)\ )$$

9. Let $P = \{a : a \in Z^+, a \leq 50\}$. We want to list all pairs (a, b) of integers in $P \times P$ such that a is a factor of b (which is the same as the predicate $F(a, b)$ that you have been working in the previous question).

The main constraint in this questions is that we want to build this list without directly listing all pairs (because it would be time consuming) and without ever evaluating the predicate $F(a, b)$ because it would involve performing a modulo operation. So, we have decided to start with an arbitrarily created initial set $K \subseteq P \times P$, defined as $K = \{(1, 2), (1, 4), (2, 4), (3, 9)\}$, knowing well that for all pairs of numbers in K, the first number is a factor of the second but that K does not contain all pairs of interest. Our goal is then to recursively build up the set $L \subseteq P \times P$ by:

- using some assumptions that we know to be true about integer pairs that are related by the property of one being the factor of the other, and

- only checking for membership of elements in K, P, and L but never directly evaluating the predicate $F(a, b)$.

As a starting guess, we decide to use the following assumptions:
(i) for all a, $b \in P$, if a is a factor of b then a is a factor of ab,
(ii) for all a, b, c in P such that $a \neq c$, $a \neq b$ and $b \neq c$, if a is a factor of b and b is a factor of c then a is a factor of c, and
(iii) for all a, b, c in P such that $a \neq c$, $a \neq b$ and $b \neq c$, if a is a factor of b and a is a factor of c then a is a factor of $b + c$

(a) Write a recursive definition for a set $L \subseteq P \times P$ that captures the following notion: $(a, b) \in L$ if and only if we can deduce from K and our assumptions (i), (ii), (iii) that a is a factor of b. Once again, in writing this definition, you are only allowed to check for membership of elements in the sets K, P, and L but never directly evaluating the predicate $F(a, b)$

(b) List five example elements in the set $\{(a, b) \in P \times P \mid F(a, b)\}$ that are each not in the set L as per your recursive definition in (9a). Write your answer in roster notation.

(c) Modify the recursive definition from (9a) so that that the resulting set contains all pairs of numbers in $P \times P$, where the first number is a factor of the second number. The same constraints as part (9a) apply here with the exception that in this question you may use a different set of assumptions than the ones provided as a starting guess. Write your solution in a way so that it generalizes well to any different choice of the sets P and K. Note that although non-recursive definitions are possible, you must provide a recursive definition to receive credit for this question.

(d) Critique your solution for part(c) by discussing whether it generalizes to any choice of the sets $P \subseteq \mathbb{Z}^+$ and $K \subseteq P \times P$, and if so under what assumptions.

Attributions

Some of the problems on this homework are based on questions originally created by Mia Minnes, Joe Politz, and Daniel Lokshtanov.