A NOTE ON MAPPING CYLINDER AND THE CURVATURE OF KNOTS

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AKSHAY P R

(Roll No. IMS14006)

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THIRUVANANTHAPURAM - 695551, INDIA

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DECLARATION

I, **Akshay P R** (**Roll No: IMS14006**), hereby declare that, this report entitled **"A note on mapping cylinder and the curvature of knots"** submitted to Indian Institute of Science Education and Research Thiruvananthapuram towards partial requirement of **Master of Science** in **Mathematics** is a review work carried out by me under the supervision of **Saikat Chatterjee** and has not formed the basis for the award of any degree or diploma, in this or any other institution or university. I have sincerely tried to uphold the academic ethics and honesty. Whenever an external information or statement or result is used then, that have been duly acknowledged and cited.

Thiruvananthapuram - 695 551 **Akshay P R**

April 2019

CERTIFICATE

This is to certify that the work contained in this project report entitled **"A note on mapping cylinder and the curvature of knots"** submitted by **Akshay P R** (**Roll No: Ims14006**) to Indian Institute of Science Education and Research Thiruvananthapuram towards partial requirement of **Master of Science** in **Mathematics** has been carried out by him under my supervision and that it has not been submitted elsewhere for the award of any degree.

Thiruvananthapuram - 695 551 (Dr. Saikat Chatterjee)
April 2019 Project Supervisor

ABSTRACT

In the first part of the project I reviewed the proof of Martin Fuchs theorem as given in the paper 'A note on mapping cylinders' published in the *Michigan Math J.*. A much more elaborate proof is given in 'A short note on mapping cylinders' by Alex Aguado. The proof of the same is described here with the necessary details. The basics of algebraic topology are explained briefly in a nutshell, to better understand the proof.

The second part of the project was to review the paper 'On the total Curvature of Knots', published in the *Annals of Mathematics* by J.W Milnor. The proofs given in the paper are elaborated with the necessary details to make it easier to understand.

Contents

1 Preliminaries **1**

1.1 Homotopy of paths 1

1.2 Covering spaces 6

1.3 Retracts and Fixed points 9

2 Martin Fuchs Theorem **12**

3 On the total curvature of knots **18**

3.0.1 Introduction 18

3.0.2 The crookedness of a closed curve 26

3.0.3 The curvature and crookedness of isotopy types of curves 34

Bibliography **39**

Chapter 1

Preliminaries

1.1 Homotopy of paths

Definition 1.1.1. Let f and $f^{'}$ be continuous maps from X to Y. f is said to be **homotopic** to $f^{'}$ if there is a continuous map $H : X \times I \to Y$ such that

$$H(x,0) = f(x) \quad \text{and} \quad H(x,1) = f^{'}(x). \tag{1.1}$$

In particular if $f^{'}$ is a constant map then f is said to be **nulhomotopic**.

A continuous map $f : I \to X$ where $I = [0,1]$ is the unit interval and X is a topological space is called a **path**. $f(0)$ and $f(1)$ are called the starting and ending points of the path respectively. If the continuous maps in the given definition are paths then the homotopy between the paths is called a **path homotopy**.

Example 1.1.2. Let f and g be two continuous maps from X to $\mathbb{R}^2$. The map $H(x,t) = tf(x) + (1-t)g(x)$ gives a homotopy called a **straight line homotopy**. If A is a *convex* subspace then any two paths in A are path homotopic by the straight line homotopy.

Lemma 1.1.3. *The homotopy is an equivalence relation in the set of all continuous maps from a space X to a space Y.*

Proof. Let the homotopy relation be given by the notation $\simeq$. It is clear that $f \simeq f$ given by the homotopy $H(x,t) = f(x)$. If $f \simeq g$ given by the homotopy $H(x,t)$, then $G(x,t) = H(x, 1-t)$ gives the homotopy between g and f therefore $g \simeq f$. Transitivity is verified as follows: Let $H(x,t)$ be the homotopy between f and g, and let $G(x,t)$ be the homotopy between g and h. define the map

$$F(x,t) = \begin{cases} H(x,2t) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ G(x, 2t-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{1.2}$$

This is a homotopy between f and h hence $f \simeq h$. □

The map constructed in this lemma gives a motivation for the definition of concatenation of paths.

Definition 1.1.4. If f and g are two paths in X with $f(1) = g(0)$ then define a map $f * g$ as follows

$$f * g(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{1.3}$$

This is a well defined map and is continuous by the pasting lemma. Hence $f * g$ is also a path in X. It can also be verified that if $f \simeq f'$ and $g \simeq g'$

then $f * g \simeq f^{'} * g^{'}$. Therefore we can define $[f * g] = [f] * [g]$.

Now it is natural to check whether the operation $*$ follows the common properties such as associativity, existence of inverse and so on.

Theorem 1.1.5. *The operation $*$ have following properties:*

(1) (Associativity) If $([f] * [g]) * [h]$ is defined, so is $[f] * ([g] * [h])$ and both are equal.

(2)(left and right identities)Let f be a path from x_0 to x_1. Let $e_x : I \to X$ be a constant path with the image x.

$$[f] * [e_{x_1}] = [f] \quad \textit{and} \quad [e_{x_0}] * [f] = [f] \tag{1.4}$$

(3)Given path f in X from x_0 to x_1. Let the reverse path $\bar{f}$ be defined as $\bar{f} = f(1-s)$ for values of s in the interval I then

$$[f] * [\bar{f}] = [e_{x_0}] \quad \textit{and} \quad [\bar{f}] * [f] = [e_{x_1}] \tag{1.5}$$

It is interesting to note that the operation $*$ follows some of the properties of a group operation. In fact the properties satisfied by $*$ is that of a groupoid. $*$ cannot operate on any two arbitrary paths, but it can operate on two different paths with the property that they have the same final and initial points respectively. Hence to consider a set of paths such that $*$ can act on any two of these paths, implies that $f * f$ is well defined or in other words the initial and final points of f are the same for all f in the set. This motivates the definition of a loop.

Definition 1.1.6. A **loop** is a path in X which begins and ends at a common point x_0.

From the theorem it is clear that the set of equivalence classes of loops follows the properties of a group under the operation $*$. Hence this motivates the following definition:

Definition 1.1.7. X is a topological space. The path homotopy classes of loops at a point x_0 forms a group with respect to the operation $*$ and the group is called the **fundamental group** at the base point x_0 and is denoted by the symbol $\pi_1(X, x_0)$.

Example 1.1.8. Let $\mathbb{R}^n$ be the n dimensional euclidean space. The fundamental group at any point of this space is trivial. For any loop at x_0 is homotopic to the constant path e_{x_0} by the straight line homotopy. In particular a ball $B^n = \{\mathbf{x} | x_1^2 + \cdots + x_n^2 \leq 1\}$ has trivial fundamental group since it is a convex subspace of the euclidean space.

The definition of fundamental group defined above is depended on the base point under consideration, it is to be seen how the fundamental group depends on the base point.

Definition 1.1.9. Let α be a path in X from x_0 to x_1. Define a map as follows

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \tag{1.6}$$

where $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha] \quad \text{and} \quad \bar{\alpha} = \alpha(1 - s)$.

Theorem 1.1.10. *The map $\hat{\alpha}$ is an isomorphism.*

Proof. Homomorphism can be easily verified. Let $\beta(s) = \alpha(1-s)$ then $\hat{\beta}$ is the left and right inverse of $\hat{\alpha}$. □

Corollary 1.1.11. *If X is a path connected space then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$ for any two points x_0 and x_1.*

Path connectedness is an equivalence relation. So given a space it is possible to divide the space into path components. Hence it is desired to consider only the path connected spaces and to consider the fundamental group of the space.

Definition 1.1.12. If X is a path connected space with a trivial fundamental group then X is called a **simply connected space**.

Lemma 1.1.13. *In a simply connected space any two paths with the same initial and final points are homotopic.*

Proof. Let α and β be two paths in a simply connected space X with x_0 and x_1 as the initial and final points. Then $\alpha * \bar{\beta}$ is a loop at x_0 hence is nulhomotopic. $[\alpha * \bar{\beta}] * [\beta] = [\alpha] * [\bar{\beta}] * [\beta] = [e_{x_0}] * [\beta]$ from which it follows that $[\alpha] = [\beta]$, hence α is homotopic to β. □

Given a continuous map from the space X to Y, it is possible to relate the fundamental groups of the spaces by mapping the equivalence class of a loop to the equvivalence class formed by the image of the loop under f.

Definition 1.1.14. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map such that $h(x_0) = y_0$. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(X, y_0) \tag{1.7}$$

$$h_*([f]) = [h \circ f] \tag{1.8}$$

The map h_* is a homomorphism since it can be verified that $h \circ (f * g) = (h \circ f) * (h \circ g)$

Theorem 1.1.15. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous maps. Then $(k \circ h)_* = k_* \circ h_*$, if $i : (X, x_0) \to (X, x_0)$ is the identity map then i_* is the identity homomorphism.*

Proof. $(k \circ h)_*([f]) = [(k \circ h) \circ f] = [k \circ (h \circ f)] = [k \circ h_*([f])] = k_*(h_*([f])) = (k_* \circ h_*)([f])$. $i_*([f]) = [i \circ f] = [f]$ □

1.2 Covering spaces

The fundamental group of $\mathbb{R}^n$ was calculated to be trivial. To calculate the fundamental groups of more spaces the concept of covering spaces is important.

Let $p : E \to B$ be a continuous, surjective map. Let U be an open set in Y. f is said to **evenly cover** U if $p^{-1}(U)$ can be written as a disjoint union of open sets V_α such that $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Example 1.2.1. The map $p : \mathbb{R}_+ \to \mathbb{S}^1$ given by $p(x) = (cos(2\pi x), sin(2\pi x))$ is continuous and surjective. For the point $b_0 = (1, 0)$ there is no neighbourhood such that $p(x)$ evenly covers it.

This example leads to the definition of a covering map.

Definition 1.2.2. Let $p : E \to B$ be a continuous and surjective map. If every point b in B has a neighbourhood U which is evenly covered by p. Then p is a **covering map** and E is a **covering space**.

Each $p^{-1}(b)$ is a discrete space. Since there exists a neighbourhood U of b such that $p^{-1}(U)$ is a disjoint union of open sets V_α and each $V_\alpha \cap p^{-1}(b)$ is a singleton set. p is also an open map, for an open set A in E and $y \in p(A)$ there is a neighbourhood U of y such that $p^{-1}(U)$ is a disjoint union of open sets V_α. There exists an $x \in V_\beta \cap A$ such that $p(x) = y$. $p(V_\beta \cap A)$ is open in B and is contained in $p(A)$.

Example 1.2.3. The map $p : \mathbb{R} \to \mathbb{S}^1$ defined by $p(x) = (cos(2\pi x), sin(2\pi x))$ is a covering map of $\mathbb{S}^1$.

It is possible to construct more covering maps from existing covering maps by taking the product of maps.

Theorem 1.2.4. *If $p : E \to B$ and $p' : E' \to B'$ are covering maps then $p \times p' : E \times E' \to B \times B'$ is a covering map.*

The study of fundamental groups is related to the study of covering spaces. In order to fully exploit the concept of covering spaces we need to define 'lifting of maps'.

Definition 1.2.5. Let $p : E \to B$ be a map. If $f : X \to B$ is a continuous map then $\bar{f}$ is a **lifting** of f if $p \circ \bar{f} = f$.

Example 1.2.6. Consider the covering map p of the example 1.2.3. The path $f(s) = (cos(\pi s), sin(\pi s))$ in $\mathbb{S}^1$ lifts to a map $g(s) = \frac{s}{2}$.

Lemma 1.2.7. *Let $p : E \to B$ be a covering map such that $p(e_0) = b_0$. Any path $f : I \to B$ beginning at b_0 lifts to a unique path $\bar{f} : I \to E$ beginning at e_0.*

This extension lemma also holds for homotopy as given in the next theorem.

Lemma 1.2.8. *If $F : I \times I \to B$ is a continuous map such that $F(0,0) = b_0$. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$, then there exists a unique continuous lifting $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. In particular if F is a path homotopy then $\tilde{F}$ is also a path homotopy.*

This lemma leads to the extension of homotopic paths to the covering space, hence the following theorem.

Theorem 1.2.9. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1. Let $\tilde{f}$ and $\tilde{g}$ be the corresponding liftings. Then $\tilde{f}$ and $\tilde{g}$ end at a same point in E.*

Definition 1.2.10. If $p : E \to B$ is a covering map with $p(e_0) = b_0$. Let $[f]$ be an element of $\pi_1(B, b_0)$ then define $\phi([f]) = \tilde{f}(1)$

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0) \tag{1.9}$$

Then ϕ is a **lifting correspondence** derived from p.

Theorem 1.2.11. *Let $p : E \to B$ be covering map with $p(e_0) = b_0$. If E is path connected then ϕ is surjective. If E is simply connected then ϕ is a bijection.*

Example 1.2.12. Let p be the covering map of the example 1.2.3, $e_0 = 0$ and $p(0) = b_0$ then the lifting correspondence is $\phi : \pi_1(\mathbb{S}^1, b_0) \to \mathbb{Z}$ where $p^{-1}(b_0) = \mathbb{Z}$. Since here $E = \mathbb{R}$ which is a simply connected space the map

ϕ is a bijection. It is possible to prove that ϕ is also a homomorphism hence the fundamental group of a circle is isomorphic to $(\mathbb{Z}, +)$.

1.3 Retracts and Fixed points

Definition 1.3.1. Let $A \subset X$, a **retraction** is a continuous map $f : X \to A$ such that $f|_A$ is the identity map. The subspace A is called a **retract** of X.

Theorem 1.3.2. *If A is a retract of the space X then the homomorphism of the fundamental groups induced by the inclusion map $j : A \to X$ is injective.*

Proof. Let $r : X \to A$ be a retraction onto A. The map $r \circ j$ is the identity map. Hence $(r \circ j)_* = r_* \circ j_* = i_*$, i_* is injective implies that j_* is also injective. $\square$

Lemma 1.3.3. *Let $h : \mathbb{S}^1 \to X$ be a continuous map. Then the following are equivalent: (1)h is nulhomotopic. (2)h extends to a continuous map $k : B^2 \to X$. (3)h_* is a trivial homomorphism.*

Theorem 1.3.4 (No hair theorem). *Given a non vanishing vector field on B^2, there exists a point on $\mathbb{S}^1$ where the field points directly inwards and directly outwards.*

Proof. Let $v(x) : B^2 \to \mathbb{R}^2 - 0$ be a non vanishing vector field. Assume that $v(x)$ does not point inwards on $\mathbb{S}^1$. Let w be the restriction of $v(x)$ on $\mathbb{S}^1$. Then $w(x)$ is homotopic to the inclusion map $j : \mathbb{S}^1 \to \mathbb{R}^2 - 0$ given by the homotopy $F(x, t) = tx + (1-t)w(x)$. Since $w(x)$ extends to a continuous map on B^2 it is nulhomotopic, hence j is nulhomotopic which is a contradiction since j_* is injective.

$\square$

Theorem 1.3.5 (Brouwers fixed point theorem). *If $f : B^2 \to B^2$ is a continuous map, then there is a point on B^2 such that $f(x) = x$.*

Proof. The proof is by contradiction. If $f(x) \neq x$ for all x, define $v(x) = f(x) - x$. This is a non vanishing vector field on B^2. There is a point x such that $v(x) = ax$ where a is positive. This implies $f(x) = (1 + a)x$ which is a contradiction since f lies inside the unit ball. $\square$

Theorem 1.3.6 (Fundamental theorem of algebra). *A polynomial equation $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with complex coefficients has atleast one solution.*

Proof. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be defined as $f(z) = z^n$. First it will be shown that f induces an injective homomorphism. Let $p_0 : I \to \mathbb{S}^1$ be the restriction of the standard covering map. It generates the fundamental group of $\mathbb{S}^1$. $f_*([P_0]) = [P_0^n] = [(cos(2\pi ns), sin(2\pi ns))]$. P_0^n lifts to the map $s \to ns$ in the covering space $\mathbb{R}$. Hence under the standard isomorphism of $\pi(\mathbb{S}^1, b_0)$ with the additive group of integers, $[p_0^n]$ corresponds to the integer n. Hence f_* is equivalent to the map $1 \to n$ hence f_* is an injection. Let $g : \mathbb{S}^1 \to \mathbb{R}^2 - 0$ be the map $g(z) = z^n$. Then $g = j \circ f$ where $j : \mathbb{S}^1 \to \mathbb{R}^2 - 0$ is the injection map. $g_* = j_* \circ f_*$ hence g_* is an injective map. Hence g is not nulhomotopic. Let $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with $|a_{n-1}| + |a_{n-2}| + \cdots + |a_0| < 1$ have no solution in B^2. Let $k : B^2 \to \mathbb{R}^2 - 0$ be defined as $k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Let h be the restriction of k to $\mathbb{S}^1$. h is nulhomotopic. On the other hand we can define a homotopy between h and the map g hence arriving at a contradiction. Define $F(z,t) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$. $|F(x,t)| \geq |z^n| - |t(a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots a_0| \geq 1 - t(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|) > 0$. $F(x,t)$ is the required homotopy. For general case consider $x = cy$, hence the equation $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ reduces to $y^n + \frac{a_{n-1}}{c}y^{n-1} + \cdots + \frac{a_0}{c^n} = 0$. Now choose c such that$|\frac{a_{n-1}}{c}| + |\frac{a_{n-2}}{c^2}| + \cdots + |\frac{a_0}{c^n}| < 1$. □

Chapter 2

Martin Fuchs Theorem

This chapter focuses on proving the Martin Fuchs theorem. Covering spaces are useful for finding the fundamental groups of topological spaces. There are other ways to determine the fundamental groups, one way is to define the notion of *homotopy type*. First we begin by defining the notion of deformation of a one space into another space.

Definition 2.0.1. A space A is called a **deformation retract** of another space X if the identity map $1_X : X \to X$ is homotopic to a retraction of X onto A. The homotopy between 1_X and the retraction is called a **deformation retraction**.

Definition 2.0.2. A space A is called a **strong deformation retract** of a space X, if A is a deformation retract of X and the values of A remain fixed by the deformation retraction.

Example 2.0.3. The space $X = \mathbb{R}^2 - \{p, q\}$ can be deformed into the eight figure $(S^1 \times b_0) \bigcup (b_0 \times \mathbb{S}^1)$ where $b_0 = (1, 0)$. The theta space $\mathbb{S}^1 \bigcup (\{0\} \times$

$[-1,1]$) is also a deformation retract of the space X.

Theorem 2.0.4. *Let space A be a deformation retract of another space X then $j_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism.*

Proof. For the proof one requires the lemma that if h and k are homotopic with the homotopy preserving the base point then $h_* = k_*$. $\square$

Definition 2.0.5. Let $f : X \to Y$ and $g : Y \to X$ be two continuous maps such that $f \circ g : Y \to Y$ is homotopic to 1_Y and $g \circ f : X \to X$ is homotopic to 1_X then f and g are called **homotopy equivalences** and each is a **homotopy inverse** of the other. Also the spaces X and Y are said to be of same **homotopy type**.

Given continuous maps $f : X \to Y$ and $g : Y \to Z$ which are homotopy equivalences, then the map $g \circ f : X \to Z$ gives a homotopy equivalence. Hence homotopy equivalence gives an equivalence relation on the set of all topological spaces.

Example 2.0.6. The theta space and eight figure are of same homotopy type but are not deformation retract of one another.

This example clearly indicates that homotopy type is more general concept than the deformation retraction. The implication in the example 2.0.3 is much more general. It can be shown that X and Y are of same homotopy type if and only if they are common deformation retract of a topological space Z. In particular Z is a space called a *mapping cylinder.*

Definition 2.0.7. Given X and Y be two topological spaces. $f : X \to Y$ is a continuous map. The **mapping cylinder** M_f is the quotient space

$((X \times I) \bigcup Y)/\sim$ obtained by the relation $f(x) \sim (x, 0)$ where $(X \times I) \bigcup Y$ has the smallest topology such that $X \times I$ and Y are open.

Theorem 2.0.8 (**Martin fuchs theorem**). *Topological spaces X and Y are of same homotopy type iff they are strong deformation retracts of a common topological space Z. In fact Z is the mapping cylinder as described above.*

Proof. Let $q : (X \times I) \bigcup Y \to M_f$ denote the quotient map. Let $\widetilde{X} = q(X \times I)$ and $\widetilde{Y} = q(Y)$. If X and Y are deformation retracts of Z. Then X and Y have same homotopy type as Z. Homotopy type is an equivalence relation hence X and Y have the same homotopy type.

Conversely suppose X and Y are of same homotopy type. By definition there exists a homotopy equivalence $f : X \to Y$. Let $g : Y \to X$ be the homotopy inverse. Let $F : X \times I \to X$ be the homotopy between $g \circ f$ and 1_X.Let $G : Y \times I \to Y$ be the homotopy between $f \circ g$ and 1_Y. It will be shown that X and Y are deformation retracts of the mapping cylinder M_f.

Define $H_1 : M_f \times I \to M_f$ as follows

$$\begin{cases} ([x,t], s) \to [x, t(1-s)] \\ ([y], s) \to [y] \end{cases} \tag{2.1}$$

H_1 is a deformation retraction of M_f onto Y. This is a homotopy between 1_X and $r : M_f \to M_f$ defined as $r([x,t]) = [x, 0]$ and $r([y]) = [y]$. H_1 is the desired strong deformation retraction of M_f onto Y.

Use f and G to define a homotopy $H_2 : M_f \times I \to M_f$ as follows

$$\begin{cases} ([x,t], s) \to [G(f(x), 1-s)] \\ ([y], s) \to [G(y, 1-s)] \end{cases} \tag{2.2}$$

This gives a homotopy between maps r and $h : M_f \to M_f$, here h is defined by $h([x,t]) = [g \circ f(x), 0]$ and $h([y]) = [g(y), 0]$.

Use g and F to define a homotopy $H_3 : M_f \times I \to M_f$ as follows

$$\begin{cases} ([x,t],s) \to [F(x,st),s] \\ ([y],s) \to [g(y),s] \end{cases} \tag{2.3}$$

This gives a homotopy between map h and $r^{'} : M_f \times I \to M_f$, where $r^{'}$ is defined by $r^{'}([x,t]) = [F(x,t),1]$ and $r^{'}([y]) = [g(y),1]$.
The concatenation $H = H_1 * H_2 * H_3$ gives a homotopy. Hence 1_X is homotopic to the map $r^{'}$.where $r^{'}$ is a retraction of M_f onto $\widetilde{X}$.
It only needs to prove the strong deformation retraction of M_f onto $\widetilde{X}$, the construction of the same is given as follows.
Define a retraction $R : M_f \times I \times I \to (M_f \times I \times 0) \bigcup (\widetilde{X} \times I \times I)$ by defining a retraction $\varphi : I^2 \to (I \times \{0\}) \bigcup (\{1\} \times I)$

$$\varphi(u,v) = \begin{cases} (\frac{2u}{2-v}, 0) & \text{if } v \leq 2-2u \\ (1, \frac{2u+v-2}{u}) & \text{if } v \geq 2-2u \end{cases} \tag{2.4}$$

Let φ_1 and φ_2 denote the components of $\varphi$and the retraction is defined by:

$$\begin{cases} ([x,t],s,l) \to ([x,\varphi_1(t,l)],s,\varphi_2(t,l)) \\ ([y],s,l) \to ([y],s,0) \end{cases} \tag{2.5}$$

Let a point in M_f be denoted by p and the point in $\widetilde{X}$ be denoted by $\widetilde{p}$.
The homotopy H constructed before needs to be modified so that it leaves the points of $\widetilde{X}$ fixed.

Let $K : M_f \times I \to M_f$ be defined as follows

$$K(p,s) = \begin{cases} H(p,2s) & \text{if } 0 \le s \le \frac{1}{2} \\ H(r'(p), 2(1-s)) & \text{if } \frac{1}{2} \le s \le 1 \end{cases} \tag{2.6}$$

Here K is a homotopy between 1_{M_f} and r'. It's restriction to $\widetilde{X} \times I$ is a homotopy beginning and ending with $1_{\widetilde{X}}$ since $r'(\widetilde{p}) = \widetilde{p}$.
Let $L : (\widetilde{X} \times I) \times I \to M_f$ as follows

$$L(\widetilde{p}, s, u) = \begin{cases} K(\widetilde{p}, s) & \text{if } u \le |2s-1| \\ k(\widetilde{p}, \frac{1-u}{2}) & \text{if } 2s - u \le 1 \le 2s + u \end{cases} \tag{2.7}$$

This is a homotopy from $K|_{\widetilde{X}\times I}$ to a map sending $(\widetilde{p}, s)$ to $\widetilde{p}$. This is a well defined homotopy since $K(\widetilde{p}, 1-s) = k(\widetilde{p}, s)$. Moreover $L(\widetilde{p}, s, u) = \widetilde{p}$ for $(s, u) \in \partial I \times I \bigcup I \times 1$. This map is to be extended to whole of mapping cylinder. Define a map $K_0 : M_f \times I \times 0 \to M_f$ by $K_0(p, s, 0) = K(p, s)$ and combine L to give a map as follows

$$(K_0, L) = \begin{cases} K_0(p,s,0) & \text{for } (p,s,0) \in M_f \times I \times 0 \\ L(\widetilde{p}, s, u) & \text{for } (\widetilde{p}, s, u) \in \widetilde{X} \times I \times I \end{cases} \tag{2.8}$$

Define L' as the composition of R and (K_0, L) as given

$$M_f \times I \times I \xrightarrow{R} M_f \times I \times 0 \bigcup \widetilde{X} \times I \times I \xrightarrow{(K_0,L)} M_f \tag{2.9}$$

This map extends L and satisfies $L^{'}(p, s, 0) = K(p, s)$ and define $\gamma : M_f \times I \to M_f$ as follows

$$\gamma(p, s) = \begin{cases} L^{'}(p, 0, 3s) & 0 \leq s \leq \frac{1}{3} \\ L^{'}(p, 3s - 1, 1) & \frac{1}{3} \leq s \leq \frac{2}{3} \\ L^{'}(p, 1, 3 - 3s) & \frac{2}{3} \leq s \leq 1 \end{cases} \tag{2.10}$$

By calculation it can be verified that

$$\gamma(p, 0) = L^{'}(p, 0, 0) = K(p, 0) = p \tag{2.11}$$

$$\gamma(p, 1) = L^{'}(p, 1, 0) = k_0(p, 1, 0) = k(p, 1) = r^{'}(p) \tag{2.12}$$

$$\gamma(\widetilde{p}, s) = \widetilde{p} \tag{2.13}$$

Therefore this is a homotopy between 1_{M_f} and $r^{'}$ with the elements of the space $\widetilde{X}$ remains fixed under this map. Hence this is the desired strong deformation retract.

$\square$

Chapter 3

On the total curvature of knots

3.0.1 Introduction

This chapter is a review of the paper 'On the total curvature of knots' by J.W Milnor, published in the Annals of Mathematics. As a historical note the total curvature of a closed curve was studied by W.Fenchel, who showed that the total curvature of a curve $\mathfrak{x}(s)$ in three dimensional space satisfies the inequality; $\int_c |\mathfrak{x}''(s)|\, ds \geq 2\pi$. Further, K. Borsuk extended this result to n dimensional space and conjectured that the total curvature of a knot in three dimensional space must exceed 4π. The proof of the same is given in this paper.

Definition 3.0.1. A *closed polygon* in R^n is defined as the set of points $a_0, a_1 ... a_{m-1}, a_m = a_0$ and the line segments $a_i a_{i+1}$ such that $a_i \neq a_{i+1}$ for $i = 0, 1, ..m-1$.

Definition 3.0.2 (*Total curvature of a polygon*)**.** Let α_i be the angle between $a_{i+1} - a_i$ and $a_i - a_{i-1}$. The total curvature $K(P)$ of a polygon P is given by $\sum \alpha_i$.

Adjoining a new point to a polygon can change its curvature. The interesting result is that the curvature cannot decrease by this addition of a new point.

Lemma 3.0.3. *The adjunction of a new point to a closed polygon cannot decrease it's total curvature. The curvature may remain constant if either the new point a_j and the adjacent points a_{j-1} and a_{j+1} are collinear or $a_{j-2}, a_{j-1}, a_j, a_{j+1}, a_{j+2}$ are coplanar. Otherwise it increases.*

Proof. Let $P^{'}$ be the closed polygon with vertices $a_1, a_2 ... a_{j-1}, a_{j+1} .. a_m$(Figure 3.1). Let P be the closed polygon with vertices $a_1, a_2 ... a_{j-1}, a_j, a_{j+1} .. a_m$. Denote $\alpha_i^{'}$ the exterior angles of $P^{'}$ and α_i the exterior angles of P for $i = 1, 2..m$. Denote by β^- the angle between $a_j - a_{j-1}$ and $a_{j+1} - a_{j-1}$. Denote by β^+ the angle between $a_{j+1} - a_{j-1}$ and $a_{j+1} - a_j$. We have by the triangle inequality for spherical triangles $\alpha_{j-1} + \beta^- \geq \alpha_{j-1}^{'}$, where equality holds only if a_{j-2}, a_{j-1}, a_j and a_{j+1} are coplanar. Similarly $\alpha_{j+1} + \beta^+ \geq \alpha_{j+1}^{'}$, where the equality holds only if $a_{j-1}, a_j, a_{j+1}, a_{j+2}$ are coplanar. From the triangle a_j, a_{j-1}, a_{j+1} we have $\beta^- + \beta^+ = \alpha_j$. Therefore

$$K(P) - K(P^{'}) = (\alpha_{j-1} - \alpha_{j-1}^{'}) + \alpha_j + (\alpha_{j+1} - \alpha_{j+1}^{'}) \tag{3.1}$$

$$\geq \alpha_j - \beta^- - \beta^+ = 0. \tag{3.2}$$

Therefore $K(P) \geq K(P^{'})$ equality holds only if $a_{j-2}, a_{j-1}, a_j, a_{j+1}$ and a_{j+2} are coplanar or a_{j-1}, a_j, a_{j+1} are collinear.

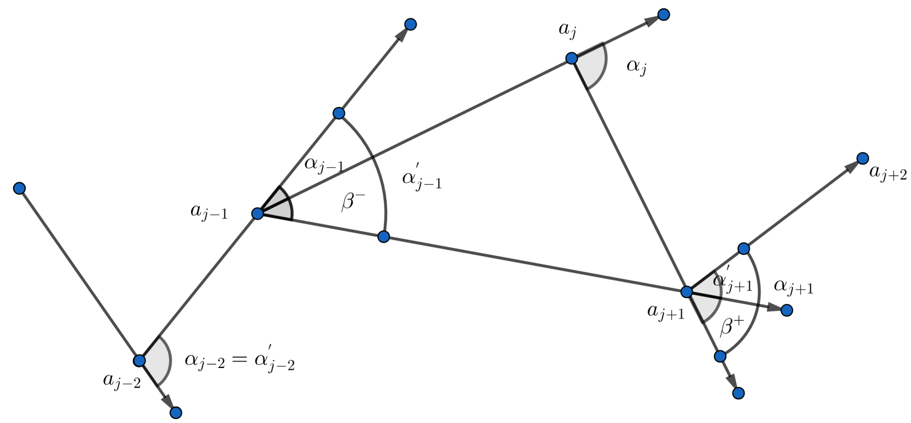

Figure 3.1: polygon P' with a new vertex a_j

$\square$

Corollary 3.0.4. *If the vertices $a_{j-2}, a_{j-1}, a_j, a_{j+1}$ of a closed polygon are not coplanar and the vertex a_j is replaced by vertex a'_j which lies on the line a_ja_{j+1} then $K(P)$ is decreased.*

Now we define a closed curve and simple closed curve in R^n.

Definition 3.0.5. A **closed curve** in R^n is a continuous function $\mathfrak{x}(s) = (\mathfrak{x}_1(s), ..., \mathfrak{x}_n(s))$ of period l which is not constant in any s interval.
In particular the condition $\mathfrak{x}(s_1) = \mathfrak{x}(s_2)$ holds iff $(s_1 - s_2)/l$ is an integer, then the curve is called a **simple closed curve**.

A closed polygon P with the vertices $a_1, ...a_m$ is said to be **inscribed in**

a closed curve $\mathfrak{x}(s)$ if there are a set of parameter values s_i for $i = 1, .., m$ such that $s_i < s_{i+1}$, $s_{i+m} = s_i + l$, and $a_i = \mathfrak{x}(s_i)$.

Lemma 3.0.6. *For any closed polygon P, $K(P) = l.u.b\{K(P^{'})\}$, where $P^{'}$ ranges over all polygons inscribed in P.*

Proof. Let $P_0^{'}$ be a representative inscribed polygon whose vertices include all but m of the vertices of P. We adjoin the remaining vertices one by one to $P_0^{'}$ producing a sequence of polygons $P_0^{'}, P_1^{'}...P_m^{'}$. Therefore $K(P_0^{'}) \leq K(P_1^{'}) \leq ....K(P_m^{'})$ by the lemma 3.0.3 ; But $K(P_m^{'}) = K(P)$. Therefore $K(P) = l.u.b\{K(P^{'})\}$, where $P^{'}$ ranges over all the polygons inscribed in the given polygon P. □

From this lemma it is natural to define the curvature of a closed curve as follows.

Definition 3.0.7. The total curvature of a closed curve $K(C)$ is defined as $K(C) = l.u.b\{K(P)\}$ where P ranges over all polygons inscribed in C.

Note by our definition a polygon is also a closed curve and the definition3.0.7 takes into account the lemma 3.0.6. It is possible to express the total curvature of a closed curve in terms of an integral. This makes the definition of curvature easier for calculation.

Theorem 3.0.8. *Let C be a closed curve of class C^2 parametrized by arclength s then $K(C) = \int_c |\mathfrak{x}''(s)| \, ds$*

Proof. If $a_1^m = \mathfrak{x}(s_1^m)...a_m^m = \mathfrak{x}(s_m^m)$ are the vertices of polygon P_m inscribed in the closed curve C such that $\lim_{m\to\infty} \max_i \left\{ \left(s_{i+1}^m - s_i^m \right) \right\} = 0$. It will be shown that $\lim_{m\to\infty} K\left(P_m\right) = \int_c |\mathfrak{x}''(s)| \, ds$. Define $\overline{s}_i^m = \frac{1}{2}\left(s_i^m + s_{i+1}^m \right)$ for every i and m. Denote by θ_i^m the angle between $\mathfrak{x}'(\overline{s}_{i-1}^m)$ and $\mathfrak{x}'(\overline{s}_i^m)$. The vector $\mathfrak{x}'(s)$ defines a curve L of length $\int_C |\mathfrak{x}''(s)|\, ds$, since $\mathfrak{x}(s)$ is arc length parametrized the curve $\mathfrak{x}'(s)$ lies on the unit sphere S^{n-1}. The vectors $\mathfrak{x}'(\overline{s}_i^m)$ form the vertices of a spherical polygon of length $\sum_{i=1}^m \theta_i^m$ which is inscribed in L. Therefore $\lim_{m\to\infty} \sum_{i=1}^m \theta_i^m = \int_C |\mathfrak{x}''(s)|\, ds$.

Since $\mathfrak{x}''(s)$ is uniformly continuous, for $\epsilon > 0$ there exists a $\delta > 0$ such that $\left|\mathfrak{x}''(u) - \mathfrak{x}''(v)\right| < \epsilon$ for all $|u - v| < \delta$. From the identity

$$\mathfrak{x}\left(s_{i+1}^m\right) - \mathfrak{x}\left(s_i^m\right) = \left(s_{i+1}^m - s_i^m\right)\mathfrak{x}'\left(\overline{s}_i^m\right) + \int_{\overline{s}_i^m}^{s_{i+1}^m}\int_{\overline{s}_i^m}^{v}\left[\mathfrak{x}''(u) - \mathfrak{x}''\left(\overline{s}_i^m\right)\right] du dv + \int_{s_i^m}^{\overline{s}_i^m}\int_{v}^{\overline{s}_i^m}\left[\mathfrak{x}''\left(\overline{s}_i^m\right) - \mathfrak{x}''(u)\right] du dv$$

we have

$$\left|\frac{\mathfrak{x}\left(s_{i+1}^m\right) - \mathfrak{x}\left(s_i^m\right)}{s_{i+1}^m - s_i^m} - \mathfrak{x}'\left(\overline{s}_i^m\right)\right| < \left(s_{i+1}^m - s_i^m\right)\frac{\epsilon}{4}$$

whenever $\max_i \left\{\left(s_{i+1}^m - s_i^m\right)\right\} < \delta$.

If φ_i^m is the angle between $\mathfrak{x}\left(s_{i+1}^m\right) - \mathfrak{x}\left(s_i^m\right)$ and $\mathfrak{x}'\left(\overline{s}_i^m\right)$ then $sin\varphi_i^m < \left(s_{i+1}^m - s_i^m\right)\frac{\epsilon}{4}$ (figure 3.2).

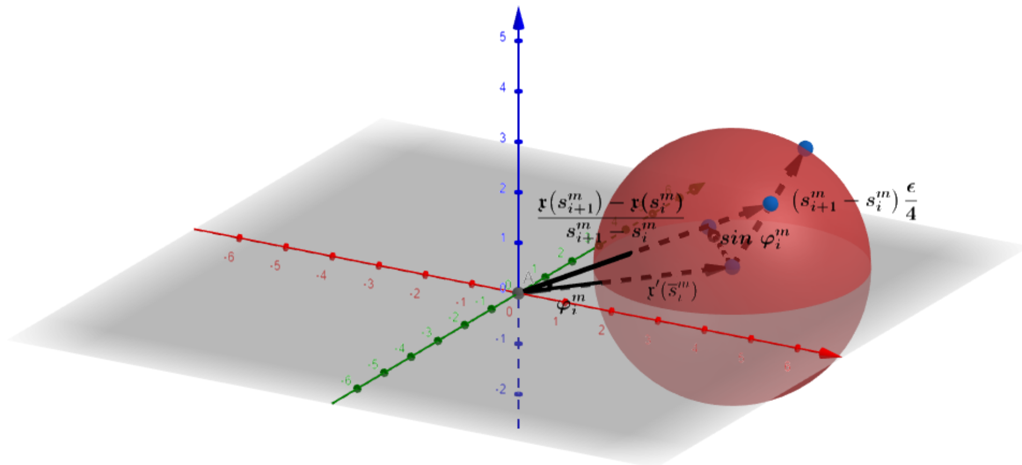

Figure 3.2:

For sufficiently small ϵ we have $\varphi_i^m < 2sin\varphi_i^m < \left(s_{i+1}^m - s_i^m\right)\frac{\epsilon}{2}$. The angle between $\mathfrak{x}\left(s_{i+1}^m\right) - \mathfrak{x}\left(s_i^m\right)$ and $\mathfrak{x}\left(s_i^m\right) - \mathfrak{x}\left(s_{i-1}^m\right)$ is α_i^m , while that between $\mathfrak{x}'\left(\overline{s}_i^m\right)$ and $\mathfrak{x}'\left(\overline{s}_{i-1}^m\right)$ is θ_i^m. Apply the spherical triangle inequality for the angles between the vectors A,B,C of the figure 3.3,

$$\alpha_i^m \leq \varphi_{i-1}^m + \delta$$

Similarly for the angles between the vectors B,C,D we have

$$\delta \leq \theta_i^m + \varphi_i^m$$

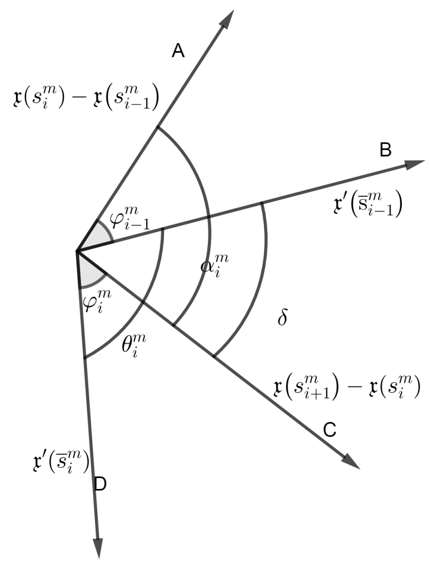

Figure 3.3:

Hence

$$\alpha_i^m \leq \varphi_{i-1}^m + \theta_i^m + \varphi_i^m$$

Using triangle inequality for the angles between the vectors B,C and D we obtain

$$\theta_i^m \leq \delta + \varphi_i^m$$

Similarly for the angles between the vectors A,B and C we have $\delta \leq \alpha_i^m + \varphi_{i-1}^m$.

Therefore

$$\theta_i^m \leq \alpha_i^m + \varphi_{i-1}^m + \varphi_i^m$$

Therefore $|\alpha_i^m - \theta_i^m| \leq \varphi_{i-1}^m + \varphi_i^m < \left(s_{i+1}^m - s_{i-1}^m\right)\frac{\epsilon}{2}$, and $|\sum_{i=1}^m \alpha_i^m - \sum_{i=1}^m \theta_i^m| < l\epsilon$, where l is the length of C. Therefore

$$\lim_{m\to\infty} K\left(P_m\right) = \lim_{m\to\infty} \sum_{i=1}^{m} \alpha_i^m = \lim_{m\to\infty} \sum_{i=1}^{m} \theta_i^m = \int_c |\mathfrak{x}''(s)|\, ds \tag{3.3}$$

In order to show $\int_C |\mathfrak{x}''(s)|\, ds = l.u.b\{K(P)\}$ for P inscribed in C, we only need to be show that $K(P) \leq \int_C |\mathfrak{x}''(s)|\, ds$.

Given any polygon P_k inscribed in C we form the sequence of polygons P_m for $m = k, k{+}1, ..$ by adjoining vertices to P_k so that $\lim_{m\to\infty} \max_i \left\{\left(s_{i+1}^m - s_i^m\right)\right\} = 0$. By the lemma 3.0.3 $K(P_k) \leq k(P_{k+1}) \leq ..$, but $\lim_{m\to\infty} K\left(P_m\right) = \int_c |\mathfrak{x}''(s)|\, ds$, and therefore $K(P_k) \leq \int_C |\mathfrak{x}''(s)|\, ds$. $\square$

3.0.2 The crookedness of a closed curve

For a closed curve C and each unit vector b, define $\mu(C, b)$ to be the number of local maxima of the function $b.\mathfrak{x}(s)$ for s in the fundamental period. Define the **crookedness** of a closed curve C as $\mu(C) = \min_b\{\mu(C, b)\}$.
For $a_{i+1} - a_i$ in R^n define $b_i = \frac{a_{i+1}-a_i}{||a_{i+1}-a_i||}$ as the **spherical image** of the vector $a_{i+1} - a_i$ in the unit sphere S^{n-1}.

Definition 3.0.9 (*Spherical polygon*)**.** Given a polygon P with the vertices $a_1, a_2 ... a_m$ and let $b_1, b_2, ..b_m$ be the spherical images of $a_2 - a_1, a_3 - a_2.., a_1 - a_m$ respectively. Let $0 \leq \alpha_i \leq \pi$ be the angle between b_{i-1} and b_i for $i = 2, 3.., m$. A spherical polygon Q is formed on S^{n-1} by joining b_{i-1} to b_i by a great circle arc of length α_i. This spherical polygon Q is also called the **spherical image** of P.

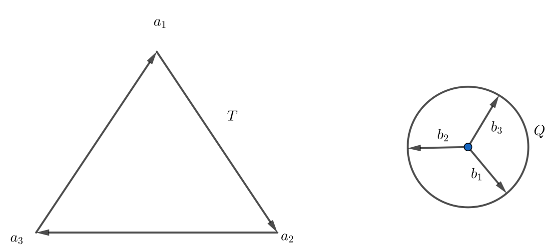

Figure 3.4: Triangle T and its spherical image Q

We will see that the spherical polygon is a useful concept in determining the local maxima and minima of the function $b.\mathfrak{x}(s)$, where $\mathfrak{x}(s)$ is the given polygon.

Theorem 3.0.10. *Any closed curve C in R^n, $n \geq 2$, the lebesgue integral $\int_{S^{n-1}} \mu(C,b)dS$ exists and is equal to $\frac{M_{n-1}K(C)}{2\pi}$ where $M_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the measure of S^{n-1}.*

Proof. Consider the case in which the curve is a polygon P. Let Q be its spherical polygon. For every point b in S^{n-1}, S_b^{n-2} denote the great sphere of S^{n-1} which has pole at b.
An edge $b_{j-1}b_j$ of Q crosses S_b^{n-2} iff $b.(a_{j+1} - a_j)$ and $b.(a_j - a_{j-1})$ have opposite signs.

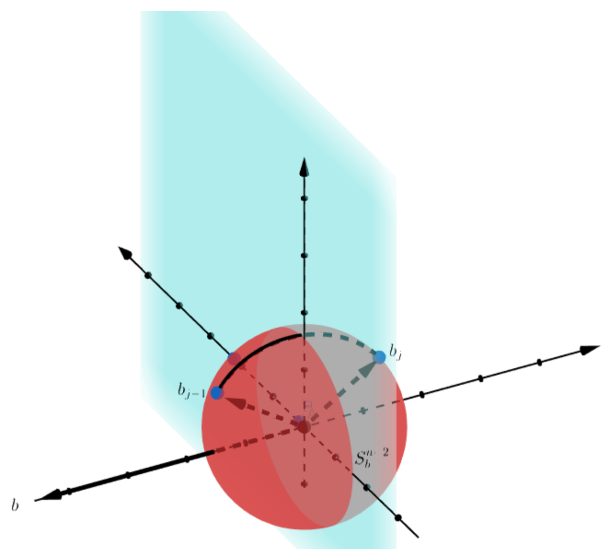

Figure 3.5: The great shpere S_b^{n-2}

Also the local maxima or minima of the function $b.\mathfrak{x}(s)$, where $\mathfrak{x}(s)$ is the polygon P, are the corners or all the points of an edge. Hence a_j is a point of local maxima or a local minima.

Also for a polygon P the function $b.\mathfrak{x}(s)$ has an even number of extrema; with equal number of maxima and minima. Therefore, if S_b^{n-2} has no vertex of Q then the number of intersections of Q with S_b^{n-2} is the number of extrema which is equal to $2\mu(P,b)$. The set of points b for which S_b^{n-2} contains some vertex of Q is the union of $S_{b_i}^{n-2}$. In each component of $S^{n-1}\setminus\bigcup_i S_{b_i}^{n-2}$ the function $2\mu(P,b)$ is constant. This follows since, $S_{b_i}^{n-2}=\{x\in S^{n-1}|x.b_i=0\}$. $S^{n-1}\setminus\bigcup_i S_{b_i}^{n-2}=\{x\in S^{n-1}|x.b_i\neq 0,\forall i\}$. Therefore the components of this space are $\{x\in S^{n-1}|x.b_1>0,x.b_2>0,..,x.b_{m-1}>0\}$, $\{x\in S^{n-1}|x.b_1>0,x.b_2>0,...,x.b_{m-1}<0\}$, .. etc. Let $x_1\in\{x\in S^{n-1}|x.b_1>0,x.b_2>0,..,x.b_{m-1}>0\}$ from which we get the inequalities $x_1.a_2>x_1.a_1$, $x_1.a_3>x_1.a_2$, .. etc. Hence the number of maxima of $x_1.\mathfrak{x}(s)$ is determined. Here the choice of x_1 was arbitrary and hence the number of maxima is independent of x_1, hence $\mu(P,x)$ is constant in the component $\{x\in S^{n-1}|x.b_1>0,x.b_2>0,..,x.b_{m-1}>0\}$. Hence the integral $\int_{S^{n-1}}2\mu(P,b)dS$ is also defined.

The set of points b for which S_b^{n-2} meets the arc $b_{i-1}b_i$ of length $0\leq\alpha\leq\pi$ is a 'double lune' bounded by $S_{b_{i-1}}^{n-2}$ and $S_{b_i}^{n-2}$ as shown in the figure 3.6.

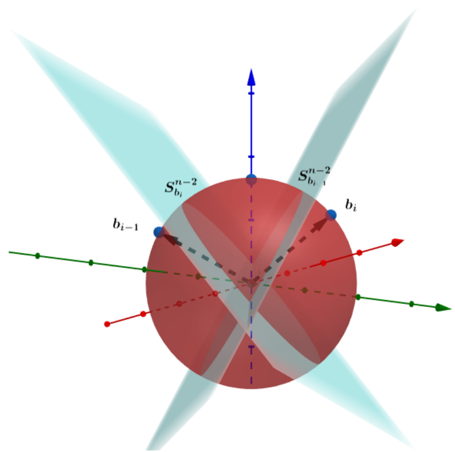

Figure 3.6: Double lune bounded by S_{b_i} and $S_{b_{i-1}}$

The contribution of $b_{i-1}b_i$ to $2\mu(P,b)$ is 1 if b is inside the double lune and 0 outside. The measure of this lune is $\frac{2\alpha_i M_{n-1}}{2\pi}$. Hence we have

$$\int_{S^{n-1}} 2\mu(P,b)dS = \frac{M_{n-1}}{\pi}\sum_{i=1}^{m}\alpha_i = \frac{M_{n-1}K(P)}{\pi}$$

For the curve C, let $\{P_m\}$ be the set of inscribed polygons $\mathfrak{x}_m(s)$ with the vertices $a_1^m = \mathfrak{x}(s_1^m)$, .., $a_m^m = \mathfrak{x}(s_m^m)$ such that each P_m contains vertices of P_{m-1}, satisfying $\lim_{m\to\infty} K\left(P_m\right) = K(C)$ and $\lim_{m\to\infty}\max_i\left\{\left(s_{i+1}^m - s_i^m\right)\right\} = 0$.

We first show that $\mu(C,b) = \lim_{m\to\infty}\mu(P_m,b)$. First note the inequality $\mu(C,b) \geq \mu(P_m,b) \geq \mu(P_{m-1},b)$. $\mu(P_m,b) \geq \mu(P_{m-1},b)$ can be obtained from the figure 3.7. Let a_r be a point of local maxima of $b.\mathfrak{x}(s)$. Where $\mathfrak{x}(s)$ represents the polygon P_{m-1}. Either $b.a_m \leq b.a_r$ or $b.a_r \leq b.a_m$. In the first case a_r remains a point of local maxima and in the second case the point of local maxima shifts from a_r to a_m, hence the number of maxima cannot decrease.

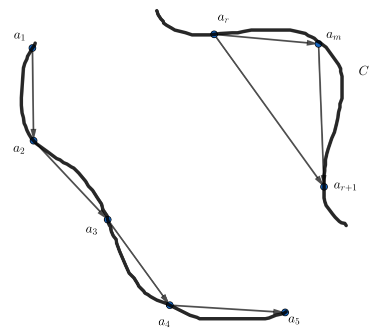

Figure 3.7: Curve C with inscribed polygons P_m and P_{m-1}

The inequality $\mu(C, b) \geq \mu(P_m, b)$ follows by observing that if a_r is a point of local maxima for the polygon then there exists a point of local maxima for the curve between $a_{r-1}a_{r+1}$. If $\mu(C, b) < \infty$, it is possible to select a neighbourhood of each $\mu(C, b)$ maxima of $b.\mathfrak{x}(s)$ and minima sufficiently small so that polygon with a vertex in each of these neighbourhoods must have at least $\mu(C, b)$ maxima; indeed this is true for P_m for large enough m due to the way in which sequence of polygons P_m are selected. If $\mu(C, b) = \infty$, values of s for which $b.\mathfrak{x}(s)$ is maximum must contain denumerable subset $\{s_{2i}\}$ such that $s_0 < s_2 < ... \lim_{i\to\infty} s_{2i} < s_0 + l$ or $s_0 > s_2 > .. > \lim_{i\to\infty} s_{2i} > s_0 - l$. In either case we can select s_{2i+1} such that $b.\mathfrak{x}(s_{2i}) > b.\mathfrak{x}(s_{2i+1})$ and s_{2i-1} such that $b.\mathfrak{x}(s_{2i}) > b.\mathfrak{x}(s_{2i-1})$. Given $2j < \infty$, select neighbourhoods of $\mathfrak{x}(s)$ for $i < 2j$, small enough such that any polygon with at least one vertex in each neighbourhood has atleast $j - 1$ maxima, which is true for large m. therefore $\mu(P_m, b)$ increases without bounds as $m \to \infty$. Hence in both cases $\mu(C, b) = \lim_{m\to\infty} \mu(P_m, b)$.

Since the integral $\int_{S^{n-1}} \mu(P_m, b)dS$ exists and nondecreasing sequence of positive functions $\mu(P_m, b)$ approaches $\mu(C, b)$. The integral $\int_{S^{n-1}} \mu(C, b)dS$ exists and equals

$$\lim_{m\to\infty} \int_{S^{n-1}} \mu(P_m, b)dS = \lim_{m\to\infty} \frac{(M_{n-1})}{2\pi} K\left(P_m\right) = \frac{(M_{n-1})}{2\pi} K(C).$$

□

Corollary 3.0.11. *$K(C) \geq 2\pi\mu(C)$, where $\mu(C)$ is the crookedness of the closed curve C.*

Proof. Since

$$\frac{M_{n-1}K(C)}{2\pi} = \int_{S^{n-1}} \mu(C, b)dS \geq \int_{S^{n-1}} \mu(C)dS = M_{n-1}\mu(C).$$

The result follows from this inequality. □

Definition 3.0.12 (*Convex curve*)**.** A closed convex curve is a plane closed curve described by $\mathfrak{x}(s)$, such that any line intersects $\mathfrak{x}(s)$ either for not more than two values of s within the fundamental period or contains $\mathfrak{x}(s)$ for all values of s within some interval.

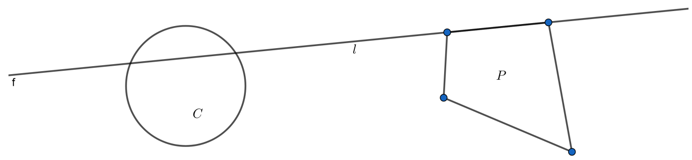

Figure 3.8: Convex curves C and P

Lemma 3.0.13. *The necessary and sufficient condition that a closed polygon P in R^2 be convex is that for every b either $\mu(P, b) = 1$ or $\mu(P, b) = \infty$.*

Theorem 3.0.14. *For any closed curve C in R^n, $K(C) \geq 2\pi$. The equality holds if and only if C is convex.*

Proof. Given a curve C choose an inscribed triangle P in C. Hence $\mu(C, b) \geq \mu(P, b) \geq \min_b\{\mu(P, b)\} = 1$. Therefore $\mu(C) \geq 1$ for any closed curve C. Therefore by the corollary3.0.11 the result follows.

Note that any curve which is not convex has an inscribed polygon which is also not convex. Also a polygon inscribed in a convex curve must be convex. Since if the polygon inscribed in the convex curve is non convex, i.e if there exists a line which intersects the polygon at more than two points, then this line will also intersect the curve at more than two points (figure3.9)

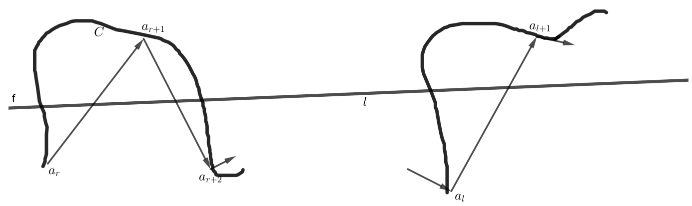

Figure 3.9: Curve C with a non convex polygon P

It only needs to prove the second part of the theorem for polygons. Let us assume that the second part of the theorem is true for polygons. If $K(C) = 2\pi$ and C is non convex curve, then there exists a non convex polygon P inscribed in C. Hence $K(P) < K(C) = 2\pi$. Here inequality is strict, since P is non convex. This is a contradiction to the first part of the theorem. Hence C has to be a convex curve.

Similarly let C be a convex curve, then by the definition of $K(C)$ and since all the inscribed polygons in C are convex, we have that $K(C) = 2\pi$.

To prove the result for polygons we note that it is proved in plane geometry that the sum of exterior angles of a convex polygon is 2π. Also if P is a non planar polygon with $K(P) = 2\pi$, then select four consecutive non coplanar vertices. Hence we can form a polygon P' with these vertices (figure3.10) such that $K(P') < K(P) = 2\pi$ which is impossible due to first part of our theorem.

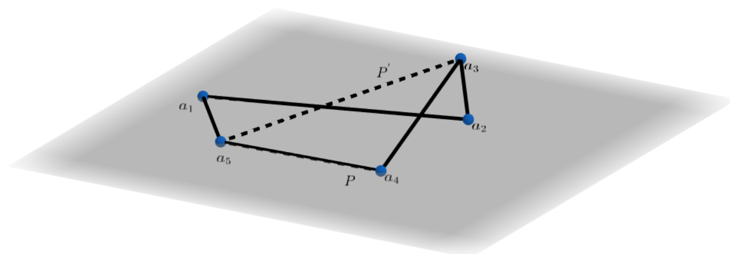

Figure 3.10: Polygon P and inscribed polygon P'

Similarly if there were a non convex planar polygon P such that $K(P) = 2\pi$ then by the lemma 3.0.13 there exists b such that $1 < \mu(P, b) < \infty$, but there exists a neighbourhood around b such that $\mu(P, b)$ is constant. Hence $K(P) = \int_{S^1} \mu(P, b) dS > \int_{S^1} dS = 2\pi$, which is again impossible. Hence for a non convex polygon $K(P) \neq 2\pi$, therefore we have our result.

$\square$

3.0.3 The curvature and crookedness of isotopy types of curves

Definition 3.0.15. An **isotopy** of R^n onto itself is a continuous map $H : R^n \times I \to R^n$ such that for each fixed t, $H(x, t) : R^n \to R^n$ is an embedding onto R^n. Also $H(x, 0) = x$ for all x in R^n.

Definition 3.0.16. A closed curve $\mathfrak{x}(s)$ of period l, and a closed curve given by $\bar{\mathfrak{x}}(s)$ of period $\bar{l}$ are said to be **equivalent by isotopy** if there is an isotopy

$H : R^n \times I \to R^n$, such that

$$H(\mathfrak{r}(ul), 0) = \mathfrak{r}(ul) \quad \text{and} \quad H(\mathfrak{r}(ul), 1) = \bar{\mathfrak{r}}(u\bar{l})$$

for all u.

Equivalence by isotopy between closed curves gives an equivalence relation. An equivalence class of closed curves is called a **curve type** and is denoted by the symbol $\mathfrak{C}$.

Definition 3.0.17. A curve type $\mathfrak{C}$ is called **simple** if the closed curves in $\mathfrak{C}$ are simple.

It should be noted that if any element of $\mathfrak{C}$ is simple, then by the definition of isotopy the curve type $\mathfrak{C}$ is also simple.

Definition 3.0.18. A simple curve type $\mathfrak{C}$ and its members are called **unknotted** if $\mathfrak{C}$ contains all circles. If a simple curve type $\mathfrak{C}$ contains no circle then it is called a **knotted** curve type.

Definition 3.0.19. A curve type $\mathfrak{C}$ and its members are **tame** if $\mathfrak{C}$ contains a polygon. Otherwise it is said to be **wild**.

Example 3.0.20. In R^2 every simple closed curve C is unknotted.
For $n > 3$, every simple tame curve is unknotted.

Lemma 3.0.21. *For each c and p in R^{n-1} such that $|c - p| < r$, there is an isotopy, $f_u^{cp}(x)$, $0 \le u \le 1$, of R^{n-1} onto R^{n-1} which transforms c into p and leaves fixed all the points of R^{n-1} outside the $n-2$ circle of radius r and center c, such that $f_u^{cp}(x)$ is a continuous function of u, r, c and p.*

Proof.

$$f_u^{cp}(x) = \begin{cases} x + u\left[1 - \frac{|x-c|}{r}\right](p-c) & |x-c| \leq r \\ x & |x-c| \geq r \end{cases}$$

□

Definition 3.0.22. For a curve type $\mathfrak{C}$, define $K(\mathfrak{C}) = g.l.b\{K(C)\}$ and $\mu(\mathfrak{C}) = min\{\mu(C)\}$ where C ranges over all members of the curve type.

Theorem 3.0.23. *For any simple closed curve C, such that $\mu(C) < \infty$, there exists a polygon P inscribed in C which is equivalent to C by isotopy.*

Proof. Let the curve be a triangle P on R^2(figure 3.11), note that $\mu(P) < \infty$. Let the triangle formed by the points a_1, $a_2^{'}$ and a_3 be denoted as $P^{'}$. Let b be a unit vector along y axis. Then it is clear that $a_1 = \mathfrak{x}(t_1)$ is the point of local maxima and $a_3 = \mathfrak{x}(t_2)$ is the point of local minima of the function $b.\mathfrak{x}(s)$, where $\mathfrak{x}(s)$ is the polygon P. About each extrema draw a line $Z_i^1(0)$ along b such that it intersects the polygon at two points $\mathfrak{x}(t_i^-)$ and $\mathfrak{x}(t_i^+)$, where $t_i^- \leq t_i \leq t_i^+$, for i=1,2. Note that in the figure 3.11 the point $t_1^+ = t_1$ and $t_2^+ = t_2 = t_2^-$. It is clear that there exists an isotopy which transforms the polygon P into the polygon $P^{'}$, such that the points $\mathfrak{x}(s)$ for $t_1^- \leq t_1 \leq t_1^+$ are transformed into the points on the line $Z_1^1(0)$ as depicted by arrows u, v, w in the figure.

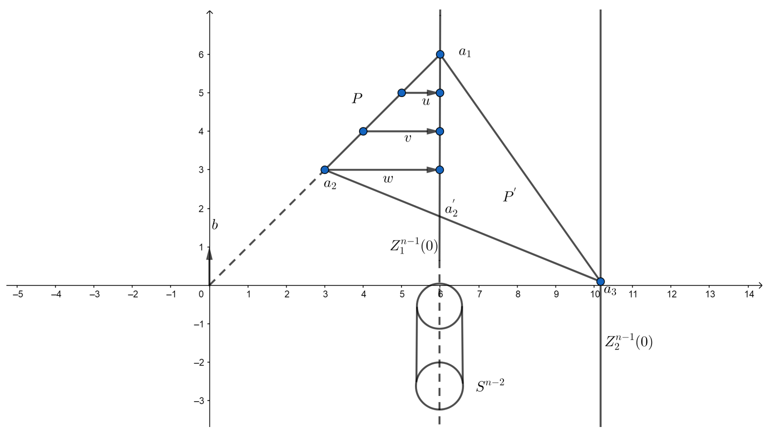

Figure 3.11: The polygon P and the equivalent polygon P'

Hence P' is the required inscribed polygon equivalent to P. A similar argument along the lines of this proof can be given for any curve C in R^n. Here the lines $Z_i^1(0)$ should be replaced by cylinders $Z_i^{n-1}(0)$ with the points $\mathfrak{x}(t_i^+)$ and $\mathfrak{x}(t_i^-)$ lying on a base of the cylinder.

□

Corollary 3.0.24. *The necessary and sufficient condition that a simple curve type $\mathfrak{C}$ is tame is that $\mu(\mathfrak{C}) < \infty$.*

Proof. If $\mu(\mathfrak{C}) < \infty$, then there exists a curve C in $\mathfrak{C}$ such that $\mu(C) < \infty$. Hence by the previous theorem there exists a polygon P inscribed in C such that P belongs to $\mathfrak{C}$. Hence $\mathfrak{C}$ is a tame curve type.
Conversely, if $\mathfrak{C}$ is a tame curve type then there exists a polygon P in $\mathfrak{C}$. Since $\mu(P) < \infty$ and by the definition of $\mu(\mathfrak{C})$ we have our result. □

Corollary 3.0.25. *The total curvature of a tame knot cannot equal the curvature of its type.*

Proof. Assume that C is a tame knot of type $\mathfrak{C}$ such that $K(C) = K(\mathfrak{C})$. Note that $K(\mathfrak{C}) = K(C) \geq 2\pi\mu(C)$. Since $K(\mathfrak{C}) = g.l.b\{K(C)\}$ and $\mathfrak{C}$ has a polygon P (C is a tame curve), $K(\mathfrak{C}) < \infty$. Therefore $\mu(C) < \infty$. By the previous theorem there exists a polygon P inscribed in C such that P is of type $\mathfrak{C}$. Since $\mathfrak{C}$ is knotted, polygon P cannot lie on any plane. We select four consecutive non planar vertices (by ignoring vertices with $\alpha_i = 0$). It is possible to select a new poygon $P^{'}$ inscribed in P such that it is of type $\mathfrak{C}$, and $K(P^{'}) < K(P) \leq K(C) = K(\mathfrak{C})$; which is impossible by the definition of $K(\mathfrak{C})$. □

Corollary 3.0.26. *The crookedness of any knot is greater than or equal to 2. Hence the curvature of any knot is greater than 4π.*

Proof. If C is a closed curve such that $\mu(C) = 1$, then by the proof of the theorem 3.0.23 we can show that C is isotopic to a plane quadrilateral, which is unknotted. Therefore $\mu(C) \geq 2$. Since $K(C) \geq 2\pi\mu(C)$ we have that $K(C) \geq 4\pi$. □

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