Solutions To HW2

Zixuan Chen
X313572
zixuanchen@ucsb.edu

June 2024

Solution To Question 1

(a) Let s represent Sam. The logical expression is:

$$\forall y\,(y \in S_t \to O(s,y))$$

(b) The logical expression is:

$$\exists x\,(x \in S_t \land \exists y\,(y \in S_f \land O(x,y)))$$

Solution To Question 2

(a) Use a counterexample to prove that the statement $\forall x\ (E(x) \to M(x))$ is false.

Proof: Let $x = 2$:

- $E(2)$ is true because 2 is even.
- $M(2)$ is false because 2 is not a multiple of 4.

Since $E(2)$ is true and $M(2)$ is false, $E(2) \to M(2)$ is false. So $\forall x\ (E(x) \to M(x))$ is false.

(b) Use a witness to prove that the statement $\exists x\ (T(x) \land M(x))$ is true.

Proof: Let $x = 12$:

- $T(12)$ is true because 12 is a multiple of 3.
- $M(12)$ is true because 12 is a multiple of 4.

Since both $T(12)$ and $M(12)$ are true, $\exists x\ (T(x) \land M(x))$ is true.

(c) Translate each of the statements in the previous two parts to English.

1. $\forall x\ (E(x) \to M(x))$**:**
    - **For all** integers x, if x is even, then x is a multiple of 4.

- **Counterexample**: We disproved this statement by showing that 2 is even but not a multiple of 4.

2. $\exists x\ (T(x) \wedge M(x))$**:**

   - **There exists** an integer x that is both a multiple of 3 and a multiple of 4.
   - **Witness**: We proved this statement by showing that 12 is both a multiple of 3 and a multiple of 4.

Solution To Question 3

Let's assume a school with **only two departments**: Computer Science (CS) and Electrical Engineering (EE).

Define the predicates:

- $P(x)$: x is a CS student.
- $Q(x)$: x is an EE student.

$$\forall x(P(x) \vee Q(x))$$
$$(\forall x P(x)) \vee (\forall x Q(x))$$

evaluate $\forall x(P(x) \vee Q(x))$:

$$\begin{array}{l} \forall x(P(x) \vee Q(x)) \\ \text{For every student } x, \text{ either } P(x) \text{ or } Q(x) \text{ is true since every student is either in CS or EE.} \\ \therefore \forall x(P(x) \vee Q(x)) = \text{True} \end{array}$$

evaluate $(\forall x P(x)) \vee (\forall x Q(x))$:

$$\begin{array}{l} \text{Assume there are two students } a \text{ and } b\text{:} \\ \text{a is a CS student.} \\ \text{b is an EE student.} \\ P(a) = \text{True} \\ Q(a) = \text{False} \\ P(b) = \text{False} \\ Q(b) = \text{True} \\ \therefore \forall x P(x) = \text{False since } b \text{ is not a CS student} \\ \therefore \forall x Q(x) = \text{False since } a \text{ is not an EE student} \\ (\forall x P(x)) \vee (\forall x Q(x)) = \text{False} \end{array}$$

Conclusion

Now we can see:

$$\forall x(P(x) \vee Q(x)) = \text{True}$$
$$(\forall x P(x)) \vee (\forall x Q(x)) = \text{False}$$

Thus, our friend's initial assertion that these two statements are logically equivalent is **incorrect**.

Solution To Question 4

(a) : Simplification

- P : Kangaroos live in Australia.
- Q : Kangaroos are marsupials.

$$\frac{P \land Q}{Q}$$

(b) : Disjunctive Syllogism

- P : It is hotter than 100 degrees today.
- Q : The pollution is dangerous.

$$\frac{\begin{array}{c}\neg P \\ P \lor Q\end{array}}{Q}$$

(c) : Modus Ponens

- P : Linda is an excellent swimmer.
- Q : Linda can work as a lifeguard.

$$\frac{\begin{array}{c}P \\ P \to Q\end{array}}{Q}$$

(d) : Hypothetical Syllogism

- P : I work all night on this homework.
- Q : I can answer all the exercises.
- R : I will understand the material.

$$\frac{\begin{array}{c}Q \to R \\ P \to Q\end{array}}{P \to R}$$

Solution To Question 5

Argument:

- I will buy a new stereo system and new sunglasses only if I get a promotion.
- I am not going to get promoted.
- I will buy new sunglasses.
- Therefore, I will not buy a new stereo system.

Propositions:

- P : I get a promotion.
- Q : I will buy a new stereo system.
- R : I will buy new sunglasses.

Logical Form:

- Premise 1: $(Q \wedge R) \rightarrow P$
- Premise 2: $\neg P$
- Premise 3: R
- Conclusion: $\neg Q$

Argument:

$$((Q \wedge R) \rightarrow P), \neg P, R \longrightarrow \neg Q$$

Truth Table:

P	Q	R	$(Q \wedge R) \rightarrow P$	$\neg P$	R	$\neg Q$
T	T	T	T	F	T	F
T	T	F	T	F	F	F
T	F	T	T	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	T

From the truth table, we can see there is no row where all the premises are true but the conclusion is false.

Conclusion: The argument is valid.

Solution To Question 6

(a)

$$\forall x\, \exists y\, \neg F(x,y)$$

$$\neg(\forall x\, \exists y\, \neg F(x,y)) \to \exists x\, \forall y\, F(x,y)$$

There exists someone who is a friend of everyone.

(b)

$$\forall x\, F(x,x)$$

$$\neg(\forall x\, F(x,x)) \to \exists x\, \neg F(x,x)$$

There exists someone who is not their own friend.

(c)

$$\exists x\, \exists y\, (x \neq y \land F(x,y))$$

$$\neg(\exists x\, \exists y\, (x \neq y \land F(x,y))) \to \forall x\, \forall y\, (x = y \lor \neg F(x,y))$$

For all people, either they are the same person or they are enemies.

(d)

$$\forall x\, \forall y\, \forall z\, (\neg F(x,y) \land \neg F(y,z) \to F(x,z))$$

$$\neg(\forall x\, \forall y\, \forall z\, (\neg F(x,y) \land \neg F(y,z) \to F(x,z))) \to \exists x\, \exists y\, \exists z\, (\neg F(x,y) \land \neg F(y,z) \land \neg F(x,z))$$

There exist at least three people such that one is an enemy of the second, the second is an enemy of the third, and the first is also an enemy of the third.

Solution To Question 7

Premises:

1. $\forall x(P(x) \lor Q(x))$ (1)
2. $\forall x((\neg P(x) \land Q(x)) \to R(x))$ (2)

To prove:

$$\forall x(\neg R(x) \to P(x))$$

Proof:

Let x be arbitrary.

1. From (1), instantiate the universal quantifier:
$$P(x) \lor Q(x)$$
(3)
2. From (2), instantiate the universal quantifier:
$$(\neg P(x) \land Q(x)) \to R(x)$$
(4)
3. Assume $\neg R(x)$ to prove $\neg R(x) \to P(x)$. (Assumption)
4. From (4), use the contrapositive:
$$\neg R(x) \to \neg(\neg P(x) \land Q(x))$$
(5)
5. Apply De Morgan's law on (5):
$$\neg R(x) \to (P(x) \lor \neg Q(x))$$
(6)
6. We have $P(x) \lor Q(x)$ from (3) and $P(x) \lor \neg Q(x)$ from (6).
7. Since $P(x) \lor Q(x)$ and $P(x) \lor \neg Q(x)$, we deduce $P(x)$. (7)

Since x was arbitrary, we have:

$$\forall x(\neg R(x) \to P(x))$$

Solution To Question 8

(a)

The proof shows that for any integer e, there exist integers $y = 1$ and $c = e$ such that $e = c \cdot y$. This corresponds to statement (vii):

$$\forall y \in \mathbb{Z} \exists x \in \mathbb{Z}^{\neq 0}(F(x, y))$$

Reason:

The proof begins by choosing e to be an arbitrary integer, indicating the statement should start with $\forall$.

Then it picks a non-zero number y, indicating that the statement include $\exists$ and the domain does not include 0.

Actually we can already know that the proof is for (vii) because only statement (vii) starts with $\forall$ and $\exists$ with the domain that not includes 0 , but I was confused by (vi) at first. Why not (vi)?

If we want to prove (vi), we will first find a witness y but not let e to be arbitrary. (But the witness $y = 1$ here is also a suitable witness for (vi).)

(b)

The proof uses a counterexample with $x = -1$ and $c = -1$ to show that $F(-1, 1)$ is true, disproving statement (iv):

$$\forall x \in \mathbb{Z}^{\neq 0}(\neg F(x, 1))$$

Reason:

"We choose -1, which is a nonzero integer so in the domain." indicates that the domain should not include 0.

"We need to show $F(-1, 1)$." means that there is only one variable in the statement so there should be something like $F(x, 1)$ in the statement.

Now, the only choices left are (ii) and (iv)

If we want to prove (ii), we need to check every x. However, if we want to prove (iv), we only need to find a counterexample x so that x is a factor of 1 and therefore in the domain. And that's what the given proof has done. So (iv) is proved by the proof.

(c)

Translation: For all non-zero integers x and y, if x is a factor of y, then x is also a factor of $x + y$.

Proof:

First, let x and y be arbitrary non-zero integers.

Given: $F(x, y)$

- This means x is a factor of y.
- By definition, there exists an integer k such that $y = k \cdot x$.

To Prove: $F(x, x + y)$

- We need to show that x is a factor of $x + y$.

Rewrite $x + y$ using the given information:

$$x + y = x + k \cdot x = x(1 + k)$$

- Here, $(1 + k)$ is an integer because k is an integer.

Conclusion:

- Since $x(1 + k)$ is in the form of $x \cdot m$ where m is an integer, x is indeed a factor of $x + y$.

Thus, the statement $\forall x \in \mathbb{Z}^{\neq 0} \forall y \in \mathbb{Z}^{\neq 0}(F(x, y) \to F(x, x + y))$ is proved to be true.

Solution To Question 9

(a) **Recursive Definition:**

BASIC STEP: $(1, 2) \in L, (1, 4) \in L, (2, 4) \in L, (3, 9) \in L$

RECURSIVE STEP:

- if $(a, b) \in L$ and $(a, ab) \in P \times P \quad$ (or $ab \in P$), then $(a, ab) \in L$.
- if $(a, b) \in L$ and $(b, c) \in L$, then$(a, c) \in L$. $[a \neq b \neq c]$
- if $(a, b) \in L$, $(a, c) \in L$ and $(a, b + c) \in P \times P \quad$ (or $(b + c) \in P$), then $(a, b + c) \in L$. $[a \neq b \neq c]$

(b) **Example Elements:**

$$\{(1, 3), (1, 7), (1, 5), (3, 6), (12, 48)\}$$

(c) **Modified Recursive Definition:**

BASIC STEP: $\forall a$, if $a \in P$, then $(a, a) \in L$.

RECURSIVE STEP:

- if $(a, b) \in L, (b, c) \in L$, then $(a, c) \in L$.
- if $(a, b) \in L$, then $(a, kb) \in L$. $(k \in \mathbb{Z}$ and $kb \in P)$
- if $(a, b) \in L, (a, c) \in L$, then $(a, b + c) \in L$ and $(a, b - c) \in L$. $(b \geq c$ and $b + c \in P, b - c \in P)$

(d) **Critique:**

To verify that it generalizes well, we choose arbitrary $e \in P, f \in P$ and p is a factor of f. Therefore, (e, f) should be in L. According to the basic step, $(e, e), (f, f) \in L$, as e is a factor of f, $\exists k \in \mathbb{Z}, f = ke$. From the recursive step, we know: $(e, e) \in L$, then $(e, ke) \in L$. Therefore, $(e, f) \in L$.

ASSUMPTION SET:

- Every number is the factor of itself.
- For all a, b in P, if a is a factor of b, then a is a factor of kb. (kb in P, k in $\mathbb{Z}$)
- if a is a factor of b, b is a factor of c, then a is a factor of c. (as a is a factor of itself, we allow a=b=c)
- if a is a factor of b and c $(b > c)$, then a is a factor of $b + c$ and $b - c$.