MF IN FINTECH

谢泽健
11810105

HW3

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Q.1 Prove by induction that, for any integer n ≥ 2:

Let $P(n)$ be:

$$\prod_{i=2}^{n} 1 - \frac{1}{i^2} = \frac{n+1}{2\,n} \tag{1}$$

Basis step: $P(2)$ is true because:

$$1 - \frac{1}{2^2} = \frac{3}{4} = \frac{2+1}{2\times 2} = \frac{n+1}{2\,n} \tag{2}$$

Inductive step: assume $P(k)$ is true:

$$\prod_{i=2}^{k} 1 - \frac{1}{i^2} = \frac{k+1}{2\,k} \tag{3}$$

Then for $P(k+1)$:

$$\prod_{i=2}^{k+1} 1 - \frac{1}{i^2} = \left(\prod_{i=2}^{k} 1 - \frac{1}{i^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2\,k}\,\frac{(k+2)\,k}{(k+1)^2} = \frac{k+2}{2\,(k+1)} \tag{4}$$

is also true.

Q.2 Prove by induction that, for any sets A1, A2, · · · , An, De Morgan's law can be generalized to

Let $P(n)$ be:

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i} \tag{5}$$

Basis step: $P(2)$ is true because, from De morgan's law

$$\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2} \tag{6}$$

Inductive step: assume $P(k)$ is true:

$$\overline{\bigcup_{i=1}^{k} A_i} = \bigcap_{i=1}^{k} \overline{A_i} \tag{7}$$

for $P(k+1)$:

$$\overline{\bigcup_{i=1}^{k+1} A_i} = \overline{\bigcup_{i=1}^{k} A_i \cup A_{k+1}} = \overline{\bigcup_{i=1}^{k+1} A_i} \cap \overline{A_{k+1}} = \left(\bigcap_{i=1}^{k} \overline{A_i}\right) \cap (\overline{A_{k+1}}) = \bigcap_{i=1}^{k+1} \overline{A_i} \tag{8}$$

is also true. It's trivial that $P(1)$ is true, so $\forall\, n\, P(n)$.

Q.3 Use induction to prove that 3 divides n^3 + 2n whenever n is a positive integer.

Let $P(n)$ be:

$$3 \mid n^3 + 2\,n \tag{9}$$

Basis step: $P(1)$ is true because:

$$n^3 + 2\,n = 3 \tag{10}$$

Inductive step: assume $P(k)$ is true:

$$3 \,\middle|\, k^3 + 2\,k \tag{11}$$

Then for $P(k+1)$:

$$(k+1)^3 + 2\,(k+1) = 3 + 5\,k + 3\,k^2 + k^3 = \left(k^3 + 2\,k\right) + 3\,k^2 + 3\,k + 3 \tag{12}$$

3 divides each term, so $P(k+1)$ is true.

Q.4 Let x ∈ R and x 6= 1. Using mathematical induction, prove that for all integers n ≥ 0,

Let $P(n)$ be:

$$\sum_{i=0}^{n} x^i = \left(x^{n+1} - 1\right) / (x - 1) \tag{13}$$

Basis step: $P(0)$ is true because:

$$1 = \frac{x-1}{x-1} \tag{14}$$

Inductive step: assume $P(k)$ is true:

$$\sum_{i=0}^{k} x^i = \left(x^{k+1} - 1\right) / (x - 1) \tag{15}$$

Then for $P(k+1)$:

$$\sum_{i=0}^{k+1} x^i = \left(x^{k+1} - 1\right) / (x - 1) + x^{k+1} = \left(x^{k+1} - 1 + x^{k+1}(x-1)\right) / (x - 1) = \left(x^{k+2} - 1\right) / (x - 1) \tag{16}$$

is also true.

Q.5 Prove that if h > −1, then $1 + \mathrm{nh} \le (1 + h)\, n$ *for all nonnegative integers. his is called Bernoulli's inequality.*

Let $P(n)$ be:

$$\forall\, h > -1,\;\; 1 + n\,h \le (1+h)^n \tag{17}$$

Basis step: $P(0)$ is true because:

$$1 \le (1+h)^0 = 1 \tag{18}$$

Inductive step: assume $P(k)$ is true:

$$\forall\, h > -1,\;\; 1 + k\,h \le (1+h)^n \tag{19}$$

Then for $P(k+1)$:

$$1 + k\,h + h \le (1+h)^k\,(1+h) = (1+h)^k + h(1+h)^k \tag{20}$$

consider inequality:

$$h \le h(1+h)^k \tag{21}$$

when $h \le 0$

$$(1+h)^k \le 1 \Rightarrow h \le h(1+h)^k \tag{22}$$

when $h > 0$:

$$(1+h)^k > 1 \Rightarrow h \le h(1+h)^k \tag{23}$$

Hence $P(k+1)$ is also true.

Q.6 Suppose that a and b are real numbers with 0 < b < a. Prove that if n is a positive integer, then an − bn ≤ nan− 1(a − b).

Consider function:

$$f(x) = x^n \tag{24}$$

for $n \geq 1$:

$$f'(x) = n x^{n-1} > 0 \tag{25}$$

$$f''(x) = n(n-1)\, x^{n-2} > 0 \tag{26}$$

by mean value theorem, $\exists\, \xi \in (b, a)$, such that

$$f'(\xi) = (f(a) - f(b)) / (a - b) \tag{27}$$

since $\xi < a$, $f'(\xi) < f'(a)$

hence

$$f'(\xi) = (f(a) - f(b)) / (a - b) < f'(a) = n a^{n-1} \tag{28}$$

i.e.

$$(f(a) - f(b)) / (a - b) \leq n a^{n-1} \Rightarrow a^n - b^n \leq n a^{n-1} (a - b) \tag{29}$$

Q.7 Let P(n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for n ≥ 18.

a.

P(18) :

$$18 = (1,\ 2) \cdot (4,\ 7) \tag{30}$$

P(19):

$$19 = (3,\ 1) \cdot (4,\ 7) \tag{31}$$

P(20):

$$20 = (5,\ 0) \cdot (4,\ 7) \tag{32}$$

P(21):

$$21 = (0,\ 3) \cdot (4,\ 7) \tag{33}$$

b.

Assume $P(k)$ is true

c.

Prove $\bigwedge_{i=18}^{k} P(k) \Rightarrow P(k+1)$ is true.

d.

Assume $P(j)$ is true for all j with $18 \leq j \leq k$, where k is a fixed integer greater than or equal to 21. hence

$$P(k-3) \text{ is true} \tag{34}$$

suppose for $n = k - 3$, there is $4\,x + 7\,y = k - 3$, then

$$4\,(x+1) + 7\,y = k + 1 \tag{35}$$

so $P(k+1)$ is true. i.e.

$$\bigwedge_{i=18}^{k} P(k) \Rightarrow P(k+1) \tag{36}$$

e.

$$P(18),\ P(19),\ P(20),\ P(21) \text{ are true} \tag{37}$$

$$\text{for } k \geq 21,\ \ P(1),\ P(2) \ldots,\ P(k) \Rightarrow P(k+1) \tag{38}$$

$$\text{for } n \geq 18\ P(n) \tag{39}$$

Q.8 A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction

since both 25 and 10 are multiples of 5, let $P(n)$ be: we can form $5\,n$ using gift certificates.

we can achieve the following value of n:

$$4 = 2 \times 2 \tag{40}$$

$$5 = 5 \tag{41}$$

so $P(4) \wedge P(5)$ is true.

Assume $P(j)$ is true for all j with $4 \leq j \leq k$, where k is a fixed integer greater than or equal to 5. hence

$$P(k-1) \text{ is true} \tag{42}$$

suppose for $n = k - 1$, there is $2\,x + 5\,y = k - 1$, then

$$2\,(x+1) + 5\,y = k + 1 \tag{43}$$

so $P(k+1)$ is true. i.e.

$$\bigwedge_{i=5}^{k} P(i) \Rightarrow P(k+1) \tag{44}$$

then we conclude that

$$\forall\, n \geq 4,\ P(n) \tag{45}$$

i.e. the set of amounts we can form is

$$\{x \mid x = 5\,n(n \geq 4)\} \bigcup \{10\}$$

Q.9 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Suppose a statement hold for $n = 1$, mathematical induction shows that

$$P(k) \Rightarrow P(k+1) \tag{46}$$

while strong induction shows

$$\bigwedge_{i=1}^{k} P(i) \Rightarrow P(k+1) \tag{47}$$

Note

$$\bigwedge_{i=1}^{k} P(i) \Rightarrow P(k) \tag{48}$$

hence

$$(P(k) \Rightarrow P(k+1)) \Rightarrow \left(\bigwedge_{i=1}^{k} P(i) \Rightarrow P(k+1) \right) \tag{49}$$

Then we prove:

$$\left(\bigwedge_{i=1}^{k} P(i) \Rightarrow P(k+1) \right) \Rightarrow (P(k) \Rightarrow P(k+1)) \tag{50}$$

since

$$\left(\bigwedge_{i=1}^{k-1} P(i) \Rightarrow P(k) \right) \tag{51}$$

if $P(k)$ is true, it follows $\bigwedge_{i=1}^{k-1} P(i)$ is true, hence

$$P(k) \Rightarrow \bigwedge_{i=1}^{k-1} P(i) \Rightarrow \bigwedge_{i=1}^{k} P(i) \Rightarrow P(k+1) \tag{52}$$

which is we desired.

Q.10

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procedure result(a,n)
if n=0 then return a
else return result(a,n-1)*result(a,n-1)
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Q.11 Suppose that the function f satisfies the recurrence relation f(n) =2f($\sqrt{n}$) + log n whenever n is a perfect square greater than 1 and f(2) = 1

a.

$$f(16) = 2\,f(4) + \ln 16 = 2\,(2\,f(2) + \ln 4) + \ln 16 = 4 + 8 \ln 2 \tag{53}$$

b.

suppose $n = 2^{(2^a)}$,

$$f\left(2^{2^a}\right) = 2^a + \ln 2^{2^a} + 2 \ln 2^{2^{a-1}} + 2^2 \left(\ln 2^{2^{a-2}}\right) + \ldots + 2^{a-1} \left(\ln 2^2\right) = a 2^a (\ln 2) + 2^a \tag{54}$$

since

$$a = \ln(\ln n - \ln 2)/\ln 2 \tag{55}$$

hence

$$f(n) = (\ln n(1 + \ln(\ln n - \ln 2))/(\ln 2) = O(\ln x) \tag{56}$$

Q.12 Find f(n) when n = 4k, where f satisfies the recurrence relation f(n) =5f(n/4) + 6n, with f(1) = 1.

Suppose sequence $\{a_n\}$, where

$$\{a_n = f(4^n)\} \tag{57}$$

then we have:

$$a_n = 5\,a_{n-1} + 6 \times 4^n \tag{58}$$

Note:

$$a_n + 6 \times 4^{n+1} = 5\,a_{n-1} + 30 \times 4^n = 5\,(a_{n-1} + 6 \times 4^n) \tag{59}$$

since $a_0 = 1$,

$$a_n + 6 \times 4^{n+1} = 5^{n+2} \tag{60}$$

hence:

$$a_n = 5^{n+2} - 3 \times 2^{2\,n+3} \tag{61}$$

$$f\left(4^k\right) = 5^{k+2} - 3 \times 2^{2\,k+3} \tag{62}$$

Q.13 Find f(n) when n = 2k, where f satisfies the recurrence relation f(n) =8f(n/2) + n2 with f(1) =

Suppose sequence $\{a_n\}$, where

$$\{a_n = f(2^n)\} \tag{63}$$

then we have:

$$a_n = 8\,a_{n-1} + 4^n \tag{64}$$

Note:

$$a_n + 4^n = 8\, a_{n-1} + 2\times 4^n = 8\left(a_{n-1} + 4^{n-1}\right) \tag{65}$$

since $a_0 = 1$,

$$a_n + 4^n = 5\times 8^{n-1} \tag{66}$$

hence:

$$f\left(2^k\right) = 5\times 8^{k-1} - 4^k \tag{67}$$

Q.14 The running time of an algorithm A is described by the following recurrence relation:

a.

Let $a_k = S\left(2^k\right)$

$$a_k = 9\, a_{k-1} + 4^k \tag{68}$$

since $a_0 = b$:

$$a_k = \frac{1}{5}\left(-4^{1+k} + 4\times 9^k + 5\times 9^k\, b\right) \tag{69}$$

assume $n = 2^k$, i.e. $k = \log_2 n$:

$$S(n) = b\, 9^{\frac{\log(n)}{\log(2)}} - \frac{4}{5}\left(4^{\frac{\log(n)}{\log(2)}} - 9^{\frac{\log(n)}{\log(2)}}\right) \tag{70}$$

b.

Let $x_k = S\left(4^k\right)$

$$x_k = a\, x_{k-1} + 16^k \tag{71}$$

since $a_0 = c$:

$$a_k = \left(-16^{1+k} + 16\, a^k - 16\, a^k\, c + a^{1+k}\, c\right) \Big/ (-16 + a) \tag{72}$$

assume $n = 4^k$, i.e. $k = \log_4 n$:

$$T(n) = \left(-16\,(c-1)\, a^{\frac{\log(n)}{\log(4)}} + c\, a^{\frac{\log(4\, n)}{\log(4)}} - 16^{\frac{\log(4\, n)}{\log(4)}}\right) \Big/ (a - 16) \tag{73}$$

c.

From master theorem:

$$n^2 = O\left(n^{\log_2 9 - \epsilon}\right) \tag{74}$$

$$n^2 = O\left(n^{\log_4 a - \epsilon}\right)(a > 16) \tag{75}$$

hence

$$S(n) = \Theta\left(n^{\log_2 9}\right) \tag{76}$$

$$T(n) = \Theta\left(n^{\log_4 a}\right) \tag{77}$$

if

$$T(n) = O(S(n)) \Rightarrow \log_4 a \le \log_2 9 \tag{78}$$

i.e.

$$16 < a \le 81 \tag{79}$$