Quiz 2

● Graded

Student

Zixuan Chen

Total Points

17 / 17 pts

Question 1

(no title) **0** / 0 pts

+ 0 pts Incorrect

✔ **+ 0 pts** Correct

Question 2

(no title) **1** / 1 pt

+ 0 pts Incorrect

✔ **+ 1 pt** Correct

Question 3

(no title) **1** / 1 pt

+ 0 pts Incorrect

✔ **+ 1 pt** Correct

Question 4

(no title) **2** / 2 pts

✔ **+ 2 pts** Correct

+ 0 pts Incorrect

Question 5

(no title) **1** / 1 pt

+ 0 pts Incorrect

✔ **+ 1 pt** Correct

Question 6

(no title) **2** / 2 pts

6.1 (no title) **1** / 1 pt

+ **0 pts** Incorrect

✔ + **1 pt** Correct

6.2 (no title) **1** / 1 pt

+ **0 pts** Incorrect

✔ + **1 pt** Correct

Question 7

(no title) **4.5** / 4.5 pts

7.1 (no title) **1.5** / 1.5 pts

+ **0 pts** Incorrect

✔ + **1.5 pts** Correct

7.2 (no title) **1.5** / 1.5 pts

+ **0 pts** Incorrect

✔ + **1.5 pts** Correct

7.3 (no title) **1.5** / 1.5 pts

+ **0 pts** Incorrect

✔ + **1.5 pts** Correct

Question 8

(no title) **1** / 1 pt

+ **0 pts** Incorrect

✔ + **1 pt** Correct

Question 9

(no title) **1.5** / 1.5 pts

✔ + **1.5 pts** Correct

+ **0 pts** Incorrect

Question 10

(no title) **3** / 3 pts

10.1 (no title) **1.5** / 1.5 pts

+ **0 pts** Incorrect

✓ + **1.5 pts** Correct

10.2 (no title) **1.5** / 1.5 pts

+ **0 pts** Incorrect

✓ + **1.5 pts** Correct

Q1

0 Points

Solve the problems for this quiz individually – collaboration is **not** allowed. Course materials (textbook, your notes, materials on Gauchospace) are allowed during the exam. You are **not** allowed to use the internet for any other purpose than to access/submit the quiz, and access the course materials. No Googling!

Do not make public posts about the quiz on Piazza.

Any instances of academic dishonesty will be reported as specified in the course syllabus and will result in being downgraded by at least one letter grade in the course.

Do you take the pledge to complete the quiz with honesty?

- (●) Yes
- ( ) No

Q2

1 Point

Given $T = \{A, GAAU, CCAGUAA, AAUGCCCAA\}$ be the set of four RNA strands constructed from the basis elements $B = \{A, C, G, U\}$.

The function $basecount : T \times B \to \mathbb{N}$ computes the number of a given base $b \in B$ appearing in a RNA strand $s \in T$

The witnesses $s_1 = GAAU, s_2 = GAAU$ can be used to prove which of the following claims true

SELECT ALL THAT APPLY

- [x] $\exists s_1 \in T, \exists s_2 \in T(\ basecount(s_1, A) = basecount(s_2, A)\ )$
- [ ] $\exists s_1 \in T, \exists s_2 \in T(\ (s_1 \neq s_2) \wedge (basecount(s_1, A) = basecount(s_2, A))\ )$
- [ ] $\neg\forall s_1 \in T, \forall s_2 \in T(\ (s_1 = s_2) \vee (basecount(s_1, A) \neq basecount(s_2, A))\ )$
- [ ] $\exists s_1 \in T, \exists s_2 \in (T - \{s_1\})(basecount(s_1, A) = basecount(s_2, A)\ )$

Q3

1 Point

A non-zero integer x is a factor of y if there exists an integer k such that $y = kx$. Other ways of stating x is a factor of y are: x divides y or y is divisible by x.

A *prime* is a positive integer p that has no other factors other than 1 and p. Consider the following theorem (it is known as Fermat's little Theorem)

Theorem: *Let p be a prime and let a be a positive integer not divisible by p. Then p divides $a^{p-1} - 1$.*

Your task: Which of the following statements follow from an invocation of the theorem? Note that if a statement is true but does not follow from the theorem you should not select it. Select **all** that apply.

- [ ] $21^6 - 1$ is divisible by 7
- [x] $3^4 - 1$ is divisible by 5
- [ ] $2^5 - 1$ is divisible by 3
- [ ] $13^{12} - 1$ is divisible by 3
- [x] $12^{12} - 1$ is divisible by 13

Q4

2 Points

Recall the Netflix example from class: Consider a four movie database. We denote the set of possible ratings $\{-1, 0, 1\} \times \{-1, 0, 1\} \times \{-1, 0, 1\} \times \{-1, 0, 1\}$ as R_4. We have the function

$d_{1,4} : R_4 \times R_4 \to \mathbb{N}$, where

$$d_{1,4}(\ (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)\ ) = \max_{1 \leq i \leq 4} |x_i - y_i|$$

$d_{2,4} : R_4 \times R_4 \to \mathbb{N}$, where

$$d_{2,4}(\ (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)\ ) = \sqrt{\sum_{i=1}^{4} (x_i - y_i)^2}$$

Your task: Given the following proof, deduce what is being proved

Proof: Towards a proof by universal generalization, let e_1 be an arbitrary element of R_4 and let e_2 be an arbitrary element of R_4.

Towards a direct proof, we assume $e_1 = e_2$ and we need to prove that $\neg((d_{1,4}(e_1, e_2) < d_{2,4}(e_1, e_2))\ )$.

Calculating, $d_{1,4}(e_1, e_2) = d_{1,4}(e_1, e_1)$ by assumption that e_1 and e_2 are equal, and therefore (since each element in the sequence over which we are taking max in the definition of $d_{1,4}$ will be 0 because it is the absolute value of the difference of a number with itself), $d_{1,4}(e_1, e_2) = 0$. Similarly, $d_{2,4}(e_1, e_2) = d_{2,4}(e_1, e_1)$ by assumption, and therefore (since each term in the sum in the definition of $d_{2,4}$ will be 0 because it is the square of the difference of a number with itself), $d_{2,4}(e_1, e_2) = 0$. Thus, $d_{1,4}(e_1, e_2) = d_{2,4}(e_1, e_2)$ so $d_{1,4}(e_1, e_2) < d_{2,4}(e_1, e_2)$ is False as required.

This completes the proof.

Your task: Given the above proof, deduce what is being proved by selecting one of the following options:

- ◯ $\neg(\ \exists r_1 \in R_4\ \exists r_2 \in R_4\ (\ (r_1 = r_2) \to \neg\,(d_{1,4}(r_1, r_2) < d_{2,4}(r_1, r_2))\ )\ )$
- ◉ $\forall r_1 \in R_4\ \forall r_2 \in R_4\ (\ (r_1 = r_2) \to \neg\,(d_{1,4}(r_1, r_2) < d_{2,4}(r_1, r_2))\ )$
- ◯ $\forall r_1 \in R_4\ \forall r_2 \in R_4\ (\ (d_{1,4}(r_1, r_2) < d_{2,4}(r_1, r_2))\ \to (r_1 = r_2))$
- ◯ $\forall r_1 \in R_4\ \exists r_2 \in R_4\ (\ (r_1 = r_2) \to \neg\,(d_{1,4}(r_1, r_2) < d_{2,4}(r_1, r_2))\ )$

Q5

1 Point

Your task: Fill in the blank in the proof of the theorem below. Select the option that applies.

A non-zero integer x is a factor of y if there exists an integer k such that $y = kx$. A *prime* is a positive integer p that has no other factors other than 1 and p .

Theorem: *Let a, b, c, d be primes such that $a \leq b \leq c \leq d$. If $ab + cd$ is odd then $a = 2$.*

Proof: We proceed by ____ . Suppose $a \neq 2$, then $2 < a \leq b \leq c \leq d$ and therefore a, b, c, d are all odd (here we use without proof that 2 is the only even prime). Then there exist integers p, q, r, s such that $a = 2p + 1$, $b = 2q + 1$, $c = 2r + 1$ and $d = 2s + 1$. Then $ab + cd = (2p + 1)(2q + 1) + (2r + 1)(2s + 1)$, which simplifies to $2(2pq + 2rs + p + q + r + s + 1)$. Since $2pq + 2rs + p + q + r + s + 1$ is an integer we conclude that $ab + cd$ is even, and therefore not odd (here we use without proof that an even integer is not odd). This completes the proof.

- ◉ proof by contrapositive.
- ◯ a proof by contradiction.
- ◯ proof by cases.
- ◯ a direct proof of the implication.

Q6
2 Points

A robot on an infinite 2-dimensional integer grid starts at $(0, 0)$ and moves to one of three possible adjacent grids at every time step.
The **set of positions** the robot can visit P is defined by:

$$\begin{array}{ll} \text{Basis Step:} & (0,0) \in P \\ \text{Recursive Step:} & \text{If } (x,y) \in P, \text{ then} \\ & (x-1, y+1), (x, y-1), (x+1, y+1) \in P \end{array}$$

Q6.1
1 Point

Which of the following positions can the robot visit? SELECT ALL THAT APPLY

- [x] (0, 1)
- [x] (-1, 1)
- [x] (1, 1)
- [x] (-1, -1)

Q6.2
1 Point

We want to show that the robot can visit any position of the form $(a, a-1)$ for an arbitrary integer a.

Which of the following claims would be a helpful lemma to use in our proof? SELECT ALL THAT APPLY:

- [x] $\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}, (\ (x = y) \rightarrow (x, y) \in P\ )$
- [ ] $\forall (x, y) \in P, ((x + y) \text{ is even })$
- [ ] $\forall (x, y) \in P, ((x + y) \text{ is odd })$
- [ ] $\forall (x, y) \in P, (|x - y| \leq 1)$

Q7

4.5 Points

Assume that have the same robot set up as the previous question.
Answer the questions that follow about modeling the path traveled by the robot and the total euclidean distance traveled for a given path.

Here are some definitions we'll use throughout:

1. Definition of path: The path traveled by the robot is the sequence of points visited by it starting from the origin.

2. Definition of Euclidean distance: Recall the Euclidean distance between any two points in 2D grid is defined by the function $d_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$:

$$d_2(\ (x_1, y_1), (x_2, y_2)\ ) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

3. Definition of distance traveled over a path: The distance traveled by the robot over a path is the sum of the Euclidean distances between all consecutive points visited on that path.

Q7.1
1.5 Points

Defining the set of all possible robot paths
We will model the path as a linked-list where the most recently visited position is at the head of the list.

Version 1:

The set of all possible paths, L is recursively defined as:

$$\begin{array}{ll} \text{Basis Step:} & ((0,0),[]) \in L, [] \in L \\ \text{Recursive Step:} & \text{If } p \in P \text{ and } l \in L, \text{ then } (p,l) \in L \end{array}$$

Suppose
$a = (0,0), b = (-1,1), c = (0,-1), d = (1,1), e = (-1,0)$

According to the above model, which of the following are possible paths of the robot? SELECT ALL THAT APPLY

- [x] $(\ a, (\ a, (\ a, []) \ ) \ )$
- [x] $(\ c, (\ b, (\ a, []) \ ) \ )$
- [x] $(\ e, (\ b, (\ a, []) \ ) \ )$
- [x] $(\ c, (\ a, (\ b, []) \ ) \ )$
- [ ] $(\ a, b, c, d\ )$

Q7.2
1.5 Points

Defining the set of all possible robot paths
We will model the path as a linked-list where the most recently visited position is at the head of the list.

Version 2:

The set of all possible paths, L is recursively defined as:

$$\begin{array}{ll} \text{Basis Step:} & ((0,0),[]) \in L, [] \in L \\ \text{Recursive Step:} & \text{If } (x,y) \in P, l \in L \text{ and } ((x,y),l) \in L, \text{then} \\ & ((x-1,y+1),((x,y),l)) \in L, \\ & ((x,y-1),((x,y),l)) \in L, \\ & ((x+1,y+1),((x,y),l)) \in L \end{array}$$

Suppose
$a = (0,0), b = (-1,1), c = (0,-1), d = (1,1), e = (-1,0)$

According to the above model, which of the following are possible paths of the robot? SELECT ALL THAT APPLY

- [ ] $(\ a, (\ a, (\ a, []) \ ) \ )$
- [ ] $(\ a, b, c, d\ )$
- [ ] $(\ c, (\ b, (\ a, []) \ ) \ )$
- [x] $(\ e, (\ b, (\ a, []) \ ) \ )$
- [ ] $(\ c, (\ a, (\ b, []) \ ) \ )$

Q7.3
1.5 Points

Given: L is the set of all possible paths traveled by the robot starting at the origin.

Complete the recursive definition of the function $d : L \to \mathbb{R}$ that computes the total distance traveled by the robot over a given path by filling in the blank.

$$\begin{array}{ll} \text{Basis Step:} & \text{If } l = ((0,0),[]), d(l) = 0 \\ \text{Recursive Step:} & \text{If } p \in P, q \in P, \text{ and } l \in L, \quad \text{then ——}\textbf{Blank}\text{—} \end{array}$$

- ○ $d((p,q)) = d(l) + d_2(p,q)$
- ○ $d((p,(q,l))) = d(l) + d_2(p,q)$
- ◉ $d((p,(q,l))) = d((q,l)) + d_2(p,q)$
- ○ $d((p,q)) = d_2(p,q)$

Q8
1 Point

Select the expression that is equivalent to the following statement:
Among any two consecutive positive integers, there is at least one integer that is not prime.

- ○ If x and y are positive integers, then x is not prime or y is not prime.
- ○ If x is a positive integer, then x is not prime and x+1 is not prime.
- ○ If x and y are positive integers, then x is not prime and y is not prime.
- ◉ If x is a positive integer, then x is not prime or x+1 is not prime.

Q9

1.5 Points

Theorem: For any real number x , if $x^2 - 6x + 5 > 5$, then $x \geq 5$ or $x \leq 1$.

Which facts are assumed and which facts are proven in a **proof by contrapositive** of the theorem?

- ○ Assume: $(x \geq 5) \vee (x \leq 1)$, Show: $x^2 - 6x + 5 \leq 5$
- ○ Assume: $(x < 5) \vee (x > 1)$, Show: $x^2 - 6x + 5 \leq 5$
- ◉ Assume: $1 < x < 5$, Show: $x^2 - 6x + 5 \leq 5$
- ○ Assume: $(x \geq 5) \wedge (x \leq 1)$, Show: $x^2 - 6x + 5 \leq 5$

Q10
3 Points

Consider the following recursive definition of a subset X of $\mathbb{Z} \times \mathbb{Z}$.

(1) $(0, 0) \in X$
(2) If (i, j) in X then $(i + 1, j + 2i + 1) \in X$.

Theorem: For all $(i, j) \in X$ it holds that $j = i^2$.

Proof: For the base case we have that __ **Blank 1** __ . For the inductive step, suppose that $(i, j) \in X$ and that __ **Blank 2** __ . We then have that $j + 2i + 1 = i^2 + 2i + 1 = (i + 1)^2$. Hence, by structural induction, the statement of the theorem follows.

Q10.1
1.5 Points

Fill in **Blank 1**

- ○ $j^2 = i$ for all $j \leq k$.
- ◉ $0 = 0^2$.
- ○ $j = i^2$ for all $j \leq k$.
- ○ $1 = 1^2$.

Q10.2
1.5 Points

Fill in **Blank 2**

- ○ $j = i^2$ for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.
- ○ $j = i^2$ for all elements (i, j) of X that have been considered so far.
- ◉ $j = i^2$.
- ○ $j = i^2$ for all $(i, j) \in X$.