MF IN FINTECH

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HW4

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Q.1 Suppose that n ≥ 1 is an integer

a.

For any element in $\{1, 2, \cdots, n\}$, there are 3 possible mappings. Hence there are 3^n functions.

b.

$$\begin{cases} 3 & n = 1 \\ 6 & n = 2 \\ 6 & n = 3 \\ 0 & n > 3 \end{cases} \tag{1}$$

c.

When $n < 3$, there are no onto functions

When $n \geq 3$, there are $3^n - 3\times 2^n - 3$ onto functions

Q.2 How many functions are there from the set {1, 2, . . . , n}, where n is a positive integer, to the set {0, 1}

a.

$$\begin{cases} 2 & n = 1 \\ 2 & n = 2 \\ 0 & n > 2 \end{cases} \tag{2}$$

b

For any element in $\{2, \cdots, n-1\}$, there are 2 possible mappings. Hence there are 2^{n-2} functions.

c.

Select a integers less than n has $n-1$ possibility. So there are $n-1$ this kind functions

Q.3 Suppose that p and q are prime numbers and that n = pq. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding n that are relatively prime to n, i.e., the Euler function $\varphi(n)$.

Suppose $E(a)$ is the positive integers that are relatively prime to a and less than n.

For $E(p)$, there are only $p,\ 2p,\ 3\,p \ldots n$ not in $E(p)$

$$|\,(E(p)\,| = n - q \tag{3}$$

Similarly

$$|\,(E(q)\,| = n - p \tag{4}$$

There are only n not in $E(p) \bigcup E(q)$, hence

$$\left| E(p) \bigcup E(q) \right| = n - 1 \tag{5}$$

Then we conclude that

$$\phi(n) = (n-q) + (n-p) - (n-1) = (p-1)\,(q-1) \tag{6}$$

Q.4 How many bit strings of length 6 have at least one of the following properties:

The number of the bit strings start with 010 is:

$$2^3 = 8 \tag{7}$$

The number of the bit strings start with 11 is:

$$2^4 = 16 \tag{8}$$

The number of the bit strings end with 00 is:

$$2^4 = 16 \tag{9}$$

The number of the bit strings start with 010 and start with 11 is:

$$0 \tag{10}$$

The number of the bit strings start with 010 and end with 00 is:

$$2 \tag{11}$$

The number of the bit strings start with 11 and end with 00 is:

$$2^2 = 4 \tag{12}$$

hence

$$8 + 16 + 16 - 0 - 2 - 4 + 0 = 34 \tag{13}$$

Q.5 Consider all permutations of the letters A, B, C, D, E, F, G

a.

Permutations of (ABC), (DE), F, G are:

$$P(4,\ 4) = 24 \tag{14}$$

b.

The number of permutations that A precede B is equal to B precede A, and either A precede B or B precede A:

$$P(7,\ 7)/2 = 2520 \tag{15}$$

Q.6 Alice is going to choose a selection of 12 chocolates. There are 25 different brands of them and she can have as many as she wants of each brand (but can only choose 12 pieces). How many ways can she make this selection?

$$\binom{12 + 25 - 1}{12} = 1\,251\,677\,700 \tag{16}$$

Q.7 16 points are chosen inside a 5×3 rectangle. Prove that two of these points lie within $\sqrt{2}$ of each other.

Divide the box into 15 1×1 square, there are at least 1 square contains 2 points, the two points lie within $\sqrt{2}$ of each other.

Q.8 Let (xi, yi), i = 1, 2, 3, 4, 5, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integers coordinates

Classify the points with integer coordinates by (odd, even), (even, odd), (even, even), (odd, odd), there are 2 points in the same set, their midpoint has integer coordinates.

Q.9 Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Firstly assume A knows B then B knows A.

For any people, the number of people he may knows is from 1 to $n - 1$, but there are n people, so there are two people know the same number of people.

Q.10 Show that if p is a prime and k is an integer such that $1 \leq k \leq p - 1$, then p divides pk.

Note

$$\binom{p}{k} = \frac{p}{k}\binom{p-1}{k-1} \tag{17}$$

hence

$$p \,\Big|\, k \times \binom{p}{k} \tag{18}$$

since GCD(p, k) = 1, then

$$p \,\Big|\, \binom{p}{k} \tag{19}$$

Q.11 Prove the hockeystick identit

a.

Select r element from $n + r + 1$ elements set $\{a_1, a_2, a_3, \cdots a_n\}$, there are $\binom{n+r+1}{r}$ ways.

exclude a_1, there are $\binom{n+r}{r}$ ways, exclude a_2 but include a_1, there are $\binom{n+r-1}{r-1}$ ways, exclude a_3 but include a_1 and a_2, there are $\binom{n+r-2}{r-2}$ ways, and so on, until exclude a_{r+1}.

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r} \tag{20}$$

b.

Note

$$\binom{n+r+1}{r} = \binom{n+r}{r} + \binom{n+r}{r-1} \tag{21}$$

Then

$$\binom{n+r}{r-1} = \binom{n+r-1}{r-1} + \binom{n+r-1}{r-2} \tag{22}$$

$$\binom{n+r-1}{r-2} = \binom{n+r-2}{r-2} + \binom{n+r-2}{r-3} \tag{23}$$

And so on, until

$$\binom{n+1}{0} = \binom{n}{0} \tag{24}$$

Hence

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r} \tag{25}$$

Q.12 Solve the recurrence relation

$$a_n = -2\, a_{n-3} + a_{n-2} + 2\, a_{n-1} \tag{26}$$

its characteristic equation is

$$x^3 = -2 + x + 2\, x^2 \tag{27}$$

the solution is

$$\left(\begin{array}{c} r_1 = -1 \\ r_2 = 1 \\ r_3 = 2 \end{array} \right) \tag{28}$$

hence

$$a_n = \alpha(-1)^n + \beta + \gamma 2^n \tag{29}$$

We know that

$$\begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_3 = 7 \end{cases} \tag{30}$$

hence

$$a_n = \frac{1}{2}\left(-5 + 3\,(-1)^n + 2^{2+n}\right) \tag{31}$$

Q.13 Solve the recurrence relation an = 4an−2, with initial conditions a0 = 3, a1 = 2.

$$a_n = 4\,a_{n-2} \tag{32}$$

its characteristic equation is

$$x^2 = 4 \tag{33}$$

the solution is

$$\begin{cases} r_1 = -2 \\ r_2 = 2 \end{cases} \tag{34}$$

hence

$$a_n = \alpha(-2)^n + \beta 2^n \tag{35}$$

We know that

$$\begin{cases} a_0 = 3 \\ a_1 = 2 \end{cases} \tag{36}$$

hence

$$a_n = (-2)^n + 2^{1+n} \tag{37}$$

Q.14
(a) Find all solutions of the recurrence relation an = 2an−1 + 2n2.
(b) Find the solution of the recurrence relation in part (a) with initial condition a1 = 4.

a.

Consider recurrence relation

$$a_n = 2\,a_{n-1} \tag{38}$$

its characteristic equation is

$$x = 2 \tag{39}$$

the solution is

$$r = 2 \tag{40}$$

hence

$$a_n = \alpha 2^n \tag{41}$$

A particular solution of this recurrence relation is

$$a_n = -2\,n^2 - 8\,n - 12 \tag{42}$$

So all the solution is

$$a_n = -2\,n^2 - 8\,n - 12 + \alpha 2^n \tag{43}$$

b.

$$a_1 = 2\,\alpha - 22 = 4 \Rightarrow \alpha = 13 \tag{44}$$

$$a_n = -2\,n^2 - 8\,n - 12 + 13 \times 2^n \tag{45}$$

Q.15
Use generating functions to prove Pascal's identity: C(n, r) = C(n − 1, r) + C(n − 1, r − 1) when n and r are positive integers with r < n. [Hint:Use the identity (1 + x)n = (1 + x)n−1 + x(1 + x)n−1.]

Note

$$(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1} \tag{46}$$

The coefficient of x^r in $(1+x)^n$ is $C(n, r)$.

The coefficient of x^r in $(1+x)^{n-1}$ is $C(n-1, r)$.

The coefficient of x^r in $x(1+x)^{n-1}$ in $(1+x)^{n-1}$ is $C(n-1, r-1)$.

hence

$$C(n, r) = C(n-1, r) + C(n-1, r-1) \tag{47}$$

Q.16 How many relations are there on a set with n elements that are

a.

Consider the set contains all element pair, any subset of which can create a symmetric relations:

$$2^{\binom{n}{2}+n} = 2^{\frac{n(n+1)}{2}} \tag{48}$$

b.

The antisymmetric relations contain any number of

$$(1, 1), (2, 2) \ldots.(n, n) \tag{49}$$

For any element pair

$$(a, b), (b, a) \tag{50}$$

A antisymmetric can contain at most one of them. Hence, there are

$$2^n 3^{\binom{n}{2}} = 2^n 3^{\frac{n(n-1)}{2}} \tag{51}$$

c.

This kind of relations contains any number of element pair

$$(a, b)$$

$$2^{2\times\binom{n}{2}} = 2^{n(n-1)} \tag{52}$$

d.

For any element pair

$$(a, b), (b, a) \tag{53}$$

This kind of relations contain exactly one of them. Hence, there are

$$2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}} \tag{54}$$

e.

$$\text{neither reflexive nor irreflexive} = \overline{\text{reflexive}} \cap \overline{\text{irreflexive}} = \overline{\text{reflexive} \cup \text{irreflexive}} = 2^{n^2} - 2^{n(n-1)} - 2^{n(n-1)} = 2^{n^2} - 2^{n(n-1)+1} \tag{55}$$

Q.17 Suppose that the relation R is symmetric. Show that R∗is symmetric.

Consider any Integer n, prove that R^n is symmetric

Assume

$$\forall R^i (i \le m)\, R^i \text{ is symmetric} \tag{56}$$

For any a and b, such that

$$a\, R^{m+1}\, b \tag{57}$$

There exist i and j smaller than m, and exist c, such that

$$aR^i c,\ cR^j b\ (i + j = m + 1) \tag{58}$$

We know that R^i and R^j are symmetric, hence

$$cR^i a,\ bR^j c\ (i + j = m + 1) \Rightarrow b\, R^{m+1}\, c \tag{59}$$

so R^{m+1} is also symmetric.

Then we prove R^2 is symmetric:

For any a and b, such that

$$aR^2\, b \tag{60}$$

There exist some c

$$aRc,\ cRb \tag{61}$$

then

$$cRa,\ bRc \Rightarrow bR^2\, a \tag{62}$$

Then for any integer i

$$R^i \text{ is symmetric} \tag{63}$$

Note

$$R^* = \bigcup_{i=1}^{\infty} R^i$$

hence R^* is symmetric

Q.18 Suppose that the relation R is irreflexive. Is the relation R2 necessarily irreflexive?

No

Suppose

$$R = \{(1,\ 2),\ (2,\ 1)\} \tag{64}$$

Then

$$R^2 = \{(1,\ 1)\} \tag{65}$$

we can that R is irreflexive but R^2 isn't irreflexive

Q.19 Let R be the relation on the set of ordered pairs of positive integers such that ((a, b),(c, d)) ∈ R if and only if ad = bc. Show that R is an equivalence relation

$$ad = bc \Leftrightarrow \frac{a}{b} = \frac{c}{d} \tag{66}$$

R is reflexive:

$$\frac{a}{b} = \frac{a}{b} \Rightarrow ((a,\ b),\ (a,\ b)) \in R \tag{67}$$

R is symmetric:

$$((a,\ b),\ (c,\ d)) \in R \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow ((c,\ d),\ (a,\ b)) \in R \tag{68}$$

R is transitive:

$$((a,\ b),\ (c,\ d)) \in R \land ((c,\ d),\ (m,\ n)) \in R \Rightarrow \frac{a}{b} = \frac{c}{d} \land \frac{c}{d} = \frac{m}{n} \Rightarrow \frac{a}{b} = \frac{m}{n} \Rightarrow ((a,\ b),\ (m,\ n)) \in R \tag{69}$$